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Group Method Analysis of the Potential Equation in Triangular Regions

Y.Z. BOUTROS †, M.B. ABD-EL-MALEK †,
I.A. EL-AWADI ‡ and S.M.A. EL-MANSI ‡

† *Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt*
E-mail: mina@alex.eun.eg

‡ *Department of Physical Sciences, Faculty of Engineering, Mansoura University, Mansoura, Egypt*

Dedicated to the memory of Prof. Dr. W. Fushchych, for his valuable contributions in the field and the sincere friendship

Abstract

The group transformation theoretic approach is applied to present an analytic study of the steady state temperature distribution in a general triangular region, Ω, for given boundary conditions, along two boundaries, in a form of polynomial functions in any degree "n", as well as the study of heat flux along the third boundary. The Laplace's equation has been reduced to a second order linear ordinary differential equation with appropriate boundary conditions. Analytical solution has been obtained for different shapes of Ω and different boundary conditions.

1 Introduction

The Laplace's equation arises in many branches of physics attracts a wide band of researchers. Electrostatic potential, temperature in the case of a steady state heat conduction, velocity potential in the case of steady irrotational flow of ideal fluid, concentration of a substance that is diffusing through solid, and displacements of a two-dimensional membrane in equilibrium state, are counter examples in which the Laplace's equation is satisfied.

The mathematical technique used in the present analysis is the parameter-group transformation. The group methods, as a class of methods which lead to reduction of the number of independent variables, were first introduced by Birkhoff [4] in 1948, where he made use of one-parameter transformation groups. In 1952, Morgan [6] presented a theory which has led to improvements over earlier similarity methods. The method has been applied intensively by Abd-el-Malek et al. [1–3].

In this work, we present a general procedure for applying a one-parameter group transformation to the Laplace's equation in a triangular domain. Under the transformation, the partial differential equation with boundary conditions in polynomial form, of any degree, is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is then solved analytically for some forms of the triangular domain and boundary conditions.

2 Mathematical formulation

The governing equation, for the distribution of temperature $T(x, y)$, is given

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x, y) \in \Omega \tag{2.1}$$

with the following boundary conditions:

$$\begin{aligned} &(i) \quad T(x, y) = \alpha x^n, \quad (x, y) \in L_1, \\ &(ii) \quad T(x, y) = \beta x^n, \quad (x, y) \in L_2, \end{aligned} \tag{2.2}$$

We seek for the distribution of the temperature $T(x, y)$ inside the domain Ω and the heat flux across L_3 with

$$\begin{aligned} &L_1 : \; y = x \tan \Phi_1, \\ &L_2 : \; y = -x \tan \Phi_2, \\ &L_3 : \; y = x \tan \Phi_3 + b, \quad b \neq 0, \end{aligned}$$

$n \in \{0, 1, 2, 3, \ldots\}$, α, β are constants.

Write

$$T(x, y) = w(x, y)q(x), \qquad q(x) \not\equiv 0 \text{ in } \Omega.$$

Hence, (2.1) and (2.2) take the form:

$$q(x)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2\frac{\partial w}{\partial x}\frac{dq}{dx} + w\frac{d^2 q}{dx^2} = 0 \tag{2.3}$$

with the boundary conditions:

$$\begin{aligned} &(i) \quad w(x, y) = \frac{\alpha x^n}{q(x)}, \qquad (x, y) \in L_1, \\ &(ii) \quad w(x, y) = \frac{\beta x^n}{q(x)}, \qquad (x, y) \in L_2. \end{aligned} \tag{2.4}$$

3 Solution of the problem

The method of solution depends on the application of a one-parameter group transformation to the partial differential equation (2.1). Under this transformation, two independent variables will be reduced by one and the differential equation (2.1) transforms into an ordinary differential equation in only one independent variable, which is the similarity variable.

3.1 The group systematic formulation

The procedure is initiated with the group G, a class of transformation of one-parameter "a" of the form

$$G : \; \overline{S} = C^s(a)S + K^s(a), \tag{3.1}$$

where S stands for x, y, w, q and the C's and K's are real-valued and at least differentiable in the real argument "a".

3.2 The invariance analysis

To transform the differential equation, transformations of the derivatives are obtained from G via chain-rule operations:

$$\overline{S}_{\overline{i}} = \left(\frac{C^S}{C^i}\right), \qquad \overline{S}_{\overline{i}\,\overline{j}} = \left(\frac{C^S}{C^i C^j}\right) S_{ij}, \qquad i = x, y; \quad j = x, y,$$

where S stands for w and q.

Equation (2.3) is said to be invariantly transformed whenever

$$\overline{q}(\overline{w}_{\overline{x}\,\overline{x}} + \overline{w}_{\overline{y}\,\overline{y}}) + 2\overline{w}_{\overline{x}}\overline{q}_{\overline{x}} + \overline{w}\overline{q}_{\overline{x}\,\overline{x}} = H_1(a)\left[q(w_{xx} + w_{yy}) + 2w_x q_x + w q_{xx}\right] \tag{3.2}$$

for some function $H_1(a)$ which may be a constant.

Substitution from equations (3.1) into equation (3.2) for the independent variables, the functions and their partial derivatives yields

$$\begin{aligned} &q\left(\left[\frac{C^q C^w}{(C^x)^2}\right] w_{xx} + \left[\frac{c^q C^w}{(C^y)^2}\right] w_{yy}\right) + 2\left[\frac{C^q C^w}{(C^x)^2}\right] w_x q_x + \left[\frac{C^q C^w}{(C^x)^2}\right] w q_{xx} + \xi_1(a) = \\ &\quad H_1(a)\left[q(w_{xx} + w_{yy}) + 2w_x q_x + w q_{xx}\right], \end{aligned} \tag{3.3}$$

where

$$\xi_1(a) = (K^q C^w)\left(\frac{w_{xx}}{(C^x)^2} + \frac{w_{yy}}{(C^x)^2}\right) + \left[\frac{K^w C^q}{(C^x)^2}\right] q_{xx}.$$

The invariance of (3.3) implies $\xi_1(a) \equiv 0$. This is satisfied by putting

$$K^q = K^w = 0$$

and

$$\left[\frac{C^q C^w}{(C^x)^2}\right] = \left[\frac{C^q C^w}{(C^y)^2}\right] = H_1(a),$$

which yields

$$C^x = C^y.$$

Moreover, the boundary conditions (2.4) are also invariant in form, that implies

$$K^x = K^q = K^w = 0, \qquad \text{and} \quad C^q C^w = (C^x)^n.$$

Finally, we get the one-parameter group G which transforms invariantly the differential equation (2.3) and the boundary conditions (2.4). The group G is of the form

$$G : \begin{cases} \overline{x} = C^x x \\ \overline{y} = C^x y + K^y \\ \overline{w} = C^w w \\ \overline{q} = \left[\dfrac{(C^x)^n}{C^w}\right] q \end{cases}$$

3.3 The complete set of absolute invariants

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation (similarity representation) in a single independent variable (similarity variable). Then we have to proceed in our analysis to obtain a complete set of absolute invariants. In addition to the absolute invariants of the independent variable, there are two absolute invariant of the dependent variables w and q.

If $\eta \equiv \eta(x, y)$ is the absolute invariant of independent variables, then

$$q_j(x, y; wq) = F_j[\eta(x, y)]; \qquad j = 1, 2,$$

are two absolute invariants corresponding to w and q. The application of a basic theorem in group theory, see [5], states that: *function $g(x, y; w, q)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation*

$$\sum_{i=1}^{4} (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0, \tag{3.4}$$

where S_i stands for x, y, w and q, respectively, and

$$\alpha_i = \frac{\partial C^{S_i}}{\partial a}(a^0) \qquad \text{and} \qquad \beta_i = \frac{\partial K^{S_i}}{\partial a}(a^0), \qquad i = 1, 2, 3, 4,$$

where a^0 denotes the value of "a" which yields the identity element of the group.

From which we get: $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_3 = \beta_4 = 0$. We take $\beta_2 = 0$.

At first, we seek the absolute invariants of independent variables. Owing to equation (3.4), $\eta(x, y)$ is an absolute invariant if it satisfies the first-order partial differential equation

$$x\frac{\partial \eta}{\partial x} + y\frac{\partial \eta}{\partial y} = 0,$$

which has a solution in the form

$$\eta(x, y) = \frac{y}{x}. \tag{3.5}$$

The second step is to obtain the absolute invariant of the dependent variables w and q. Applying (3.4), we get $q(x) = R(x)\theta(\eta)$.

Since $q(x)$ and $R(x)$ are independent of y, while η is a function of x and y, then $\theta(\eta)$ must be a constant, say $\theta(\eta) = 1$, and from which

$$q(x) = R(x), \tag{3.6}$$

and the second absolute invariant is:

$$w(x, y) = \Gamma(x)F(\eta). \tag{3.7}$$

4 The reduction to an ordinary differential equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariants are used to an obtain ordinary differential equation. Generally, the absolute invariant $\eta(x, y)$ has the form given in (3.5).

Substituting from (3.6), (3.7) into equation (2.3) yields

$$(\eta^2+1)\frac{d^2F}{d\eta^2} - 2\eta\left[\left(\frac{1}{\Gamma}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{dR}{dx}\right)x - 1\right]\frac{dF}{d\eta} + \left[\frac{1}{\Gamma}\frac{d^2\Gamma}{dx^2} + \frac{2}{R\Gamma}\frac{dR}{dx}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{d^2R}{dx^2}\right]x^2F = 0. \tag{4.1}$$

For (4.1) to be reduced to an expression in the single independent invariant η, the coefficients in (4.1) should be constants or functions of η. Thus,

$$\left(\frac{1}{\Gamma}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{dR}{dx}\right)x = C_1, \tag{4.2}$$

$$\left(\frac{1}{\Gamma}\frac{d^2\Gamma}{dx^2} + \frac{2}{R\Gamma}\frac{dR}{dx}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{d^2R}{dx^2}\right)x^2 = C_2. \tag{4.3}$$

It follows, then, from (4.2) that:

$$\Gamma(x)R(x) = C_3 x^{C_1}.$$

Also, from (4.2) and (4.3) we can show that

$$C_2 = C_1(C_1 - 1).$$

By taking $C_3 = 1$ and $C_1 = n$, we get

$$(\eta^2+1)F'' - 2\eta(n-1)F' + n(n-1)F = 0. \tag{4.4}$$

Under the similarity variable η, the boundary conditions are:

$$F(\tan\Phi_1) = \alpha, \qquad F(-\tan\Phi_2) = \beta, \tag{4.5}$$

such that the boundary L_1 or L_2 does not coincide with the vertical axis.

5 Analytic solution

Solution corresponds to: $n = 0$

Equation (4.4) takes the form:

$$(\eta^2+1)F'' + 2\eta F' = 0.$$

Its solution with the aid of boundary conditions (4.5) is presented as

$$F(\eta) = \frac{1}{\Phi_1+\Phi_2}\left[(\alpha-\beta)\tan^{-1}\eta + \beta\Phi_1 + \alpha\Phi_2\right],$$

and from which

$$T(x,y) = \frac{1}{\Phi_1+\Phi_2}\left[(\alpha-\beta)\tan^{-1}\left(\frac{y}{x}\right)+\beta\Phi_1+\alpha\Phi_2\right].$$

Heat flux across L_3:

$$\frac{\partial T}{\partial n}(x,y)\Big|_{L_3} = -\frac{\partial T}{\partial x}\sin\Phi_3 + \frac{\partial T}{\partial y}\cos\Phi_3.$$

Hence, we get:

$$\frac{\partial T}{\partial n}(x,y)\Big|_{L_3} = -\sin\Phi_3\left(\frac{\beta-\alpha}{\Phi_1+\Phi_2}\right)\left(\frac{y}{x^2+y^2}\right)+\cos\Phi_3\left(\frac{\alpha-\beta}{\Phi_1+\Phi_2}\right)\left(\frac{x}{x^2+y^2}\right).$$

Solution corresponds to: $n \geq 1$

$$F(\eta) = b_0\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}\eta^{2k} + \frac{b_1}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}\eta^{2k+1} \tag{5.1}$$

and from which we get

$$T(x,y) = b_0\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}y^{2k}x^{n-2k} + \frac{b_1}{n}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}y^{2k+1}x^{n-2k-1}$$

$$\frac{\partial T}{\partial n}(x,y)\Big|_{L_3} = b_0[-\sin\Phi_3 M_{0,1}+\cos\Phi_3 M_{0,2}] + \frac{b_1}{n}[-\sin\Phi_3 M_{1,1}+\cos\Phi_3 M_{1,2}].$$

Applying the boundary conditions (4.5), we get:

$$\alpha = b_0 z_{0,1} + \frac{b_1}{n}z_{1,1}, \tag{5.2}$$

$$\beta = b_0 z_{0,2} + \frac{b_1}{n}z_{1,2} \tag{5.3}$$

where

$$M_{0,1} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k(n-2k)\binom{n}{2k}y^{2k}x^{n-2k-1},$$

$$M_{1,1} = \sum_{k=0}^{\left[\frac{n-2}{2}\right]}(-1)^k(n-2k-1)\binom{n}{2k+1}y^{2k+1}x^{n-2k-2},$$

$$M_{0,2} = \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k(2k)\binom{n}{2k}y^{2k-1}x^{n-2k},$$

$$M_{1,2} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k(2k+1)\binom{n}{2k+1}y^{2k}x^{n-2k-1},$$

$$z_{0,1} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} \tan^{2k} \Phi_1, \qquad z_{1,1} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k+1} \tan^{2k+1} \Phi_1,$$

$$z_{0,2} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} \tan^{2k} \Phi_2, \qquad z_{1,2} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k+1} \tan^{2k+1} \Phi_2.$$

Solving (5.2) and (5.3) for the given value of "n" we get b_0 and b_1.

6 Special Cases

Case 1: Boundary conditions with different degrees of polynomials

The governing equation is given as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x, y) \in \Omega,$$

with the following boundary conditions:

$$\begin{aligned} (i) \quad & T(x,y) = \alpha x^6, \quad (x,y) \in L_1, \\ (ii) \quad & T(x,y) = \beta x^5, \quad (x,y) \in L_2, \end{aligned}$$

and $\Phi_1 = 60°$, $\Phi_2 = 45°$, $\Phi_3 = 0°$.

From the principle of superposition, write

$$T(x,y) = T_1(x,y) + T_2(x,y),$$

where the boundary conditions for $T_1(x,y)$ are:

$$\begin{aligned} (i) \quad & T(x,y) = \alpha x^6, \quad (x,y) \in L_1, \\ (ii) \quad & T(x,y) = 0, \quad (x,y) \in L_2, \end{aligned}$$

and the boundary conditions for $T_2(x,y)$ are:

$$\begin{aligned} (i) \quad & T(x,y) = 0, \quad (x,y) \in L_1, \\ (ii) \quad & T(x,y) = \beta x^5, \quad (x,y) \in L_2. \end{aligned}$$

Setting $n = 6$ in the general solution (5.1), we get:

$$T_1(x,y) = \frac{\alpha}{64}(x^6 - 15y^2x^4 + 15y^4x^2 - y^6),$$

$$\left.\frac{\partial T_1}{\partial n}(x,y)\right|_{L_3} = -\frac{3\alpha b}{32}(5x^4 - 10b^2x^2 + b^4).$$

Setting $n = 5$ in the general solution (5.1), we get:

$$T_2(x,y) = \frac{\beta}{4(1-\sqrt{3})}\left[\sqrt{3}(x^5 - 10y^2x^3 + 5y^4x) + (5yx^4 - 10y^3x^2 + y^5)\right],$$

$$\left.\frac{\partial T_2}{\partial n}(x,y)\right|_{L_3} = \frac{5\beta}{4(1-\sqrt{3})}\left[-4b\sqrt{3}(x^3 - b^2x) + (x^4 - 6b^2x^2 + b^4)\right].$$

Hence, the analytic solution has the form

$$T(x,y) = \frac{\alpha}{64}(x^6 - 15y^2x^4 + 15y^4x^2 - y^6) + \\ \frac{\beta}{4(1-\sqrt{3})}\left[\sqrt{3}(x^5 - 10y^2x^3 + 5y^4x) + (5yx^4 - 10y^3x^2 + y^5)\right],$$

and

$$\frac{\partial T}{\partial n}(x,y)\Big|_{L_3} = -\frac{3\alpha b}{32}(5x^4 - 10b^2x^2 + b^4) + \\ \frac{5\beta}{4(1-\sqrt{3})}\left[-4b\sqrt{3}(x^3 - b^2x) + (x^4 - 6b^2x^2 + b^4)\right].$$

Case 2: One of the boundaries is vertical

The governing equation is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x,y) \in \Omega \tag{6.1}$$

with the following boundary conditions:

$$\begin{aligned} (i) \quad & T(x,y) = \alpha y^n, \quad (x,y) \in L_1, \\ (ii) \quad & T(x,y) = \beta x^n, \quad (x,y) \in L_2, \end{aligned} \tag{6.2}$$

and $\Phi_1 = \dfrac{\pi}{2}$.

Write

$$T(x,y) = w(x,y)q(y), \qquad q(y) \not\equiv 0 \quad \text{in} \quad \Omega.$$

Hence, (6.1) and (6.2) take the form:

$$q(y)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2\frac{\partial w}{\partial y}\frac{dq}{dy} + w\frac{d^2 q}{dy^2} = 0 \tag{6.3}$$

with the boundary conditions:

$$\begin{aligned} (i) \quad & w(x,y) = \frac{\alpha y^n}{q(y)}, \qquad (x,y) \in L_1, \\ (ii) \quad & w(x,y) = \frac{\beta x^n}{q(y)}, \qquad (x,y) \in L_2. \end{aligned}$$

Applying the invariant analysis, we get:

$$G : \begin{cases} \overline{x} = C^x x \\ \overline{y} = C^x y \\ \overline{w} = C^w w \\ \overline{q} = \dfrac{(C^x)^n}{C^w} q \end{cases}$$

and the absolute invariant $\eta(x,y)$ is:

$$\eta(x,y) = \frac{x}{y}. \tag{6.4}$$

The complete set of the absolute invariant corresponding to w and q are:

$$q(y) = R(y), \tag{6.5}$$

$$w(x, y) = \Gamma(y)F(\eta). \tag{6.6}$$

Substituting (6.4)–(6.6) in (6.3), with $\Gamma(y)R(y) = y^n$, we get:

$$(\eta^2 + 1)F'' - 2\eta(n-1)F' + n(n-1)F = 0. \tag{6.7}$$

Under the similarity variable η, the boundary conditions are:

$$\begin{aligned} &F(0) = \alpha, \\ &F(-\cot\Phi_2) = (-\cot\Phi_2)^n\beta. \end{aligned} \tag{6.8}$$

For $n = 0$**:** Solution of (6.7) with the boundary conditions (6.8) is:

$$T(x, y) = \frac{\beta - \alpha}{\frac{\pi}{2} + \Phi_2}\tan^{-1}\left(\frac{x}{y}\right) + \alpha.$$

The heat flux across L_3 is:

$$\frac{\partial T}{\partial n}\Big|_{L_3} = \frac{\alpha - \beta}{\frac{\pi}{2} + \Phi_2}\frac{1}{x^2 + y^2}(y\sin\Phi_3 + x\cos\Phi_3).$$

For $n \geq 1$**:** Solution of (6.7) with the boundary conditions (6.8) is:

$$T(x, y) = b_0\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\left(\frac{x}{y}\right)^{2k}y^n + \frac{b_1}{n}\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}\left(\frac{x}{y}\right)^{2k+1}y^n,$$

where

$$b_0 = \alpha, \tag{6.9}$$

$$\beta(-\cot\Phi_2)^n = b_0\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\cot^{2k}\Phi_2 + \frac{b_1}{n}\sum_{k=0}^{[\frac{n}{2-1}]}(-1)^{k+1}\binom{n}{2k+1}\cot^{2k+1}\Phi_2. \tag{6.10}$$

Solving (6.9) and (6.10) for the given value of "n" we get b_0 and b_1.

The heat flux across L_3 is:

$$\frac{\partial T}{\partial n}(x, y)\Big|_{L_3} = b_0\left[-\sin\Phi_3 N_{0,1} + \cos\Phi_3 N_{0,2}\right] + \frac{b_1}{n}\left[-\sin\Phi - 3N_{1,1} + \cos\Phi_3 N_{1,2}\right],$$

where

$$N_{0,1} = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k(2k)\binom{n}{2k}x^{2k-1}y^{n-2k},$$

$$N_{1,1} = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k(2k+1)\binom{n}{2k+1}x^{2k}y^{n-2k-1},$$

$$N_{0,2} = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k(n-2k)\binom{n}{2k}x^{2k}y^{n-2k-1},$$

$$N_{1,2} = \sum_{k=0}^{[\frac{n-2}{2}]}(-1)^k(n-2k-1)\binom{n}{2k+1}x^{2k+1}y^{n-2k-2}.$$

Case 3: $\Phi_1 = \Phi_2 = \dfrac{\pi}{4}$**,** $\Phi_3 = 0$

From (5.2), (5.3) and $\Phi_1 = \Phi_2 = \dfrac{\pi}{4}$, we find that $n = 2$.

The governing equation is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x, y) \in \Omega \tag{6.11}$$

with the following boundary conditions:

$$\begin{array}{ll} (i) & T(x, y) = \alpha x^2, \qquad (x, y) \in L_1, \\ (ii) & T(x, y) = -\alpha x^2, \qquad (x, y) \in L_2. \end{array} \tag{6.12}$$

Write

$$T(x, y) = w(x, y) q(x), \qquad q(x) \not\equiv 0 \quad \text{in} \quad \Omega.$$

Hence, (6.11) and (6.12) take the form:

$$q(x)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2\frac{\partial w}{\partial x}\frac{dq}{dx} + w\frac{d^2 q}{dx^2} = 0 \tag{6.13}$$

with the boundary conditions:

$$\begin{array}{ll} (i) & w(x, y) = \dfrac{\alpha x^2}{q(x)}, \qquad (x, y) \in L_1, \\ (ii) & w9x, y) = -\dfrac{\alpha x^2}{q(x)}, \qquad (x, y) \in L_2. \end{array}$$

Applying the invariant analysis, we get:

$$G : \begin{cases} \overline{x} = C^x x \\ \overline{y} = C^x y + K^y \\ \overline{w} = C^w w \\ \overline{q} = \dfrac{(C^x)^2}{C^w} q \end{cases}$$

and the absolute invariant $\eta(x, y)$ is:

$$\eta(x, y) = \frac{y}{x}. \tag{6.14}$$

The complete set of the absolute invariants corresponding to w and q is:

$$q(x) = R(x), \tag{6.15}$$

$$w(x, y) = \Gamma(x) F(\eta). \tag{6.16}$$

Substituting (6.14)–(6.16) in (6.13), with $\Gamma(x) R(x) = x^2$, we get:

$$(\eta^2 + 1) F'' - 2\eta F' + 2F = 0. \tag{6.17}$$

Under the similarity variable η, the boundary conditions are:

$$\begin{aligned} &F(-1) = -\alpha, \\ &F(1) = \alpha. \end{aligned} \tag{6.18}$$

It is clear that two conditions in (6.18) are identical. Hence, to find the second condition, assume that the heat flux across L_3 takes the form:

$$\left.\frac{\partial T}{\partial y}\right|_{L_3} = \gamma + \alpha x, \tag{6.19}$$

where γ is a constant.

Solution of (6.17) with the boundary conditions (6.18) and (6.19) is:

$$T(x, y) = \frac{\gamma}{2b}(y^2 - x^2) + \alpha xy.$$

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