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Hill's Potentials and Their Spectral Gaps

On the complex Hilbert space $L^2(\mathbb{R})$ we study the Hill-Schrödinger operators

$$S(q)u := -u'' + q(x)u, \quad x \in \mathbb{R}, \tag{1}$$

with 1-periodic, real-valued, distribution potentials

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k)e^{ik2\pi x} \in H^{-1}(\mathbb{T}, \mathbb{R}), \quad \mathbb{T} := \mathbb{R}/\mathbb{Z}, \tag{2}$$

that is

$$\sum_{k\in\mathbb{N}} k^{-2}|\widehat{q}(k)|^2 < \infty, \quad \text{and} \quad \widehat{q}(k) = \overline{\widehat{q}(-k)}, \quad k \in \mathbb{Z}.$$

Under the assumptions (2) the Hill-Schrödinger operators (1) can be well defined on $L^2(\mathbb{R})$ in the following basic ways: (a) as form-sum operators; (b) as quasi-differential operators; (c) as limits of operators with smooth 1-periodic potentials in the norm resolvent sense. Equivalence of all these definitions was established in [2].

The operators $S(q)$, $q \in H^{-1}(\mathbb{T}, \mathbb{R})$, are lower semibounded and self-adjoint, their spectra are absolutely continuous and have a zone structure, as in the classical case of $L^2(\mathbb{T}, \mathbb{R})$-potentials, see [2] and references therein. Spectra $\operatorname{spec}(S(q))$ are completely defined by the location of the endpoints of spectral gaps $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ [2, Theorem C].

We are going to characterize the behaviour of the lengths of spectral gaps

$$\gamma_q(n) := \lambda_n^+(q) - \lambda_n^-(q), \quad n \in \mathbb{N}$$

of the Hill-Schrödinger operators $S(q)$ in terms of the behaviour of the Fourier coefficients $\{\widehat{q}(n)\}_{n\in\mathbb{N}}$ of the potentials q with respect to the appropriate weight spaces.

Theorem ([3, 4])**.** *Let the weight sequence $\omega = \{\omega(k)\}_{k\in\mathbb{N}}$ satisfy conditions:*

$$k^s \ll w(k) \ll k^{1+2s}, \quad s \in (-1, 0],$$
$$k^s \ll w(k) \ll k^{1+s}, \quad s \in [0, \infty).$$

Then

$$q \in H^{\omega}(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^{\omega}(\mathbb{N}), \quad i.e., \quad \sum_{k\in\mathbb{N}} \omega^2(k)|\widehat{q}(k)|^2 < \infty \Leftrightarrow \sum_{k\in\mathbb{N}} \omega^2(k)\gamma_q^2(k).$$

The case $q \in L^2(\mathbb{T})$, and $\omega(k) = k^s$, $s \in \mathbb{Z}_+$, was studied in [1].

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