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Higher Symmetries of the Wave Equation with Scalar and Vector Potentials

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Abstract

Higher order symmetry operators for the wave equation with scalar and vector potentials are investigated. All scalar potentials which admit second order symmetry operators are found explicitly.

Symmetries of partial differential equations are used for separation of variables [1], description of conservation laws [2], construction of exact solutions [3], etc. Before can be applied, symmetries have to be found, therefore search for symmetries attracts attention of many investigators.

We say a linear differential operator of order n is a symmetry operator (or simply a symmetry) of a partial differential equation if it transforms any solution of this equation into a solution. If $n = 1$, then the symmetry is nothing but a generator of a Lie group being a symmetry group of the equation considered. If $n > 1$, the related symmetries are referred as non-Lie or higher symmetries.

The symmetry aspects of the Schrödinger equation have been investigated by many authors (see [1] and references cited in), the higher symmetries of this equation where investigated in [4–9]. In contrast, the higher symmetries of the relativistic wave equation have not been studied well yet.

In this paper, we investigate the higher symmetries of the wave equation with an arbitrary scalar potential

$$L\psi \equiv (\partial_\mu \partial^\mu - V)\psi = 0 \tag{1}$$

where $\mu = 0, 1, \ldots, m$. We deduce equations describing both potentials and coefficients of the corresponding symmetry operators of order n and present the complete list of potentials admitting symmetries for the case $m = 1$, $n = 2$, which completes the results of papers [10, 11]. In addition, we find all the possible scalar potentials V and vector potentials A_μ such that the equation

$$\widehat{L}\psi \equiv (D_\mu D^\mu - V)\psi = 0, \qquad D_\mu = \partial_\mu - eA_\mu, \quad \mu = 0, 1 \tag{2}$$

admits any Lie symmetry. Moreover, we consider the case of time-dependent potential V and present a constructive test in order to answer the question if the corresponding wave equation admits any Lie symmetry.

Let us represent a differential operator of arbitrary order n in the form

$$Q_n = \sum_{j=0}^{n} \left[[\cdots [K^{a_1 a_1 \ldots a_j}, \partial_{a_1}]_+, \partial_{a_2}]_+, \cdots \partial_{a_j} \right]_+, \tag{3}$$

where $K^{a_1 a_1 \ldots a_j}$ are arbitrary functions of $x = (x_0, x_1, \ldots, x_m)$, $[A, B]_+ = AB + BA$. We say Q_n is a symmetry operator of equation (1) if it satisfies the invariance condition [13]

$$[Q_n, L] = \alpha_{n-1} L, \tag{4}$$

where α_{n-1} is a differential operator of order $n-1$ which we represent in the form

$$\alpha_n = \sum_{j=0}^{n-1} \left[[\cdots [\alpha^{a_1 a_2 \ldots a_j}, \partial_{a_1}]_+, \partial_{a_2}]_+, \cdots \partial_{a_j} \right]_+. \tag{5}$$

To find the determining equations for coefficients of a symmetry operator of arbitrary order, it is sufficient to equate coefficients of the same differentials in (4). We start with the case $m = 1$ in order to verify the old results [10]. For the first order symmetries

$$Q = [K^a, \partial^a]_+ + K \tag{6}$$

we obtain the following determining equations

$$\partial^{(a} K^{b)} = \frac{1}{2} g^{ab} \alpha, \tag{7}$$

$$\partial^a K = -\frac{1}{2} \partial^a \alpha, \tag{8}$$

$$2K^a \partial^a V = -\alpha V - \frac{1}{4} (\partial^b \partial_b \alpha). \tag{9}$$

Formula (7) defines the equation for a conformal Killing vector whose general form is

$$K^0 = \varphi(x - t) + f(x + t) + c, \qquad K^1 = \varphi(x - t) - f(x + t), \tag{10}$$

where φ and f are arbitrary functions, c is an arbitrary constant. Moreover, in accordance with (7), (8), (10), $\alpha = 4(\varphi' - f')$, $K = \alpha/2 + C$, and the remaining condition (9) takes the form

$$\partial^0 (K^0 V) + \partial^1 (K^1 V) = 0. \tag{11}$$

Using the fact that K^1 satisfies the wave equation and changing $V = 1/U^2$, we come to the following differential consequence of (11):

$$U'' = \omega^2 U + C, \qquad \omega = \text{const}, \qquad C = 0, \tag{12}$$

which is clearly simply the compatibility condition for system (6)–(9).

Nonequivalent solutions of (12) have the form

$$V = C, \qquad V = \frac{C}{x^2}, \qquad V = C\exp(-2\omega x),$$
$$V = \frac{C}{\cos^2(\omega x)}, \qquad V = \frac{C}{\cosh^2(\omega x)}, \qquad V = \frac{C}{\sinh^2(\omega x)}, \tag{13}$$

where C and ω are arbitrary real constants. These potentials are defined up to equivalence transformations, $x \to ax + b$, where a, b are constants.

In comparison with the list of symmetry adopting potentials present in [10], formula (13) includes one additional potential (the last one) which was not found in [10].

We see the list of potentials admitting Lie symmetries is very restricted and is exhausted by the list of potentials enumerated in (13). The corresponding symmetry operators are easily calculated using relations (6)–(11).

We notice that the determining equations (6)-(9) are valid in the case of time-dependent potentials also. Moreover, we again come to equation (11) where K^0 and K^1 are functions defined in (10). These functions can be excluded by a consequent differentiation of (11). As a result, we obtain the following relation for V:

$$\Box\left\{\ln[\ln(\Box \ln V)_\eta - (\ln V)_\eta]\right\} = \Box\left\{\ln[\ln(\Box \ln V)_\zeta - (\ln V)_\zeta]\right\},$$
$$\Box = \partial_t^2 - \partial_x^2, \qquad (\cdot)_\eta = \partial_\eta(\cdot), \qquad \eta = x + t, \qquad \zeta = x - t, \tag{14}$$

which is a necessary and sufficient condition for the corresponding equation (1) to admit a Lie symmetry. It is the case, e.g., if the potential V satisfies the wave equation or has the form $V = f(\eta)\varphi(\zeta)$.

For the classification of the potentials admitting invariance algebras of dimension $k > 1$, refer to [12].

The second order symmetries are searched in the form (see (3))

$$Q = [[K^{ab}, \partial^a,]_+, \partial^b]_+ + [K^a, \partial^a]_+ + K, \qquad \widehat{\alpha} = [\alpha^a, \partial^a]_+ + \alpha \tag{15}$$

which leads to the following determining equations:

$$\partial^{(a}K^{bc)} = -\frac{1}{4}\alpha^{(a}g^{bc)}, \qquad \partial^{(a}K^{b)} = \frac{1}{2}(\partial^{(a}\alpha^{b)} - g^{ab}\alpha),$$
$$\partial^a K = \frac{1}{2}\partial^a\alpha + \alpha^a V - 4K^{ab}\partial^b V, \qquad K^a\partial^a V = \frac{1}{2}(\alpha^a\partial^a V + \alpha V). \tag{16}$$

These equations can be solved by analogy with (6)–(12). Omitting straightforward but cumbersome calculations, we notice that the compatibility condition for system (15) again has the form (12) where C is an arbitrary constant. Moreover, $V = U/2(U')$. The corresponding general solution for the potential is given by formulas (13) and (17):

$$V = Cx, \qquad V = C_1 + \frac{C_2}{x}, \qquad V = C\exp(\omega x)x + C\exp(2\omega x),$$
$$V = C_1\frac{\cos(\omega x) + C_2}{\sin^2(\omega x)}, \qquad V = C_1\frac{\sinh(\omega x) + C_2}{\cosh^2(\omega x)}, \qquad V = C_1\frac{\cosh(\omega x) + C_2}{\sinh^2(\omega x)}. \tag{17}$$

The list of solutions (17) includes all the potentials found in [11] and one additional potential given by the last term. We notice that potentials (13), (17) make it possible to separate variables in equation (1) [14]. In other words, the potentials admitting a separation of variables are exactly the same as the potentials admitting second order symmetries.

Let us consider equation (2). Without loss of generality, we set $A_1 = 0$, bearing in mind ambiguities arising due to gauge transformations. Then the corresponding determining equations for the first-order symmetries reduce to the form

$$\begin{aligned}
&\partial^{(a}K^{b)} = -\frac{1}{2}g^{ab}\alpha, \quad \partial^0 K - 2iA_1\dot{K}^0 + 2iA_1'K^1 = \frac{1}{2}\dot{\alpha} + i\alpha A_1,\\
&K' + 2i\dot{K}^1\ A_1 = \frac{1}{2}\alpha', \quad 2iA_0\dot{K} - 2VF^1 = -\alpha(V + A_1^2) + i\dot{\alpha}(V + A_1^2) + \frac{1}{4}\Box\alpha.
\end{aligned} \tag{18}$$

These equations are valid for time-dependent as well as time-independent potentials. Restricting ourselves to the last case, we find the compatibility condition for this system can be represented in the form (12), moreover,

$$k_1 A_1' = k_2 V = \frac{1}{U^2} \tag{19}$$

where k_1, k_2 are arbitrary constants. Using (13), we find the admissible vector potentials in the form

$$A_1 = \widetilde{C}x, \tag{20a}$$

$$A_1 = \frac{\widetilde{C}}{x}, \tag{20b}$$

$$A_1 = \widetilde{C}\exp(-2\omega x), \tag{20c}$$

$$A_1 = \widetilde{C}\tan(\omega x), \tag{20d}$$

$$A_1 = \widetilde{C}\tanh(\omega x), \tag{20e}$$

$$A_1 = \widetilde{C}\coth(\omega x). \tag{20f}$$

The list (20) exhausts all time independent vector potentials which admit Lie symmetries. Let us present explicitly the corresponding symmetry operators:

$$Q_1 = \partial_t, \qquad Q_2 = \partial_x - i\widetilde{C}t, \qquad Q_3 = t\partial_x - x\partial_t - i\widetilde{C}(x^2 + t^2); \tag{21a}$$

$$Q = \partial_t, \qquad Q_2 = t\partial_t + x\partial_x, \qquad Q_3 = (t^2 + x^2)\partial_t + 2tx\partial_x - 2i\widetilde{C}x; \tag{21b}$$

$$\begin{aligned}
&Q_1 = \partial_t, \qquad Q_2 = \exp[\omega(x+t)](\partial_t + \partial_x) + i\widetilde{C}\exp[\omega(x-t)],\\
&Q_3 = \exp[\omega(x-t)](\partial_t - \partial_x)] + i\widetilde{C}\exp[-\omega(x+t)];
\end{aligned} \tag{21c}$$

$$\begin{aligned}
&Q_1 = \partial_t,\\
&Q_2 = \sin(\omega t)\sin(\omega t)\partial_t - \cos(\omega t)\cos(\omega x)\partial_x + \widetilde{C}\sin(\omega t)\cos(\omega x);\\
&Q_3 = \cos(\omega t)\sin(\omega x)\partial_t + \sin(\omega t)\cos(\omega x)\partial_x - \widetilde{C}\cos(\omega t)\cos(\omega x);
\end{aligned} \tag{21d}$$

$$\begin{aligned}
&Q_1 = \partial_t,\\
&Q_2 = \sinh(\omega t)\sinh(\omega x)\partial_t + \cosh(\omega t)\cosh(\omega x)\partial_x - i\widetilde{C}\sinh(\omega t)\cosh(\omega x),\\
&Q_3 = \cosh(\omega t)\sinh(\omega x)\partial_t + \cosh(\omega t)\sinh(\omega t)\partial_x - i\widetilde{C}\cosh(\omega t)\cosh(\omega x);
\end{aligned} \tag{21e}$$

$$
\begin{aligned}
&Q_1 = \partial_t,\\
&Q_2 = \cosh(\omega t)\cosh(\omega x)\partial_t + \sinh(\omega t)\sinh(\omega x)\partial_x - i\widetilde{C}\cosh(\omega t)\sinh(\omega x),\\
&Q_3 = \sinh(\omega t)\cosh(\omega x)\partial_t + \cosh(\omega t)\sinh(\omega x)\partial_x - i\widetilde{C}\sinh(\omega t)\sinh(\omega x).
\end{aligned}
\tag{21f}
$$

Operators (21a) and (21b) form the Lie algebra isomorphic to $AO(1,2)$ while the algebras of operators (21c)–(21f) are isomorphic to $AE(2)$. The corresponding potentials V have to satisfy relation (20). We notice that formulae (21) define symmetry operators for equation (1) with potentials (14) also if we set $\ddot{C} = 0$. These results are in accordance with the general classification scheme present in [12].

In conclusion, we return to equation (1) and present the determining equations for the symmetry operators (3) of arbitrary order n in a space of any dimension $m+1$:

$$
\begin{aligned}
&\partial^{(a_{n+1}} K^{a_1 a_2 \ldots a_n)} = \frac{1}{4} g^{(a_n a_{n+1}} \alpha^{a_1 a_2 \ldots a_{n-1})},\\
&\partial^{(a_n} K^{a_1 a_2 \ldots a_{n-1})} = \frac{1}{4} g^{(a_n a_{n-1}} \alpha^{a_1 a_2 \ldots a_{n-2})} - \frac{1}{2}\partial^{(a_n}\alpha^{a_1 a_2 \ldots a_{n-1})},\\
&\partial^{(a_{m-n+1}} K^{a_1 a_2 \ldots a_{n-m})} = -\frac{1}{4}\partial^b\partial_b \alpha^{a_1 a_2 \ldots n-m+1} +\\
&\quad + \sum_{k=0}^{\left[\frac{n-m}{2}\right]} (-1)^k \frac{2(n-m+2+2k)!}{(2k+1)!(n-m+1)!} U^{a_1 a_2 \ldots a_{n-m+1}} + W^{a_1 a_2 \ldots a_{n-m+1}},\\
&\sum_{p=0}^{\left[\frac{n-1}{2}\right]} (-1)^{p+1} K^{b_1 b_2 \ldots b_{2p+1}} \partial_1^b \partial_2^b \ldots \partial_{2p+1}^b V + \sum_{k=0}^{n-1} \alpha^{b_1 b_2 \ldots b_k} \partial_1^b \partial_2^b \ldots \partial_k^b V = 0.
\end{aligned}
\tag{22}
$$

Here,

$$
\begin{aligned}
&U^{a_1 a_2 \ldots a_{n-m+1}} = K^{a_1 a_2 \ldots a_{n-m+1} b_1 b_2 \ldots b_{2k+1}} \partial^{b_1}\partial^{b_2} \ldots \partial^{b_{2k+1}} V,\\
&W^{a_1 a_2 \ldots a_{n-1}} = \alpha^{a_1 a_2 \ldots a_{n-1}} V,\\
&W^{a_1 a_2 \ldots a_{n-2}} = -(n-1)\alpha^{a_1 a_2 \ldots a_{n-2} b}\partial^b V - \alpha^{a_1 a_2 \ldots a_{n-2}} V,\\
&W^{a_1 a_2 \ldots a_{n-2q-1}} = -\sum_{k=0}^{q-1} (-1)^k \frac{(n-2k-2q)!}{(2k+1)!(n-2q-1)!} \times\\
&\quad \times \alpha^{a_1 a_2 \ldots a_{n-2q-1} b_1 b_2 \ldots b_{2k+1}} \partial^{b_1}\partial^{b_2} \ldots \partial^{b_{2k+1}} V - \alpha^{a_1 a_2 \ldots a_{n-2p-1}} V -\\
&\quad -\frac{1}{2}\sum_{k=0}^{q-1} (-1)^{k+q} \frac{(n-2k-1)!}{(n-2p-1)!(2p-2k-1)!(p-k)} \times\\
&\quad \times \alpha^{a_1 a_a \ldots a_{n-2k-1} b_1 b_2 \ldots b_{2p-2k}} \partial^{b_1}\partial^{b_2} \ldots \partial^{b_{2p-2k}} V,\\
&q = 1, 2, \ldots \left[\frac{n-2}{2}\right],\\
&W^{a_1 a_2 \ldots a_{n-2q}} = -\sum_{k=0}^{q-1} (-1)^k \frac{(n-2q+2k+1)!}{(2k+1)!(n-2q)!} \times
\end{aligned}
$$

$$\times\alpha^{a_1a_2\ldots a_{n-2q+2k+1}b_1b_2\ldots b_{2k+1}}\partial^{b_1}\partial^{b_2}\ldots\partial^{b_{2k+1}}V-\alpha^{a_1a_2\ldots a_{n-2q}}V-$$

$$-\frac{1}{2}\sum_{k=0}^{q-2}(-1)^{k+q}\frac{(n-2k-2)!}{(n-2q)!(2q-2k-3)!(q-k-1)}\times$$

$$\times\alpha^{a_1a_2\ldots a_{n-2k}b_1b_2\ldots b_{2q-2k-2}}\partial^{b_1}\partial^{b_2}\ldots\partial^{b_{2q-2k-2}}V,$$

$$q=2,3,\ldots,\left[\frac{n-1}{2}\right].$$

Equations (22) define the potentials admitting non-trivial symmetries of order n, and the coefficients $K^{\cdots}$ of the corresponding symmetry operators, as well.

For $V=0$, equations (22) reduce to the following form

$$\partial^{(a_{j+1}}K^{a_1a_2\ldots a_j)}=\frac{2}{m+2j-1}\partial^bK^{b(a_1a_2\ldots a_{j-1}}g^{a_ja_{j+1})}=0, \tag{23}$$

$$\alpha^{a_1a_2\ldots a_{j-1}}=\frac{2}{m+2j-1}\partial^bK^{ba_1a_2\ldots a_{j-1}}, \tag{24}$$

where $K^{\cdots}$ and $\alpha^{\cdots}$ are symmetric and traceless tensors.

Thus, in the above case, the system of determining equations is decomposed into the uncoupled subsystems (24), which are the equations for a conformal Killing tensor that can be integrated for any m [8]. The corresponding symmetry operators reduce to polynomials in generators of the conformal group, moreover, the number of linearly independent symmetries for $m=2$ and $m=3$ [8] is given by the formulas:

$$N_2=\frac{1}{3}(n+1)(2n+1)(2n+3),\qquad N_3=\frac{1}{12}(n+1)^2(n+2)^2(2n+3). \tag{25}$$

Formulae (25) take into account n-order symmetries but no symmetries of order $j<n$.

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