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Symmetry Reduction of a Generalized Complex Euler Equation for a Vector Field

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Abstract

The procedure of constructing linear ansatzes is algorithmized. Some exact solutions of a generalized complex Euler equation for a vector field, invariant under subalgebras of the Poincaré algebra $AP(1,3)$ are found.

In this article, we consider the equation

$$\frac{\partial \Sigma_k}{\partial x_0} + \Sigma_l \frac{\partial \Sigma_k}{\partial x_l} = 0, \quad \Sigma_k = E_k + iH_k \quad (k,l = 1,2,3). \tag{1}$$

It was proposed by W. Fushchych [1] to describe vector fields. This equation can be considered as a complex generalization of the Euler equation for ideal liquid [2]. Equation (1) is equivalent to the system of real equations for $\vec{E} = (E_1, E_2, E_3)$ and $\vec{H} = (H_1, H_2, H_3)$:

$$\begin{cases} \dfrac{\partial E_k}{\partial x_0} + E_l \dfrac{\partial E_k}{\partial x_l} - H_l \dfrac{\partial H_k}{\partial x_l} = 0, \\ \dfrac{\partial H_k}{\partial x_0} + H_l \dfrac{\partial E_k}{\partial x_l} + E_l \dfrac{\partial H_k}{\partial x_l} = 0. \end{cases} \tag{2}$$

It was established in paper [1] that the maximal invariance algebra of system (2) is a 24-dimensional Lie algebra containing the affine algebra $AIGL(4,\mathbb{R})$ with the basis elements

$$\begin{aligned} & P_\alpha = \frac{\partial}{\partial x_\alpha} \quad (\alpha = 0,1,2,3), \quad \Gamma_{a0} = -x_0 \frac{\partial}{\partial x_a} - \frac{\partial}{\partial E_a}, \\ & \Gamma_{00} = -x_0 \frac{\partial}{\partial x_0} + E_l \frac{\partial}{\partial E_l} + H_l \frac{\partial}{\partial H_l} \quad (l = 1,2,3), \\ & \Gamma_{aa} = -x_a \frac{\partial}{\partial x_a} - E_a \frac{\partial}{\partial E_a} - H_a \frac{\partial}{\partial H_a} \ (\text{no sum over } a), \\ & \Gamma_{0a} = -x_a \frac{\partial}{\partial x_0} + (E_a E_k - H_a H_k) \frac{\partial}{\partial E_k} + (E_a H_k + H_a E_k) \frac{\partial}{\partial H_k}, \\ & \Gamma_{ac} = -x_c \frac{\partial}{\partial x_a} - E_c \frac{\partial}{\partial E_a} - H_c \frac{\partial}{\partial H_a} \quad (a \neq c; \ a,c = 1,2,3). \end{aligned} \tag{3}$$

The algebra $AIGL(4,\mathbb{R})$ contains as a subalgebra the Poincaré algebra $AP(1,3)$ with the basis elements

$$J_{0a} = -\Gamma_{0a} - \Gamma_{a0}, \quad J_{ab} = \Gamma_{ba} - \Gamma_{ab}, \quad P_\alpha \quad (a,b = 1,2,3; \ \alpha = 0,1,2,3).$$

The purpose of our investigation is to construct invariant solutions to system (2) by reducing this system to systems of ordinary differential equations on subalgebras of the algebra $AP(1,3)$.

On the basis of Proposition 1 [3] and the necessary existence condition for nondegenerate invariant solutions [4], we obtain that to perform the reduction under consideration, we need the list of three-dimensional subalgebras of the Poincaré algebra $AP(1,3)$ with only one main invariant of the variables x_0, x_1, x_2, x_3. We can consider subalgebras of the algebra $AP(1,3)$ up to affine conjugacy.

Let us denote $G_a = J_{0a} - J_{a3}$ $(a = 1,2)$.

Proposition 1. *Up to affine conjugacy, three-dimensional subalgebras of the algebra $AP(1,3)$, having only one main invariant depending on the variables x_0, x_1, x_2, x_3, are exhausted by the following subalgebras:*

$$\langle P_1,P_2,P_3\rangle,\ \langle J_{12}+\alpha J_{03},P_0,P_3\rangle,\ \langle J_{12}+\alpha J_{03},P_1,P_2\rangle\ (\alpha\neq 0),\ \langle J_{03},P_1,P_2\rangle,$$
$$\langle G_1,P_0+P_3,P_2+\alpha P_1\rangle,\ \langle G_1,G_2,P_0+P_3\rangle,\ \langle G_1,J_{03},P_2\rangle,\ \langle J_{12},J_{03},P_0+P_3\rangle,$$
$$\langle J_{03}+P_1,P_0,P_3\rangle,\ \langle G_1,G_2,J_{12}+\alpha J_{03}\rangle\ (\alpha>0),\ \langle J_{12}+P_0,P_1,P_2\rangle,\ \langle G_1,G_2,J_{03}\rangle,$$
$$\langle J_{03}+\gamma P_1,P_0+P_3,P_2\rangle\ (\gamma=0,1),\ \langle G_1+P_2,P_0+P_3,P_1\rangle,$$
$$\langle G_1+P_0-P_3,P_0+P_3,P_2\rangle,\ \langle G_1+P_0-P_3,P_0+P_3,P_1+\alpha P_2\rangle,$$
$$\langle G_1,G_2+P_2,P_0+P_3\rangle,\ \langle G_1,J_{03}+\alpha P_1+\beta P_2,P_0+P_3\rangle,$$
$$\langle G_1+P_2,G_2-P_1+\beta P_2,P_0+P_3\rangle,\ \langle G_1,G_2,J_{12}+P_0+P_3\rangle.$$

To obtain this list of subalgebras, we should apply the affine conjugacy to the list of subalgebras of the algebra $AP(1,3)$, considered up to $P(1,3)$-conjugacy [5]. In so doing, in particular, we may identify all one-dimensional subspaces of the translation space $\langle P_0,P_1,P_2,P_3\rangle$.

The linear span Q of a system of operators, obtained from basis (3) by excluding the operators Γ_{0a} $(a = 1,2,3)$, forms a Lie subalgebra of the algebra $AIGL(4,\mathbb{R})$. Each operator $Y \in Q$ can be presented as

$$Y = a_\alpha(x)\frac{\partial}{\partial x_\alpha} + b_{ij}\left(E_j\frac{\partial}{\partial E_i} + H_j\frac{\partial}{\partial H_i}\right) + c_i\frac{\partial}{\partial E_i}, \qquad (4)$$

where $x = (x_0,x_1,x_2,x_3)$; b_{ij}, c_i are real numbers; $\alpha = 0,1,2,3$; $i,j = 1,2,3$.

Definition. An invariant of a subalgebra L of the algebra Q, that is a linear function in the variables E_a, H_a $(a = 1,2,3)$, is called *linear*. A vector function $\vec F(\vec E,\vec H)$ is called a *linear invariant* of a subalgebra L if its components are linear invariants of this subalgebra.

Let

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix},\ \vec C = \begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix},\ U = \begin{pmatrix} u_{11}(x) & u_{12}(x) & u_{13}(x)\\ u_{21}(x) & u_{22}(x) & u_{23}(x)\\ u_{31}(x) & u_{32}(x) & u_{33}(x)\end{pmatrix},\ \vec V = \begin{pmatrix} v_1(x)\\ v_2(x)\\ v_3(x)\end{pmatrix}.$$

Theorem. *The vector function $U\vec E + \vec V$ is a linear invariant of the operator Y if and only if*

$$a_\alpha(x)\frac{\partial U}{\partial x_\alpha} + UB = 0, \quad a_\alpha(x)\frac{\partial \vec V}{\partial x_\alpha} + U\vec C = \vec 0.$$

The vector function $U\vec{H}$ is a linear invariant of the operator Y if and only if

$$a_\alpha(x)\frac{\partial U}{\partial x_\alpha} + UB = 0.$$

Proposition 2. *Let*

$$Y_j = a_\alpha^{(j)}(x)\frac{\partial}{\partial x_\alpha} + \sum_{i,k=1}^{3} b_{ik}^{(j)}\left(E_k\frac{\partial}{\partial E_i} + H_k\frac{\partial}{\partial H_i}\right) + \sum_{i=1}^{3} c_i^{(j)}\frac{\partial}{\partial E_i} \quad (j = 1, 2, 3) \tag{5}$$

be operators of the form (4) and their corresponding matrices B_1, B_2, B_3 be linearly independent and satisfy the commutation relations

$$[B_3, B_j] = B_j \quad (j = 1, 2), \quad [B_1, B_2] = 0.$$

The vector function $U\vec{H}$ with the matrix $U = \prod_{i=1}^{3} \exp[f_i(x)B_i]$ is a linear invariant of the algebra $\langle Y_1, Y_2, Y_3\rangle$ if and only if

$$a_\alpha^{(i)}(x)\frac{\partial f_j}{\partial x_\alpha} + g_{ij}(x) = 0, \tag{6}$$

where $i, j = 1, 2, 3$ and $(g_{ij}) = diag\,[\mathrm{e}^{f_3}, \mathrm{e}^{f_3}, 1]$.

Proof. On the basis of the Campbell-Hausdorff formula, we have

$$\exp(\theta B_3)\cdot B_j\cdot\exp(-\theta B_3) = B_j + \frac{\theta}{1!}B_j + \frac{\theta^2}{2!}B_j + \cdots = \mathrm{e}^{\theta}\, B_j \quad (j = 1, 2).$$

Therefore,

$$B_j \exp(\theta B_3) = \mathrm{e}^{-\theta} \exp(\theta B_3) B_j \quad (j = 1, 2).$$

For this reason

$$a_\alpha^{(j)}(x)\frac{\partial U}{\partial x_\alpha} = a_\alpha^{(j)}(x)\frac{\partial f_1}{\partial x_\alpha}\exp(f_1B_1)B_1\exp(f_2B_2)\exp(f_3B_3) +$$
$$+a_\alpha^{(j)}(x)\frac{\partial f_2}{\partial x_\alpha}\exp(f_1B_1)\exp(f_2B_2)B_2\exp(f_3B_3) +$$
$$+a_\alpha^{(j)}(x)\frac{\partial f_3}{\partial x_\alpha}\exp(f_1B_1)\exp(f_2B_2)\exp(f_3B_3)fB_3 =$$
$$= a_\alpha^{(j)}(x)\frac{\partial f_1}{\partial x_\alpha}\,\mathrm{e}^{-f_3}\,UB_1 + a_\alpha^{(j)}(x)\frac{\partial f_2}{\partial x_\alpha}\,\mathrm{e}^{-f_3}\,UB_2 + a_\alpha^{(j)}(x)\frac{\partial f_3}{\partial x_\alpha}UB_3.$$

It follows from this that $a_\alpha^{(1)}(x)\frac{\partial U}{\partial x_\alpha} + UB_1 = 0$ if and only if

$$a_\alpha^{(1)}(x)\frac{\partial f_1}{\partial x_\alpha}\,\mathrm{e}^{-f_3} + 1 = 0, \quad a_\alpha^{(1)}(x)\frac{\partial f_2}{\partial x_\alpha} = 0, \quad a_\alpha^{(1)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0.$$

Similarly, $a_\alpha^{(2)}(x)\frac{\partial U}{\partial x_\alpha} + UB_2 = 0$ if and only if

$$a_\alpha^{(2)}(x)\frac{\partial f_1}{\partial x_\alpha} = 0, \quad a_\alpha^{(2)}(x)\frac{\partial f_2}{\partial x_\alpha}\,\mathrm{e}^{-f_3} + 1 = 0, \quad a_\alpha^{(2)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0.$$

Finally, $a_\alpha^{(3)}(x)\dfrac{\partial U}{\partial x_\alpha} + UB_3 = 0$ if and only if

$$a_\alpha^{(3)}(x)\frac{\partial f_1}{\partial x_\alpha} = 0, \quad a_\alpha^{(3)}(x)\frac{\partial f_2}{\partial x_\alpha} = 0, \quad a_\alpha^{(3)}(x)\frac{\partial f_3}{\partial x_\alpha} + 1 = 0.$$

The proposition is proved.

Proposition 3. *Let Y_j $(j = 1, 2, 3)$ be operators (5) and their corresponding matrices B_1, B_2, $B_3 = B_3' + B_3''$ be linearly independent and satisfy the commutation relations*

$$[B_3', B_j] = \rho B_j \quad (j = 1, 2), \quad [B_3'', B_1] = -B_2, \quad [B_3'', B_2] = B_1,$$

$$[B_3', B_3''] = 0, \quad [B_1, B_2] = 0.$$

The vector function $U\vec{H}$ with the matrix $U = \prod\limits_{i=1}^{3} \exp[f_i(x)B_i]$ is a linear invariant of the algebra $\langle Y_1, Y_2, Y_3\rangle$ if and only if functions $f_1(x)$, $f_2(x)$, $f_3(x)$ satisfy system (6), where

$$(g_{ij}) = \begin{pmatrix} e^{\rho f_3}\cos f_3 & -e^{\rho f_3}\sin f_3 & 0 \\ e^{\rho f_3}\sin f_3 & e^{\rho f_3}\cos f_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Let us apply the Campbell-Hausdorff formula:

$$\begin{aligned}
&\exp(\theta B_3)(\gamma B_1 + \delta B_2)\exp(-\theta B_3) = \\
&\quad = \exp(\theta B_3'')\exp(\theta B_3')(\gamma B_1 + \delta B_2)\exp(\theta B_3')\exp(\theta B_3'') = \\
&\quad = e^{\rho\theta}\left\{\gamma B_1 + \delta B_2 + \frac{\theta}{1!}(-\gamma B_2 + \delta B_1) + \frac{\theta^2}{2!}(-\gamma B_1 - \delta B_2) + \cdots\right\} = \\
&\quad = e^{\rho\theta}\left\{(\gamma B_1 + \delta B_2)\cos\theta + (-\gamma B_2 + \delta B_1)\sin\theta\right\} = \\
&\quad = e^{\rho\theta}\left\{(\gamma\cos\theta + \delta\sin\theta)B_1 + (\delta\cos\theta - \gamma\sin\theta)B_2\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\gamma B_1 + \delta B_2)\exp(\theta B_3) = \\
&\qquad = e^{-\rho\theta}\exp(\theta B_3)\left\{(\gamma\cos\theta - \delta\sin\theta)B_1 + (\delta\cos\theta + \gamma\sin\theta)B_2\right\}.
\end{aligned}$$

On the basis of the formula obtained, we get

$$\begin{aligned}
a_\alpha^{(j)}(x)\frac{\partial U}{\partial x_\alpha} &= a_\alpha^{(j)}(x)\frac{\partial f_1}{\partial x_\alpha}U\,e^{-\rho f_3}(\cos f_3 B_1 + \sin f_3 B_2) + \\
&+ a_\alpha^{(j)}(x)\frac{\partial f_2}{\partial x_\alpha}U\,e^{-\rho f_3}(-\sin f_3 B_1 + \cos f_3 B_2) + a_\alpha^{(j)}(x)\frac{\partial f_3}{\partial x_\alpha}UB_3.
\end{aligned}$$

It follows from this that

$$a_\alpha^{(1)}(x)\frac{\partial U}{\partial x_\alpha} + UB_1 = 0$$

if and only if

$$\begin{cases} a^{(1)}_\alpha(x)\dfrac{\partial f_1}{\partial x_\alpha}\cos f_3 - a^{(1)}_\alpha(x)\dfrac{\partial f_2}{\partial x_\alpha}\sin f_3 + \mathrm{e}^{\rho f_3} = 0, \\ a^{(1)}_\alpha(x)\dfrac{\partial f_1}{\partial x_\alpha}\sin f_3 + a^{(1)}_\alpha(x)\dfrac{\partial f_2}{\partial x_\alpha}\cos f_3 = 0, \\ a^{(1)}_\alpha(x)\dfrac{\partial f_3}{\partial x_\alpha} = 0. \end{cases}$$

If we consider the obtained system as a linear inhomogeneous system in the variables $a^{(1)}_\alpha(x)\dfrac{\partial f_1}{\partial x_\alpha}$, $a^{(1)}_\alpha(x)\dfrac{\partial f_2}{\partial x_\alpha}$, $a^{(1)}_\alpha(x)\dfrac{\partial f_3}{\partial x_\alpha}$, then it is equivalent to the system

$$a^{(1)}_\alpha(x)\frac{\partial f_1}{\partial x_\alpha} = -\,\mathrm{e}^{\rho f_3}\cos f_3, \quad a^{(1)}_\alpha(x)\frac{\partial f_2}{\partial x_\alpha} = \mathrm{e}^{\rho f_3}\sin f_3, \quad a^{(1)}_\alpha(x)\frac{\partial f_3}{\partial x_\alpha} = 0.$$

Reasoning similarly, we obtain that

$$a^{(2)}_\alpha(x)\frac{\partial U}{\partial x_\alpha} + UB_2 = 0$$

if and only if

$$\begin{cases} a^{(2)}_\alpha(x)\dfrac{\partial f_1}{\partial x_\alpha}\cos f_3 - a^{(2)}_\alpha(x)\dfrac{\partial f_2}{\partial x_\alpha}\sin f_3 = 0, \\ a^{(2)}_\alpha(x)\dfrac{\partial f_1}{\partial x_\alpha}\sin f_3 + a^{(2)}_\alpha(x)\dfrac{\partial f_2}{\partial x_\alpha}\cos f_3 + \mathrm{e}^{\rho f_3} = 0, \\ a^{(2)}_\alpha(x)\dfrac{\partial f_3}{\partial x_\alpha} = 0 \end{cases}$$

or

$$a^{(2)}_\alpha(x)\frac{\partial f_1}{\partial x_\alpha} = \mathrm{e}^{\rho f_3}\sin f_3, \quad a^{(2)}_\alpha(x)\frac{\partial f_2}{\partial x_\alpha} = \mathrm{e}^{\rho f_3}\cos f_3, \quad a^{(2)}_\alpha(x)\frac{\partial f_3}{\partial x_\alpha} = 0.$$

The equality

$$a^{(3)}_\alpha(x)\frac{\partial U}{\partial x_\alpha} + UB_3 = 0$$

holds if and only if

$$\begin{cases} a^{(3)}_\alpha(x)\dfrac{\partial f_1}{\partial x_\alpha}\cos f_3 - a^{(3)}_\alpha(x)\dfrac{\partial f_2}{\partial x_\alpha}\sin f_3 = 0, \\ a^{(3)}_\alpha(x)\dfrac{\partial f_1}{\partial x_\alpha}\sin f_3 + a^{(3)}_\alpha(x)\dfrac{\partial f_2}{\partial x_\alpha}\cos f_3 = 0, \\ a^{(3)}_\alpha(x)\dfrac{\partial f_3}{\partial x_\alpha} + 1 = 0 \end{cases}$$

or

$$a^{(3)}_\alpha(x)\frac{\partial f_1}{\partial x_\alpha} = 0, \quad a^{(3)}_\alpha(x)\frac{\partial f_2}{\partial x_\alpha} = 0, \quad a^{(3)}_\alpha(x)\frac{\partial f_3}{\partial x_\alpha} + 1 = 0.$$

The proposition is proved.

Using subalgebras from Proposition 1, we construct ansatzes of the form

$$U\vec{E}+\vec{V}=\vec{M}(\omega),\quad U\vec{H}=\vec{N}(\omega) \tag{7}$$

or

$$\vec{E}=U^{-1}\vec{M}(\omega)-U^{-1}\vec{V},\quad \vec{H}=U^{-1}\vec{N}(\omega), \tag{8}$$

where $\vec{M}(\omega)$, $\vec{N}(\omega)$ are unknown three-component functions, the matrices U, $\vec{V}$ are known, and $\det U\neq 0$ in some domain of the point x space.

Ansatzes of the form (7) or (8) are called *linear*.

Since the generators G_1, G_2, J_{03} are nonlinear differential operators, we act on subalgebras containing them by the inner automorphism corresponding to the element $g=\exp\left(\frac{\pi}{4}X\right)$, where

$$X=-\Gamma_{03}+\Gamma_{30}=x_3\frac{\partial}{\partial x_0}-x_0\frac{\partial}{\partial x_3}+(H_3H_k-E_3E_k)\frac{\partial}{\partial E_k}-(E_3E_k+H_3H_k)\frac{\partial}{\partial H_k}-\frac{\partial}{\partial E_3}.$$

Let us denote $J'_{\alpha\beta}=gJ_{\alpha\beta}g^{-1}$, $P'_\alpha=gP_\alpha g^{-1}$, $G'_\alpha=\frac{\sqrt{2}}{2}gG_\alpha g^{-1}$. It is not difficult to verify that

$$G'_a=J'_{0a}-J'_{a3}=x_0\frac{\partial}{\partial x_a}+x_a\frac{\partial}{\partial x_3}+E_a\frac{\partial}{\partial E_3}+H_a\frac{\partial}{\partial H_3}+\frac{\partial}{\partial E_a}\quad (a=1,2),$$

$$J'_{12}=J_{12}=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}+E_2\frac{\partial}{\partial E_1}-E_1\frac{\partial}{\partial E_2}+H_2\frac{\partial}{\partial H_1}-H_1\frac{\partial}{\partial H_2},$$

$$J'_{03}=-x_0\frac{\partial}{\partial x_0}+x_3\frac{\partial}{\partial x_3}+\sum_{k=1}^{2}\left(E_k\frac{\partial}{\partial E_k}+H_k\frac{\partial}{\partial H_k}\right)+2E_3\frac{\partial}{\partial E_3}+2H_3\frac{\partial}{\partial H_3},$$

$$P'_0=\frac{\sqrt{2}}{2}(P_0+P_3),\quad P'_3=-\frac{\sqrt{2}}{2}(P_0-P_3),\quad P'_1=P_1,\quad P'_2=P_2.$$

Employing some subalgebras from Proposition 1, let us perform the reduction of system (2) to systems of ODEs, on solutions of which we construct the corresponding solutions of system (2). For each of these subalgebras, we write out its corresponding ansatz, the reduced system, particular or general solution of the reduced system, and the corresponding solution of system (2).

1. $\langle G'_1,G'_2,J'_{03}\rangle:\ E_1=\frac{x_1}{x_0}+\frac{1}{x_0}M_1(\omega),\ E_2=\frac{x_2}{x_0}+\frac{1}{x_0}M_2(\omega),$

$$E_3=\frac{x_1^2+x_2^2}{2x_0^2}+\frac{x_1}{x_0^2}M_1(\omega)+\frac{x_2}{x_0^2}M_2(\omega)+\frac{1}{x_0^2}M_3(\omega),\ H_1=\frac{1}{x_0}N_1(\omega),$$

$$H_2=\frac{1}{x_0}N_2(\omega),\ H_3=\frac{x_1}{x_0^2}N_1(\omega)+\frac{x_2}{x_0^2}N_2(\omega)+\frac{1}{x_0^2}N_3(\omega),\ \omega=x_1^2+x_2^2-2x_0x_3,$$

$$\begin{cases}\dot{M}_1(\omega-2M_3)+2N_3\dot{N}_1=0,\quad \dot{M}_2(\omega-2M_3)+2N_3\dot{N}_2=0,\\ \dot{M}_3(\omega-2M_3)-2M_3+M_1^2+M_2^2-N_1^2-N_2^2+2N_3\dot{N}_3=0,\\ \dot{N}_1(\omega-2M_3)-2N_3\dot{M}_1=0,\quad \dot{N}_2(\omega-2M_3)-2N_3\dot{M}_2=0,\\ \dot{N}_3(\omega-2M_3)-2N_3+2M_1N_1+2M_2N_2-2N_3\dot{M}_3=0.\end{cases}$$

The reduced system has the solution

$$M_1 = C_1, \quad M_2 = C_2, \quad M_3 = \frac{1}{2}(C_1^2 + C_2^2 - C_3^2 - C_4^2),$$
$$N_1 = C_3, \quad N_2 = C_4, \quad N_3 = C_1C_3 + C_2C_4,$$

where C_i $(i = \overline{1,4}\,)$ are arbitrary constants. The corresponding invariant solution of system (2) is of the form

$$E_1 = \frac{x_1 + C_1}{x_0}, \quad E_2 = \frac{x_2 + C_2}{x_0},$$
$$E_3 = \frac{x_1^2 + x_2^2 + 2(C_1x_1 + C_2x_2) + C_1^2 + C_2^2 - C_3^2 - C_4^2}{2x_0^2},$$
$$H_1 = \frac{C_3}{x_0}, \quad H_2 = \frac{C_4}{x_0}, \quad H_3 = \frac{C_3x_1 + C_4x_2 + C_1C_3 + C_2C_4}{x_0^2}.$$

2. $\langle G_1', P_3, P_2 + \alpha P_1\rangle : \ E_1 = \dfrac{x_1 - \alpha x_2}{x_0} + M_1(\omega), \ E_2 = M_2(\omega),$

$$E_3 = \frac{(x_1 - \alpha x_2)^2}{2x_0^2} + \frac{x_1 - \alpha x_2}{x_0}M_1(\omega) + M_3(\omega), \quad H_1 = N_1(\omega), \quad H_2 = N_2(\omega),$$

$$H_3 = \frac{x_1 - \alpha x_2}{x_0}N_1(\omega) + N_3(\omega), \ \omega = x_0,$$

$$\begin{cases} \omega\dot{M}_1 + M_1 - \alpha M_2 = 0, \quad \dot{M}_2 = 0, \quad \omega\dot{M}_3 + M_1^2 - \alpha M_1M_2 - N_1^2 + \alpha N_1N_2 = 0, \\ \omega\dot{N}_1 + N_1 - \alpha N_2 = 0, \quad \dot{N}_2 = 0, \quad \omega\dot{N}_3 + 2M_1N_1 - \alpha M_1N_2 - \alpha M_2N_1 = 0. \end{cases}$$

The general solution of the reduced system is

$$M_1 = \frac{C_3}{\omega} + \alpha C_1, \quad M_2 = C_1, \quad M_3 = \frac{C_3^2 - C_4^2}{2\omega^2} + \frac{\alpha(C_1C_3 - C_2C_4)}{\omega} + C_5,$$
$$N_1 = \frac{C_4}{\omega} + \alpha C_2, \quad N_2 = C_2, \quad N_3 = \frac{C_3C_4}{\omega^2} + \frac{\alpha(C_1C_4 + C_2C_3)}{\omega} + C_6,$$

where C_i $(i = \overline{1,6}\,)$ are arbitrary constants.

The corresponding invariant solution of system (2) is of the form

$$E_1 = \frac{x_1 - \alpha x_2 + C_3}{x_0} + \alpha C_1, \quad E_2 = C_1, \quad H_1 = \frac{C_4}{x_0} + \alpha C_2, \quad H_2 = C_2,$$
$$E_3 = \frac{(x_1 - \alpha x_2 + C_3)^2 - C_4^2}{2x_0^2} + \frac{\alpha[C_1(x_1 - \alpha x_2 + C_3) - C_2C_4]}{x_0} + C_5,$$
$$H_3 = \frac{C_4(x_1 - \alpha x_2 + C_3)}{x_0^2} + \frac{\alpha[C_2(x_1 - \alpha x_2 + C_3) + C_1C_4]}{x_0} + C_6.$$

3. $\langle G_1', G_2', J_{12}' + P_3\rangle : \ E_1 = \dfrac{x_1}{x_0} + M_1(\omega)\cos f_3 - M_2(\omega)\sin f_3,$

$$E_2 = \frac{x_2}{x_0} + M_1(\omega)\sin f_3 + M_2(\omega)\cos f_3, \ E_3 = \frac{x_1^2 + x_2^2}{2x_0^2} +$$

$$+\frac{1}{x_0}(x_1\cos f_3+x_2\sin f_3)M_1(\omega)-\frac{1}{x_0}(x_1\sin f_3-x_2\cos f_3)M_2(\omega)+M_3(\omega),$$

$$H_1=N_1(\omega)\cos f_3-N_2(\omega)\sin f_3,\ \ H_2=N_1(\omega)\sin f_3+N_2(\omega)\cos f_3,$$

$$H_3=\frac{1}{x_0}(x_1\cos f_3+x_2\sin f_3)N_1(\omega)-\frac{1}{x_0}(x_1\sin f_3-x_2\cos f_3)N_2(\omega)+N_3(\omega),$$

$$\omega=x_0,\quad f_3=\frac{x_1^2+x_2^2}{2x_0}-x_3.$$

This ansatz reduces system (2) to the system

$$\begin{cases}\dot M_1+M_2M_3-N_2N_3+\dfrac{M_1}{\omega}=0, & \dot M_2-M_1M_3+N_1N_3+\dfrac{M_2}{\omega}=0,\\ M_1^2+M_2^2-N_1^2-N_2^2=0, & \dot N_3=0,\\ \dot N_1+M_3N_2+M_2N_3+\dfrac{N_1}{\omega}=0, & \dot N_2-M_3N_1-M_1N_3+\dfrac{N_2}{\omega}=0.\end{cases}$$

The reduced system has the following particular solution:

$$M_1=N_1=\frac{C_1}{\omega},\quad M_2=N_2=\frac{C_2}{\omega},\quad M_3=N_3=0,$$

where C_1, C_2 are integration constants.

The corresponding solution of system (2), invariant with respect to the subalgebra $\langle G_1', G_2', J_{12}'+P_3\rangle$, is of the form

$$E_1=\frac{x_1}{x_0}+\frac{C_1}{x_0}\cos f_3-\frac{C_2}{x_0}\sin f_3,\ \ E_2=\frac{x_2}{x_0}+\frac{C_1}{x_0}\sin f_3+\frac{C_2}{x_0}\cos f_3,$$

$$E_3=\frac{x_1^2+x_2^2}{2x_0^2}+\frac{C_1}{x_0^2}(x_1\cos f_3+x_2\sin f_3)-\frac{C_2}{x_0^2}(x_1\sin f_3-x_2\cos f_3),$$

$$H_1=\frac{C_1}{x_0}\cos f_3-\frac{C_2}{x_0}\sin f_3,\ \ H_2=\frac{C_1}{x_0}\sin f_3+\frac{C_2}{x_0}\cos f_3,$$

$$H_3=\frac{C_1}{x_0^2}(x_1\cos f_3+x_2\sin f_3)-\frac{C_2}{x_0^2}(x_1\sin f_3-x_2\cos f_3),$$

where $f_3=\dfrac{x_1^2+x_2^2}{2x_0}-x_3$.

References

[1] Fushchych W., Tsyfra I. and Boyko V., Nonlinear representations for Poincaré and Galilei algebras and nonlinear equations for electromagnetic fields, *J. Nonlin. Math. Phys.*, 1994, V.1, N 2, 210–221.

[2] Rosen G., Conformal transformation matrix for fields, *Ann. Phys.*, 1973, V.77, N 2, 452–453.

[3] Fushchych W.I. and Barannyk L.L., Symmetry reduction on subalgebras of the Poincaré algebra of a nonlinear system of differential equations for a vector field, *Dopovidi Akad. Nauk Ukr.*, 1997, N 8, 50–57 (in Ukrainian).

[4] Ovsyannikov L.V., Group Analysis of Differential Equations, New York, Academic Press, 1982.

[5] Fushchych W.I., Barannyk L.F. and Barannyk A.F., Subgroup Analysis of Poincaré, Galilei Groups and Reduction of Nonlinear Equations, Kyiv, Naukova Dumka, 1991.