ATTRACTING AND ATTRACTED SETS

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Let T be a continuous single-valued transformation of a compact space E into itself. Every point $x \in E$ generates an iterating sequence $\{T^j x\}_{j=0}^{\infty}$. If $x, Tx, \cdots, T^{k-1}x$ are pairwise distinct and $T^{k-1}x = x$, then $x, Tx, \cdots, T^{k-1}x$ form a cycle of order k.

The point y is called an *ω-limit point of the sequence* $\{T^j x\}$, if for each neighborhood U of y and every $n > 0$ there exists an $m \geq n$ such that $T^m x \in U$. We denote the set of all ω-limit points of $\{T^j x\}$ by Ω_x.

1. The set $\Omega = \Omega_x$ is closed and $T\Omega = \Omega$. The following two theorems characterize the transformation T on Ω.

Theorem 1. *If U is an open set in Ω, such that $U \neq \Omega$, then the closure of TU is not contained in U.*

The following two corollaries follow from Theorem 1:

Corollary 1. *If $\Omega' \subset \Omega$ is such that $T\Omega' = \Omega'$, then Ω' can not be both closed and open in Ω.*

Corollary 2. *If Ω is finite then the points of Ω form a cycle.*

Corollary 3. *If Ω is infinite then every point of a cycle contained in Ω is a limit point of points of Ω.*

Theorem 2. *If Ω is infinite and $T^j x \in \Omega$ for $j \geq j_0$, then Ω is the derived set of $\bigcup_{j=0}^{\infty} T^j U$ for every set U which is open in Ω.*

If Ω has interior points in E, then there always exists a j_0 such that $T^j x \in \Omega$ for $j \geq j_0$.

Theorems 1 and 2 characterize the transformation on Ω. By this remark we mean the following.

A. If T is a continuous transformation of E into itself such that every closed set Ω with $T\Omega = \Omega$ satisfies the assertion of Theorem 2, then there exists a point $x \in \Omega$ such that $\Omega_x = \Omega$.

B. If a closed set Ω does not have interior points in E, Ω does not contain isolated points of E and T is given on Ω so that $T\Omega = \Omega$ and Theorem 1 is satisfied, then the transformation T can be extended to a closed set E', $\Omega \subset E' \subseteq E$ in such a way that T is continuous on E' and there exists a point $x \in E'$ such that $\Omega_x = \Omega$.

The problem of the structure of ω-limit sets of an iterating sequence reduces to the following: what is the structure of a closed set $\Omega \subseteq E$ if one can define a continuous transformation on Ω such that $T\Omega = \Omega$ and such that Theorems 1 and 2 hold for Ω.

One may mention the following result: If the compact space E is locally connected and Ω has interior points in E, then $\Omega = \bigcup_{j=1}^{k} \Omega^{(j)}$, $1 \leq k < \infty$, where $\Omega^{(1)}, \cdots, \Omega^{(k)}$ are connected closed sets which are mutually disjoint and $T\Omega^{(j)} = \Omega^{(j+1)}$, $j = 1, 2, \cdots, k-1$, $T\Omega^{(k)} = \Omega^{(1)}$.

2. We shall say that the point $x \in E$ is *attracted to the set* Ω, if Ω is the set of ω-limit points of the sequence. The set of points of the compactum which are attracted to a given set Ω will be denoted by $P(\Omega)$. From now on we shall denote by the letter Ω those and only those sets (with indices

or without indices) for which $P(\Omega)$ is not empty.

Theorem 3. *$\bigcup_{\Omega' \supseteq \Omega} P(\Omega')$ is a set of type G_δ.*

Let Σ be a system of open sets σ_i, $i = 1, 2, \cdots$, such that: 1) $\sigma_i \cap \Omega \neq 0$, $i = 1, 2, \cdots$; 2) for every neighborhood U of a point $x \in \Omega$ there exists a $\sigma \in \Sigma$ which is contained in U. For every $\sigma_i \in \Sigma$ we construct an open set $S(\sigma_i)$, consisting of all inverse images of σ_i: $x \in S(\sigma_i)$ if and only if there exists an integer $k \geq 0$ such that $T^k x \in \sigma_i$. Then $\bigcap_{i=1}^{\infty} S(\sigma_i) = \bigcup_{\Omega' \supseteq \Omega} P(\Omega')$. Indeed, if $x \in P(\Omega')$, where $\Omega' \supseteq \Omega$, then $x \in S(\sigma_i)$, $i = 1, 2, \cdots$. If $x \in P(\Omega'')$ and Ω'' does not contain Ω, i.e., if there exists a point $y \in \Omega$ which does not belong to Ω'', then there exists $\sigma_{i'} \ni y$ which does not contain any of the points of the sequence $\{T^j x\}_{j=0}$. Hence $x \overline{\in} S(\sigma_{i'})$ and $x \overline{\in} \bigcup_{\Omega' \supseteq \Omega} P(\Omega')$.

Theorem 4. *$\bigcup_{\Omega' \subseteq \Omega} P(\Omega')$ is a set of type $F_{\sigma\delta}$.*

Indeed, if F is an arbitrary closed set, then the set $p(F)$ consisting of the points $x \in E$ such that $T^j x \in F$ for $j \geq j_0$ (j_0 depends on x) is a set of type F_σ. Let us consider a sequence of open sets $U_1 \supset U_2 \supset U_3 \cdots$ such that $\bigcap_{i=1}^{\infty} U_i = \Omega$ and let F_i be the closure of U_i. Let us construct $p(F_i)$, $i = 1, 2, \cdots$. Then $\bigcap_{i=1}^{\infty} p(F_i)$ is also the set $\bigcup_{\Omega' \subseteq \Omega} P(\Omega')$. Indeed, if $x \in P(\Omega')$, $\Omega' \subseteq \Omega$, then $x \in p(F_i)$, $i = 1, 2, \cdots$. If $x \in P(\Omega'')$ and there exists a point $y \in \Omega''$ which does not belong to Ω, then there exists an i_0 such that $y \overline{\in} F_{i_0}$ and then $x \overline{\in} p(F_i)$, $i \geq i_0$.

Corollary. *$P(\Omega)$ is a set of type $F_{\sigma\delta}$.*

If $\Omega \cap P(\Omega)$ is not empty, then it follows from Theorem 3 that $P(\Omega)$ is a set in Ω of type G_δ of second category.

The problem arises as to whether there exist transformations for which the set $P(\Omega)$ is a set of type G_δ or $F_{\sigma\delta}$ but is not a set of a simpler type. The theorems formulated below give an affirmative answer to this question.

3. Let us consider the case when E is a segment of the real line.

Theorem 5. *If Ω is infinite, then $P(\Omega)$ is a set of class ≥ 1 in the Baire-de la Vallée-Poussin classification.*

Corollary. *If Ω is infinite and there does not exist a set $\Omega' \supset \Omega$ then $P(\Omega)$ is a G_δ set but not an F_σ set.*

Theorem 5 follows from Lemma 1.

Lemma 1. *If the conditions of Theorem 5 are fulfilled then* 1) *in every neighborhood of each point $X \in P(\Omega)$ there is an $x' < x$, $x' \in P(\Omega)$, (if x is not the left end point of E) and an $x'' > x$, $x'' \in P(\Omega)$ (if x is not the right end point of E);* 2) *if the interval $(a, b) \subset P(\Omega)$, then also $a, b \in P(\Omega)$.*

Theorem 6. *If Ω contains a cycle and there exists an $\Omega' \supset \Omega$, then $\bigcup_{\Omega' \supseteq \Omega} P(\Omega')$ is a set of second class.*

The proof of Theorem 6 depends on Lemmas 1 and 2.

Lemma 2. *If $(a, b) \subset \bigcup_{\Omega' \supseteq \Omega} P(\Omega')$ and Ω contains a cycle, then there exists an Ω' such that $(a, b) \subset P(\Omega')$.*

Lemma 2 is probably also valid even if Ω does not contain cycles.

Theorem 7. *If Ω contains a cycle and* 1) *there exists an $\Omega' \supset \Omega$;* 2) *in each neighborhood U of Ω there exists an $x \in P(\Omega)$, $x \overline{\in} \Omega$, $T^j x \in U$, $j = 0, 1, 2, \cdots$, then $P(\Omega)$ is a set of third class in the Baire-de la Vallée-Poussin classification.*

Thus $P(\Omega)$ is (under the conditions of the theorem) an $F_{\sigma\delta}$ set and not a $G_{\delta\sigma}$ set.

For instance, if the transformation is $Tx = x - x \sin(1/x)$ defined on $[0, 1]$ then the set of points x such that $T^j x \to 0$ as $j \to \infty$ is exactly an $F_{\sigma\delta}$ set.

If Ω is infinite and contains at least one isolated point (and hence also a denumerable number of isolated points), then condition 2) of the theorem holds.

The following is the summary of the proof of Theorem 7: 1) We choose a subset J of the set $\bigcup_{\Omega' \supseteq \Omega} P(\Omega')$ which is homeomorphic to the set of irrational points [1]; 2) we prove that $P(\Omega) \cap J$ can be obtained by the same means and from the same elements as a Baire set of third class [2].

We obtain Theorem 8 as a corollary of Theorems 6 and 7.

Theorem 8. *If the conditions of Theorem 7 are fulfilled then* $\bigcup_{\Omega' \subseteq \Omega} P(\Omega')$ *is a set of third class.*

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