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Lévy Approximation of Impulsive Recurrent Process with Semi-Markov Switching

We consider the space $\mathbf{R}^d$ endowed with a norm $|\cdot|$ $(d \geq 1)$, and $(E, \mathcal{E})$, a *standard phase space*. Let $C_3(\mathbf{R}^d)$ be a measure-determining class of real-valued bounded functions, such that $g(u)/|u|^2 \to 0$, as $|u| \to 0$ for $g \in C_3(\mathbf{R}^d)$.

The impulsive processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$ on $\mathbf{R}^d$ in the series scheme with small series parameter $\varepsilon \to 0$, $(\varepsilon > 0)$ are defined by the sum

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \sum_{k=1}^{\nu(t/\varepsilon^2)} \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon), \quad t \geq 0.$$

For any $\varepsilon > 0$, and any sequence $z_k, k \geq 0$, of elements of $\mathbf{R}^d \times E$, the random variables $\alpha_k^\varepsilon(z_{k-1}), k \geq 1$ are supposed to be independent. Let us denote by $G_{u,x}^\varepsilon$ the distribution function of $\alpha_k^\varepsilon(x)$, that is,

$$G_{u,x}^\varepsilon(dv) := P(\alpha_k^\varepsilon(u, x) \in dv), k \geq 0, \varepsilon > 0, x \in E, u \in \mathbf{R}^d.$$

The switching semi-Markov process $x(t), t \geq 0$ on the standard phase space $(E, \mathcal{E})$, is defined by the semi-Markov kernel

$$Q(x, B, t) = P(x, B)F_x(t), x \in E, B \in \mathcal{E}, t \geq 0,$$

which defines the associated Markov renewal process $x_n, \tau_n, n \geq 0$:

$$Q(x, B, t) = P(x_{n+1} \in B, \theta_{n+1} \leq t | x_n = x) = P(x_{n+1} \in B | x_n = x)P(\theta_{n+1} \leq t | x_n = x).$$

Finally we should denote ξ_n^ε: $\xi_n^\varepsilon := \xi(\varepsilon^2 \tau_n) = \xi_0^\varepsilon + \sum_{k=1}^n \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_{k-1}^\varepsilon)$.

The Lévy approximation of Markov impulsive process is considered under the following conditions.

C1: The semi-Markov process $x(t), t \geq 0$ is uniformly ergodic with the stationary distribution

$$\pi(dx)q(x) = q\rho(dx), q(x) := 1/m(x), q := 1/m, m(x) := \mathbf{E}\theta_x = \int_0^\infty \overline{F}_x(t)dt,$$

$$m := \int_E \rho(dx)m(x), \rho(B) = \int_E \rho(dx)P(x, B), \rho(E) = 1.$$

C2: *Lévy approximation.* The family of impulsive processes $\xi^\varepsilon(t), t \geq 0$ satisfies the Lévy approximation conditions.

L1: Initial value condition: $\sup\limits_{\varepsilon > 0} E|\xi_0^\varepsilon| \leq C < \infty$.

L2: Approximation of the mean values:

$$a^\varepsilon(u;x) = \int_{\mathbf{R}^d} vG^\varepsilon_{u,x}(dv) = \varepsilon a_1(u;x) + \varepsilon^2[a(u;x) + \theta^\varepsilon_a(u;x)],$$

$$c^\varepsilon(u;x) = \int_{\mathbf{R}^d} vv^* G^\varepsilon_{u,x}(dv) = \varepsilon^2[c(u;x) + \theta^\varepsilon_c(u;x)].$$

L3: Poisson approximation condition for intensity kernel

$$G^\varepsilon_g(u;x) = \int_{\mathbf{R}^d} g(v)G^\varepsilon_{u,x}(dv) = \varepsilon^2[G_g(u;x) + \theta^\varepsilon_g(u;x)]$$

for all $g \in C_3(\mathbf{R}^d)$, and the kernel $G_g(u;x)$ is bounded for all $g \in C_3(\mathbf{R}^d)$.

Here $G_g(u;x) = \int_{\mathbf{R}^d} g(v)G_{u,x}(dv), \quad g \in C_3(\mathbf{R}^d)$.

Negligible terms satisfy the condition $\sup_{x\in E} |\theta^\varepsilon_\cdot(u;x)| \to 0, \quad \varepsilon \to 0$.

L4: *Balance condition*: $\int_E \rho(dx)a_1(u;x) = 0$.

In addition the following conditions are used:

C3: *Uniform square-integrability*: $\lim_{c\to\infty} \sup_{x\in E} \int_{|v|>c} vv^* G_{u,x}(dv) = 0$, where the kernel $G_{u,x}(dv)$ is defined on the measure determining class $C_3(\mathbf{R}^d)$ by the relation

$$\Gamma_g(u;x) = \int_{\mathbf{R}^d} g(v)\Gamma(u,dv;x), \quad g \in C_3(\mathbf{R}^d).$$

C4: *Linear growth*: there exists a positive constant L such that

$$|a(u;x)| \le L(1+|u|), \quad \text{and} \quad |c(u;x)| \le L(1+|u|^2),$$

and for any real-valued non-negative function $f(v), v \in \mathbf{R}^d$, such that $\int_{\mathbf{R}^d\setminus\{0\}}(1+f(v))|v|^2 dv < \infty$, we have $|G_{u,x}(v)| \le Lf(v)(1+|u|)$.

Theorem. *Under conditions* **C1** $-$ **C4** *the weak convergence*

$$\xi^\varepsilon(t) \Rightarrow \xi^0(t), \quad \varepsilon \to 0$$

takes place.

The limit process $\xi^0(t), t \ge 0$ is a Lévy process defined by the generator $\mathbf{L}$ *as follows*

$$\mathbf{L}\varphi(u) = (\widehat{a}(u) - \widehat{a}_0(u))\varphi'(u) + \frac{1}{2}\sigma^2(u)\varphi''(u) + \lambda(u)\int_{\mathbf{R}^d}[\varphi(u+v) - \varphi(u)]G^0_u(dv),$$

where:

$$\widehat{a}(u) = q\int_E \rho(dx)a(u;x), \widehat{a}_0(u) = \int_E vG_u(dv), G_u(dv) = q\int_E \rho(dx)G_{u,x}(dv),$$

$$\widehat{a^2_1}(u) = q\int_E \rho(dx)a^2_1(u;x), \quad \tilde{a}_1(u;x) := q(x)\int_E P(x,dy)a_1(u;x)$$

$$\sigma^2(u) = 2\int_E \pi(dx)[\tilde{a}_1(u;x)\widetilde{R}_0\tilde{a}^*_1(u;x) + \frac{1}{2}c(x)] - \widehat{a^2_1}(u), \quad \sigma^2(u) \ge 0$$

$$\lambda(u) = G_u(\mathbf{R}^d), G^0_u(dv) = G_u(dv)/\lambda(u),$$

$\widetilde{R}_0$ *is the potential operator of the embedded Markov chain.*
