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On Duality for a Braided Cross Product

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Abstract

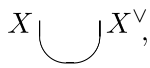
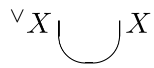
In this note, we generalize a result of [4] (see also [9]) and set the isomorphism between the iterated cross product algebra $H^\vee\#(H\#A)$ and braided analog of an A-valued matrix algebra $H^\vee\otimes A\otimes H$ for a Hopf algebra H in the braided category $\mathcal{C}$ and for an H-module algebra A. As a preliminary step, we prove the equivalence between categories of modules over both algebras and category whose objects are Hopf H-modules and A-modules satisfying certain compatibility conditions.

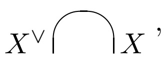
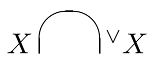
Introduction and preliminaries

A purpose of this note is to generalize the result of [4] (see also [9]) about the isomorphism between the iterated cross product algebra $H^*\#(H\#A)$ and A-valued matrix algebra $\mathrm{M}(H)\otimes A$ (for an H-module algebra A) to the fully braided case.

Throughout this paper, the symbol $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ denotes a strict monoidal category with braiding Ψ. For convenience of the reader, we recall the necessary facts about braided monoidal categories and Hopf algebras in them.

For object $X \in \mathcal{C}$, we say that $X^\vee$ and ${}^\vee X \in \mathcal{C}$ are dual objects if evaluation and coevaluation morphisms

$\mathrm{ev} : X \otimes X^\vee \to \mathbb{1} =$  , $\mathrm{ev} \ : {}^\vee X \otimes X \to \mathbb{1} =$  ,

$\mathrm{coev} : \mathbb{1} \to X^\vee \otimes X =$  , $\mathrm{coev}: \mathbb{1} \to X \otimes {}^\vee X =$ 

can be chosen so that the compositions

$$X = X\otimes\mathbb{1} \xrightarrow{1\otimes\mathrm{coev}} X\otimes(X^\vee\otimes X) = (X\otimes X^\vee)\otimes X \xrightarrow{\mathrm{ev}\otimes 1} \mathbb{1}\otimes X = X,$$
$$X = \mathbb{1}\otimes X \xrightarrow{\mathrm{coev}\otimes 1} (X\otimes {}^\vee X)\otimes X = X\otimes({}^\vee X\otimes X) \xrightarrow{1\otimes\mathrm{ev}} X\otimes\mathbb{1} = X,$$
$$X^\vee = \mathbb{1}\otimes X^\vee \xrightarrow{\mathrm{coev}\otimes 1} (X^\vee\otimes X)\otimes X^\vee = X^\vee\otimes(X\otimes X^\vee) \xrightarrow{1\otimes\mathrm{ev}} X^\vee\otimes\mathbb{1} = X^\vee,$$
$${}^\vee X = {}^\vee X\otimes\mathbb{1} \xrightarrow{1\otimes\mathrm{coev}} {}^\vee X\otimes(X\otimes {}^\vee X) = ({}^\vee X\otimes X)\otimes {}^\vee X \xrightarrow{\mathrm{ev}\otimes 1} \mathbb{1}\otimes {}^\vee X = {}^\vee X$$

are all identity morphisms.

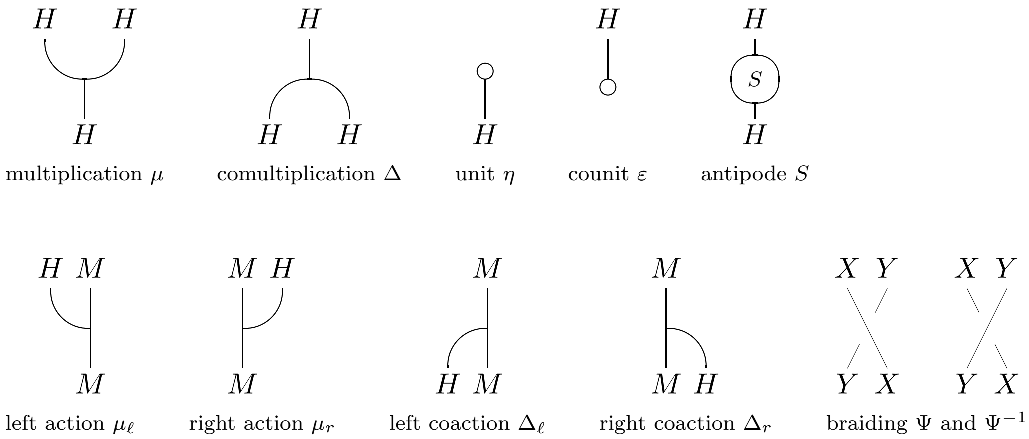


Figure 1: Graphical notations

Recall that a Hopf algebra $H \in \mathcal{C}$ [7] is an object $H \in \operatorname{Obj} \mathcal{C}$ together with an associative multiplication $m : H \otimes H \to H$ and an associative comultiplication $\Delta : H \to H \otimes H$, obeying the bialgebra axiom

$$\begin{aligned}&\big(H \otimes H \xrightarrow{m} H \xrightarrow{\Delta} H \otimes H\big)\\ &\quad = \big(H \otimes H \xrightarrow{\Delta\otimes\Delta} H \otimes H \otimes H \otimes H \xrightarrow{H\otimes\Psi\otimes H} H \otimes H \otimes H \otimes H \xrightarrow{m\otimes m} H \otimes H\big),\end{aligned}$$

which possesses the unit $\eta : \mathbb{1} \to H$, the counit $\varepsilon : H \to \mathbb{1}$, the antipode $S : H \to H$, and the inverse antipode $S^{-1} : H \to H$ (definitions are the same as in the classical case).

A left (resp., right) module over an algebra H is an object $M \in \mathcal{C}$ equipped with an associative action $\mu_\ell : H \otimes M \to M$ (resp., $\mu_r : M \otimes H \to M$). The category of left (resp., right) H-modules will be denoted by ${}_H\mathcal{C}$ (resp., $\mathcal{C}_H$). A left (resp., right) comodule over a coalgebra H is an object $M \in \mathcal{C}$ equipped with an associative coaction $\Delta_\ell : M \to H \otimes M$ (resp., $\Delta_r : M \to M \otimes H$). The category of left (resp., right) H-comodules will be denoted by ${}^H\mathcal{C}$ (resp., $\mathcal{C}^H$).

If $(\mathcal{C}, \otimes, \mathbb{1}, \Psi)$ is a braided monoidal category, then $\overline{\mathcal{C}} = (\mathcal{C}, \otimes, \mathbb{1}, \overline{\Psi})$ denotes the same monoidal category with the mirror-reversed braiding $\overline{\Psi}_{X,Y} := {\Psi_{Y,X}}^{-1}$. For a Hopf algebra H in $\mathcal{C}$, we denote by H^{op} (resp., H_{op}) the same coalgebra (resp., algebra) with opposite multiplication μ^{op} (resp., opposite comultiplication Δ^{op}) defined through

$$\mu^{\mathrm{op}} := \mu \circ {\Psi_{H\,H}}^{-1} \qquad \big(\text{resp., } \Delta^{\mathrm{op}} := {\Psi_{H\,H}}^{-1} \circ \Delta\big)\ . \tag{1}$$

It is easy to see that H^{op} and H_{op} are Hopf algebras in $\overline{\mathcal{C}}$ with antipode S^{-1}. We will always consider H^{op} and H_{op} as objects of the category $\overline{\mathcal{C}}$. In what follows, we often use a graphical notation for morphisms in monoidal categories [1, 5, 6, 8]. The graphics and notation for (co-)multiplication, (co-)unit, antipode, left and right (co-)action, and braiding are given in Fig. 1, where H is a Hopf algebra and M is an H-module (H-comodule).

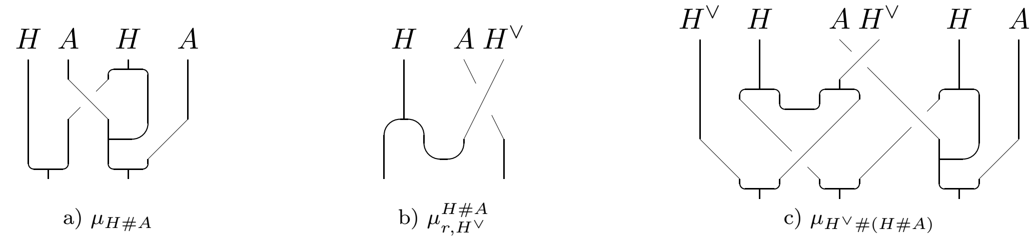


Figure 2:

Duality results

Let H be a Hopf algebra with an invertible antipode in a braided monoidal category $\mathcal{C}$, A be an algebra in the monoidal category $\mathcal{C}_H$ of right H-modules. For these data, one can equip the object $H\otimes A$ with a structure of algebra in $\mathcal{C}$ [8]. Multiplication $\mu_{H\#A}$ in this cross product algebra $H\#A$ is given by the diagram in Fig. 2a. The object $H\#A$ equipped with the right $H^\vee$-module structure $\mu^{H\#A}_{r,H^\vee}$ given in Fig. 2b becomes an algebra in the category $\overline{\mathcal{C}}_{(H^\vee)_{\mathrm{op}}}$. Multiplication $\mu_{H^\vee\#(H\#A)}$ in the cross product algebra $H^\vee\#(H\#A)$ is given by the diagram in Fig. 2c.

Let us consider the category $\mathcal{C}^H_{H,A}$ whose objects X are right Hopf H-modules (i.e., right H-modules and right H-comodules satisfying the compatibility condition presented in Fig. 3a) and right A-modules in $\mathcal{C}_H$ (i.e., action $\mu^X_{r,A} : X\otimes A \to X$ is an H-module morphism as shown in Fig. 3b) with the additional connection between A-action and H-coaction given in Fig. 3c.

Proposition 1. *There exists an isomorphism between categories $\mathcal{C}^H_{H,A}$ and $\mathcal{C}_{H^\vee\#(H\#A)}$. Functors that set this equivalence are identical on underlying objects and morphism from $\mathcal{C}$. For given $(X, \mu^X_{r,H}, \Delta^X_{r,H}, \mu^X_{r,A}) \in \mathrm{Obj}(\mathcal{C}^H_{H,A})$, the structure of the $(H^\vee\#(H\#A))$-module on X is given by the composition*

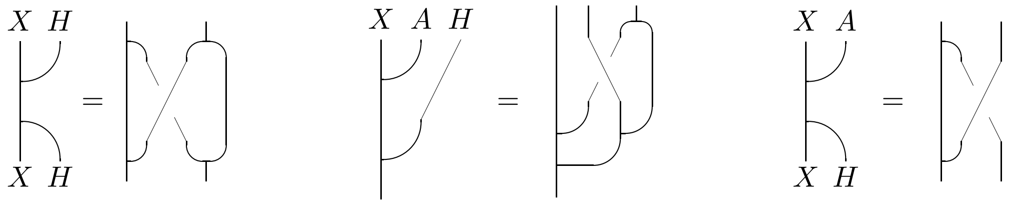
$$\mu^X_{r,H} := \big\{X\otimes H^\vee\otimes H\otimes A \xrightarrow{\Delta^X_{r,H}\otimes\mathrm{id}_{H^\vee\otimes H\otimes A}} X\otimes H\otimes H^\vee\otimes H\otimes A \xrightarrow{\mathrm{id}_X\otimes\mathrm{ev}\otimes\mathrm{id}_{H\otimes A}} X\otimes H\otimes A \xrightarrow{\mu^X_{r,H}\otimes\mathrm{id}_A} X\otimes A \xrightarrow{\mu^X_{r,A}} X\big\}\,.$$

An "A-valued matrix algebra" is an object $H^\vee\otimes A\otimes H$ equipped with multiplication given by the composition

$$H^\vee\otimes A\otimes H\otimes H^\vee\otimes A\otimes H \xrightarrow{\mathrm{id}_{H^\vee\otimes A}\otimes\mathrm{ev}\otimes\mathrm{id}_{A\otimes H}} H^\vee\otimes A\otimes A\otimes H \xrightarrow{\mathrm{id}_{H^\vee}\otimes\mu_A\otimes\mathrm{id}_H} H^\vee\otimes A\otimes H\,.$$

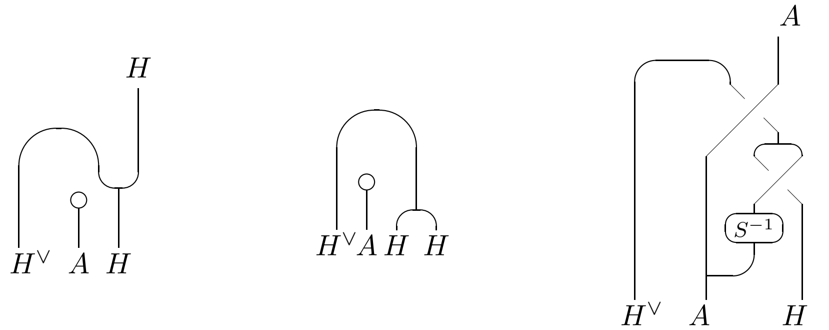
For a Hopf module X, endomorphism

$$\Pi(X) := \big\{X \xrightarrow{\Delta^X_r} X\otimes H \xrightarrow{\mathrm{id}_X\otimes S} X\otimes H \xrightarrow{\mu^X_r} X\big\}$$



a) Hopf module axiom b) A-module in $\mathcal{C}_H$ c)

Figure 3:



$\phi^H_{H^\vee\otimes A\otimes H}$ $\phi_{H^\vee\otimes A\otimes H,H}$ $\phi^A_{H^\vee\otimes A\otimes H}$

Figure 4:

is an idempotent. This idempotent plays a key role in the theory of Hopf modules [2] and integration on braided Hopf algebras [3].

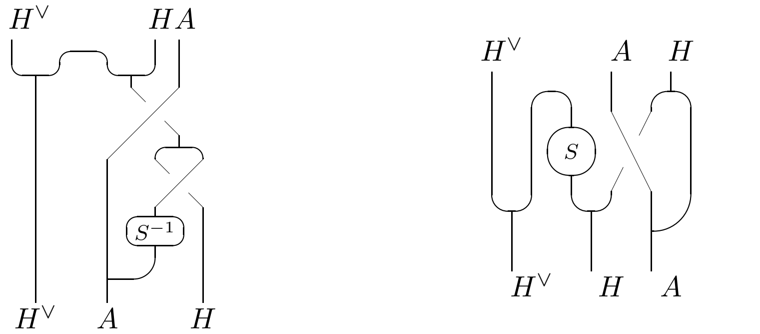
Proposition 2. *There exists an isomorphism between categories $\mathcal{C}^H_{H,A}$ and $\mathcal{C}_{H^\vee\otimes A\otimes H}$. Functors that set this equivalence are identical on underlying objects and morphism from $\mathcal{C}$. For given $(X,\mu^X_{r,H},\Delta^X_{r,H},\mu^X_{r,A})\in\mathrm{Obj}\big(\mathcal{C}^H_{H,A}\big)$, the structure of the $(H^\vee\otimes A\otimes H)$-module on X is given by the composition*

$$\mu^X_{r,H^\vee\otimes A\otimes H}:=\big\{X\otimes H^\vee\otimes A\otimes H\xrightarrow{\Delta^X_{r,H}\otimes\mathrm{id}_{H^\vee\otimes A\otimes H}}X\otimes H\otimes H^\vee\otimes A\otimes H$$
$$\xrightarrow{\Pi(X)\otimes\mathrm{ev}\otimes\mathrm{id}_{A\otimes H}}X\otimes A\otimes H\xrightarrow{\mu^X_{r,A}\otimes\mathrm{id}_H}X\otimes H\xrightarrow{\mu^X_{r,H}}X\big\}\,.$$

Conversely, for a given right $(H^\vee\otimes A\otimes H)$-module $(X,\mu^X_{r,H^\vee\otimes A\otimes H})$, one can turn X into an object of $\mathcal{C}^H_{H,A}$ equipped with (co)actions

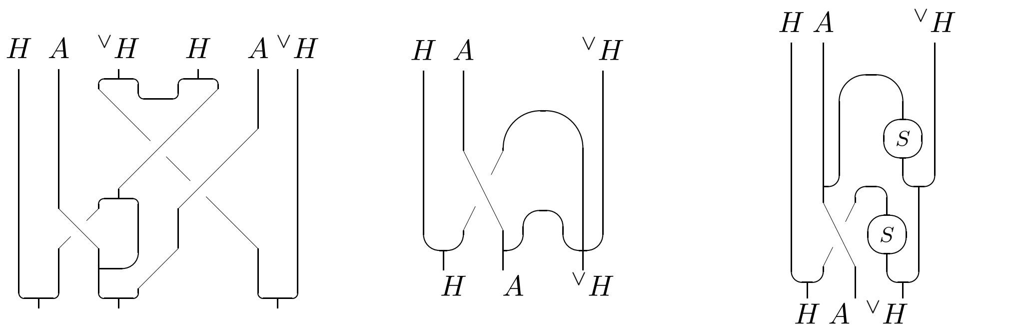
$$\mu^X_{r,H}\;:=\;\big\{X\otimes H\xrightarrow{\phi^H_{H^\vee\otimes A\otimes H}}X\otimes H^\vee\otimes A\otimes H\xrightarrow{\mu^X_{r,H^\vee\otimes A\otimes H}}X\big\}\,,$$
$$\mu^X_{r,A}\;:=\;\big\{X\otimes A\xrightarrow{\phi^A_{H^\vee\otimes A\otimes H}}X\otimes H^\vee\otimes A\otimes H\xrightarrow{\mu^X_{r,H^\vee\otimes A\otimes H}}X\big\}\,,$$
$$\Delta^X_{r,H}\;:=\;\big\{X\xrightarrow{\mathrm{id}_X\otimes\phi_{H^\vee\otimes A\otimes H,H}}X\otimes H^\vee\otimes A\otimes H\otimes H\xrightarrow{\mu^X_{r,H^\vee\otimes A\otimes H}\otimes\mathrm{id}_H}X\otimes H\big\}\,,$$

where morphisms $\phi^H_{H^\vee\otimes A\otimes H}$, $\phi_{H^\vee\otimes A\otimes H,H}$, $\phi^A_{H^\vee\otimes A\otimes H}$ are presented in Fig. 4.



$\phi : H^\vee\#(H\#A) \to H^\vee\otimes A\otimes H$ $\phi^{-1} : H^\vee\otimes A\otimes H \to H^\vee\#(H\#A)$

Figure 5:



a) $\mu_{(H\#A)\#{}^\vee H}$ b) $(H\#A)\#{}^\vee H \to H\otimes A\otimes{}^\vee H$ c) $H\otimes A\otimes{}^\vee H \to (H\#A)\#{}^\vee H$

Figure 6:

Corollary 3. *There exists an algebra isomorphism $\phi : H^\vee\#(H\#A) \to H^\vee\otimes A\otimes H$ shown in Fig. 5 such that the corresponding isomorphism of categories $\mathcal{C}_{H^\vee\otimes A\otimes H} \overset{\phi^*}{\simeq} \mathcal{C}_{H^\vee\#(H\#A)}$ is given by the compositions of functors from Propositions 1, 2.*

Proof. We put

$$\phi := \mu^{(3)}_{H^\vee\otimes A\otimes H} \circ \left(\phi^{H^\vee}_{H^\vee\otimes A\otimes H}\otimes\phi^{H}_{H^\vee\otimes A\otimes H}\otimes\phi^{A}_{H^\vee\otimes A\otimes H}\right),$$

where

$$\phi^{H^\vee}_{H^\vee\otimes A\otimes H} := \left\{H^\vee \xrightarrow{\phi_{H^\vee\otimes A\otimes H,H}\otimes\mathrm{id}_{H^\vee}} H^\vee\otimes A\otimes H\otimes H\otimes H^\vee \xrightarrow{\mathrm{id}_{H^\vee\otimes A\otimes H}\otimes\mathrm{ev}_H} H^\vee\otimes A\otimes H\right\}$$

and $\mu^{(3)} := \mu \circ (\mu\otimes\mathrm{id})$. Consideration of the regular $(H^\vee\otimes A\otimes H)$-module implies that ϕ is an algebra isomorphism.

In the special case $A = \mathbb{1}$, we obtain the braided Heisenberg double $\mathcal{H}(H) := H^\vee\#H$, which is isomorphic to the matrix algebra $H^\vee\otimes H$ (with multiplication $\mathrm{id}_{H^\vee}\otimes\mathrm{ev}_H\otimes\mathrm{id}_H$), and an isomorphism between the category $\mathcal{C}_H^H$ of right Hopf H-modules and the category $\mathcal{C}_{\mathcal{H}(H)}$ of $\mathcal{H}(H)$-modules. See [3] for this special case and connection with integration on braided Hopf algebras.

Remark. In a similar way, one can obtain another variant of the above construction, which does not involve a skew antipode S^{-1}. Let $A \in \mathrm{Obj}(\mathcal{C}_H)$ be a right H-module algebra. One can turn the corresponding cross product algebra $H\#A$ into a left ${}^\vee H$-module algebra with action $\mu_{r,{}^\vee H}^{H\#A} := (\mathrm{ev}\otimes\mathrm{id}_{H\otimes A})\circ(\mathrm{id}_{{}^\vee H}\otimes\Delta_H\otimes\mathrm{id}_A)$. Multiplication in the corresponding cross product algebra $(H\#A)\#{}^\vee H$ is given in Fig. 6a. Isomorphism between this algebra and the "A-valued matrix algebra" $H\otimes A\otimes{}^\vee H$ and its inverse is given in Fig. 6b,c.

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