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q-Deformed Inhomogeneous Algebras $U_q(iso_n)$ and Their Representations

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Abstract

Proceeding from the nonstandard q-deformed algebras $U'_q(so_{n+1})$ and their finite-dimensional representations in a q-analog of the Gel'fand-Tsetlin basis, we obtain, by means of the contraction procedure, the corresponding q-deformed inhomogeneous algebras $U'_q(iso_n)$ in a uniform fashion for all $n \geq 2$ as well as their infinite-dimensional representations.

1. Introduction

Lie algebras of inhomogeneous orthogonal or pseudoorthogonal Lie groups are important for various problems of theoretical and mathematical physics. Recently, certain efforts were devoted to the problem of constructing quantum, or q-deformed, analogs of inhomogeneous (Euclidean) algebras [1–3]. Practically, all these works exploit as starting point the standard deformations $U_q(B_r)$, $U_q(D_r)$, given by Jimbo and Drinfeld [4], of Lie algebras of the orthogonal groups $SO(2r+1)$ and $SO(2r)$. In addition to the fact that most of papers [1–3] concern Euclidean algebras *of low dimension* $2, 3, 4$, their q-analogs may be examined from the viewpoint of (non-)possessing the following two characteristic features:

(i) after deformation, a rotation subalgebra remains closed;

(ii) both rotation and translation subalgebras in the q-analog are nontrivially deformed.

Examination shows that a rotation subalgebra may become nonclosed within a specific approach (cf. Celeghini et al. in [2]) to q-deformation; moreover, in most of the examples of q-deformed Euclidean algebras [1–3], either the whole (rotation or translation) subalgebra remains nondeformed (i.e., coincides with classical one) or at least some from the set of translations are still commuting.

The purpose of this contribution is to describe a certain nonstandard version of the q-deformed inhomogeneous algebras $U_q(iso_n)$ (i.e., q-Euclidean algebras) as well as their representations, obtained by a simple contraction procedure from the nonstandard q-deformed algebras $U'_q(so_{n+1})$, and their representations which were proposed and studied in [5]. As will be seen, our q-analogs are obtained in a uniform fashion for all values $n \geq 2$, and the same concerns their representations. Other viable features are: the homogeneous (rotation) subalgebra remains closed and becomes completely deformed; moreover, the translation generators are all mutually noncommuting (in fact, they q-commute).

2. q-Deformed inhomogeneous algebras $U_q'(iso_n)$

The well-known connection of the inhomogeneous (Euclidean) algebras $iso(n)$ to the Lie algebras $so(n+1)$ of orthogonal Lie groups performed by means of the procedure of contraction [6] is applied here to the non-standard q-deformed algebras $U_q'(so_{n+1})$ (studied in [5]) in order to obtain the corresponding version $U_q'(iso_n)$ of q-deformed inhomogeneous algebras.

A. Nonstandard q-deformed algebras $U_q'(so_{n+1})$

We consider a q-deformation of the orthogonal Lie algebras $so(n, C)$ that essentially differs from the standard quantum algebras $U_q(B_r)$, $U_q(D_r)$ given by Jimbo and Drinfeld [4]. It is known that, in order to describe explicitly finite-dimensional irreducible representations (irreps) of the q-deformed algebras $U_q(so(n,C))$ and their compact real forms, one needs a q-analog of the Gelfand-Tsetlin (GT) basis and the GT action formulas that require the existence of canonical embeddings q-analogous to the chain $so(n,C) \supset so(n-1,C) \supset \cdots \supset so(3,C)$. Evidently, such embeddings do not hold for the standard quantum algebras $U_q(B_r)$ and $U_q(D_r)$. Another feature is the restricted set of possible noncompact real forms admitted by Drinfeld-Jimbo's q-algebras, which exclude the Lorentz signature in multidimensional cases. On the contrary, the nonstandard q-deformation $U_q'(so(n,C))$ does admit [5] all noncompact real forms that exist in the classical case. Moreover, validity of the chain of embeddings $U_q(so(n,1)) \supset U_q(so(n)) \supset U_q(so(n-1)) \supset \cdots \supset U_q(so(3))$ allows us to construct and analyze infinite-dimensional representations of the q-Lorentz algebras $U_q'(so_{n,1})$.

According to [5], the nonstandard q-deformation $U_q'(so(n,C))$ of the Lie algebra$so(n,C)$ is given as a complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \dots, I_{n,n-1}$ obeying the defining relations (denote $q + q^{-1} \equiv [2]_q$)

$$I_{j,j-1}^2 I_{j-1,j-2} + I_{j-1,j-2} I_{j,j-1}^2 - [2]_q \; I_{j,j-1} I_{j-1,j-2} I_{j,j-1} = -I_{j-1,j-2}, \tag{1}$$

$$I_{j-1,j-2}^2 I_{j,j-1} + I_{j,j-1} I_{j-1,j-2}^2 - [2]_q \; I_{j-1,j-2} I_{j,j-1} I_{j-1,j-2} = -I_{j,j-1}, \tag{2}$$

$$[I_{i,i-1}, I_{j,j-1}] = 0 \qquad \text{if} \quad |\, i-j \,| > 1. \tag{3}$$

The compact and noncompact (of the Lorentz signature) real forms $U_q(\mathrm{so}_n)$and $U_q(\mathrm{so}_{n-1,1})$ are singled out from the complex q-deformed algebra $U_q(\mathrm{so}(n,C))$ by means of appropriate $*$-structures [5] which read in the compact case:

$$I_{j,j-1}^* = -I_{j,j-1}, \qquad j = 2, \dots, n. \tag{4}$$

Besides the definition in terms of trilinear relations, *one can also give a 'bilinear' presentation* (useful for comparison to other approaches). To this end, one introduces the generators (here, $k > l+1, \;\; 1 \le k, l \le n$)

$$I_{k,l}^{\pm} \equiv [I_{l+1,l}, I_{k,l+1}^{\pm}]_{q^{\pm 1}} \equiv q^{\pm 1/2} I_{l+1,l} I_{k,l+1}^{\pm} - q^{\mp 1/2} I_{k,l+1}^{\pm} I_{l+1,l}$$

together with $I_{k+1,k} \equiv I_{k+1,k}^{+} \equiv I_{k+1,k}^{-}$. Then (1)–(3) imply

$$\begin{aligned}
&[I_{lm}^+, I_{kl}^+]_q = I_{km}^+, \quad [I_{kl}^+, I_{km}^+]_q = I_{lm}^+, \quad [I_{km}^+, I_{lm}^+]_q = I_{kl}^+ \quad \text{if} \;\; k > l > m, \\
&[I_{kl}^+, I_{mn}^+] = 0 \qquad \text{if} \;\; k > l > m > n \qquad \text{or} \;\; \text{if} \;\; k > m > n > l; \\
&[I_{kl}^+, I_{mn}^+] = (q - q^{-1})(I_{ln}^+ I_{km}^+ - I_{kn}^+ I_{ml}^+) \qquad \text{if} \;\; k > m > l > n.
\end{aligned} \tag{5}$$

An analogous set of relations is obtained for I^-_{kl} combined with $q \to q^{-1}$ (denote this alternative set by (5′)). When $q \to 1$, i.e., in the 'classical' limit, both relations (5) and (5′) go over into those of $so(n+1)$.

B. Contraction of $U_q(so_{n+1})$ into $U_q(iso_n)$ in terms of trilinear relations

To obtain the deformed algebra $U_q(iso_n)$, we apply the contraction procedure in its usual form [6] first to the q-deformed algebra $U_q(so_{n+1})$ given in terms of the trilinear relations (1)–(3): replacing $I_{n+1,n} \to \rho\, P_n$, with the trivial replacement $I_{k+1,k} \to \tilde{I}_{k+1,k}$ for $1 \le k \le n-1$, and sending $\rho \to \infty$, we arrive at the relations [7]

$$\tilde{I}^2_{n,n-1} P_n + P_n \tilde{I}^2_{n,n-1} - [2]_q\ \tilde{I}_{n,n-1} P_n \tilde{I}_{n,n-1} = -P_n, \tag{6}$$

$$P_n^2 \tilde{I}_{n,n-1} + \tilde{I}_{n,n-1} P_n^2 - [2]_q\ P_n \tilde{I}_{n,n-1} P_n = 0, \tag{7}$$

$$[\tilde{I}_{k,k-1}, P_n] = 0 \qquad \text{if} \quad k < n, \tag{8}$$

which together with the rest of relations (that remain intact and form the subalgebra $U'_q(so_n)$) define the q-deformed inhomogeneous algebra $U'_q(iso_n)$. Of course, this real form of the complex inhomogeneous algebra $U'_q(iso(n, C)$ requires that the involution

$$I^*_{j,j-1} = -I_{j,j-1}, \quad j = 2, \ldots, n, \qquad P^*_n = -P_n, \tag{9}$$

be imposed (compare with (4)).

Observe that in the formulation just given, we have only a single ('senior') component of translation generators. The whole set of translations emerges when one uses the 'bilinear' approach discussed right below.

C. Contraction into $U'_q(iso_n)$ of the bilinear version of $U'_q(so_{n+1})$

Now let us contract relations (5), (5′). Set $I^{\pm}_{n+1,k} = \rho\, P^{\pm}_k$ for $1 \le k \le n$ as well as $I^{\pm}_{kl} = \tilde{I}^{\pm}_{kl}$ for $1 \le l < k \le n$, and then send $\rho \to \infty$. As a result, we get the equality

$$[P^{\pm}_l, P^{\pm}_m]_{q^{\pm 1}} = 0, \qquad 1 \le m < l \le n, \tag{10}$$

as well as the rest of relations that remain unchanged (formally, i.e., modulo replacement $I^{\pm}_{kl} \to \tilde{I}^{\pm}_{kl}$ and $I^{\pm}_{n+1,k} \ \to\ P^{\pm}_k$): those which form the subalgebra $U'_q(so_n)$ and those which characterize the transformation property of $P^{\pm}_k$ with respect to $U'_q(so_n)$.

If $q \to 1$, the set of relations defining $U'_q(iso_n)$ turns into commutation relations of the 'classical' algebra $iso(n)$. In what follows, we shall omit tildas over I_{kl} and the prime in the notation of q-deformed algebras.

Remark 1. The algebra $U_q(iso_n)$ contains the subalgebra $U_q(so_n)$ in canonical way, i.e., similarly to the embedding of nondeformed algebras: $so(n) \subset iso(n)$.

Remark 2. The generators of translations in $U_q(iso_n)$ are noncommuting: as seen from (10), they q-commute.

Remark 3. It can be proved that the element

$$C_2(U_q(iso_n)) \equiv \sum_{k=1}^{n} ([2]/2)^{k-1}\ \ (1/2)\{P^+_k, P^-_k\}$$

is central for the inhomogeneous q-algebra $U_q(iso_n)$. When $q \to 1$, it reduces to the Casimir C_2 of the classical $iso(n)$.

Let us quote examples of $U_q(iso_n)$ for small values $n = 2,\ 3$.

n = 2

$$[I_{21}, P_2]_q = P_1^+ \qquad [P_1^+, I_{21}]_q = P_2 \qquad [P_2, P_1^+]_q = 0 \tag{11}$$

n = 3

$$\begin{aligned} &[I_{21}, I_{32}]_q = I_{31}^+ && [I_{21}, P_3] = 0 \\ &[I_{32}, I_{31}^+]_q = I_{21} && [I_{32}, P_1^+] = 0 \\ &[I_{31}^+, I_{21}]_q = I_{32} && [I_{31}^+, P_2^+] = (q - q^{-1})(P_1^+ I_{32} - P_3 I_{21}) \end{aligned} \tag{12}$$

$$\begin{aligned} &[I_{32}, P_3]_q = P_2^+ && [I_{21}, P_2^+]_q = P_1^+ && [I_{31}^+, P_3]_q = P_1^+ \\ &[P_2^+, I_{32}]_q = P_3 && [P_1^+, I_{21}]_q = P_2^+ && [P_1^+, I_{31}^+]_q = P_3 \\ &[P_3, P_2^+]_q = 0 && [P_2^+, P_1^+]_q = 0 && [P_3, P_1^+]_q = 0 \end{aligned} \tag{13}$$

Remark 4. The Euclidean q-algebra $U_q(iso_3)$ contains the homogeneous subalgebra $U_q(so_3)$, given by the left column in (12), isomorphic to the (cyclically symmetric, Cartesian) q-deformed Fairlie-Odesskii algebra [8]. Besides, three columns in (13) represent three distinct *inhomogeneous subalgebras* of $U_q(iso_3)$, each isomorphic to $U_q(iso_2)$, conf. (11). This feature extends to higher n: the algebra $U_q(iso_n)$ contains n distinct subalgebras isomorphic to $U_q(iso_{n-1})$.

3. Representations of inhomogeneous algebras $U_q(iso_n)$

We proceed with finite-dimensional representations of the algebras $U_q(so_{n+1})$. These representations denoted by $T_{m_{n+1}}$ are given by 'highest weights' m_{n+1} consisting of $[\frac{n+1}{2}]$ components $m_{1,n}, m_{2,n}, \ldots, m_{[\frac{n+1}{2}],n+1}$ (here $[r]$ means the integer part of r) which are all integers or all half-integers satisfying the dominance conditions

$$m_{1,2k+1} \geq m_{2,2k+1} \geq \ldots \geq m_{k,2k+1} \geq 0 \qquad \text{if} \qquad n = 2k, \tag{14}$$

$$m_{1,2k} \geq m_{2,2k} \geq \ldots \geq m_{k-1,2k} \geq |m_{k,2k}| \qquad \text{if} \qquad n = 2k-1. \tag{15}$$

When restricted to subalgebra $U_q(so_n)$, the representation $T_{m_{n+1}}$ contains with multiplicity 1 those and only those irreps T_{m_n} for which the inequalities ('branching rules') similar to the nondeformed case [9] are satisfied:

$$m_{1,2k+1} \geq m_{1,2k} \geq m_{2,2k+1} \geq m_{2,2k} \geq \ldots \geq m_{k,2k+1} \geq m_{k,2k} \geq -m_{k,2k+1}, \tag{16}$$

$$m_{1,2k} \geq m_{1,2k-1} \geq m_{2,2k} \geq m_{2,2k-1} \geq \ldots \geq m_{k-1,2k-1} \geq | m_{k,2k} | \,. \tag{17}$$

For a basis in the representation space, we take (q-analogue of) the GT basis [9]. Its elements are labelled by the GT schemes

$$\{\xi_{n+1}\} \equiv \{m_{n+1}, m_n, \ldots, m_2\} \equiv \{m_{n+1}, \xi_n\} \equiv \{m_{n+1}, m_n, \xi_{n-1}\} \tag{18}$$

and denoted as $|\{\xi_{n+1}\}\rangle$ or simply $|\xi_{n+1}\rangle$.

We use the notation $[x] \equiv \dfrac{q^x - q^{-x}}{q - q^{-1}}$ for the q-number corresponding to a real number x. In what follows, q is not a root of unity.

The infinitesimal generator $I_{2k+1,2k}$ in the representation $T_{m_{2k+1}}$ of $U_q(so_{2k+1})$ acts upon the basis elements (18) according to (here $\beta \equiv \xi_{n-1}$)

$$\begin{aligned} T_{m_{2k+1}}(I_{2k+1,2k})|m_{2k+1}, m_{2k}, \beta\rangle &= \sum_{j=1}^{k} A_{2k}^{j}(m_{2k})|m_{2k+1}, m_{2k}^{+j}, \beta\rangle \\ &- \sum_{j=1}^{k} A_{2k}^{j}(m_{2k}^{-j})|m_{2k+1}, m_{2k}^{-j}, \beta\rangle \end{aligned} \tag{19}$$

and the generator $I_{2k,2k-1}$ in the representation $T_{m_{2k}}$ of $U_q(so_{2k})$ acts as

$$\begin{aligned} T_{m_{2k}}(I_{2k,2k-1})|m_{2k}, m_{2k-1}, \beta\rangle = \sum_{j=1}^{k-1} B_{2k-1}^{j}(m_{2k-1})|m_{2k}, m_{2k-1}^{+j}, \beta\rangle \\ - \sum_{j=1}^{k-1} B_{2k-1}^{j}(m_{2k-1}^{-j})|m_{2k}, m_{2k-1}^{-j}, \beta\rangle + \ i\ C_{2k-1}(m_{2k-1})|m_{2k}, m_{2k-1}, \beta\rangle. \end{aligned} \tag{20}$$

In these formulas, $m_n^{\pm j}$ means that the j-th component $m_{j,n}$ of the highest weight m_n is to be replaced by $m_{j,n} \pm 1$; matrix elements A_{2k}^{j}, B_{2k-1}^{j}, C_{2k-1} are given in terms of 'l-coordinates' $l_{j,2k+1} = m_{j,2k+1} + k - j + 1$, $\quad l_{j,2k} = m_{j,2k} + k - j$ by the expressions

$$\begin{aligned} A_{2k}^{j}(\xi) = \left(\frac{[l_{j,2k}][l_{j,2k}+1]}{[2l_{j,2k}][2l_{j,2k}+2]} \right)^{\frac{1}{2}} \Bigg| \frac{\prod_{i=1}^{k} [l_{i,2k+1} + l_{j,2k}][l_{i,2k+1} - l_{j,2k} - 1]}{\prod_{i\neq j}^{k} [l_{i,2k} + l_{j,2k}][l_{i,2k} - l_{j,2k}]} \\ \times \frac{\prod_{i=1}^{k-1} [l_{i,2k-1} + l_{j,2k}][l_{i,2k-1} - l_{j,2k} - 1]}{\prod_{i\neq j}^{k} [l_{i,2k} + l_{j,2k} + 1][l_{i,2k} - l_{j,2k} - 1]} \Bigg|^{\frac{1}{2}}, \end{aligned} \tag{21}$$

$$\begin{aligned} B_{2k-1}^{j}(\xi) = \Bigg| \frac{\prod_{i=1}^{k} [l_{i,2k} + l_{j,2k-1}][l_{i,2k} - l_{j,2k-1}]}{[2l_{j,2k-1}+1][2l_{j,2k-1}-1] \prod_{i\neq j}^{k-1} [l_{i,2k-1} + l_{j,2k-1}][l_{i,2k-1} - l_{j,2k-1}]} \\ \times \frac{\prod_{i=1}^{k-1} [l_{i,2k-2} + l_{j,2k-1}][l_{i,2k-2} - l_{j,2k-1}]}{[l_{j,2k-1}]^2 \prod_{i\neq j}^{k-1} [l_{i,2k-1} + l_{j,2k-1} - 1][l_{i,2k-1} - l_{j,2k-1} - 1]} \Bigg|^{\frac{1}{2}}, \end{aligned} \tag{22}$$

$$C_{2k-1}(\xi) = \frac{\prod_{s=1}^{k} [l_{s,2k}] \prod_{s=1}^{k-1} [l_{s,2k-2}]}{\prod_{s=1}^{k-1} [l_{s,2k-1}][l_{s,2k-1} - 1]}. \tag{23}$$

The detailed proof that the representation operators defined by (19)–(23) satisfy the basic relations (1)–(3) of the algebra $U_q(\mathrm{so}_n)$ for $n = 2k+1$ and $n = 2k$ is given in [10]. It can be verified that the $*$-condition $T(I_{j,j-1})^* = -T(I_{j,j-1})$, $j = 2, \ldots, n+1$ (compare with (4)), for representation operators given in (19)-(23) is fulfilled if $q \in \mathbf{R}$ or $q = \exp \mathrm{i}h$, $h \in \mathbf{R}$. Therefore, the action formulas for $T_{m_{n+1}}(I_{n+1,n})$ together with similar formulas for the operators $T_{m_{n+1}}(I_{i,i-1})$, $i < n+1$, give irreducible infinitesimally unitary (or $*$-) representations of the algebra $U_q(\mathrm{so}_{n+1})$.

Representations of $U_q(iso_n)$

Representations of inhomogeneous algebras $U_q(iso_n)$ are obtained from the representations of $U_q(so_{n+1})$ given above in a manner similar to that followed by Chakrabarti [6] for obtaining irreps of Euclidean algebras $iso(n)$ from finite-dimensional irreps of rotation algebras $so(n+1)$.

The representations of $U_q(iso_n)$ are characterized by a complex number a and the set $\tilde{m}_{n+1} \equiv \{m_{2,n+1}, m_{3,n+1}, \ldots, m_{[\frac{n+1}{2}],n+1}\}$ of numbers which are all integers or all half-integers. Due to validity of the chain of inclusions

$$U_q(\mathrm{iso}_n) \supset U_q(\mathrm{so}_n) \supset \ldots \supset U_q(\mathrm{so}_4) \supset U_q(\mathrm{so}_3), \tag{24}$$

the representation space $\mathcal{V}_{a,\tilde{m}_{n+1}}$ for a representation of $U_q(iso_n)$ is taken as a direct sum of the representation spaces of the q-rotation subalgebra $U_q(so_n)$ given by m_n, whose components $m_{2,n}, \ldots, m_{[\frac{n}{2}],n}$ satisfy the inequalities (16)-(17) and the first component $m_{1,n}$ is bounded only from below, $\infty \geq m_{1,n} \geq m_{2,n+1}$. In this way, one is led to infinite-dimensional representations of the inhomogeneous algebras $U_q(iso_n)$.

The representation operators $T_{a,\tilde{m}_{2k+1}}(I_{j,j-1})$ that correspond to the generators $I_{j,j-1}$ of the compact subalgebra $U_q(so_n)$ act according to formulas coinciding with (19)–(23).

The representation operator $T_{a,\tilde{m}_{2k+1}}(P_{2k})$ which corresponds to the translation generator P_{2k} of the algebra $U_q(iso_{2k})$ acts upon basis elements (12) according to (here $\beta \equiv \xi_{n-1}$)

$$\begin{aligned} T_{a,\tilde{m}_{2k+1}}(P_{2k})|\tilde{m}_{2k+1}, m_{2k}, \beta\rangle = &\sum_{j=1}^{k} \mathcal{A}^j_{2k}(m_{2k})|\tilde{m}_{2k+1}, m^{+j}_{2k}, \beta\rangle \\ &- \sum_{j=1}^{k} \mathcal{A}^j_{2k}(m^{-j}_{2k})|\tilde{m}_{2k+1}, m^{-j}_{2k}, \beta\rangle \end{aligned} \tag{25}$$

and the representation operator $T_{a,\tilde{m}_{2k}}(P_{2k-1})$ which corresponds to the translation generator P_{2k-1} of the algebra $U_q(iso_{2k-1})$ acts as

$$\begin{aligned} &T_{a,\tilde{m}_{2k}}(P_{2k-1})|\tilde{m}_{2k}, m_{2k-1}, \beta\rangle = \sum_{j=1}^{k-1} \mathcal{B}^j_{2k-1}(m_{2k-1})|\tilde{m}_{2k}, m^{+j}_{2k-1}, \beta\rangle \\ &- \sum_{j=1}^{k-1} \mathcal{B}^j_{2k-1}(m^{-j}_{2k-1})|\tilde{m}_{2k}, m^{-j}_{2k-1}, \beta\rangle + \ i\, \mathcal{C}_{2k-1}(m_{2k-1})|\tilde{m}_{2k}, m_{2k-1}, \beta\rangle \end{aligned} \tag{26}$$

where

$$\mathcal{A}^j_{2k}(\xi) = a\left(\frac{[l_{j,2k}][l_{j,2k}+1]}{[2l_{j,2k}][2l_{j,2k}+2]}\right)^{\frac{1}{2}} \left|\frac{\prod\limits_{i=2}^{k}[l_{i,2k+1}+l_{j,2k}][l_{i,2k+1}-l_{j,2k}-1]}{\prod\limits_{i\neq j}^{k}[l_{i,2k}+l_{j,2k}][l_{i,2k}-l_{j,2k}]}\right. \times \left.\frac{\prod\limits_{i=1}^{k-1}[l_{i,2k-1}+l_{j,2k}][l_{i,2k-1}-l_{j,2k}-1]}{\prod\limits_{i\neq j}^{k}[l_{i,2k}+l_{j,2k}+1][l_{i,2k}-l_{j,2k}-1]}\right|^{\frac{1}{2}}, \tag{27}$$

$$\mathcal{B}^j_{2k-1}(\xi) = a\left|\frac{\prod\limits_{i=2}^{k}[l_{i,2k}+l_{j,2k-1}][l_{i,2k}-l_{j,2k-1}]}{[2l_{j,2k-1}+1][2l_{j,2k-1}-1]\prod\limits_{i\neq j}^{k-1}[l_{i,2k-1}+l_{j,2k-1}][l_{i,2k-1}-l_{j,2k-1}]}\right. \times \left.\frac{\prod\limits_{i=1}^{k-1}[l_{i,2k-2}+l_{j,2k-1}][l_{i,2k-2}-l_{j,2k-1}]}{[l_{j,2k-1}]^2\prod\limits_{i\neq j}^{k-1}[l_{i,2k-1}+l_{j,2k-1}-1][l_{i,2k-1}-l_{j,2k-1}-1]}\right|^{\frac{1}{2}}, \tag{28}$$

$$\mathcal{C}_{2k-1}(\xi) = a\frac{\prod\limits_{s=2}^{k}[l_{s,2k}]\prod\limits_{s=1}^{k-1}[l_{s,2k-2}]}{\prod\limits_{s=1}^{k-1}[l_{s,2k-1}][l_{s,2k-1}-1]}. \tag{29}$$

The representation operators given by formulas (25)–(29) (together with formulas (19)–(23) for the subalgebra $U_q(\mathrm{so}_n)$) can be proved to satisfy the defining relations (6)–(8), in complete analogy to the proof [10] in the case of homogeneous algebra $U_q(\mathrm{so}_n)$. Moreover, it can be verified that these representations are $*$-representations (satisfy $*$-relations (9)), for q real or the pure phase if a is real in formulas (25)–(29).

4. Class 1 representations of the inhomogeneous algebra $U_q(iso_n)$

We use the term class 1 (or C1)representation for those representations of either $U_q(so_{n+1})$ or $U_q(iso_n)$ which contain the trivial (identical) representation of the maximal compact subalgebra $U_q(\mathrm{so}_n)$. Note that among representations of $U_q(\mathrm{so}_3)$ all irreps T_l are given by an integer l and only these are C1 representations with respect to the Abelian subalgebra generated by I_{21}.

The particular case T_a (C1 representations of $U_q(iso_n)$ characterized by a single complex number a) was considered in [11]. These special representations are obtainable from our general formulas (19)–(23), (25)–(29) if we set $m_{2,n+1} = m_{3,n+1} = \cdots = m_{[\frac{n+1}{2}],n+1} = 0$.

The carrier space V_a of T_a is composed of carrier spaces $V_{\mathbf{m}_n}$ of irreps $T_{\mathbf{m}_n}$ of the subalgebra $U_q(\mathrm{so}_n)$ with the signatures $(m_{1,n}, 0, \ldots, 0)$, $\infty \geq m_{1,n} \geq 0$ (which in turn are

C1 irreps w.r.t. $U_q(\mathrm{so}_{n-1})$). Accordingly, the basis in V_a is composed as the union of G-T bases of such subspaces $V_{\mathbf{m}_n}$. We denote basis elements in the representation space $V_{\mathbf{m}_n}$ by $|m_n, m_{n-1}, \cdots, m_3, m_2\rangle$.

The operator $T_a(I_{21})$ and operators $T_a(I_{i,i-1})$, $3 \le i \le n$, representing generators of subalgebra $U_q(iso_n)$ act in this basis by the formulas

$$\begin{aligned} &T_a(I_{21})|m_n, m_{n-1}, \ldots, m_2\rangle = i[m_2]|m_n, m_{n-1}, \ldots, m_2\rangle, \\ &T_a(I_{i,i-1})|m_n, m_{n-1}, \ldots, m_2\rangle = \\ &\quad = \left([m_i^{(+)} + i - 2][m_i^{(-)}]\right)^{1/2} \ R(m_{i-1})|m_n, \ldots, m_{i-1}+1, \ldots, m_2\rangle \\ &\quad - \left([m_i^{(+)} + i - 3][m_i^{(-)} + 1]\right)^{1/2} R(m_{i-1}-1)|m_n, \ldots, m_{i-1}-1, \ldots, m_2\rangle, \end{aligned} \tag{30}$$

where $m_i^{(\pm)} \equiv m_i \pm m_{i-1}$ and

$$R(m_i) = \left(\frac{[m_i^{(+)} + i - 2][m_i^{(-)} + 1]}{[\ 2\ m_i + i - 2][\ 2\ m_i + i]} \right)^{1/2}.$$

The operator $T_a(P_n)$ of the representation T_a of $U_q(iso_n)$, $n \ge 2$, corresponding to the translation P_n is given by the formula

$$\begin{aligned} T_a(P_n)|m_n, m_{n-1}, \cdots, m_2\rangle &= a\ R(m_n)|m_n+1, m_{n-1}, m_{n-2}, \cdots m_2\rangle \\ &\quad -a\ R(m_n - 1)|m_n - 1, m_{n-1}, m_{n-2}, \cdots, m_2\rangle \end{aligned} \tag{31}$$

In summary, we have presented the nonstandard q-deformed inhomogeneous algebras $U_q(iso_n)$ defined in a uniform manner for all $n \ge 2$, for which both 'trilinear' and 'bilinear' presentations were given. All the infinite-dimensional representations of $U_q(iso_n)$ that directly correspond to well-known irreducible representations of the classical limit iso_n are obtained and illustrated with the particular case of class 1 irreps. It is an interesting task to analyze $U_q(iso_n)$ representations for cases where discrete components characterizing representations are not all integers or all half-integers as well as the cases of q being roots of 1.

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