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Equational maps of subdirectly-closed abstract classes of algebras

Let $i = 1, 2$, $\mathcal{L}_i$ an infinitary algebraic signature, K_i a class of $\mathcal{L}_i$-algebras and $\mathbf{C}_i$ ($\mathbf{S}_i$) the category constituted by members of K_i as objects and by (respectively, surjective) homomorphisms between them as morphisms.

An *equational map from* K_1 *to* K_2 [1] is any $e : \mathsf{K}_1 \to \mathsf{K}_2$ preserving carriers such that:

- for each $f \in \mathcal{L}_2$ of rank r such that $r > 0$ whenever $\mathcal{L}_1$ has no constant, there is an $\mathcal{L}_1$-term $t(\bar{x})$, where $\bar{x}$ is a sequence of pairwise-distinct variables of length r, such that, for every $\mathcal{A} \in \mathsf{K}_1$, $t^{\mathcal{A}} = f^{e(\mathcal{A})}$;
- in case $\mathcal{L}_1$ has no constant, it holds that, for each constant $c \in \mathcal{L}_2$, there is an $\mathcal{L}_1$-term $s(x)$ with a single variable x such that, for every $\mathcal{A} \in \mathsf{K}_1$ and all $a \in \mathcal{A}$, $s^{\mathcal{A}}(a) = c^{e(\mathcal{A})}$.

Next, K_1 and K_2 are said to be *rationally equivalent* [2] provided there are mutually-inverse equational maps from K_1 to K_2 and from K_2 to K_1.

Theorem 1 *Suppose K_1 is an abstract class closed under formation of subdirect products of non-empty systems. Then, equational maps from K_1 to K_2 are exactly object components of those functors from $\mathbf{C}_1$ ($\mathbf{S}_1$) to $\mathbf{C}_2$ (respectively, $\mathbf{S}_2$) which commute with forgetful set functor.*

Corollary 1 *Assume both K_1 and K_2 are abstract classes closed under formation of subdirect products of non-empty systems. Then, K_1 and K_2 are rationally equivalent iff there is an isofunctor between $\mathbf{C}_1$ ($\mathbf{S}_1$) and $\mathbf{C}_2$ (respectively, $\mathbf{S}_2$) commuting with forgetful set functor.*

The particular cases of Theorem 1 and Corollary 1 not involving the categories $\mathbf{S}_1$ and $\mathbf{S}_2$ are proved in [1] and [2], respectively, for hereditary multiplicative abstract classes.

Neither Theorem 1 nor Corollary 1 can be extended to multiplicative abstract classes. For instance, when K_1 is the class of all bounded distributive lattices having complement and K_2 is the variety of all Boolean algebras, the equational map from K_2 to K_1 that assigns complement-less reducts to Boolean algebras is the object component of an isofunctor between $\mathbf{C}_2$ and $\mathbf{C}_1$ commuting with forgetful set functor whereas there is no equational map from K_1 to K_2 because the only unary polynomial operations of any bounded distributive lattice are the diagonal and the constants zero and unit while the complement operation of any non-trivial Boolean algebra is neither diagonal nor constant.

[1] Felscher W. Equational maps // Contributions to Mathematical Logic. — Amsterdam: North-Holland Publishing Company, 1968. — P. 121–161.

[2] Мальцев А.И. Структурная характеристика некоторых классов алгебр // Доклады академии наук СССР. — 1958. — Т. 120. — N. 1. — С. 29–32.