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Reduction operators of the linear rod equation

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We study reduction operators (called also nonclassical or conditional symmetries) of the (1+1)-dimensional linear rod equation. In particular, we prove and illustrate a new theorem on linear reduction operators of linear partial differential equations.

1 Introduction

For linear partial differential equations, there exist well-developed classical methods of their analytical solution, which, in particular, includes the separation of variables, different integral transforms, Fourier series and their generalizations. At the same time, the study of symmetry properties of such equations is important, first of all, for the development of methods of symmetry analysis itself.

In this paper we consider the (1+1)-dimensional constant-coefficient linear rod equation $u_{tt} + \lambda u_{xxxx} = 0$, where $\lambda > 0$, for unknown function u of the two independent variables t and x. This equation describes transverse vibrations of elastic rods. It is a special case of the Euler–Bernoulli beam equations, corresponding to constant values of parameters. Lie symmetries and the general equivalence problem for the class of Euler–Bernoulli beam equations were studied in [5, 6, 11]. By simple scaling of t or x, without loss of generality we can set $\lambda = 1$, i.e., it is sufficient to consider the equation

$$u_{tt} + u_{xxxx} = 0. \tag{1}$$

Some simple exact solutions of this equation are presented in [9, Section 9.2.2].[1] The maximal Lie invariance algebra of equation (1) is

$$\mathfrak{g} = \langle \partial_t,\ \partial_x,\ 2t\partial_t + x\partial_x,\ u\partial_u,\ h(t,x)\partial_u \rangle,$$

where $h = h(t, x)$ is an arbitrary solution of equation (1).

We study reduction operators (called also nonclassical or conditional symmetries) of the (1+1)-dimensional linear rod equation (1). First, in Section 2 we

[1] See also http://eqworld.ipmnet.ru/en/solutions/lpde/lpde501.pdf.

prove a theorem on linear reduction operators of general linear partial differential equations. This is why the notation in this section is different from the other part of the paper. The consideration of the next two sections illustrates both the statement and the proof of the theorem. The description of singular reduction operators of (1) in Section 3 is exhaustive. In contrast to this, only particular classes of regular reduction operators of (1) are found in Section 4. Possible generalizations of results obtained in the paper are discussed in the conclusion. We list interesting symmetry properties of equation (1) and additionally indicate the relation between the (1+1)-dimensional linear rod equation (1) and the (1+1)-dimensional free Schrödinger equation.

2 Linear reduction operators of linear equation

In order to present a theoretical background on reduction operators, based on [1–4, 10, 12], we first consider a general rth order differential equation $\mathcal{L}$ of the form $L(x, u_{(r)}) = 0$ for the unknown function u of the independent variables $x = (x_1, \dots, x_n)$. Here, $u_{(r)}$ denotes the set of all the derivatives of the function u with respect to x of order not greater than r, including u as the derivative of order zero. Any vector field Q in the foliated space of the n independent variables x and the single dependent variable u takes the form

$$Q = \xi^i(x, u)\partial_i + \eta(x, u)\partial_u,$$

where the coefficients ξ^i and η are smooth functions of x and u. The first-order differential function $Q[u] = \eta - \xi^i u_i$ is called the characteristic of Q.

Here and in what follows the index i runs from 1 to n, and we use the summation convention for repeated indices, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $\alpha_i \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and δ_i is the multi-index whose ith entry equals 1 and whose other entries are zero. Subscripts of functions denote differentiation with respect to the corresponding variables, $\partial_i = \partial/\partial x_i$ and $\partial_u = \partial/\partial u$. The variable u_α of the rth order jet space $J^r = J^r(x|u)$ corresponds to the derivative $\partial^{|\alpha|}u/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, and $u_i \equiv u_{\delta_i}$. All considerations are in the local smooth setting. Then the equation $\mathcal{L}$ can be viewed as an algebraic equation in the jet space J^r and is identified with the manifold of its solutions in J^r:

$$\mathcal{L} = \{(x, u_{(r)}) \in J^r \mid L(x, u_{(r)}) = 0\}.$$

We use the same symbol $\mathcal{L}$ for this manifold and write $\mathcal{Q}_{(r)}$ for the manifold defined by the set of all the differential consequences of the characteristic equation $Q[u] = 0$ in J^r, i.e.,

$$\mathcal{Q}_{(r)} = \{(x, u_{(r)}) \in J^r \mid D_1^{\alpha_1} \cdots D_n^{\alpha_n} Q[u] = 0, \ \ \alpha_i \in \mathbb{N} \cup \{0\}, \ \ |\alpha| < r\},$$

where $D_i = \partial_{x_i} + u_{\alpha+\delta_i}\partial_{u_\alpha}$ is the operator of total differentiation with respect to the variable x_i.

Definition 1. The differential equation $\mathcal{L}$ is called *conditionally invariant* with respect to the vector field Q if the relation $Q_{(r)}L(x,u_{(r)})|_{\mathcal{L}\cap\mathcal{Q}_{(r)}}=0$ holds. This relation is called the *conditional invariance criterion* [1–3, 12]. Then Q is called a *conditional symmetry* (or Q-conditional symmetry, or nonclassical symmetry, etc.) operator of the equation $\mathcal{L}$.

In this definition, $Q_{(r)}$ denotes the standard rth prolongation of Q [7, 8]:

$$Q_{(r)}=Q+\sum_{0<|\alpha|\leqslant r}\eta^\alpha\partial_{u_\alpha},\quad \text{where}\quad \eta^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}Q[u]+\xi^i u_{\alpha+\delta_i}.$$

The equation $\mathcal{L}$ is conditionally invariant with respect to the vector field Q if and only if an ansatz constructed with Q reduces $\mathcal{L}$ to a differential equation with $n-1$ independent variables [12]. Thus, we will briefly call a conditional symmetry operator of the equation $\mathcal{L}$ a *reduction operator* of this equation.

Reduction operators $\widetilde{Q}$ and Q are called *equivalent*, $\widetilde{Q}\sim Q$, if they differ by a multiplier which is a nonvanishing function of x and u: $\widetilde{Q}=\lambda Q$, where $\lambda=\lambda(x,u)\neq 0$. Reduction operators Q and $\widetilde{Q}$ are called equivalent with respect to a group G of point transformations if there exists $g\in G$ for which the operators Q and $g_*\widetilde{Q}$ are equivalent, where g_* is the mapping induced by g on the set of vector fields.

Now consider an rth order linear differential equation $\mathcal{L}$ of the form

$$L[u]:=\sum_{|\alpha|\leqslant r}a^\alpha(x)u_\alpha=0$$

for the unknown function u of the independent variables $x=(x_1,\dots,x_n)$, where some coefficient a^α with $|\alpha|=r$ does not vanish.

Among Lie symmetries of linear differential equations, a distinguished role is played by symmetries associated with first-order linear differential operators acting on $u=u(x)$. If $n\geqslant 2$ and $r\geqslant 2$ or $n=1$ and $r\geqslant 3$, the system of determining equations SDE($\mathcal{L}$) for the coefficients of vector fields from the maximal Lie invariance algebra $\mathfrak{g}^{\max}$ of $\mathcal{L}$ necessarily implies the equations $\xi^i_u=0$ and $\eta_{uu}=0$. In other words, any of such vector fields can be represented as

$$Q=\xi^i(x)\partial_i+(\eta^1(x)u+\eta^0(x))\partial_u, \tag{2}$$

and the system SDE($\mathcal{L}$) additionally gives that η^0 is an arbitrary solution of $\mathcal{L}$. The vector fields $\eta^0(x)\partial_u$, where η^0 runs through the set of solutions of the equation $\mathcal{L}$, form an ideal of the algebra $\mathfrak{g}^{\max}$ and generate point symmetries that are associated with the linear superposition principle. Up to the equivalence in $\mathfrak{g}^{\max}$ that is generated by adjoint actions of elements from the ideal, we can assume $\eta^0=0$ in (2) if at least one of the coefficients ξ^i or η^1 does not vanish.

The purpose of the further consideration in this section is to extend the last claim to reduction operators of the form (2), which will be called *linear reduction* *operators.* Note that general conditions when a linear differential equation admits only reduction operators which are equivalent to linear ones are not known.

Additionally recall that a vector field Q is called *(weakly) singular* for the differential equation $\mathcal{L}$: $L[u]=0$ if there exists a differential function $\tilde L=\tilde L[u]$ of an order less than r and a nonvanishing differential function $\lambda=\lambda[u]$ of an order not greater than r such that $L|_{\mathcal{Q}_{(r)}}=\lambda\tilde L|_{\mathcal{Q}_{(r)}}$. Otherwise Q is called a *(weakly) regular* vector field for $\mathcal{L}$. A vector field Q is *ultra-singular* for the equation $\mathcal{L}$ if this equation is satisfied by any solution of the characteristic equation $Q[u]:=\eta-\xi^iu_i=0$. See [1, 4] for theoretical background on singular reduction operators.

Theorem 1. *Let a linear partial differential equation $\mathcal{L}$ possess a reduction operator Q of the form* (2)*. Then the coefficient η^0 is represented as $\eta^0=\xi^i\zeta^0_i-\eta^1\zeta^0$, where $\zeta^0=\zeta^0(x)$ is a solution of $\mathcal{L}$. Hence, up to equivalence generated by action of the Lie symmetry group of $\mathcal{L}$ on the set of reduction operators of $\mathcal{L}$, the coefficient η^0 can be set equal to zero. Any vector field of the form $\xi^i\partial_i+(\eta^1u+\xi^i\zeta_i-\eta^1\zeta)\partial_u$, where $\zeta=\zeta(x)$ is an arbitrary solution of $\mathcal{L}$, is a reduction operator of $\mathcal{L}$.*

Proof. Since Q is a reduction operator, at least one of the coefficients ξ^i does not vanish. Consider the vector field $\hat Q=\xi^i(x)\partial_i+\eta^1(x)u\partial_u$. Let $X^1(x),\dots,X^{n-1}(x)$ be functionally independent solutions of the equation $\xi^iv_i=0$, let $X^n(x)$ be a particular solution of the equation $\xi^iv_i=1$ and let $U(x)$ be a nonvanishing solution of the equation $\xi^iv_i+\eta^1v=0$. We introduce the notation $X=(X^1,\dots,X^n)$. Then the components of X and the function $U(x)u$ are functionally independent in total as functions of (x,u). This means that the change of variables $\mathcal{T}$: $\tilde x=X(x)$, $\tilde u=U(x)u$ is well defined.

We carry out this change of variables and represent all objects and relations in the new variables $(\tilde x,\tilde u)$. Thus, the vector field $\hat Q$ coincides with the generator of shifts with respect to the variable $\tilde x_n$, $\hat Q=\partial_{\tilde x_n}$, and hence $Q=\partial_{\tilde x_n}+\tilde\eta^0(\tilde x)\partial_{\tilde u}$, where $\tilde\eta^0(\tilde x)=U(x)\eta^0(x)$. Then the characteristic equation associated with the vector field Q in the new variables is $\tilde u_{\tilde x_n}=\tilde\eta^0$. The change of variables $\mathcal{T}$ also preserves the linearity of the equation $\mathcal{L}$, which takes the form

$$\tilde L[\tilde u]=\sum_{|\alpha|\leqslant r}\tilde a^\alpha(\tilde x)\tilde u_\alpha=0, \tag{3}$$

where each coefficient $\tilde a^\alpha$ are expressed in terms of the coefficients $a^{\alpha'}$, $|\alpha'|\geqslant|\alpha|$, and derivatives of X^i and U. The variable $\tilde u_\alpha$ of the jet space J^r corresponds to the derivative $\partial^{|\alpha|}\tilde u/\partial\tilde x_1^{\alpha_1}\dots\partial\tilde x_n^{\alpha_n}$. Up to nonvanishing multiplier, a coefficient $\tilde a^{\alpha^0}$, where $|\alpha^0|=r$, can be assumed to be identically equal to 1.

We denote an antiderivative of $\tilde\eta^0$ with respect to $\tilde x_n$ by $\tilde\zeta^0$,

$$\tilde\eta^0=\tilde\zeta^0_{\tilde x_n}.$$

We separately consider two cases depending on whether or not the reduction operator Q is ultra-singular for $\mathcal{L}$, and show that in each of these cases there exists an antiderivative $\tilde\zeta^0$ of $\tilde\eta^0$ satisfying the representation (3) of the equation $\mathcal{L}$ in the new variables, $\tilde L[\tilde\zeta^0]=0$.

Suppose that the reduction operator Q is ultra-singular for $\mathcal{L}$. As the property of ultra-singularity is not affected by changes of variables, this means that the representation $\tilde L[\tilde u]=0$ of the equation $\mathcal{L}$ in the new variables is satisfied by any solution of the characteristic equation $\tilde u_{\tilde x_n}=\tilde\eta^0$, i.e.,

$$\sum_{|\alpha|\leqslant r,\alpha_n\neq0}\tilde a^\alpha\tilde\eta^0_{\alpha-\delta_n}+\sum_{|\alpha|\leqslant r,\alpha_n=0}\tilde a^\alpha\tilde u_\alpha=0,$$

where the derivatives $\tilde u_\alpha$ with $\alpha_n=0$ are not constrained. Splitting with respect to them, we obtain the system of equations $\tilde a^\alpha=0$ for α running the set of multi-indices with $|\alpha|\leqslant r$ and $\alpha_n=0$ and an equation for the coefficient $\tilde\eta^0$,

$$\sum_{|\alpha|\leqslant r,\alpha_n\neq0}\tilde a^\alpha\tilde\eta^0_{\alpha-\delta_n}:=\sum_{|\alpha|\leqslant r,\alpha_n\neq0}\tilde a^\alpha\tilde\zeta^0_\alpha=0.$$

So, the summation in equation (3) is in fact for the values of the multi-index α with $\alpha_n\neq0$ and hence the function $\tilde\zeta^0$ satisfies this equation.

Suppose that the reduction operator Q is not ultra-singular for $\mathcal{L}$. As the rth prolongation of Q is given by $Q_{(r)}=\partial_{\tilde x_n}+\sum_{|\alpha|\leqslant r}\tilde\eta^0_\alpha(\tilde x)\partial_{\tilde u_\alpha}$, the conditional invariance criterion implies for this case that

$$Q_{(r)}\tilde L[\tilde u]=\sum_{|\alpha|\leqslant r}(\tilde a^\alpha_{\tilde x_n}\tilde u_\alpha+\tilde a^\alpha\tilde\eta^0_\alpha)=0\tag{4}$$

for all points of the jet space J^r where $\tilde L[\tilde u]=0$ and $\tilde u_{\alpha'}=\tilde\eta^0_{\alpha'-\delta_n}$ with $|\alpha'|\leqslant r$ and $\alpha_n>0$. As $\tilde a^{\alpha^0}=1$, the differential function $Q_{(r)}\tilde L[\tilde u]$ does not depend on the derivative $\tilde u_{\alpha^0}$. Hence the constraint $\tilde L[\tilde u]=0$ is not essential in the course of confining to the manifold $\mathcal{L}\cap\mathcal{Q}_{(r)}$. The derivatives $\tilde u_\alpha$ with $\alpha_n=0$ are not constrained. Splitting with respect to them in (4) gives the system of equations $\tilde a^\alpha_{\tilde x_n}=0$ for α running the set of multi-indices with $|\alpha|\leqslant r$ and $\alpha_n=0$ as a necessary condition for the equation $\mathcal{L}$ to admit the reduction operator Q. Then on the manifold $\mathcal{Q}_{(r)}$ we get

$$\begin{aligned}Q_{(r)}\tilde L[\tilde u]&=\sum_{|\alpha|\leqslant r,\alpha_n=0}\tilde a^\alpha_{\tilde x_n}\tilde u_\alpha+\sum_{|\alpha|\leqslant r,\alpha_n\neq0}\tilde a^\alpha_{\tilde x_n}\tilde u_\alpha+\sum_{|\alpha|\leqslant r}\tilde a^\alpha\tilde\eta^0_\alpha\\&=\sum_{|\alpha|\leqslant r,\alpha_n\neq0}\tilde a^\alpha_{\tilde x_n}\tilde\eta^0_{\alpha-\delta_n}+\sum_{|\alpha|\leqslant r}\tilde a^\alpha\tilde\eta^0_\alpha\\&=\sum_{|\alpha|\leqslant r,\alpha_n=0}\tilde a^\alpha_{\tilde x_n}\tilde\zeta^0_\alpha+\sum_{|\alpha|\leqslant r,\alpha_n\neq0}\tilde a^\alpha_{\tilde x_n}\tilde\zeta^0_\alpha+\sum_{|\alpha|\leqslant r}\tilde a^\alpha\tilde\zeta^0_{\alpha+\delta_n}\\&=\left(\sum_{|\alpha|\leqslant r}\tilde a^\alpha\tilde\zeta^0_\alpha\right)_{\tilde x_n}=0.\end{aligned}$$

The integration of the last equality with respect to $\tilde{x}_n$ gives that the function $\tilde{\zeta}^0 = \tilde{\zeta}^0(x)$ satisfies the inhomogeneous linear equation

$$\tilde{L}[\tilde{\zeta}^0] := \sum_{|\alpha|\leqslant r} \tilde{a}^\alpha \tilde{\zeta}^0_\alpha = g(x_1, \dots, x_{n-1}) \tag{5}$$

for some smooth function $g = g(x_1, \dots, x_{n-1})$. As in this case the reduction operator Q is not ultra-singular for $\mathcal{L}$, there exists the multi-index α with $|\alpha| \leqslant r$ and $\alpha_n = 0$ such that $\tilde{a}^\alpha \neq 0$. Hence equation (5) has a particular solution h that does not depend on $\tilde{x}_n$, $h = h(x_1, \dots, x_{n-1})$.[2] The function $\tilde{\zeta}^0 - h$ is also an antiderivative of $\tilde{\eta}^0$ with respect to $\tilde{x}_n$ and, at the same time, it satisfies the corresponding homogeneous linear equation, $\tilde{L}[\tilde{\zeta}^0 - h] = 0$. Therefore, without loss of generality we can assume that the antiderivative $\tilde{\zeta}^0$ itself is a solution of equation (3), $\tilde{L}[\tilde{\zeta}^0] = 0$.

We carry out the inverse change of the variables in the equality $\tilde{\eta}^0 = \tilde{\zeta}^0_{\tilde{x}_n} = \hat{Q}\tilde{\zeta}^0$ and introduce the function $\zeta^0 = \tilde{\zeta}^0/U$, which satisfies the equation $\mathcal{L}$ in the old variables (x, u). We have $U\eta^0 = \xi^i(U\zeta^0)_i = U\xi^i\zeta^0_i + (\xi^i U_i)\zeta^0 = U(\xi^i\zeta^0_i - \eta^1\zeta^0)$, i.e., $\eta^0 = \xi^i\zeta^0_i - \eta^1\zeta^0$. Here we use that $\xi^i U_i = -\eta^1 U$. The mapping generated by the point symmetry transformation $\bar{x} = x$, $\bar{u} = u - \zeta^0(x)$ of $\mathcal{L}$ on the set of reduction operators of $\mathcal{L}$ maps the vector field Q to the vector field $\hat{Q}$, for which the coefficient η^0 is zero. This means that $\hat{Q}$ is a reduction operator of $\mathcal{L}$. Applying the similar mapping generated by the point symmetry transformation $\bar{x} = x$, $\bar{u} = u + \zeta(x)$ with an arbitrary solution $\zeta = \zeta(x)$ of $\mathcal{L}$, we obtain that any vector field of the form $\xi^i\partial_i + (\eta^1 u + \xi^i\zeta_i - \eta^1\zeta)\partial_u$ is a reduction operator of $\mathcal{L}$. □

An ansatz constructed for the unknown function u with the vector field Q is

$$u = \frac{1}{U(x)}\varphi(\omega_1, \dots, \omega_{n-1}) + \zeta^0(x),$$

where φ is the invariant dependent variable, $\omega_1 = X^1(x)$, …, $\omega_{n-1} = X^{n-1}(x)$ are invariant independent variables, and we use the notation from the proof of the theorem. The corresponding reduced equation is

$$\sum_{|\alpha|\leqslant r, \alpha_n=0} \tilde{a}^\alpha(\omega_1, \dots, \omega_{n-1}) \frac{\partial^{|\alpha|}\varphi}{\partial\omega_1^{\alpha_1} \dots \partial\omega_{n-1}^{\alpha_{n-1}}} = 0.$$

It is obvious that the form of the reduced equation does not depend on the parameter-function $\zeta^0(x)$. The substitution of an arbitrary solution of $\mathcal{L}$ instead of $\zeta^0(x)$ gives the same reduced equation.

[2] If $n > 2$, then for the guaranteed existence of such a classical solution we suppose that all functions are analytical. In the case $n = 2$ or for specific linear equations the requested smoothness of functions can be lowered.

3 Singular reduction operators of the rod equation

For the linear rod equation (1), i.e., $\mathcal{L}$: $u_{tt}+u_{xxxx}=0$, the general form of reduction operators is

$$Q=\tau(t,x,u)\partial_t+\xi(t,x,u)\partial_x+\eta(t,x,u)\partial_u,$$

where the coefficients τ, ξ and η are smooth functions of (t,x,u) with $(\tau,\xi)\neq(0,0)$. Similarly to the evolution equations, a vector field Q is singular for the linear rod equation (1) if and only if the coefficient τ identically vanishes. Note that vector fields that are weakly singular for this equation are also strongly singular for it. Then $\xi\neq 0$ and hence up to usual equivalence of reduction operators we can set $\xi=1$. In other words, for the exhaustive study of singular reduction operators of the linear rod equation (1) it suffices to consider vector fields of the form

$$Q=\partial_x+\eta(t,x,u)\partial_u.$$

The manifold $\mathcal{L}\cap\mathcal{Q}_{(4)}$ is defined by the equations

$$u_x=\eta,\quad u_{xx}=\eta_x+\eta\eta_u,\quad u_{xxx}=(\partial_x+\eta\partial_u)^2\eta,\quad u_{xxxx}=(\partial_x+\eta\partial_u)^3\eta,$$
$$u_{tt}=-u_{xxxx}=-(\partial_x+\eta\partial_u)^3\eta.$$

Hence the conditional invariance criterion implies that

$$\eta_{tt}+2\eta_{tu}u_t+\eta_{uu}u_t^2-\eta_u(\partial_x+\eta\partial_u)^3\eta+(\partial_x+\eta\partial_u)^4\eta=0.$$

Collecting coefficients of different powers of the unconstrained derivative u_t and splitting with respect to it, we derive the system of three determining equations for the coefficient η:

$$\eta_{uu}=0,\quad \eta_{tu}=0,\quad \eta_{tt}-\eta_u(\partial_x+\eta\partial_u)^3\eta+(\partial_x+\eta\partial_u)^4\eta=0.$$

Thus, in contrast to a (1+1)-dimensional evolution equation, where there is a single determining equation for the coefficient η of singular reduction operators and this equation is reduced, in a certain sense, to the evolution equation under consideration, finding singular reduction operators of the linear rod equation is not a no-go problem. The equations $\eta_{uu}=0$ and $\eta_{tu}=0$ give the expression

$$\eta=\eta^1(x)u+\eta^0(t,x)$$

for the coefficient η, where $\eta^1=\eta^1(x)$ and $\eta^0=\eta^0(t,x)$ are smooth functions of their variables. Theorem 1 implies that, up to equivalence generated by the maximal Lie symmetry group $G^{\max}$ of the linear rod equation on the set of reduction operators of this equation, we can set $\eta^0=0$. We also show this directly.

After substituting the expression for η into the last determining equation, we can additionally split with respect to u to obtain

$$\partial_x(\partial_x+\eta^1)^3\eta^1=0,\quad \eta^0_{tt}-\eta^1\eta^{03}+\eta^{04}=0,$$

where the functions η^{03} and η^{04} are defined by the recurrent relation $\eta^{00}:=\eta^0$ and $\eta^{0k}=\eta^{0,k-1}_x+\eta^0(\partial_x+\eta^1)^{k-1}\eta^1$, $k=1,2,3,4$. We make the differential substitution

$$\eta^1=\frac{\theta_x}{\theta},\quad \eta^0=\zeta_x-\frac{\theta_x}{\theta}\zeta,$$

where $\theta=\theta(x)$ and $\zeta=\zeta(t,x)$ are the new unknown functions. It is possible to show by induction that

$$\eta^{0k}=\frac{\partial^{k+1}\zeta}{\partial x^{k+1}}-\frac{\zeta}{\theta}\frac{d^{k+1}\theta}{dx^{k+1}},\quad k=1,2,\dots .$$

Hence the differential substitution reduces the system for η^1 and η^0 to a system for θ and ζ,

$$\left(\frac{\theta_{xxxx}}{\theta}\right)_x=0,\quad \zeta_{ttx}-\frac{\theta_x}{\theta}\zeta_{tt}-\frac{\theta_x}{\theta}\zeta_{xxxx}+\frac{\theta_x\theta_{xxxx}}{\theta^2}\zeta+\zeta_{xxxxx}-\frac{\theta_{xxxxx}}{\theta}\zeta=0.$$

Integrating once the first equation, we get the constant-coefficient linear ordinary differential equation $\theta_{xxxx}=\kappa\theta$, where κ is the integration constant. The second equation can be represented as

$$\left(\frac{\zeta_{tt}+\zeta_{xxxx}}{\theta}\right)_x-\left(\frac{\theta_{xxxx}}{\theta}\right)_x\zeta=0,\quad \text{hence}\quad \left(\frac{\zeta_{tt}+\zeta_{xxxx}}{\theta}\right)_x=0.$$

The integration of the last equation with respect to x results in the equation $\zeta_{tt}+\zeta_{xxxx}=\rho(t)\theta$, where ρ is a smooth function of t. The function ζ is defined up to the transformation $\tilde\zeta=\zeta+\sigma\theta$, where σ is an arbitrary smooth function of t. This transformation allows us to set $\rho=0$. Indeed, $\tilde\zeta_{tt}+\tilde\zeta_{xxxx}=\rho\theta+\sigma_{tt}\theta+\sigma\kappa\theta=0$ if $\sigma_{tt}+\kappa\sigma=-\rho$. In other words, we can assume that the function ζ satisfies the linear rod equation (1). Then the mapping generated by the point symmetry transformation $\bar t=t$, $\bar x=x$, $\bar u=u-\zeta(t,x)$ of equation (1) on the set of reduction operators of this equation maps the vector field Q to the vector field of the same form, where $\zeta=0$ and hence $\eta^0=0$.

Proposition 1. *Up to equivalence generated by symmetry transformations of linear superposition, the set of singular reduction operators of the linear rod equation* (1) *is exhausted by the vector fields of the form*

$$Q_{\rm s}=\partial_x+\frac{\theta_x}{\theta}u\partial_u,$$

where the function $\theta=\theta(x)$ satisfies the ordinary differential equation $\theta_{xxxx}=\kappa\theta$ for some constant κ.

An ansatz constructed with the reduction operator Q is $u = \theta(x)\varphi(\omega)$, where $\omega = t$ is the invariant independent variable and φ is the invariant dependent variable. The corresponding reduced equation is $\varphi_{\omega\omega} + \kappa\varphi = 0$. As an interpretation, we can say that the reduction operator $Q_{\rm s}$ is related to separation of variables in the linear rod equation (1). It is obvious that the reduction operator $Q_{\rm s}$ is equivalent to a Lie symmetry operator only if $\theta_x/\theta = {\rm const}$.

4 Regular reduction operators of the rod equation

Consider regular reduction operators of the linear rod equation (1), for which the coefficient τ does not vanish. Up to usual equivalence of reduction operators we can set $\tau = 1$, i.e., it suffices to consider vector fields of the form

$$Q = \partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u.$$

Essential among the equations defining the manifold $\mathcal{L} \cap \mathcal{Q}_{(4)}$ are the equations

$$\begin{aligned}&u_t = \eta - \xi u_x, \quad u_{tx} = \eta_x + \eta u_x - \xi_x u_x - \xi_u u_x^2 - \xi u_{xx},\\ &u_{tt} = -u_{xxxx} = \eta_t + \eta_u(\eta - \xi u_x) - (\xi_t + \xi_u(\eta - \xi u_x))u_x\\ &\qquad -\xi(\eta_x + \eta u_x - \xi_x u_x - \xi_u u_x^2 - \xi u_{xx}).\end{aligned}$$

Collecting coefficients of $u_{xx}u_{xxx}$ in the condition following from the conditional invariance criterion, we obtain the equation $\xi_u = 0$. Other terms with u_{xxx} give the equations $\eta_{uu} = 0$ and $\eta_{xu} = \frac{3}{2}\xi_{xx}$. Therefore, we have

$$\xi = \xi(t,x), \quad \eta = \eta^1(t,x)u + \eta^0(t,x), \quad \text{where} \quad \eta^1 := \frac{3}{2}\xi_x + \gamma(t)$$

with a smooth function $\gamma = \gamma(t)$. The other determining equations reduce to

$$2\xi_t\xi + 5\xi_{xxx} + 4\xi^2\xi_x = 0, \tag{6}$$

$$\xi_{tt} + \xi_{xxxx} + 2(\eta^1\xi)_t + 2\xi_t\xi_x - 4\eta^1_{xxx} + 8\xi\xi_x\eta^1 - 4\xi{\xi_x}^2 = 0, \tag{7}$$

$$\eta^1_{tt} + \eta^1_{xxxx} + 2\eta^1\eta^1_t - 2\xi_t\eta^1_x + 4\xi_x(\eta^1_t + \eta^1\eta^1 - \xi\eta^1_x) = 0, \tag{8}$$

$$\eta^0_{tt} + \eta^0_{xxxx} + 2\eta^0\eta^1_t - 2\xi_t\eta^0_x + 4\xi_x(\eta^0_t + \eta^1\eta^0 - \xi\eta^0_x) = 0, \tag{9}$$

where every appearance of η^1 should be replaced by $\frac{3}{2}\xi_x + \gamma(t)$.

Similarly to singular reduction operators, Theorem 1 again implies that, up to equivalence generated by the maximal Lie symmetry group $G^{\max}$ of the linear rod equation on the set of reduction operators of this equation, we can set $\eta^0 = 0$. We show that the direct proof of this fact is not trivial. Indeed, let the function ζ be defined by the relation $\eta^0 = \zeta_t + \xi\zeta_x - \eta^1\zeta$. As it is a first-order quasi-linear partial differential equation with respect to ζ, such a function ζ exists. We use this relation to substitute for η^0 into equation (9). Taking into account equations (6)–(8) and $\eta_{xu} = \frac{3}{2}\xi_{xx}$, we derive the following equation for the function ζ:

$$(\partial_t + \xi\partial_x - \eta^1 + 4\xi_x)(\zeta_{tt} + \zeta_{xxxx}) = 0,$$

i.e., $\zeta_{tt}+\zeta_{xxxx}=h(t,x)$, where the function $h=h(t,x)$ satisfies the equation

$$h_t+\xi h_x+(-\eta^1+4\xi_x)h=0.$$

The function $h=h(t,x)$ can be set to zero. Indeed, the function ζ is defined up to summand that is a solution of the equation $g_t+\xi g_x-\eta^1 g=0$. Any such solution is represented as $g=g^0(t,x)\varphi(\omega)$, where g^0 is a fixed solution of the same equation, φ ia an arbitrary function of ω, and $\omega=\omega(t,x)$ is a nonconstant solution of the equation $\omega_t+\xi\omega_x=0$. Then $\chi={\omega_x}^4$ satisfies the equation

$$\chi_t+\xi\chi_x+4\xi_x\chi=0.$$

Therefore, the function h possesses the representation $h=g^0{\omega_x}^4\psi(\omega)$ for some smooth function ψ of ω. The above determining equations imply that the vector field $\partial_t+\xi\partial_x+\eta^1 u\partial_u$ is a reduction operator for the equation $u_{tt}+u_{xxxx}=0$. Hence we have

$$g_{tt}+g_{xxxx}=g^0{\omega_x}^4\varphi_{\omega\omega\omega\omega}+\cdots=g^0{\omega_x}^4(\varphi_{\omega\omega\omega\omega}+\cdots),$$

where the expression in the brackets depends merely on ω and the dots denote terms including derivatives of φ of orders less than four. This means that the ansatz $g=g^0(t,x)\varphi(\omega)$ reduces the equation $g_{tt}+g_{xxxx}=h$ to the ordinary differential equation $\varphi_{\omega\omega\omega\omega}+\cdots=\psi$, which definitely has a solution $\varphi^0=\varphi^0(\omega)$. Subtracting the corresponding function $g=g^0\varphi^0$ from the function ζ, we annihilate the function h.

Therefore, without loss of generality we can assume that the function ζ satisfies the initial equation (1). Then the mapping generated by the point symmetry transformation $\bar t=t$, $\bar x=x$, $\bar u=u-\zeta(t,x)$ of (1) on the set of reduction operators of (1) maps the vector field Q to the vector field of the same form, where $\zeta=0$ and hence $\eta^0=0$.

As a result, the study of regular reduction operators of the linear rod equation (1) reduces to the solution of the overdetermined system of nonlinear differential equations (6)–(8) for the functions $\xi=\xi(t,x)$ and $\gamma=\gamma(t)$. (Recall that $\eta^1:=\frac{3}{2}\xi_x+\gamma(t)$.) This solution appears an unexpectedly complicated problem. Hence we have considered particular cases of regular reduction operators by imposing additional constraints on the functions ξ and γ. Thus, cumbersome and tricky computations with `Maple` show that any regular reduction operator of (1) with $\gamma=0$ is equivalent to a Lie symmetry operator of this equation. The same result is true under the assumption $\xi_{xx}=0$ and $\xi\neq0$. There are no regular reduction operators with $\xi_t=0$ and $\xi_x\neq0$.

Suppose that $\xi=0$. Then equations (6) and (7) are identically satisfied and the coefficient η^1 is represented as $\eta^1=\gamma(t)$. Equation (8) implies the single ordinary differential equation $\gamma_{tt}+2\gamma\gamma_t=0$ for the function γ, which is once integrated to $\gamma_t+\gamma^2=-\kappa$, where κ is the integration constant. Hence the function γ admits the representation $\gamma=\varphi_t/\varphi$, where the function $\varphi=\varphi(t)$ is a solution of the linear ordinary differential equation $\varphi_{tt} + \kappa\varphi = 0$. The corresponding reduction operator

$$Q_{\rm r} = \partial_t + \frac{\varphi_t}{\varphi} u\partial_u,$$

results in the ansatz $u = \varphi(t)\theta(\omega)$, where $\omega = x$ is the invariant independent variable and θ is the invariant dependent variable. The corresponding reduced equation is $\theta_{\omega\omega\omega\omega} = \kappa\theta$. Therefore, similarly to the singular reduction operator $Q_{\rm s}$ from Proposition 1 the regular reduction operator $Q_{\rm r}$ is related to separation of variables in the linear rod equation (1). This operator can be considered as a regular counterpart of the operator $Q_{\rm s}$. The reduction operator $Q_{\rm r}$ is equivalent to a Lie symmetry operator only if $\varphi_t/\varphi = \mathrm{const}$.

5 Conclusion

In spite of the rod equation (1) is linear and has only obvious Lie symmetries, it is interesting from the symmetry point of view since it possesses a number of nontrivial properties related to the field of symmetry analysis. We list five of these properties:

- Equation (1) possesses both regular and singular nonclassical symmetries which are inequivalent to Lie symmetries and associated with separation of variables.
- A potential system of the rod equation (1) coincides with the (1+1)-dimensional free Schrödinger equation. Hence equation (1) possesses purely potential and nonclassical potential symmetries.
- A function is a solution of the rod equation (1) if and only if it is the real (resp. imagine) part of a solution of the (1+1)-dimensional free Schrödinger equation. This allows us to construct new families of exact solutions of (1) in an easy way.
- Equation (1) has a nonlocal recursion operator whose action on local symmetries (which necessarily are affine in derivatives of u) gives nontrivial local symmetries of higher order. As a result, for arbitrary fixed order, excluding order two, this equation possesses local symmetries of this order which do not belong to the enveloping algebras of local symmetries of lower orders.
- As the linear differential operator associated with (1) is formally self-adjoint, the space of cosymmetries and the space of characteristics of local symmetries coincides. This implies that equation (1) has conservation laws of arbitrarily high order.

A detail discussion of these properties will be a subject of a forthcoming paper. In the present paper, we have studied the first property and below we briefly present the next two properties.

The linear differential operator $L := \partial_t^2 + \partial_x^4$ associated with equation (1) is factorized to the product of the free Schrödinger operator and its formal adjoint:

$$L = (i\partial_t + \partial_x^2)(-i\partial_t + \partial_x^2).$$

This indicates that the solution of (1) is closely connected with the solution of the free (1+1)-dimensional Schrödinger equation

$$i\psi_t + \psi_{xx} = 0. \tag{10}$$

To make this connection explicit, we consider the potential system constructed for equation (1) with the conservation law having the characteristic 1:

$$v_x = u_t, \quad v_t = -u_{xxx}. \tag{11}$$

The second equation of (11) is in conserved form that allows us to introduce the potential w satisfying the conditions

$$w_x = v, \quad w_t = -u_{xx}. \tag{12}$$

Excluding v from the joint system of (11) and (12), we obtain the system

$$u_t = w_{xx}, \quad w_t = -u_{xx}. \tag{13}$$

The maximal Lie invariance algebra of system (13) is

$$\begin{aligned}\mathfrak{g}_1 = \langle &\partial_t,\, \partial_x,\, 2t\partial_t + x\partial_x,\, w\partial_u - u\partial_w,\, 2t\partial_x + xw\partial_u - xu\partial_w,\\ &4t^2\partial_x + 4tx\partial_x + \big(x^2w - 2tu\big)\partial_u - \big(x^2u + 2tw\big)\partial_w,\\ &u\partial_u + w\partial_w,\, \beta(t,x)\partial_u + \gamma(t,x)\partial_w\rangle,\end{aligned} \tag{14}$$

where $(\beta(t,x), \gamma(t,x))$ is an arbitrary solution of system (13).

System (13) implies that the complex-valued function $\psi = w + iu$ of the variables t and x satisfies equation (10) and the function w is a solution of equation (1). Finally, we have the following simple assertion.

Proposition 2. *The function $u = u(t,x)$ is a solution of equation* (1) *if and only if it is the real* (*resp. imagine*) *part of a solution of the (1+1)-dimensional free Schrödinger equation $i\psi_t + \psi_{xx} = 0$.*

A fixed solution of equation (1) corresponds to a set of solutions of equation (10) which differ by summands of the form $C_1x + C_0$, where C_0 and C_1 are arbitrary real constants. As wide families of exact solutions of equation (10) are already known, Proposition 2 gives the simplest way of finding exact solutions for equation (1).

In fact, the main result of the paper is Theorem 1 on single linear reduction operators of general linear partial differential equations. The next step is to extend this assertion to multidimensional reduction modules that are generated by linear vector fields.

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