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# Representations of a Cubic Deformation of *su(2)* and Parasupersymmetric Commutation Relations

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**Abstract**

Application of the twisted generalized Weyl construction to description of irreducible representations of the algebra generated by two idempotents and a family of graded-commuting selfadjoint unitary elements which are connected by relations of commutation and anticommutation is presented. It's also discussed $*$-representations and the theory of cubic deformation $A^+_{pq}(3,1)$ of the enveloping algebra $su(2)$.

## 1 Introduction.

In the last few years, quantum groups, different $q$-deformations of the universal algebra of Lie algebras, their $\mathbb{Z}_2$-graded analogs, superalgebras and quantum superalgebras have attracted more interest and play an important role in various branches of modern physics. For applications, in particular in particle physics, knot theory, supersymmetric models and others, it is desirable to have a well-developed representation theory.

The purpose of this paper is to study $*$-representations of some nonlinear deformation of the enveloping algebra $su(2)$ and the algebra generated by two idempotents, and a family of graded-commuting selfadjoint elements which are connected by relations of commutation and anticommutation. Namely, in Section 2, we study representations of a cubic deformation of $su(2)$ such as Witten's deformation $A^+_{pq}(3,1)$ [4]. This algebra and their $*$-representations have recently been studied in connection with some physically interesting applications (see [2, 3, 4] and references therein). There are some other non-linear generalizations of $su(2)$ intensively studied in the literature and worth to mention here: in particular, the quantum algebra $su_q(2)$ [23], Witten's first deformation [27, 19], the Higgs algebras [7, 1], the Fairlie $q$-deformation of $so(3)$ [6, 22], nonlinear $sl(2)$ algebras [1] and others. Our purpose is to describe all irreducible representations of the algebra $A^+_{pq}(3,1)$ by bounded and unbounded operators. The method of solving this problem is based on the study of some dynamical system. The important point to note here that this method allows us to give a complete classification of representations, up to a unitary equivalence, in the class of "integrable" representations and it can be applied, in particular, to the study of $*$-representations of the Higgs algebra [7] and other nonlinear $sl(2)$ algebras [1].

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Using some generalization of this method, in Section 3, we will study collections of unitary graded-commuting selfadjoint operators $(\gamma_k)_{k=1}^n$ which commute or anticommute with a pair of unitary selfadjoint operators $u$, $v$. Note that these operators determine representations of the $*$-algebra which can be considered as an algebra obtained by the twisted generalized Weyl construction investigated in [12]. The study of their representations can be reduced to the study of a dynamical system on the set of irreducible representations of some noncommutative subalgebra. Such methods of studying representations by using dynamical systems go back to the classical papers [5, 11, 17, 24]. In particular, they have been developed and extended to the families of operators satisfying relations of some special type (see [13, 16, 25, 26]), and then applied for the study of many objects important in mathematical physics ([14, 15] and references therein).

The paper is organized as follows: in Section 2, we describe briefly the method of dynamical systems and study $*$-representations of the algebra $A_{pq}^+(3,1)$. In Section 3, we review some facts on twisted generalized Weyl constructions. Then, using these results and results on representations of commutative Lie superalgebras ([21]), we will give a complete classification of irreducible representations of the algebra generated by unitary selfadjoint elements $u$, $v$ and the collection of unitary graded-commuting selfadjoint elements $(\gamma_k)_{k=1}^n$ which are connected with each other by the relations of commutation or anticommutation. Families of unitary selfadjoint operators and families of idempotents in the algebra of bounded operators in a Hilbert space were studied, in particular, in [9, 10].

# 2 Representations of a cubic deformation of $su(2)$

## 2.1 Representations of commutation relations and dynamical systems

Consider the operator relation

$$A_k B = B F_k(\mathbb{A}), \qquad (k = 1, \dots, n), \tag{1}$$

where $\mathbb{A} = (A_k)_{k=1}^n$ is a family of selfadjoint, generally speaking, unbounded operators in a complex separable Hilbert space $H$, $F_k(\cdot)$ is a real measurable function on $\mathbb{R}^n$. The method of study of operators satisfying (1) was developed in [13, 25, 26]. It has the origin in the theory of imprimitivity and induced representations of groups ([11]), on the other hand, in the theory of $C^*$-products and their representations ([5, 17, 24]). At the same time, this method has some new aspects which allow us to study many objects appearing in mathematical physics (see [15]).

We allow the operators $\mathbb{A}$, $B$ to be unbounded. Thus, we have to make precise the sense in which relations (1) hold.

*We call the operators $\mathbb{A} = (A_k)_{k=1}^n$ and $B$ a representation of (1) if*

$$E_{\mathbb{A}}(\Delta) U = U E_{\mathbb{A}}(\mathbb{F}^{-1}(\Delta)), \quad [E_{|B|}(\Delta'), E_{\mathbb{A}}(\Delta)] = 0, \quad \Delta \in \mathfrak{B}(\mathbb{R}^n), \ \Delta' \in \mathfrak{B}(\mathbb{R}), \tag{2}$$

*where $\mathbb{F}(\cdot) = (F_1(\cdot), \dots, F_n(\cdot)) : \mathbb{R}^n \to \mathbb{R}^n$, $E_{\mathbb{A}}(\cdot)$ is a joint resolution of the identity for the commuting family $\mathbb{A}$, $B = U|B|$ is a polar decomposition of the closed operator $B$.*

For bounded $A_k$, $B$, this definition and the usual pointwise definition are equivalent. We will say that the representation $(\mathbb{A}, B)$ is irreducible if any bounded operator such that $CX \subseteq XC$, $C^*X \subseteq XC^*$, where $X$ is one of the operators $A_k$, $B$, $B^*$, is a multiple of the identity operator.

The important role in the study of the commutation relations (1) is played by the dynamical system (d.s.) on $\mathbb{R}^n$ generated by $\mathbb{F}$. We will assume that $\mathbb{F}$ is bijective. It follows from (2) that for any $\mathbb{F}$-invariant set $\Delta$, the operator $E_{\mathbb{A}}(\Delta)$ is a projection on an invariant subspace. If the dynamical system is *simple*, i.e., there exists a measurable set intersecting every orbit of the dynamical system exactly at one point, then any irreducible representation arises from an orbit of d.s., i.e., the spectral measure of the family $\mathbb{A}$ is concentrated on an orbit. We restrict ourselves by considering only this case.

If no conditions are imposed on the operator $B$, then the problem of unitary classification of all families $(\mathbb{A}, B)$ is a very difficult problem. It contains as a subproblem the problem of unitary classification of pairs of selfadjoint operators without any relations (see [18, 10]). We will assume that the operators $B$, $B^*$ are additionally connected by the relation

$$B^*B = \varphi(\mathbb{A}, BB^*), \tag{3}$$

where $\varphi(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function. If the operators $\mathbb{A}$, $B$, $B^*$ are bounded, (3) is equivalent to the following equality

$$|B|^2 U = U F_{n+1}(\mathbb{A}, |B|^2) \tag{4}$$

with $F_{n+1}(x_1, \ldots, x_{n+1}) = \varphi(F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n), x_{n+1})$. Relation (4) is of the form (1), hence we can give a definition of unbounded representations of relations (1) and (3). Note that the assumed condition implies the following relations for the operator $U$: operators $U^l(U^*)^l$, $(U^*)^l U^l$, $l = 1, 2, \ldots$ form a commuting family (i.e., the operator $U$ is centered).

The complete classification of all irreducible families $(\mathbb{A}, B, B^*)$ satisfying (1), (3) was given in [25]. Moreover, there was proved the structure theorem which defines the form of any such operators as a direct sum or a direct integral of irreducible ones. Let

$$\mathcal{F}(x_1, \ldots, x_{n+1}) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n), F_{n+1}(x_1, \ldots, x_{n+1})$$

be bijective. By [25], if the dynamical system is simple, any irreducible representation of relations (1), (3) arises from certain subsets of an orbit of the dynamical system on $\mathbb{R}^{n+1}$ generated by $\mathcal{F}$. Namely, any irreducible representation is unitarily equivalent to one of the following:

$$A_k e_{\mathbf{x}} = x_k e_{\mathbf{x}}, \ B e_{\mathbf{x}} = \sqrt{x_{n+1}}\, u(\mathbf{x}) e_{\mathcal{F}(\mathbf{x})}, \ \mathbf{x} = (x_1, \ldots, x_{n+1}) \in \Omega_0,$$

where $\Omega_0$ is a connected subset of some orbit of the dinamcal system on $\mathbb{R}^{n+1}$ generated by $\mathcal{F}$ [16].

## 2.2 Representations of the cubic deformation $A^+_{pq}(3,1)$ of $su(2)$

The $*$-algebra $A^+_{pq}(3,1)$ was introduced by Delbeq and Quesne ([4]) as a two-parameter nonlinear cubic deformation of $su(2)$. It is generated by generators $J_0$, $J_+$, $J_-$ satisfying the relations:

$$\begin{aligned} &[J_0, J_+] = (1 + (1-q)J_0)J_+, \qquad [J_0, J_-] = -J_-(1 + (1-q)J_0), \\ &[J_+, J_-] = 2J_0(1 + (1-q)J_0)(1 - (1-p)J_0), \end{aligned} \tag{5}$$

with involution defined as follows: $J_0^* = J_0$, $J_+^* = J_-$. We will assume that $0 < p < 1$, $0 < q < 1$.

This algebra has a Casimir operator

$$C = J_-J_+ + \frac{2(q-1)}{(q+1)(q^3-1)}J_0(J_0+1)(1+(p+q)q-(1-p)(1+q)J_0).$$

Representations of the $*$-algebra $A^+_{pp}(3,1)$ were classified in [4]. In this paper we allow parameters $p$, $q$ to be different.

It is clear that bounded representations of the $*$-algebra are defined by their value on the generators, i.e., operators $J_0^* = J_0$, $J_+ \in L(H)$ satisfying (5). An unbounded representation of $A^+_{pq}(3,1)$ is defined to be formed by unbounded operators $J_0^* = J_0$, $J_+$ satisfying (1), (3) in the sense of the definition given in the previous subsection. It follows from the definition that the spectral projections of the Casimir operator $C$ commute with the generators of the algebra, i.e., $E_C(\Delta)A \subseteq AE_C(\Delta)$, where $A$ is one of the operators $J_0$, $J_+$, $J_+^*$ and $\Delta \in \mathfrak{B}(\mathbb{R})$. Thus, given an irreducible triple $(J_0, J_+, J_-)$, the operator $C$ is a multiple of the identity operator: $C = \mu I$, where $\mu \in \mathbb{R}$. Moreover,

$$J_0J_+ = q^{-1}J_+(J_0+1), \quad J_+^*J_+ = g(J_0,\mu), \tag{6}$$

where $g(x,\mu) = \mu - \dfrac{2(q-1)}{(q+1)(q^3-1)}x(x+1)(1+(p+q)q-(1-p)(1+q)x)$.

Conversely, any irreducible representation of the algebra generated by $J_0 = J_0^*$, $J_+$, $J_+^*$ and relations (6) which is defined on a Hilbert space $H$, $\dim H \geq 2$, is an irreducible representation of $A^+_{pq}(3,1)$ (see [20], Lemma 1). In what follows, we will study representations of relations (6).

To (6) there corresponds the dynamical system generated by $\mathbb{F}(x,y) = (q^{-1}(x+1), g(q^{-1}(x+1),\mu)) : \mathbb{R}^2 \to \mathbb{R}^2$. It has the measurable section $\tau = ([q^{-1}(\delta_1+1),\delta_1) \cup \left\{\dfrac{1}{q-1}\right\} \cup (\delta_2, q^{-1}(\delta_2+1)]) \times \mathbb{R}$, where $\delta_1 < \dfrac{1}{q-1} < \delta_2$. Thus, any irreducible representation arises from an orbit of the dynamical system and can be described by the formulae given above. It is easy to show that any orbit of the dynamical system is of the form

$$\Omega_x = \left\{(q^{-n}\left(x - \frac{1}{q-1}\right) + \frac{1}{q-1}, g\left(q^{-n}\left(x-\frac{1}{q-1}\right)+\frac{1}{q-1},\mu\right) \mid n \in \mathbb{Z}\right\}.$$

In the sequel we will denote the point of the orbit by $(f_1(x,\mu,n), f_2(x,\mu,n))$. It depends on a behavior of the function $g(x,\mu)$ what kind of representations relations (6) will have. Since the calculations are rather lengthy, we leave them out and state the final result. For the deeper discussion, we refer the reader to [20]. First, let us introduce some notations.

Denote by $x_1(\mu) \leq x_2(\mu) \leq x_3(\mu)$ the real roots of the equation $g(x,\mu) = 0$. Let $a = 1+(p+q)q$, $b = (1-p)(1+q)$, $\gamma(p,q) = ab^{-1}$. Then

$$\varepsilon_1(p,q) = \frac{a-b+\sqrt{a^2+b^2+ab}}{3b}, \quad \varepsilon_2(p,q) = \frac{a-b-\sqrt{a^2+b^2+ab}}{3b}$$

are the extreme points of $g(x,\mu)$. Write $y_1(p,q) = \min g(x,0) \equiv g(\varepsilon_1(p,q),0)$, $y_2(p,q) = \max g(x,0) \equiv g(\varepsilon_2(p,q),0)$ and $\psi(\mu,p,q) = \dfrac{x_3(\mu)-(q-1)^{-1}}{x_2(\mu)-(q-1)^{-1}}$. Then

$$\max_{\mu\in[-y_2(p,q),-y_1(p,q)]}\psi(\mu,p,q) = \frac{ab^{-1}-1-2\varepsilon_2(p,q)-(q-1)^{-1}}{\varepsilon_2(p,q)-(q-1)^{-1}},$$

which will be denoted by $\psi(p,q)$.

Below we give a list of irreducible representations of the $*$-algebra $A^+_{pq}(3,1)$:

1. one-dimensional representations: $J_0 = (q-1)^{-1}$, $J_+ = \lambda$, $\lambda \in \mathbb{C}$;

2. finite-dimensional representations:

a) for any $p$, $q$ such that $\gamma(p,q) - 1 - 2\varepsilon_1(p,q) \le (q-1)^{-1}$, for any $n \ge 2$, there exists the representation of the dimension $n+1$ with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k),\ f_2(x_1(\mu),\mu,k) \mid k = 0,\dots,n\}$, where $\mu$ is uniquely defined from the equation $\kappa(\mu,p,q) = q^{-n}$; where $\kappa = \dfrac{x_2 - (q-1)^{-1}}{x_1 - (q-1)^{-1}}$.

b) for any $p$, $q$ such that $\gamma(p,q) - 1 - 2\varepsilon_1(p,q) > (q-1)^{-1}$, there exist the representations of any dimension $n \le \log_{q^{-1}}(\gamma(p,q) - 1 - 2\varepsilon_1(p,q)) + 1$ with $\Omega_0 = \{(f_1(x_1(-y_1(p,q)), -y_1(p,q),k),\ f_2(x_1(-y_1(p,q)),\ -y_1(p,q),k) \mid k = 0,\dots,n\}$;

c) for any $n \in \mathbb{N}$ and $\mu \in \mathbb{R}$ such that $x_1(\mu) > (q-1)^{-1}$ and $q^n(x_3(\mu) - (q-1)^{-1}) = x_1(\mu) - (q-1)^{-1}$ and $x_3(\mu) < q^{-1}(x_2(\mu)+1)$, there exists the representation of the dimension $n+1$ with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k), f_2(x_1(\mu),\mu,k) \mid k = 0,\dots,n\}$;

3. representations with higher weight:

a) for any $p$, $q$ such that $\gamma(p,q) - 1 - 2\varepsilon_1(p,q) \le (q-1)^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_2(-y_1(p,q)), -y_1(p,q),k), f_2(x_2(-y_1(p,q)),\ -y_1(p,q),k) \mid k \le 0\}$;

b) for any $\mu$ such that $x_1(\mu) \le (q-1)^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_2(\mu),\mu,k), f_2(x_2(\mu),\mu,k) \mid k \le 0\}$;

c) for any $p$, $q$ such that $\psi(p,q) \le q^{-1}$ and $\mu \in (-y_2(p,q), -y_1(p,q))$, there exists the representations with $\Omega_0 = \{(f_1(x_3(\mu),\mu,k), f_2(x_3(\mu),\mu,k) \mid k \le 0\}$;

d) for any $p$, $q$ such that $\psi(p,q) > q^{-1}$ and any $\mu \in (\mu_0, -y_1(p,q))$ such that $x_1(\mu) \le (q-1)^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_3(\mu),\mu,k), f_2(x_3(\mu),\mu,k) \mid k \le 0\}$, where $\mu_0$ is uniquely defined by the condition $\psi(\mu_0,p,q) = q^{-1}$;

4. representations with a lower weight:

a) for any $p$, $q$ and $\mu \in (-\infty, -y_1(p,q))$, there exists the representation with $\Omega_0 = \{(f_1(x_3(\mu),\mu,k), f_2(x_3(\mu),\mu,k) \mid k \ge 0\}$;

b) for any $p$, $q$ and $\mu \in (-y_1(p,q), +\infty)$ and $x_1(\mu) > (q-1)^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k), f_2(x_1(\mu),\mu,k) \mid k \ge 0\}$;

c) for any $p$, $q$ such that $\psi(p,q) > q^{-1}$ and $\mu \in (-y_2(p,q), \mu_0)$, there exists the representation with $\Omega_0 = \{(f_1(x_2(\mu),\mu,k), f_2(x_2(\mu),\mu,k) \mid k \ge 0\}$, where $\mu_0$ is uniquely defined by the condition $\psi(\mu_0,p,q) = q^{-1}$;

d) for any $p$, $q$ such that $\psi(p,q) < q^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_2(-y_2(p,q)), -y_2(p,q),k), f_2(x_2(-y_2(p,q)),\ -y_2(p,q),k) \mid k \ge 0\}$;

e) for any $p$, $q$ and $\mu \in (-y_2(p,q), \mu_0)$ such that there exists $n \in \mathbb{N} \cup \{0\}$ satisfying the condition $q^{-n} < \dfrac{x_2(\mu) - (q-1)^{-1}}{x_1(\mu) - (q-1)^{-1}}$, $q^{-n-1} > \dfrac{x_3(\mu) - (q-1)^{-1}}{x_1(\mu) - (q-1)^{-1}}$, we have the representation with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k), f_2(x_1(\mu),\mu,k) \mid k \ge 0\}$, where $\mu_0$ is defined as follows: $x_1(\mu_0) = (q-1)^{-1}$;

f) for any $p$, $q$ and $\mu$ such that $x_1(\mu) < (q-1)^{-1}$, there is representation with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k), f_2(x_1(\mu),\mu,k) \mid k \le 0\}$

5. nondegenerate representations:

a) for any $\lambda \in \left[\dfrac{(2q\varepsilon_1(p,q) - 1)}{q+1}, \dfrac{(2\varepsilon_1(p,q) - 1)}{q+1}\right) \setminus \{\varepsilon_1(p,q)\}$, if $\gamma - 1 - 2\varepsilon_1 < (q-1)^{-1}$, then there exists the representation with $\Omega_0 = \{(f_1(\lambda, -y_1(p,q),k), f_2(\lambda, -y_1(p,q),k) \mid k \in \mathbb{Z}\}$;

b) for any $\lambda \in \left[\dfrac{(2q\varepsilon_1(p,q)-1)}{q+1}, \dfrac{(2\varepsilon_1(p,q)-1)}{q+1}\right) \setminus \{\varepsilon_1(p,q)\}$ and $\mu \in (\mu_\lambda, -y_1(p,q))$ such that $x_1(\mu) \le (q-1)^{-1}$ there exists the representation with $\Omega_0 = \{(f_1(\lambda,\mu,k), f_2(\lambda,\mu,k) \mid k \in \mathbb{Z}\}$, where $\mu_\lambda$ is defined by the condition: $\mathrm{dist}(\{x \mid g(x,\mu_\lambda) = 0\}, \varepsilon_1(p,q)) = \mathrm{dist}(\{q^{-n}(\lambda - (q-1)^{-1}) + (q-1)^{-1} \mid n \in \mathbb{Z}\}, \varepsilon_1(p,q))$ (here $\mathrm{dist}(M,x)$ is the distance between the subset $M \subseteq \mathbb{R}$ and $x \in \mathbb{R}$).

# 3 On the structure of families of unitary selfadjoint operators

## 3.1 Twisted generalized Weyl construction.

Let $R$ be a unital $*$-algebra, $t = t^*$ a central element and $\sigma$ an automorphism such that $\sigma(r^*) = (\sigma(r))^*$. Define the $*$-algebra $\mathfrak{A}_R^1$ as the $R$-algebra generated by two elements $X$, $X^*$ subjected to the following relations:

- $Xr = \sigma(r)X$ and $rX^* = X^*\sigma(r)$ for any $r \in R$,
- $X^*X = t$ and $XX^* = \sigma(t)$.

We will say that the $*$-algebra $\mathfrak{A}_R^1$ is obtained from $R$, $\sigma$, $t$ by the twisted generalized Weyl construction. Such $*$-algebras were introduced in [12] and their Hilbert space representations were studied up to a unitary equivalence.

In this subsection, we set up a notation and give a brief exposition of results from [12] which will be needed below.

Let $H$ be a complex separable Hilbert space, $L(H)$ denotes the set of all bounded operators on $H$, $\mathcal{M}' = \{c \in L(H) \mid [c,a] = 0, a \in \mathcal{M}\}$ is the commutator of the operator algebra $\mathcal{M}$.

Assume that $R$ is an algebra of type I, i.e., the $W^*$-algebra $\{\pi(r), r \in R\}''$ is of type I for any representation $\pi$ of $R$, and, given a representation $\pi$ of $R$, the automorphism $\sigma$ can be extended to the corresponding von Neumann algebra. Let $\widehat{R}$ be the set of equivalence classes of irreducible representations of $R$. The automorphism $\sigma$ generates the dynamical system on the set $\widehat{R}$. Indeed, if $\pi$ is an irreducible representation of $R$, then so is $\pi(\sigma)$. Denote by $\Omega_\pi$ the orbit of the dynamical system, i.e., $\Omega_\pi = \{\pi(\sigma^k), k \in \mathbb{Z}\}$.

The next assumption will be needed throughout the section. Suppose that it is possible to choose the subset $\tau \subset \widehat{R}$ which meets each orbit just once in such a way that $\tau$ is a Borel subset. In this case, we will say that the dynamical system $\widehat{R} \ni \pi \to \pi(\sigma) \in \widehat{R}$ is *simple.* Then any irreducible representation $w : \mathfrak{A}_R^1 \to L(H)$ is concentrated on an orbit of the dynamical system, i.e., $H = \oplus_{\Omega_0 \subset \Omega_\pi} H_{\pi_k}$, where $H_{\pi_k}$ is invariant with respect to $w(r)$ for any $r \in R$, and $w|_{H_{\pi_k}}$, as a representation of $R$, is unitarily equivalent to $\pi(\sigma^k) \otimes I$ (here $I$ is the identity operator of dimension $n(k) \le \infty$). We will call $\Omega_0$ the support of $w$ and denote by supp$w$. Without loss of generality, we can assume that $\pi \in \Omega_0$. Then $\Omega_0 = \{\pi(\sigma^k), k \in \mathbb{Z} \mid \pi(\sigma^l(t) > 0$ for any $0 \le l \le k$ if $k \ge 0$ or $k < l < 0$ if $k < 0\}$. Denote by $\tilde{K}$ the subgroup of $\mathbb{Z}$ consisting of $k \in \mathbb{Z}$ such that $\pi(\sigma^k)$ is unitarily equivalent to $\pi$ and $\pi(\sigma^l) \in \Omega_0$ for any $0 < l < k$ if $k > 0$ or $0 > l > k$ if $k < 0$.

**Theorem 1** *Any irreducible representation $w$ of the $*$-algebra $\mathfrak{A}_\tau^1$ such that supp$w = \Omega_0$ coincides, up to a unitary equivalence, with one of the following:*

1. *If $\tilde{K} = \emptyset$, then $H = \oplus_{\Omega_0} H_{\pi_k}$*

$$w(r)|_{H_{\pi_k}} = \pi(\sigma^k(r)), \quad X : H_{\pi_k} \to H_{\pi_{k+1}}, \quad X|_{H_{\pi_k}} = \begin{cases} \pi(\sigma^k(t)), & \pi(\sigma_{k+1}) \in \Omega_0, \\ 0, & \pi(\sigma_{k+1}) \notin \Omega_0. \end{cases}$$

2. *If $\tilde{K} \neq \emptyset$ and $n \in \mathbb{N}$ is the smallest number such that $\pi(\sigma^n)$ and $\pi$ are unitarily equivalent, then $H = \oplus_{k=0}^{n-1} H_{\pi_k}$*

$$w(r)|_{H_{\pi_k}} = \pi(\sigma^k(r)), \quad X : H_{\pi_k} \to H_{\pi_{k+1}}, \quad X|_{H_{\pi_k}} = \begin{cases} \pi(\sigma^k(t)), \ k \neq n-1, \\ e^{i\varphi} W \pi(\sigma^{n-1}(t)), \ k = n-1, \end{cases}$$

*where $W^{-1}\pi W = \pi(\sigma^n)$, $\varphi \in [0, 2\pi)$.*

This technique can be applied to the study of many objects important in mathematical physics such as $Q_{ij} - CCR$ ([8]), $su_q(3)$ and others. Next section is devoted to the study of one of them.

### 3.2 Representations of $*$-algebras generated by unitary selfadjoint generators

The purpose of this subsection is to describe representations of the $*$-algebra $\mathfrak{A}$ generated by selfadjoint unitary generators $u$, $v$ and $j_k$ ($k = 1, \ldots, n$), and the relations

$$j_i j_k = (-1)^{g(i,k)} j_k j_i, \tag{7}$$

$$u j_k = (-1)^{h(k)} j_k u, \quad v j_k = (-1)^{w(k)} j_k v, \tag{8}$$

here $g(i,k) = g(k,i) \in \{0,1\}$, $g(i,i) = 0$, $h(k)$, $w(k) \in \{0,1\}$ for any $i, k = 1, \ldots, n$. Any family of elements $(j_k)_{k=1}^n$ satisfying (7) is said to be a graded-commuting family.

Any representations of $\mathfrak{A}$ is determined by representation operators corresponding to the generators. Thus, instead of representations of the $*$-algebra $\mathfrak{A}$, we will study collections of unitary selfadjoint operators $u$, $v$, $j_k$ ($k = 1, \ldots, n$) on a complex separable Hilbert space $H$ satisfying (7), (8).

We start with the study of graded-commuting selfadjoint operators $\mathcal{J} = (j_k)_{k=1}^n$ with the condition $j_k^2 = I$ ($k = 1, \ldots, n$). For these operators, the structure question was solved in [21]. We will present only main results and constructions from [21].

To collection of selfadjoint unitary operators $(j_k)_{k=1}^n$, there corresponds a simple graph $\Gamma = (S, R)$ (without loops and multiple edges). Here $S$ is element subsets of $S$ corresponding to the edges. The vertices $a_k$ and $a_m$ are connected with an edge if $\{j_k, j_m\} \equiv j_k j_m + j_m j_k = 0$ and there is no edge if the operators commute. In what follows, we will regard such a collection as selfadjoint and unitary representations of the graph $\Gamma$.

Denote by $\sigma_x$, $\sigma_y$, $\sigma_z$ and $\sigma_0$ the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the following construction of irreducible representations of the graph $\Gamma$, which are defined inductively.

1. If $\Gamma = \begin{pmatrix} a_1 & a_2 & & a_n \\ \cdot & \cdot & \dots & \cdot \end{pmatrix}$ then collections of unitary commuting selfadjoint operators form representations of the graph. Set $j_k = (-1)^{i_k}$, where $i_k \in \{0,1\}$, $k = 1,\dots,n$, and $m(\Gamma) = 0$. We get $2^n$ unitarily inequivalent representations.

2. Suppose now that at least two of vertices (without loss of generality, we can assume that these are $a_1$, $a_2$) are connected by an edge. In the space $H = \mathbb{C}^2 \otimes H_1$, consider $j_1 = \sigma_z \otimes I$, $\quad j_2 = \sigma_x \otimes I$, where $I$ is the identity operator on a Hilbert space $H_1$.

If $j_k$ commutes with $j_1$ and $j_2$, set $j_k = \sigma_0 \otimes B_k$;

if $j_k$ commutes with $j_1$ and anticommutes with $j_2$, set $j_k = \sigma_z \otimes B_k$;

if $j_k$ anticommutes with $j_1$ and commutes with $j_2$, set $j_k = \sigma_x \otimes B_k$;

if $j_k$ anticommutes with $j_1$ and $j_2$, set $j_k = \sigma_y \otimes B_k$, where $(B_k)_{k=3}^n$ is the representation of the derivative graph $\Gamma_1 = (S_1, R_1)$ which is defined in the following way:

a) the graph $\Gamma_1$ contains the vertices $(b_k)_{k=3}^n$;

b) if, in the graph $\Gamma$, the vertex $a_k$ is contained in the star of the vertex $a_1$ (i.e., it is connected with $a_1$ by an edge) and $a_m$ is contained in the star of $a_2$ but at least one of them is not in both of the stars (i.e., is not connected with both vertices), then the edge $(b_k, b_m) \in R_1$ if $(a_k, a_m) \notin R$ and conversely, $(b_k, b_m) \notin R_1$ if $(a_k, a_m) \in R$;

c) in all other cases $(b_k, b_m) \in R_1$ if $(a_k, a_m) \in R$ and $(b_k, b_m) \notin R_1$ if $(a_k, a_m) \notin R$.

Proceeding in such a manner at the end we find that either there exists $m(\Gamma) \in \mathbb{N}$ such that all vertices of the graph $\Gamma_{m(\Gamma)}$ are isolated and we get $2^{n-2m(\Gamma)}$ representations of the dimension $2^{m(\Gamma)}$, or $m(\Gamma) \equiv n \in 2\mathbb{N}$ and $\Gamma_{m(\Gamma)/2} = \emptyset$. In the last case, we get the unique representations of the dimension $2^n$.

**Theorem 2** *A simple graph $\Gamma$ with $n$ vertices has $2^{r(\Gamma)}$ $(0 \le r(\Gamma) \le n)$ unitarily inequivalent irreducible representations of the same dimension $2^{m(\Gamma)}$ with $r(\Gamma) = n - 2m(\Gamma)$. Any of them is unitarily equivalent to one defined by the construction above.*

It follows from Theorem 2 that any irreducible unitary selfadjoint representation of the graph $\Gamma$ is realized, up to a permutation and a unitary equivalence, on $H = \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$ by the formulae $j_k = i_k \sigma_{k1} \otimes \dots \otimes \sigma_{km(\Gamma)}$ $\quad (k = 1,\dots,n)$, where $\sigma_{km}$ is the Pauli matrix contained as the $m$-th factor in $j_k$, $i_1 = \dots = i_{2m(\Gamma)} = 1$, $i_k \in \{0,1\}$ for $k > 2m(\Gamma)$.

To distinguish families of unitary selfadjoint operators $(j_k)_{k=1}^n$ satisfying (7), we will say that $(j_k)_{k=1}^n$ is a family of unitary graded-commuting selfadjoint operators corresponding to $g : M \times M \to \{0,1\}$, where $M = \{1,2,\dots,n\}$.

2. Next we describe collections $(u,\ v,\ j_1,\dots,j_n)$ of selfadjoint unitary operators which satisfy the relations:

$$j_i j_k = (-1)^{g(i,k)} j_k j_i, \tag{9}$$

$$u j_k = (-1)^{h(k)} j_k u, \quad v j_k = (-1)^{h(k)} j_k v, \tag{10}$$

where $g(i,k) = g(k,i) \in \{0,1\}$, $g(i,i) = 0$, $h(k) \in \{0,1\}$ for any $i,k = 1,\dots,n$.

They determine representations of the $*$-algebra $\mathcal{A}$ generated by $u = u^*$, $v = v^*$, $j_k^* = j_k$ $(k = 1,\dots,n)$ satisfying (9), (10) and the condition $u^2 = v^2 = j_k^2 = 1$. Denote by $\mathcal{A}_0$ the subalgebra of $\mathcal{A}$ generated by $j_k$, $k = 1,\dots,n$. The $*$-algebra $\mathcal{A}$ can be treated as a $*$-algebra obtained by the twisted generalized Weyl construction. Indeed, let $X = u + iv$,

$X^* = u - iv$. It is easy to check that relations (10) and $u^2 = v^2 = 1$ are equivalent to the following ones:

$$XX^* + X^*X = 4, \quad X^2 + (X^*)^2 = 0, \\ Xj_k = (-1)^{h(k)} j_k X, \quad X^* j_k = (-1)^{h(k)} j_k X^* \tag{11}$$

Consider the unital $*$-algebra $\mathcal{A}_0 \oplus \mathbb{C}X^*X$ as the ground $*$-algebra $R$ with the central element $t = X^*X$ and the automorphism $\sigma$ which is defined in the following way: $\sigma(X^*X) = 4 - X^*X$, $\sigma(j_k) = (-1)^{h(k)} j_k$, $k = 1, \dots, n$. Then $\mathcal{A}$ is $*$-isomorphic to $\mathfrak{A}^1_R$.

**Theorem 3** *Any irreducible representation of $\mathcal{A}$ is unitarily equivalent to one of the following:*

1. $H = H_0 \otimes \mathbb{C}^2$

$$u = I \otimes \sigma_x \ v = I \otimes \begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix}, \ j_k = \begin{cases} -j'_k \otimes \sigma_y, & h(k) = 1, \\ j'_k \otimes \sigma_0, & h(k) = 0, \end{cases} \tag{12}$$

*where $\varphi \in (-\pi, \pi)$, $(j'_k)_{k=1}^n$ is an irreducible family of unitary graded-commuting selfadjoint operators on $H_0$ corresponding to $g$.*

2. $H = H_0$

$$j_k = j'_k, \ k = 1, \dots, n, \quad u = j'_{n+1}, \quad v = j'_{n+2}, \tag{13}$$

*where $(j'_k)_{k=1}^{n+2}$ is an irreducible family of unitary graded-commuting selfadjoint operators corresponding to $g'$ defined as follows:*

$$g'(k, i) = \begin{cases} g(k, i), & k, i \le n, \\ h(k), & k \le n < i, \\ 0, & \text{otherwise}. \end{cases}$$

*Proof.* To the $*$-algebra $\mathcal{A}$, there corresponds the dynamical system $\widehat{R} \ni \pi \to \pi(\sigma) \in \widehat{R}$ with $\sigma^2 = 1$. Any representation $\pi \in \widehat{R}$ is defined by the collection $(j'_1, \dots, j'_n, X^*X = \lambda I)$, where $(j'_k)_{k=1}^n$ is an irreducible family of unitary graded-commuting selfadjoint operators corresponding to $g$, and $\lambda \in [0, 4]$. Any irreducible representation arises from an orbit of the dynamical system.

If $\lambda \ne 0, 2, 4$, then the representations $\pi$, $\pi(\sigma)$ are not unitarily equivalent and and $\pi(t)$, $\pi(\sigma(t)) > 0$, hence, by Theorem 1, the corresponding irreducible representation of the $*$-algebra $\mathcal{A}$ is of the form

$$j_k = j'_k \otimes \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{h(k)} \end{pmatrix} \quad X = I \otimes \begin{pmatrix} 0 & e^{i\psi}\sqrt{4-\lambda} \\ \sqrt{\lambda} & 0 \end{pmatrix},$$

$\psi \in [0, 2\pi)$, $\lambda \in (0, 2)$. Moreover, since $X^2 + (X^*)^2 = 0$, $e^{i\psi} = \pm i$. Thus,

$$u = I \otimes \begin{pmatrix} 0 & e^{i\delta} \\ e^{-i\delta} & 0 \end{pmatrix}, \ v = I \otimes \begin{pmatrix} 0 & ie^{-i\delta} \\ -ie^{i\delta} & 0 \end{pmatrix}, \tag{14}$$

where $\lambda = 4\cos^2\delta$, $\delta \in (-\pi/4, \pi/4)$.

If $\lambda = 4$, then $\pi(\sigma(t)) = 0$, $\pi(t) = 4$ and the corresponding irreducible representation is of the form

$$j_k = j'_k \otimes \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{h(k)} \end{pmatrix}, \quad X = I \otimes \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

and $u = I \otimes \sigma_x$, $v = I \otimes (-\sigma_y)$.

If $\lambda = 0$, then $\pi(t) = 0$, $\pi(\sigma(t)) = 4$ and the corresponding irreducible representation is unitarily equivalent to one given in the previous case.

It is easy to check that these representations are unitarily equivalent to that given by (12) with the unitary operator

$$W = \frac{1}{\sqrt{2}} I \otimes \begin{pmatrix} 1 & i \\ e^{-i\delta} & -ie^{-i\delta} \end{pmatrix} \quad \text{and} \quad \varphi = 2\delta.$$

If $\lambda = 2$, then $\pi(t) = \pi(\sigma(t))$. Hence, for the corresponding irreducible representation of $\mathcal{A}$, we have $XX^* = X^*X = 2$, which is equivalent to the following $[u, v] = 0$, $u^2 = v^2 = I$. The corresponding irreducible family $(u, v, j_1, \ldots, j_n)$ is defined by (13). Moreover, any collection of the form (13) determines a representation of the $*$-algebra $\mathcal{A}$. ∎

3. Now consider representations of the $*$-algebra generated by $u$, $v$, $j_k$, $k = 1, \ldots, n$, and relations (7), (8). Suppose that there exists $k \le n$ such that $h(k) \neq w(k)$. Without loss of generality, we can assume that $h(k) = w(k)$ for some $k \le s < n$, $h(k) \neq w(k)$ if $s < k \le n$ and $h(n) = 0$, $w(n) = 1$. Consider the selfadjoint unitary generators

$$\tilde{j}_k = \begin{cases} j_k, & k \le s \text{ or } k = n, \\ i^{g(n,k)} j_k j_n, & s < k < n. \end{cases}$$

One can check that relations (7), (8) are equivalent to the following

$$\tilde{j}_k \tilde{j}_l = (-1)^{\tilde{g}(k,l)} \tilde{j}_l \tilde{j}_k \tag{15}$$

$$\tilde{j}_k u = (-1)^{\tilde{h}(k)} u \tilde{j}_k, \ \tilde{j}_k v = (-1)^{\tilde{h}(k)} v \tilde{j}_k, \ k \neq n, \tag{16}$$

$$\tilde{j}_n u = u \tilde{j}_n, \ \tilde{j}_n v = -v \tilde{j}_n, \tag{17}$$

where $\tilde{g}(k,l) = \begin{cases} g(k,l), & k < l \le s \text{ or } l = n, \\ (g(k,l) + g(k,n))(\mathrm{mod} 2), & k \le s < l < n, \\ (g(l,n) + g(k,l) + g(k,n))(\mathrm{mod} 2), & s < k < l < n, \end{cases}$

$$\tilde{g}(k,l) = \tilde{g}(l,k), \ \tilde{g}(l,l) = 0, \ \tilde{h}(k) = \begin{cases} h(k), & k \le s, \\ (h(n) + h(k))(\mathrm{mod} 2), & s < k < n. \end{cases}$$

The $*$-algebra $\mathcal{A}'$ generated by selfadjoint unitary elements $(u,\ v,\ (\tilde{j}_k)_{k=1}^n)$ satisfying (15)–(17) is $*$-isomorphic to $\mathcal{A}$ and can be considered as a $*$-algebra obtained by the twisted generalized Weyl construction from the ground ring $R$ generated by $(u,\ v,\ \tilde{j}_1, \ldots, \tilde{j}_{n-1})$ satisfying (15)–(16), the central element $t = 1$ and the automorphism $\sigma$ defined in the following way:

$$\sigma(u) = u, \ \sigma(v) = -v \ \sigma(\tilde{j}_k) = (-1)^{\tilde{g}(k,n)} \tilde{j}_k.$$

We can use the technique above to describe representations of the algebra.

**Theorem 4** *Any irreducible representation of the $*$-algebra coincides, up to a unitary equivalence, with the following ones:* 1. $H = H_0 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$u = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I, \quad v = I \otimes \begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\tilde{j}_k = \begin{cases} -\tilde{j}'_k \otimes \sigma_y \otimes \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{\tilde{g}(n.k)} \end{pmatrix}, & \tilde{h}(k) = 1, k \neq n, \\ \tilde{j}'_k \otimes \sigma_0 \otimes \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{\tilde{g}(n.k)} \end{pmatrix}, & \tilde{h}(k) = 0, k \neq n, \end{cases} \qquad j_n = I \otimes I \otimes \sigma_x, \tag{18}$$

*where $\varphi \in (-\pi, \pi)$, $(\tilde{j}'_k)_{k=1}^n$ is an irreducible family of unitary graded-commuting selfadjoint operators on $H_0$ corresponding to $\tilde{g}$.*

2. $H = H_0$

$$\tilde{j}_k = \tilde{j}'_k, \ k = 1, \ldots, n, \ u = \tilde{j}'_{n+1}, \quad v = \tilde{j}'_{n+2}, \tag{19}$$

*where $(\tilde{j}'_k)_{k=1}^{n+2}$ is an irreducible family of unitary graded-commuting selfadjoint operators corresponding to $\tilde{g}'$ defined as follows:*

$$\tilde{g}'(k,i) = \begin{cases} \tilde{g}(k,i), & k < i \le n, \\ \tilde{h}(k), & k < n < i, \\ 0, & (k,i) = (n,n+1), (n+1,n+2), \\ 1, & (k,i) = (n,n+2). \end{cases}$$

*Proof.* By Theorem 3, given an irreducible representations $\pi$ of the $*$-algebra $R$, we have either $\pi([u,v]) = 0$ or, up to a unitary equivalence, $\pi(u) = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\pi(v) = I \otimes \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$. It is easy to show that the pairs $(\pi(u), \pi(v))$ and $(\pi(\sigma(u)), \pi(\sigma(v)))$ are not unitarily equivalent. Hence, by Theorem 1, if $\pi([u,v]) \neq 0$, then the irreducible representation corresponding to the orbit $\widehat{R} \ni \pi \to \pi(\sigma) \in \widehat{R}$ is unitarily equivalent to the representation defined by (18). If $\pi([u,v]) = 0$, the corresponding irreducible representation is defined by the family of graded-commuting unitary selfadjoint operators described by (19). ∎

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