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Symmetry of Burgers-type Equations with an Additional Condition

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Abstract

We study symmetry properties of Burgers-type equations with an additional condition.

It is well known that the Burgers equation

$$u_t + uu_x + \lambda u_{xx} = 0, \qquad \lambda = \mathrm{const} \tag{1}$$

can be reduced by means of the Cole-Hopf non-local transformation

$$u = 2\lambda \frac{\psi_x}{\psi} \tag{2}$$

to the linear heat equation

$$\psi_t - \lambda \psi_{xx} = 0. \tag{3}$$

We note that the symmetry of the heat equation (3) is wider than the symmetry of the Burgers equation (1) [1].

In [2], the symmetry classification of the following generalization of the Burgers equation

$$u_t + uu_x = F\left(u_{xx}\right) \tag{4}$$

is carried out. In the general case, equation (4) with an arbitrary function $F(u_{xx})$ is invariant with respect to the Galilei algebra $AG(1,1)$. Equation (4) admits a wider symmetry only in the following cases [2]:

$$F(u_{xx}) = \lambda u_{xx}^k, \tag{5}$$

$$F(u_{xx}) = \ln u_{xx}, \tag{6}$$

$$F(u_{xx}) = \lambda u_{xx}, \tag{7}$$

$$F(u_{xx}) = \lambda u_{xx}^{1/3}, \tag{8}$$

where k, λ are arbitrary constants.

This paper contains the symmetry classification of equation (4), where $F(U_{xx})$ is determined by relations (6)-(8), with the additional condition which is a generalization of (2) of the following form

$$\psi_{xx} + f^1(u)\psi_x + f^2(u)\psi = 0. \tag{9}$$

Let us consider the system

$$\begin{aligned} &u_t + uu_x + \lambda u_{xx} = 0, \\ &\psi_{xx} + f^1(u)\psi_x + f^2(u)\psi = 0. \end{aligned} \tag{10}$$

Theorem 1. *Maximal invariance algebras of system (10) depending on functions $f^1(u)$ and $f^2(u)$ are the following Lie algebras:*
1) $\langle P_0, P_1, X_1\rangle$ if $f^1(u)$, $f^2(u)$ are arbitrary, where

$$P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad X_1 = b(t)\psi\partial_\psi;$$

2) $\langle P_0, P_1, X_1, X_2\rangle$ if $f^1(u)$ is arbitrary, $f^2 = 0$, where

$$X_2 = h(t)\partial_\psi;$$

3) $\langle P_0, P_1, X_1, X_3, X_4, X_5\rangle$ if $f^1(u) = au + b$, $f^2(u) = \frac{1}{4}a^2u^2 + \frac{1}{2}abu + d$, where

$$\begin{aligned} &X_3 = t\partial_x + \partial_u - \frac{1}{2}ax\psi\partial_\psi \qquad X_4 = 2t\partial_t + x\partial_x - u\partial_u - \frac{1}{2}bx\psi\partial_\psi, \\ &X_5 = t^2\partial_t + tx\partial_x + (x - tu)\partial_u - \frac{1}{4}(2btx + ax^2)\psi\partial_\psi, \end{aligned}$$

4) $\langle P_0, P_1, X_1, X_3, X_4, X_5, R_1, R_2\rangle$ if $f^1 = b$, $f^2 = d$ (b, d are arbitrary constants), where

$$\left.\begin{aligned} &R_1 = C_1(t)\exp\left(-\frac{1}{2}(b + \sqrt{b^2 - 4d})x\right), \\ &R_2 = C_2(t)\exp\left(-\frac{1}{2}(b - \sqrt{b^2 - 4d})x\right), \end{aligned}\right. \qquad if \quad b^2 - 4d > 0,$$

$$\left.\begin{aligned} &R_1 = C_1(t)\exp\left(-\frac{b}{2}x\right), \\ &R_2 = xC_2(t)\exp\left(-\frac{b}{2}x\right), \end{aligned}\right. \qquad if \quad b^2 - 4d = 0,$$

$$\left.\begin{aligned} &R_1 = C_1(t)\exp\left(-\frac{b}{2}x\right)\cos\frac{\sqrt{4d - b^2}}{2}x, \\ &R_2 = C_2(t)\exp\left(-\frac{b}{2}x\right)\sin\frac{\sqrt{4d - b^2}}{2}x, \end{aligned}\right. \qquad if \quad b^2 - 4d < 0.$$

Let us consider the system

$$\begin{aligned} &u_t + uu_x + \ln u_{xx} = 0, \\ &\psi_{xx} + f^1(u)\psi_x + f^2(u)\psi = 0. \end{aligned} \tag{11}$$

Theorem 2. *Maximal invariance algebras of system (11) depending on functions $f^1(u)$ and $f^2(u)$ are the following Lie algebras:*
1) $\langle P_0, P_1, X_1 \rangle$ if $f^1(u)$, $f^2(u)$ are arbitrary;
2) $\langle P_0, P_1, X_1, X_2 \rangle$ if $f^1(u)$ is arbitrary, $f^2 = 0$;
3) $\langle P_0, P_1, X_1, Q \rangle$ if $f^1(u) = au + b$, $f^2(u) = \frac{1}{4}a^2u^2 + \frac{1}{2}abu + d$, where

$$Q = t\partial_x + \partial_u - \frac{1}{2}ax\psi\partial_\psi;$$

4) $\langle P_0, P_1, X_1, X_3, X_4, X_5, R_1, R_2 \rangle$ if $f^1 = b$, $f^2 = d$ (b, d are arbitrary constants).

Let us consider the system

$$\begin{aligned} &u_t + uu_x + \lambda(u_{xx})^{1/3} = 0, \\ &\psi_{xx} + f^1(u)\psi_x + f^2(u)\psi = 0. \end{aligned} \tag{12}$$

Theorem 3. *Maximal invariance algebras of system (12) depending on functions $f^1(u)$ and $f^2(u)$ are the following Lie algebras:*
1) $\langle P_0, P_1, X_1, Y_1 \rangle$ if $f^1(u)$, $f^2(u)$ are arbitrary, where

$$Y_1 = u\partial_x;$$

2) $\langle P_0, P_1, X_1, X_2, Y_1 \rangle$ if $f^1(u)$ is arbitrary, $f^2 = 0$;
3) $\langle P_0, P_1, X_1, Y_1, Y_2, Y_3, Y_4 \rangle$ if $f^1(u) = au + b$, $f^2(u) = \frac{1}{4}a^2u^2 + \frac{1}{2}abu + d$, where

$$\begin{aligned} &Y_2 = t\partial_x + \partial_u - \frac{1}{2}ax\psi\partial_\psi, \\ &Y_3 = (t^2u - tx)\partial_x + (tu - x)\partial_u + \frac{1}{2}btx\psi\partial_\psi, \\ &Y_4 = \frac{8}{15}t\partial_t + (x - \frac{2}{3}u)\partial_x - \frac{1}{5}u\partial_u - \frac{1}{2}bx\psi\partial_\psi; \end{aligned}$$

4) $\langle P_0, P_1, X_1, Y_1, Y_2, Y_3, Y_4, R_1, R_2 \rangle$ if $f^1 = b$, $f^2 = d$ (b, d are arbitrary constants).

The proof of Theorems 1-3 is carried out by means of the classical Lie algorithm [1].

References

[1] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993.

[2] Fushchych W. and Boyko V., Galilei-invariant higher-order equations of Burgers and Korteweg-de Vries types, *Ukrain. Math. J.*, 1996, V.48, N 12, 1489–1601.