Vladimir Chilin, Botir Zakirov (National University of Uzbekistan, Tashkent Railway Engineering Institute, Tashkent, Uzbekistan)

The Radon-Nikodym-type theorem for Maharam traces

Let M be an von Neumann algebra, let $S(M)$ be the $*$-algebra of all measurable operators affiliated with to M and let $t_l(M)$ be the topology of convergence locally in measure in $S(M)$ ([1], III, §3.5). Let $\mathcal{A}$ be an arbitrary commutative von Neumann algebra and let Φ be the faithful normal $S(\mathcal{A})$-valued trace on M. An operator $x \in S(M)$ is said to be Φ-*integrable* if there exists a sequence $\{x_n\} \subset M$ such that $x_n \stackrel{t_l(M)}{\longrightarrow} x$ and $\Phi(|x_n - x_m|) \stackrel{t_l(\mathcal{A})}{\longrightarrow} 0$ as $n, m \to \infty$. In this case, there exists $\widehat{\Phi}(x) \in S(\mathcal{A})$ such that $\Phi(x_n) \stackrel{t_l(\mathcal{A})}{\longrightarrow} \widehat{\Phi}(x)$. Let $L^1(M, \Phi)$ be the set of all Φ-integrable operators and denote $\|x\|_\Phi = \widehat{\Phi}(|x|)$ for every $x \in L^1(M, \Phi)$. Then $(L^1(M, \Phi), \|\cdot\|_\Phi)$ is a (bo)-complete lattice normed space.

The trace Φ possesses *the Maharam property* if for any $x \in M_+,\ 0 \leq f \leq \Phi(x) \in S(\mathcal{A})$ there exists a positive $y \leq x$ such that $\Phi(y) = f$. A faithful normal $S(\mathcal{A})$-valued trace Φ with the Maharam property is called *Maharam trace* (compare with [2], III, 3.4.1).

Theorem 1. *$L^1(M, \Phi)$ is a Banach-Kantorovich space if and only if Φ is a Maharam trace.*

Let Φ be a $S(\mathcal{A})$-valued Maharam trace on M and let Ψ be a normal $S(\mathcal{A})$-valued trace on M. We denote by $s(a)$ the support of an element $a \in S(\mathcal{A})$. A trace Ψ is called *absolutely continuous with respect to* Φ if $s(\Psi(p)) \leq s(\Phi(p))$ for all projections $p \in M$.

The next theorem is a noncommutative version of the Radon-Nikodym-type theorem for Maharam traces (compare with [2], VI, 6.1.11).

Theorem 2. *Let Φ be a $S(\mathcal{A})$-valued Maharam trace on a von Neumann algebra M. If Ψ is a normal $S(\mathcal{A})$-valued trace on M absolutely continuous with respect to Φ, then there exists an operator $y \in L^1_+(M, \Phi) \cap S(Z(M))$ such that*

$$\Psi(x) = \widehat{\Phi}(yx)$$

for all $x \in M$ (here $Z(M)$ is the center of M).

[1] Muratov M.A., Chilin V.I. Algebras of measurable and locally measurable operators. – Kyiv, Pratsi In-ty matematiki NAN Ukraini. 2007. V. 69. – 390 p.(Russian).

[2] Kusraev A.G., Dominanted Operators, Mathematics and its Applications, 519, Kluwer Academic Publishers. — Dordrecht, 2000. – 446 p.