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THE PARTIALLY ORDERED SYSTEM OF ATTRACTING SETS

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We consider a single-valued continuous mapping T of a closed interval R of the real line into itself. The attracting set Ω_x means the set of ω-limit-points of the sequence $\{T^j x\}_{j=0}^{\infty}$, $x \in R$.

Let A be the aggregate of attracting sets Ω_x, as x runs over the whole space R. There is a natural partial ordering of A: if $\Omega', \Omega'' \in A$, $\Omega' \subset \Omega''$, then Ω' precedes Ω'' in A.

In the present note properties of the partially ordered system A are formulated.

1°. Every maximal chain in the system A possesses a minimal and a maximal element.

A set Ω is a minimal element if and only if $\Omega_x = \Omega$ for each $x \in \Omega$. A minimal element may be either a finite set of points forming a cycle of the mapping T, or a nowhere dense perfect set.

A maximal element which is not at the same time minimal will be called

1) a maximal element of the first kind if it contains no cycles,

2) a maximal element of the second kind if it contains at least one cycle.

Every maximal element of the second kind is a perfect set; every maximal element of the first kind contains a perfect part together with a countable set of points.

A maximal element Ω of the second kind possesses the following important property [1]: for any point x for which $\Omega_x \subseteq \Omega$ there exists an index j_x such that $T^{j_x} x \in \Omega$, and consequently $T^j x \in \Omega$ for $j \geq j_x$.

2°. There are at most denumerably many maximal elements of the first and second kinds.

However, there may exist continuum many maximal elements which are also minimal.

3°. Any two maximal chains that contain a common element which is not minimal have one and the same maximal element.

4°. If the mapping T only has cycles of order 2^i, $i = 0, 1, 2, \cdots$, then any two maximal chains containing a common element have one and the same minimal element; and every maximal element is either minimal or a maximal element of the first kind.

5°. If the mapping T has cycles of order $\neq 2^i$, $i = 0, 1, 2, \cdots$, then there exists at least one maximal element of the second kind.

6°. A maximal element of the second kind has no immediate predecessor in any maximal chain.

7°. The set of minimal elements that precede a maximal element of the second kind has the power of the continuum, the subset of minimal elements that are distinct from cycles also having the power of the continuum.

8°. The set of elements immediately succeeding each element that precedes a maximal element of the second kind has the power of the continuum.

9°. For any element Ω for which the set of points x such that $\Omega_x = \Omega$ is a set of class 3 in the

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* Editor's note. The present translation incorporates suggestions made by the author.

Baire-de la Vallée Poussin classification, and only for such elements, there exists a chain containing Ω in which there is no element immediately succeeding Ω.

If the mapping T has cycles of order $\neq 2^i$, $i = 0, 1, 2, \cdots$, then [1] there exists such an element, and consequently the system A does not satisfy the condition that descending chains terminate.

The proofs of assertions 1°–9° are based essentially on the results of the paper [1]. We note a number of further facts ($1^{\circ\circ}$–$3^{\circ\circ}$), following from [1] and used in the proofs of 1°–9°, which are of independent interest.

$1^{\circ\circ}$. If $\Omega', \Omega'' \in A$, the set $\Omega' \cap \Omega''$ is nonempty, and at least one point of $\Omega' \cap \Omega''$ is a limit point on the left (right) for both Ω' and Ω'', then $\Omega' \cup \Omega'' \in A$.

$2^{\circ\circ}$. If $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$, $\Omega_n \in A$, then the closure (in R) of the set $\bigcup_n \Omega_n$ is an element of the system A.

$3^{\circ\circ}$. If $\Omega' \subset \Omega$, $\Omega', \Omega \in A$, and Ω is a maximal element of the second kind, then for any point $x \in \Omega$, $x \overline{\in} \Omega'$ for which $\Omega_x \subseteq \Omega'$ and Ω_x is not a cycle, there exists a set $\Omega'' \in A$ such that $\Omega'' \supset \Omega'$, $\Omega'' \ni x$, and for any point $x' \in \Omega'' \setminus \Omega' \setminus \{T^j x\}_{j=0}^{\infty}$ there is an index m such that $T^m x' = x$, there being only one such point x' for any m.

In the case when T is a smooth mapping (at least continuously differentiable on R), for any element Ω not containing cycles there exists an element $\Omega' \supset \Omega$. Consequently, in this case every maximal element is either a maximal element of the second kind (a perfect set) or a minimal element (a finite set).

In [2] an example is constructed of a continuous, but nondifferentiable mapping having a maximal element of the first kind.

It is interesting to examine the extent to which the partially ordered system A characterizes the mapping. For example, let Ω and Ω' be maximal elements of the mappings T and T', and let A/Ω, A'/Ω' be the partially ordered systems consisting of the attracting sets contained in Ω and Ω' respectively. Will the mappings T and T' be isomorphic, considered on the respective sets Ω and Ω' (that is, $T' = STS^{-1}$ where S is a homeomorphism $\Omega \rightarrow \Omega'$) if the partially ordered systems A/Ω, A'/Ω' are isomorphic? Obviously it must be assumed in addition that the least orders of cycles contained in Ω and Ω' are the same (if there are cycles in Ω and Ω'). It is possible that an analogous result is valid if a partially ordered system is considered that includes not all attracting sets but only certain of them, selected in an appropriate fashion.

It would also be interesting to find a complete set of properties that an arbitrary partially ordered set must possess in order to be isomorphic to the partially ordered system of attracting sets for some continuous mapping.

We remark that on every maximal element Ω of the second kind the mapping T is determined by giving a system of closed subsets on the set $\Omega \subset R$, namely the attracting sets contained in Ω; where if Ω contains a cycle of order k (that is, a set consisting of k points) then there exist not more that $(2k)!$ mappings mutually isomorphic on Ω and having on Ω the given system of attracting sets.

This result is a corollary of $3^{\circ\circ}$ and the proposition:

a) if Ω contains a cycle of order k then for any point $x \in \Omega$ (except for not more than $3k$ points) there exist at least two distinct points $x', x'' \in \Omega$ such that $T^{2k}x' = T^{2k}x'' = x$;

b) for any set $\Omega' \subset \Omega$, the points $x \in \Omega$ for which $\Omega_x = \Omega'$ are everywhere dense in Ω.

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