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Interpolation with a Function Parameter and Hörmander Spaces

We consider the Hilbert Hörmander spaces

$$H^{\varphi}(\mathbf{R}^n) := \Big\{ w \in \mathcal{S}'(\mathbf{R}^n) : \|w\|_{H^{\varphi}(\mathbf{R}^n)} := \ \|\varphi((1+|\xi|^2)^{1/2})(\mathcal{F}w)(\xi)\|_{L_2(\mathbf{R}^n,d\xi)} < \infty \Big\}$$

parametrized by the function $\varphi \in \mathrm{RO}$. Here RO is the set of all Borel measurable functions $\varphi : [1,\infty) \to (0,\infty)$ such that $c^{-1} \leq \varphi(\lambda t)/\varphi(t) \leq c$ for every $t \geq 1$ and $\lambda \in [1,a]$, with the constants $a > 1$ and $c \geq 1$ depending on φ. The class RO was introduced by V. G. Avakumović. In the case where $\varphi(t) = t^s$ we have the Hilbert Sobolev space $H^{\varphi}(\mathbf{R}^n) = H^s(\mathbf{R}^n)$ of order $s \in \mathbf{R}$.

Theorem 1. *The class of spaces $\{H^{\varphi}(\mathbf{R}^n) : \varphi \in \mathrm{RO}\}$ coincides (up to equivalence of norms) with the set of all interpolation Hilbert spaces for couples of Hilbert Sobolev spaces $H^{s_0}(\mathbf{R}^n)$, $H^{s_1}(\mathbf{R}^n)$, $-\infty < s_0 < s_1 < \infty$. Each space $H^{\varphi}(\mathbf{R}^n)$ can be obtained by the interpolation with a function parameter* [1] *of a couple of Hilbert Sobolev spaces.*

The property of $H^{\varphi}(\mathbf{R}^n)$ to be an interpolation space for the couple $H^{s_0}(\mathbf{R}^n)$, $H^{s_1}(\mathbf{R}^n)$ means that each linear operator bounded on the both spaces $H^{s_j}(\mathbf{R}^n)$, $j = 0,\ 1$, should be bounded on $H^{\varphi}(\mathbf{R}^n)$ as well.

Theorem 1 remains true if we replace $\mathbf{R}^n$ with a closed (compact) C^{∞}-manifold Γ. The space $H^{\varphi}(\Gamma)$ consists of all distributions on Γ which belong in local coordinates to $H^{\varphi}(\mathbf{R}^n)$. The space $H^{\varphi}(\Gamma)$ coincides with the completion of $C^{\infty}(\Gamma)$ with respect to the norm $\|\varphi((1-\Delta)^{1/2})f\|_{L_2(\Gamma)}$, $f \in C^{\infty}(\Gamma)$. Here Δ is the Beltrami-Laplace operator.

We consider some applications of the spaces $H^{\varphi}(\Gamma)$, $\varphi \in \mathrm{RO}$. Let A be an elliptic pseudodifferential operator on Γ of a real order r, $\varrho(t) := t$, and $\log^* t := \max\{1, \log t\}$.

Theorem 2. *Operator A maps $H^{\varphi}(\Gamma)$ into $H^{\varphi\varrho^{-r}}(\Gamma)$ continuously for every $\varphi \in \mathrm{RO}$. This operator is Fredholm, its kernel and index are independent of φ.*

Theorem 3. *Assume A to be a normal operator in $L_2(\Gamma)$ with $r > 0$. Then for every function $f \in H^{\log^*}(\Gamma)$, its Fourier series in eigenfunctions of A converges almost everywhere on Γ.*

Here the eigenfunctions of A are indexed in such a way that the corresponding eigenvalues increase in modulus.

The results are obtained together with V. A. Mikhailets [1–3].

[1] Mikhailets V. A., Murach A. A. // Methods Funct. Anal. Topology. — 2008. — **14**, N 1.

[2] Mikhailets V. A., Murach A. A. // Transact. Institute Math. NAS of Ukraine. — 2008. — **5**, N 1.

[3] Mikhailets V. A., Murach A. A. // Dopovidi Nats. Acad. Nauk. Ukrainy. — 2009. — N 3.