On Reduction and
Q-conditional (Nonclassical) Symmetry

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Abstract

Some aspects of Q-conditional symmetry and of its connections with reduction and compatibility are discussed.

There are a number of examples in the history of science when a scientific theory quickly developed, gave good results and applications, and had no sufficiently good theoretical fundament. Let us remember mathematical analysis in the XVIII-th century. It is true in some respect for the theory of Q-conditional (also called non-classical) symmetry. The pioneer paper of Bluman and Cole [1] where this conception appeared was published 28 years ago. And even now, some properties of Q-conditional symmetry are not completely investigated.

1. Q-conditional symmetry being a generalization of the classical Lie symmetry, we recall briefly some results of the Lie theory.

Let

$$L(x, \underset{(r)}{u}) = 0, \quad L = (L^1, L^2, \ldots, L^q), \tag{1}$$

be a system of q partial differential equations (PDEs) of order r in n independent variables $x = (x_1, x_2, \ldots, x_n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, satisfying the maximal rank condition. Here $\underset{(r)}{u}$ denotes all derivatives of the function u up to order r.

Consider a differential operator of the first order

$$Q = \sum_{i=1}^{n} \xi^i(x, u)\partial_{x_i} + \sum_{j=1}^{m} \eta^j(x, u)\partial_{u^j}.$$

Actions of the operator Q on the functions u^j, $j = \overline{1, m}$, are defined by the formula

$$Qu^j = \eta^j - \sum_{i=1}^{n} \xi^i u_i^j, \quad j = \overline{1, m}.$$

Definition. *An operator Q is called a Lie symmetry operator of system (1) if system (1) is invariant under the local transformations generated by the operator Q.*

Theorem. *An operator Q is a Lie symmetry operator of system (1) if and only if the condition*

$$\underset{(r)}{Q} L(x, \underset{(r)}{u}) \Bigg|_{\{L(x, \underset{(r)}{u})=0\} =: K} = 0 \tag{2}$$

is satisfied where $Q\atop(r)$ *denotes the r-th prolongation of the operator Q and K does the set of differential consequences of system (1).*

Theorem. *The set of Lie symmetry operators of system (1) is a Lie algebra under the standard Lie brackets of differential operators of the first order.*

Consider the set of the Lie symmetry operators $Q_1, Q_2, \ldots, Q_s$ ($s < n$) of system (1) which satisfy the following conditions:

1. $\langle Q_1, Q_2, \ldots, Q_s \rangle$ is a Lie algebra;

2. $\operatorname{rank} ||\xi^{ki}|| \, {}_{k=1}^{s} \, {}_{i=1}^{n} = \operatorname{rank} ||\xi^{ki}, \eta^{kj}|| \, {}_{k=1}^{s} \, {}_{i=1}^{n} \, {}_{j=1}^{m} = s.$

Then, a general solution of the system

$$Q^k u^j = 0, \quad k = \overline{1,s}, \quad j = \overline{1,m}, \tag{3}$$

can be written in the form

$$W^j(x,u) = \varphi^j(\omega(x,u)), \quad j = \overline{1,m}, \quad \omega = (\omega_1, \omega_2, \ldots, \omega_{n-s}), \tag{4}$$

where φ^j, $j = \overline{1,m}$, are arbitrary differentiable functions of ω; W^j, $j = \overline{1,m}$, and ω^l, $l = \overline{1, n-s}$, are functionally independent first integrals of system (3) and the Jacobian of W under u does not vanish, that is,

$$\det ||W^j_{u^{j'}}|| \, {}_{j,j'=1}^{m} \neq 0. \tag{5}$$

Owing to condition (5) being satisfied, we can solve equation (4) with respect to u. As a result, we obtain a form to find u, also called "ansatz":

$$u = F(x, \varphi(\omega)), \tag{6}$$

where u, F, and φ have m components.

Theorem. *Substituting (6) into (1), we obtain a set of equations that is equivalent to a system for the functions φ^j ($j = \overline{1,m}$), depending only on variables ω_1, ω_2, $\ldots$, ω_{n-s} (that is called the reduced system corresponding to the initial system (1) and the algebra $\langle Q_1, Q_2, \ldots, Q_s \rangle$).*

Let us emphasize some properties of Lie symmetry and Lie reduction.

1. The *local approach,* that is, a system of PDEs is assumed a manifold in a prolongated space.

2. To find the maximal Lie invariance algebra of a system of PDEs, we have to consider *all the non-trivial differential consequences* of this system of order less than $r + 1$.

3. *Conditional compatibility,* that is, the system

$$L(x, u\atop(r)) = 0, \quad Q^k u^j = 0, \quad k = \overline{1,s}, \quad j = \overline{1,m},$$

consisting of an inverstigated system od PDEs and the surface invariant conditions is compatible if and only if the corresponding reduced system is compatible.

Note that there are a number of examples when the reduced system is not compatible. For instance, consider the equation $tu_t + xu_x = 1$. It is invariant under the dilatation operator $t\partial_t + x\partial_x$, but the corresponding reduced system $(0 = 1)$ is not compatible.

2. A direct generalization of Lie symmetry is conditional symmetry introduced by Prof. Fushchych in 1983 [2].

Definition. *Consider a system of the form (1) and an appended additional condition*

$$L'(x, \underset{(r')}{u}) = 0. \tag{7}$$

System (1), (7) being invariant under an operator Q, system (1) is called conditionally invariant under this operator.

Conditional symmetry was a favourite conception of Prof. Fushchych.

3. Let us pass to Q-conditional symmetry. Below we take $m = q = 1$, i.e., we consider one PDE in one unknown function.

It can be seemed on the face of it that Q-conditional (non-classical) symmetry is a particular case of conditional symmetry. One often uses the following definition of non-classical symmetry.

Definition 1. *A PDE of the form (1) is called Q-conditionally (or non-classical) invariant under an operator Q if the system of equation (1) and the invariant surface condition*

$$Qu = 0 \tag{8}$$

is invariant under this operator.

It is not a quitely correct definition. Indeed, equation (1) is Q-conditionally invariant under an operator Q in terms of definition 1 if the following condition is satisfied:

$$\underset{(r)}{Q} L(x, \underset{(r)}{u}) \bigg|_{\{L(x, \underset{(r)}{u})=0,\, Qu=0\} \,=:\, M} = 0, \tag{9}$$

where M denotes the set of differential consequences of system (1), (8). But the equation $\underset{(r)}{Q} L(x, \underset{(r)}{u})$ belongs to M because it is simple to see that the following formula is true:

$$\underset{(r)}{Q} L(x, \underset{(r)}{u}) = \sum_{\alpha:\, |\alpha| \leq r} \left(\partial_{u_\alpha} L(x, \underset{(r)}{u}) \right) \cdot D_\alpha(Qu) + \sum_{i=1}^{n} \xi^i D_{x_i} \left(L(x, \underset{(r)}{u}) \right),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multiindex, D is the full derivation operator. Therefore, equation (9) is an identity for an arbitrary operator Q.

We like the slightly different definition of Q-conditional symmetry that appeared in [3] and was developed in [4].

Following [4], at first, let us formulate two auxiliary definitions.

Definition. *Operators Q^k, $k = \overline{1, s}$ form an involutive set if there exist functions $f^{klp} = f^{klp}(x, u)$ satisfying the condition*

$$[Q^k, Q^l] = \sum_{p=1}^{s} f^{klp} Q^p, \quad k = \overline{1, s}, \quad where \quad [Q^k, Q^l] = Q^k Q^l - Q^l Q^k.$$

Definition. *Involutive sets of operators $\{Q^k\}$ and $\{\tilde{Q}^k\}$ are equivalent if*

$$\exists\, \lambda_{kl} = \lambda_{kl}(x, u)\ (\det ||\lambda_{kl}||\,^s_{k,l=1} \neq 0): \quad \tilde{Q}^k = \sum_{l=1}^{s} \lambda_{kl} Q^l, \quad k = \overline{1, s}.$$

Definition 2 [4]. *The PDE (1) is called Q-conditionally invariant under the involutive sets of operators $\{Q^k\}$ if*

$$Q^k_{(r)} L(x, \underset{(r)}{u}) \Bigg|_{L(x, \underset{(r)}{u})=0,\ \overline{\{Q^l u = 0,\quad l=\overline{1,s}\}} =: N} = 0, \quad k = \overline{1, s}, \tag{10}$$

where N denotes the set of differential consequences of the invariant surface conditions $Q^l u = 0$, $l = \overline{1, s}$, of order either less than or equal to $r - 1$.

In fact, namely this definition is always used to find Q-conditional symmetry. Restriction on the maximal order of the differential consequences of the invariant surface conditions is not essential.

Lemma. *If equation (1) is Q-conditionally invariant under an involutive set of operators $\{Q^k\}$ then equation (1) is Q-conditionally invariant under an arbitrary involutive set of operators $\{\tilde{Q}^k\}$ being equivalent to the set $\{Q^k\}$ too.*

It follows from Definition 2 that Q-conditional symmetry conserves the properties of Lie symmetry which are emphasized above, and the reduction theorem is proved in the same way as for Lie symmetry.

The conditional compatibility property is conserved too. Therefore, reduction under Q-conditional symmetry operators does not always follow compatibility of the system from the initial equation and the invariant surface conditions. And vice versa, compatibility of this system does not follow Q-conditional invariance of the investigated equation under the operators Q^k. For instance, the equation

$$u_t + u_{xx} - u + t(u_x - u) = 0 \tag{11}$$

is not invariant under the translation operator with respect to t (∂_t), and system of the equation (11) and $u_t = 0$ is compatible because it has the non-trivial solution $u = Ce^x$ ($C = \text{const}$).

Unfortunately, in terms of the local approach, equation (10) is not a necessary condition of reduction with respect to the corresponding ansatz, although the conceptions of Q-conditional symmetry and reduction are very similar. For example, the equation

$$u_t + (u_x + tu_{xx})(u_{xx} + 1) = 0$$

is reduced to the equation $\varphi'' + 1 = 0$ by means of the ansatz $u = \varphi(x)$ and is not invariant under the operator ∂_t. It is possible that the definition of Q-conditional invariance can be given in another way for the condition analogous to (10) to be a necessary condition of reduction.

4. Last time, Q-conditional symmetry of a number of PDEs was investigated, in particular, by Prof. Fushchych and his collaborators (see [6] for references). Consider two simple examples.

At first, consider the one-dimensional linear heat equation

$$u_t = u_{xx}. \tag{12}$$

Lie symmetries of equation (12) are well known. Its maximal Lie invariance algebra is generated by the operators

$$\partial_t, \quad \partial_x, \quad G = t\partial_x - \tfrac{1}{2}xu\partial_u, \quad I = u\partial_u, \quad D = 2t\partial_t + x\partial_x,$$
$$\Pi = 4t^2\partial_t + 4tx\partial_x - (x^2 + 2t)u\partial_u, \quad f(t,x)\partial_u, \tag{13}$$

where $f = f(t,x)$ is an arbitrary solution of (12). Firstly, Q-conditional symmetry of (12) was inverstigated by Bluman and Cole in [1].

Theorem 1 [7]. *An arbitrary Q-conditional symmetry operator of the heat equation (12) is equivalent to either the operator*

$$Q = \partial_t + g^1(t,x)\partial_x + (g^2(t,x)u + g^3(t,x))\partial_u, \quad where$$

$$g_t^1 - g_{xx}^1 + 2g_x^1 g^1 + 2g_x^2 = 0, \quad g_t^k - g_{xx}^k + 2g_x^1 g^k = 0, \quad k = 2,3, \tag{14}$$

or the operator

$$Q = \partial_x + \theta(t,x,u)\partial_u, \quad where$$

$$\theta_t + \theta_{xx} + 2\theta\theta_{xu} - \theta^2\theta_{uu} = 0. \tag{15}$$

The system of defining equations (14) was firstly obtained by Bluman and Cole [1]. Further investigation of system (14) was continued in [5] where the question of linearization of the first two equations of (14) was studied. The general solution of the problem of linearization of (14) and (15) was given in [7]. We investigated Lie symmetry properties of (14) and (15).

Theorem 2 [7]. *The maximal Lie invariance algebra (14) is generated by the operators*

$$\partial_t, \quad \partial_x, \quad G^1 = t\partial_x + \partial_{g^1} - \tfrac{1}{2}g^1\partial_{g^2} - \tfrac{1}{2}xg^3\partial_{g^3}, \quad I^1 = g^3\partial_{g^3},$$
$$D^1 = 2t\partial_t + x\partial_x - g^1\partial_{g^1} - 2g^2\partial_{g^2}, \quad (f_t + f_x g^1 - fg^2)\partial_{g^3}, \tag{16}$$
$$\Pi^1 = 4t^2\partial_t + 4tx\partial_x - 4(x - tg^1)\partial_{g^1} - (8tg^2 - 2xg^1 - 2)\partial_{g^2} - (10t + x^2)g^3\partial_{g^3},$$

where $f = f(t,x)$ is an arbitrary solution of (12).

Theorem 3 [7]. *The maximal Lie invariance algebra of (15) is generated by the operators*

$$\partial_t, \quad \partial_x, \quad G^2 = t\partial_x + -\tfrac{1}{2}xu\partial_u - \tfrac{1}{2}(x\theta + u)\partial_\theta, \quad I^2 = u\partial_u + \theta\partial_\theta,$$
$$D^2 = 2t\partial_t + x\partial_x + u\partial_u, \quad f\partial_u + f_x\partial_\theta, \tag{17}$$
$$\Pi^2 = 4t^2\partial_t + 4tx\partial_x - (x^2 + 2t)u\partial_u - (x\theta + 6t\theta - 2xu)\partial_\theta,$$

where $f = f(t,x)$ is an arbitrary solution of (12).

It is easy to see that algebras (16) (17) are similar to algebra (13). Therefore, there can exist trasformations which, in some sense, reduce (14) and (15) to (12).

Theorem 4 [7]. *System (14) is reduced to the system of three uncoupled heat equations* $z_t^a = z_{xx}^a$, $a = \overline{1,3}$, *functions* $z^a = z^a(t,x)$ *by means of the nonlocal transformation*

$$g^1 = -\frac{z_{xx}^1 z^2 - z^1 z_{xx}^2}{z_x^1 z^2 - z^1 z_x^2}, \quad g^2 = -\frac{z_{xx}^1 z_x^2 - z_x^1 z_{xx}^2}{z_x^1 z^2 - z^1 z_x^2}, \quad g^3 = z_{xx}^3 + g^1 z_x^3 - g^2 z^3, \tag{18}$$

where $z_x^1 z^2 - z^1 z_x^2 \neq 0$.

Theorem 5 [7]. *Equation (15) is reduced by means of the nonlocal change*

$$\theta = -\Phi_t/\Phi_u, \quad \Phi = \Phi(t,x,u) \tag{19}$$

and the hodograph transformation

$$y_0 = t, \quad y_1 = x, \quad y_2 = \Phi, \quad \Psi = u, \tag{20}$$

to the heat equation for the function $\Psi = \Psi(y_0, y_1, y_2)$: $\Psi_{y_0} - \Psi_{y_1 y_1} = 0$, *where the variable* y_2 *can be asssumed a parameter.*

Theorem 4 and 5 show that, in some sense, the problem of finding Q-conditional symmetry of a PDE is reduced to solving this equation, although the defining equations for the coefficient of a Q-conditional symmetry operator are more complicated.

To demonstrate the efficiency of Q-conditional symmetry, consider a generalization of the heat equation, which is a linear transfer equation:

$$u_t + \frac{h(t)}{x} + u_{xx} = 0. \tag{21}$$

Theorem 6. [8, 9]. *The maximal Lie invariance algebra of (21) is the algebra*

1) $A^1 = \langle u\partial_u, \ f(t,x)\partial_u \rangle$ *if* $h \neq \text{const}$;

2) $A^2 = A^1 + \langle \partial_t, \ D, \ \Pi \rangle$ *if* $h = \text{const}$, $h \notin \{0, -2\}$;

3) $A^3 = A^2 + \langle \partial_x + \frac{1}{2}hx^{-1}u\partial_u, \ G \rangle$ *if* $h \in \{0, -2\}$.

Here, $f = f(t,x)$ *is an arbitrary solution of (21),* $D = 2t\partial_t + x\partial_x$, $\Pi = 4t^2\partial_t + 4tx\partial_x - (x^2 + 2(1-h)t)u\partial_u$, $G = t\partial_x - \frac{1}{2}(x - htx^{-1})u\partial_u$.

Theorems being like to Theorems 1–5 were proved for equation (21) too.

It follows from Theorem 6 that the Lie symmetry of equation (21) is trivial in the case $h \neq \text{const}$. But for an arbitrary function h, equation (21) is Q-conditionally invariant, for example, under the folowing operators:

$$X = \partial_t + (h(t) - 1)x^{-1}\partial_x, \quad \tilde{G} = (2t + A)\partial_x - xu\partial_u, \quad A = \text{const}. \tag{22}$$

By means of operators (22), we construct solutions of (21):

$$u = C_2(x^2 - 2\int(h(t) - 1)dt + C_1, \quad u = C_1 \exp\left\{-\frac{x^2}{2(2t + A)} + \int\frac{h(t) - 1}{2t + A}dt\right\}$$

that can be generalized in such a way:

$$u = \sum_{k=0}^{n} T^k(t)x^{2k}, \quad u = \sum_{k=0}^{n} S^k(t)\left(\frac{x}{2t + A}\right)^{2k}\exp\left\{-\frac{x^2}{2(2t + A)} + \int\frac{h(t) - 1}{2t + A}dt\right\}.$$

The functions T^K and S^k satisfy systems of ODEs which are easily integrated.

Using Q-conditional symmetry, the nonlocal equivalence transformation in the class of equations of the form (21) are constructed too.

In conclusion, we should like to note that the connection between Q-conditional symmetry and reduction of PDEs is analogous to one between Lie symmetry and integrating ODEs of the first order. The problem of finding Q-conditional symmetry is more complicated than the problem of solving the initial equation. But constructing a Q-conditional symmetry operator by means of any additional suppositions, we obtain reduction of the initial equation. And it is possible that new classes of symmetries, which are as many as Q-conditional symmetry and will be found as simply as the Lie symmetry will be investigated in the future.

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