Symmetry in Nonlinear Mathematical Physics 1997, V.2, 488–491.

On One Problem of Automatic Control with Turning Points

Valentin LEIFURA

Mykolaiv Teachers Training Institute, Mykolaiv, Ukraine

Abstract

An algorithm for solutions of one problem of automatic control is suggested for the case of an interval containing turning points.

Let us consider an equation of the form

$$\varepsilon^2 \frac{d^2x(t,\varepsilon)}{dt^2} + \varepsilon a(\tau,\varepsilon)\frac{dx(t,\varepsilon)}{dt} + b(\tau,\varepsilon)x(t,\varepsilon) = \varepsilon h(\tau) \int\limits_{-\infty}^{t} G(t-t',\tau')x(t',\varepsilon)dt', \tag{1}$$

where ε is a small parameter, $\varepsilon \in (0;\varepsilon_0]$, $\varepsilon_0 \le 1$, $\tau = \varepsilon t$, $t \in [0, L]$. The left-hand side of the equation defines automatic control, where $G(t-t',\tau')$ is the impulse transfer function of a regulator. Functions $a(\tau,\varepsilon)$, $b(\tau,\varepsilon)$ admit the following decomposition:

$$a(\tau,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i a_i(\tau), \qquad b(\tau,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i b_i(\tau). \tag{2}$$

In papers [1–2], automatic control systems described by equation (1) were studied in detail, but turning points were not considered. It was assumed that the characteristic equation, i.e., the equation

$$\lambda_0^2(\tau) + a_0(\tau)\lambda_0(\tau) + b_0(\tau) = 0, \tag{3}$$

has two roots for all $\tau \in [0, \varepsilon L]$.

For the case where equation (1) is considered on the interval $[\alpha,\beta]$, which does not contain any turning points, the following theorem is true:

Theorem 1. *If functions $a_i(\tau)$, $b_i(\tau)$ are continuous and differentiable infinite number of times on $[\varepsilon\alpha, \varepsilon\beta]$ and $a_0^2(\tau) - 4b_0(\tau) \ne 0$, then, on the interval $[\alpha,\beta]$, equation (1) has two linearly independent formal solutions of the form*

$$x_1(t,\varepsilon) = \exp\left(\frac{1}{\varepsilon}\int\limits_{\alpha}^{t} \varphi_1(\tau',\varepsilon)dt'\right), \qquad \varphi_1(\tau,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \varphi_{1i}(\tau), \tag{4}$$

$$x_2(t,\varepsilon) = \exp\left(\frac{1}{\varepsilon}\int\limits_{\alpha}^{t} \varphi_2(\tau',\varepsilon)dt'\right), \qquad \varphi_2(\tau,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \varphi_{2i}(\tau), \tag{5}$$

where $\varphi_{1i}(\tau)$, $\varphi_{2i}(\tau)$ *are continuous and infinitely differentiable on* $[\varepsilon\alpha, \varepsilon\beta]$.

An algorithm for determination of functions $\varphi_{1i}(\tau)$, $\varphi_{2i}(\tau)$ is adduced in [2].

Along with equation (1), we consider also the following auxiliary equation

$$\frac{d^2 z(\xi,\mu)}{d\xi^2} + \widetilde{a}(\eta,\mu)\frac{dz(\xi,\mu)}{d\xi} + \widetilde{b}(\eta,\mu)z(\xi,\mu) = \\ = \mu\widetilde{h}(\eta,\mu)\int\limits_{-\infty}^{\xi} G\left((\xi-\xi')\mu, \mu^2 t_0 + \mu\eta'\right) z(\xi',\mu)d\xi', \tag{6}$$

where $\xi \in [\alpha;\beta]$, $\mu \in (0;\mu_0)$, $\eta = \xi\mu^2$, $t_0 \in [\alpha\mu;\beta\mu]$, and functions $\widetilde{a}(\eta,\mu)$, $\widetilde{b}(\eta,\mu)$, $\widetilde{h}(\eta,\mu)$ admit the following decomposititon:

$$\widetilde{a}(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i\widetilde{a}_i(\eta), \qquad \widetilde{b}(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i\widetilde{b}_i(\eta), \qquad \widetilde{h}(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i h_i(\eta).$$

Theorem 2. *If functions* $\widetilde{a}_i(\eta)$, $\widetilde{b}_i(\eta)$, $\widetilde{h}_i(\eta)$ *are infinitely differentiable on the interval* $[\alpha\mu^2\beta\mu^2]$ *and* $\widetilde{a}_0^2(\eta) - 4\widetilde{b}_0(\eta) \neq 0$, *then, on the interval* $[\alpha,\beta]$, *equation (6) has two linearly independent formal solutions of the form*

$$z_1(\xi,\mu) = \exp\left(\int\limits_{\alpha}^{\xi}\lambda_1(\eta',\mu)d\xi'\right), \qquad \lambda_1(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i\lambda_{1i}(\eta),$$

$$z_2(\xi,\mu) = \exp\left(\int\limits_{\alpha}^{\xi}\lambda_2(\eta',\mu)d\xi'\right), \qquad \lambda_2(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i\lambda_{2i}(\eta),$$

where $\lambda_{1i}(\tau)$, $\lambda_{2i}(\tau)$ *are infinitely differentiable functions on* $[\mu^2\alpha, \mu^2\beta]$.

The proof of this theorem is similar to that of Theorem 1.

Let us go back to the consideration of equation (1) under condition of the existence of a zero turning point. Let functions $a_0(\tau)$, $b_0(\tau)$ have the form $a_0 = \tau a_{10}(\tau)$, $b_0 = \tau^2 b_{10}(\tau)$. Then roots of the characteristic equation have the form

$$\lambda_{1,2}(\tau) = -\frac{1}{2}\tau\left(a_{10}(\tau) \pm \sqrt{a_{10}^2(\tau) - 4b_{10}(\tau)}\right). \tag{7}$$

If $a_{10}^2(\tau) - 4b_{10}(\tau) \neq 0$ $\forall t \in (0;L]$, then $t = 0$ is a zero turning point.

Let us consider the Cauchy problem for equation (1) with the initial conditions

$$x(\beta\sqrt{\varepsilon},\varepsilon) = y_{10}, \qquad \left.\frac{dx}{dt}\right|_{t=\beta\sqrt{\varepsilon}} = y_{20}, \tag{8}$$

where β is a real number, y_{10} and y_{20} are quantities which do not depend on ε.

First using Theorem 1 for the case $t \in [\beta\sqrt{\varepsilon}, L]$, we construct the m-approximation for a general solution of equation (1) using the formula

$$x_m^{(2)}(t,\varepsilon) = a_1^{(2)}(\varepsilon)x_{1m}(t,\varepsilon) + a_2^{(2)}(\varepsilon)x_{2m}(\varepsilon),$$

where

$$x_m^{(2)}(t,\varepsilon) = \exp\left(\frac{1}{\varepsilon}\int\limits_{\beta\sqrt{\varepsilon}}^{t}\left(\sum_{i=0}^{m}\varepsilon^i\varphi_{ki}(\tau)\right)dt'\right), \qquad k=1,2,$$

$a_k^{(2)}(\varepsilon)$, $k=1,2$ are arbitrary constants which are determined from condition (8).

Finally we construct a solution of equation (1) within the vicinity of a turning point or when $t\in[0,\beta\sqrt{\varepsilon}]$.

After the substitution $t=\xi\sqrt{\varepsilon}$ ($\xi\in[0,\beta)$) using the notations $\mu=\sqrt{\varepsilon}$, $\xi\mu^2=\eta$, $x=z(\xi,\mu)$ equation (1) takes the form

$$\begin{aligned}\frac{d^2z(\xi,\mu)}{d\xi^2} &+ \eta a_{10}(\eta\mu) + \sum_{i=1}^{\infty}\mu^{2i-1}a_i(\eta\mu)\frac{dz(\xi,\mu)}{d\xi} + \\ &+\left(\eta^2 b_{10}(\eta\mu) + \sum_{i=1}^{\infty}b_i(\eta\mu)\mu^{2i-2}\right)z(\xi,\mu) = \\ &= \mu h(\eta\mu)\int\limits_{-\infty}^{\xi}G\left((\xi-\xi')\mu,\eta'\mu\right)z(\xi',\mu)d\xi'.\end{aligned} \tag{9}$$

Let us assume that the coefficients $a_{10}(\tau)$, $b_{10}(\tau)$, $a_i(\tau), b_i(\tau)$, $i=1,\dots,8$, and function $h(\tau)$ can be decomposed into a Taylor series within the neighbourhood of the point $\tau=0$. Then

$$a_{10}(\eta\mu) = a_{10}(0) + \sum_{i=1}^{\infty}\mu^i\frac{a_{10}^{(i)}(0)}{i!}\eta^i, \qquad a_i(\eta\mu) = a_i(0) + \sum_{j=1}^{\infty}\mu^j\frac{a_i^{(j)}(0)}{j!}\eta^j,$$

$$\sum_{i=1}^{\infty}\mu^{2i-1}a_i(\eta\mu) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\mu^{2i+j-1}\frac{a_i^{(j)}(0)}{j!}\eta^j = \sum_{i=1}^{\infty}\mu^i\widetilde{a}_i(\eta),$$

whence

$$\eta a_{10}(\eta\mu) + \sum_{i=1}^{\infty}\mu^{2i-1}a_i(\eta\mu) = \eta a_{10}(0) + \sum_{i=1}^{\infty}\mu^i\widetilde{a}_i(\eta). \tag{10}$$

Similarly, we get

$$\eta^2 b_{10}(\eta\mu) + \sum_{i=1}^{\infty}\mu^{2i-2}b_i(\eta\mu) = b_{10}(0)\eta^2 + \sum_{i=1}^{\infty}\mu^i\widetilde{b}_i(\eta) + b_1(0). \tag{11}$$

Finally, for $h(\eta\mu)$, we have

$$h(\eta\mu) = \sum_{i=1}^{\infty}\frac{h^{(i)}(0)}{i!}(\eta\mu)^i = \sum_{i=1}^{\infty}\mu^i\widetilde{h}_i(\eta). \tag{12}$$

Substituting (10)–(12) into (9), we get

$$\begin{aligned}&\frac{d^2 z(\xi,\mu)}{d\xi^2}+\left(\eta a_{10}(0)+\sum_{i=1}^{\infty}\mu^i\widetilde{a}_i(\eta)\right)\frac{dz(\xi,\mu)}{d\xi}+\\&\qquad+\left(b_{10}(0)\eta^2+b_1(0)+\sum_{i=1}^{\infty}\mu^i\widetilde{b}_i(\eta)\right)z(\xi,\mu)=\\&\qquad=\mu\sum_{i=1}^{\infty}\mu^i\widetilde{h}_i(\eta)\int_{-\infty}^{\xi}G\left((\xi-\xi')\mu,\eta'\mu\right)z(\xi',\mu)d\xi'.\end{aligned}\tag{13}$$

If $\forall\xi\in[0,\beta)$ $\eta^2(a_{10}^2(0)-4b_{10}(0))-4b_1(0)\neq 0$, then, according to Theorem 2, equation (13) has the general solution, the m-approximation of which can be constructed using the formula

$$x_m^{(1)}(t,\varepsilon)=a_1^{(1)}(\xi)x_{1m}^{(1)}(t,\varepsilon)+a_1^{(1)}(\xi)x_{2m}^{(1)}(t,\varepsilon),\qquad t\in[0,\beta\sqrt{\varepsilon}],$$

where

$$x_{km}^{(1)}(t,\varepsilon)=\exp\left(\frac{1}{\sqrt{\varepsilon}}\int_{\beta\sqrt{\varepsilon}}^{t}\lambda_{km}\left(\frac{\tau'}{\sqrt{\varepsilon}},\sqrt{\varepsilon}\right)dt'\right),\qquad k=1,2,$$

$$\lambda_{km}\left(\frac{\tau}{\sqrt{\varepsilon}},\sqrt{\varepsilon}\right)=\sum_{i=m}^{m}(\sqrt{\varepsilon})^i\lambda_{ki}\left(\frac{\tau}{\sqrt{\varepsilon}}\right).$$

The constants $a_1^{(1)}(\varepsilon)$, $a_2^{(1)}(\varepsilon)$ are determined as solutions of the system $x_m^{(1)}(\beta\sqrt{\varepsilon},\varepsilon)=y_{10}$, $dx_m^{(1)}\Big|_{t=\beta\sqrt{\varepsilon}}=y_{20}$. Thus, for equation (1) with a zero turning point, we have constructed a continuous (together with its derivative) m-approximation of the solution

$$x_m(t,\varepsilon)=\begin{cases}x_m^{(1)}(t,\varepsilon), & t\in[0,\beta\sqrt{\varepsilon}],\\ x_m^{(2)}(t,\varepsilon), & t\in[\beta\sqrt{\varepsilon},L],\end{cases}$$

which satisfies the initial conditions (8).

Asymptotic character of the solution obtained (as defined in [3]) is established by using the methods suggested in [4].

References

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