Symmetry in Nonlinear Mathematical Physics 1997, V.2, 287–296.

Spinor Fields over Stochastic Loop Spaces

R. LÉANDRE

Département de Mathématiques, Institut Elie Cartan, Faculté des Sciences,
Université de Nancy I, 54000. Vandoeuvre-les-Nancy, France

Abstract

We give the construction of a line bundle over the based Brownian bridge, as well as the construction of spinor fields over the based and the free Brownian bridge.

Introduction

The Dirac operator over the free loop space is a very important object for the algebraic topology [24]; its index gives the Witten genus and it can predict the rigidity theorem of Witten: the index of some classical operator is rigid under a geometrical action of the circle over the manifold. Unfortunately, the Dirac operator over the free loop space is an hypothetical object.

In [13], we have constructed an approximation of it by considering the Brownian measure over the loop space: why is a measure important? It is to compute the adjoint of the Dirac operator, the associated Laplacian, and the Hilbert space of spinors where it acts; the choice of physicists gives a hypothetical measure over the loop space. The purpose of [13] is to replace the formal measure of physicists by a well-defined measure, that is the Brownian bridge measure. The fiber of the Dirac operator is related to the Fourier expansion. After [25, 13] the Fourier expansion has extended in an invariant by rotation way for the natural circle action over the free loop space. Unfortunately, this works only for small loops.

The problem to construct the spin bundle over the free loop space is now a well-understood problem in mathematics (see [14, 24, 26, 7, 6, 20]). In order to construct a suitable stochastic Dirac operator over the free loop space, it should be reasonable to define a Hilbert space of spinor fields over the free loop space where the operator acts. It should be nice to extend the previous work mentioned in the references above in the stochastic context. It is the subject of [17, 18] and [19]. The goal of this paper is to do a review of the results of [17, 18, 19].

In the first part, we study the problem to construct stochastic line bundles over the stochastic loop space: their transition functions are only almost surely defined. Therefore, we define the line bundle by its sections. If we consider the path space as a family of Brownian bridges, we meet the problem to glue together all the line bundles over the Brownian bridge into a line bundle over the Brownian motion. There is an obstruction which is measured in [5] for smooth loops. When the criterium of this obstruction is satisfied, the tools of the quasi-sure analysis allow one to restrict a smooth section of the line bundle over the Brownian motion into a section of the line bundle over the Brownian bridge. Moreover, if we consider the bundle associated to a given curvature (we neglect all torsion phenomena by considering the case where the loop space is simply connected) whose fiber is a circle, we cannot define it by its sections because they do not exist: we define it by its functionals. This way to define topological spaces is very useful in algebra.

In the second part, we study the case of the based Brownian bridge in order to construct spinor fields over it. This part is based upon [6] when there is no measure. We consider the case of a principal bundle $Q \to M$ over the compact manifold M. The based loop space $L_e(Q)$ is a principal bundle over the based loop space $L_x(M)$ of M with the based loop group of G as a structure group. The problem to construct spinor fields (or a string structure) is to construct a lift of $L_e(Q)$, $\tilde{L}_e(Q)$, by the basical central extension of $L_e(G)$ if G is supposed simple simply laced (see [23]). This allows us, when the first Pontryagin class of Q is equal to 0, to construct a set of transition functions with values in $\tilde{L}_{e,f}(G)$, the basical central extension of the group of finite energy based loops in G. These transition functions are almost surely defined: we can impose some rigidity by saying they belong to some Sobolev spaces. If there exists a unitary representation of $\tilde{L}_{e,f}(G)$ called $Spin_\infty$, this allows us to define the Hilbert space of sections of the associated bundle. The second part treats too the problem to construct the $\tilde{L}_{e,f}(G)$ principal bundle (and not only the associated bundle which is defined by its sections). For that, we construct a measure over $L_e(Q)$ and by using a suitable connection, we define Sobolev spaces over $L_e(Q)$. We construct a circle bundle $\tilde{L}_e(Q)$ over $L_e(Q)$ by using its functionals, called string functionals: the space of L^p string functionals is therefore defined. There is an Albeverio-Hoegh-Krohn density over $\tilde{L}_{e,f}$ associated to right and left translations by a deterministic element of $\tilde{L}_{e,2}(G)$ (the central extension of the group of loops with two derivatives), which belongs only in L^1. This shows that the stochastic gauge transform of the formal bundle $\tilde{L}_e(Q) \to L_x(M)$ operate only in $L^\infty(\tilde{L}_e(Q))$.

In the third part, we treat the free loop space case by considering the formalism of [7] related to the Chern-Simons theory. In such a case, it is possible to consider the spin representation of the free loop group when G is the finite-dimensional spinor group (see [23, 4] for instance). We construct an Hilbert space of spinor fields invariant by rotation.

1. Bundles over the Brownian bridge

Let M be a compact Riemannian manifold. Let $L_x(M)$ be the space of continuous loops over M starting from x and arriving in time 1 over x. Let $dP_{1,x}$ be the Brownian bridge measure over $L_x(M)$. Let γ_t be a loop. We consider τ_t, the parallel transport from γ_0 to γ_t. The tangent space [3, 12] of a loop γ_t is the space of sections X_t over γ_t of $T(M)$ such that

$$X_t = \tau_t H_t; \qquad H_0 = H_1 = 0 \tag{1.1}$$

with the Hilbert norm $\|X\|^2 = \int_0^1 \|H'_s\|^2 ds$. We suppose that $L_x(M)$ is simply connected in order to avoid all torsion phenomena.

Let ω be a form which is a representative of $H^3(M; Z)$. We consider the transgression

$$\tau(\omega) = \int_0^1 \omega(d\gamma_s, ., .). \tag{1.2}$$

It is the special case of a stochastic Chen form, which is closed. If we consider a smooth loop, it is Z-valued.

Let γ_i be a dense countable set of finite energy loops. Let γ_{ref} be a loop of reference. If $\gamma \in B(\gamma_i, \delta)$, the open ball of radius δ and of center γ_i for the uniform norm, we can produce a distinguished path going from γ to γ_{ref}. Between γ and γ_i, it is $s \to \exp_{\gamma_{i,s}}[t(\gamma_s - \gamma_{i,s})] = l_{i,t}(s)$ and between γ_i and γ_{ref}, it is any deterministic path. Over $B(\gamma_i, \delta)$, we say that the line bundle is trivial, and we assimilate an element α over γ to α_i over γ_{ref} by the parallel transport for the connection whose the curvature is $\tau(\omega)$: there is a choice. The consistency relation between α_i and α_j is given by the parallel transport along the path joining γ_{ref} to γ_{ref} by going from γ_{ref} to γ by l_i runned in the opposite sense and going from γ to γ_{ref} by l_j. First of all, we fulfill the small stochastic triangle γ, γ_i and γ_j by a small stochastic surface and we use for that the exponential charts as before. After we use the fact that $L_x(M)$ is supposed simply connected. We fullfill the big deterministic triangle γ_{ref}, γ_i, γ_j by a big deterministic surface. We get, if we glue the two previous surfaces, a stochastic surface $S_{i,j}(\gamma)$ constituted of the loop $l_{i,j,u,v}$. The desired holonomy should be equal to

$$\rho_{i,j}(\gamma) = \exp\left[-2i\pi \int_{S_{i,j}(\gamma)} \tau(\omega)\right].$$

Let us remark that $\int_{S_{i,j}(\gamma)} \tau(\omega)$ is well defined by using the theory of non anticipative Stratonovitch integrals. Namely, $\partial/\partial u\, l_{i,j,u,v}$ as well as the derivative with respect to v are semi-martingales. Moreover, if we choose the polygonal approximation γ^n of γ, $\rho_{i,j}(\gamma^n) \to \rho_{i,j}(\gamma)$ almost surely, we get:

Theorem 1.1. *Almost surely over $B(\gamma_i, \delta) \cap B(\gamma_j, \delta)$,*

$$\rho_{i,j}(\gamma)\rho_{j,i}(\gamma) = 1 \tag{1.3}$$

and, over $B(\gamma_i, \delta) \cap B(\gamma_j, \delta) \cap B(\gamma_k, \delta)$

$$\rho_{i,j}(\gamma)\rho_{j,k}(\gamma)\rho_{k,i}(\gamma) = 1 \tag{1.4}$$

Proof. We use the fact that $\tau(\omega)$ is Z-valued such that (1.3) and (1.4) are true surely for γ^n. It remains to pass to the limit. ♢

Moreover, $\gamma \to \rho_{i,j}(\gamma)$ belongs locally to all the Sobolev spaces.

This allows us to give the following definition:

Definition 1.2. *A measurable section ϕ of the formal bundle associated to $\tau(\omega)$ is a collection of random variables with values in C α_i over $B(\gamma_i, \delta)$ subjected to the rule*

$$\alpha_i = \rho_{i,j}\alpha_j. \tag{1.5}$$

Over $B(\gamma_i, \delta)$, we put the metric:

$$\|\alpha(\gamma)\|^2 = \mid \alpha_i \mid^2 . \tag{1.6}$$

Since the transition functions are of modulus one, this metric is consistent with the change of charts. We can give the definition:

Definition 1.3. *The L^p space of sections of the formal bundle associated to $\tau(\omega)$ is the space of measurable sections ϕ endowed with the norm:*

$$\|\phi\|_{L^p} = \| \, \|\phi\| \, \|_{L^p}. \tag{1.7}$$

We can define too the circle bundle $\tilde{L}_x(M)$ (the fiber is a circle) associated to $\tau(\omega)$ by its functionals (instead of its sections).

Definition 1.4. *A measurable functional $\tilde{F}$ of $\tilde{L}_x(M)$ is a family of random variables $F_i(\gamma, u_i)$ over $B(\gamma_i, \delta) \times S^1$ submitted to the relation*

$$F_i(\gamma, u_i) = F_j(\gamma, u_j), \tag{1.8}$$

where $u_i = u_j \rho_{j,i}$ almost surely.

Over the fiber, we put:

$$\|\tilde{F}\|_{p,\gamma} = \left(\int_{S^1} | \, F_i(\gamma, u_i) \, |^p \right)^{\frac{1}{p}}. \tag{1.9}$$

Since the measure over the circle is invariant by rotation, $\|\tilde{F}\|_{p\gamma}$ is intrinsically defined and is a random functional over the basis $L_x(M)$. We can give:

Definition 1.5. *An L^p functional over $\tilde{L}_x(M)$ is a functional $\tilde{F}$ such that $\|\tilde{F}\|_{p,\gamma}$ belongs to $L^p(L_x(M))$.*

We will give a baby model due to [5] of the problem to construct a string structure. Let $P_x(M)$ be the space of continuous applications from [0,1] into M. dP_1^x is the law of the Brownian motion starting from x and $dP_{1,x,y}$ is the law of the Brownian bridge between x and y: it is a probability measure over $L_{x,y}(M)$, the space of continuous paths starting from x and arriving in y. Let $p_t(x, y)$ be the heat kernel associated to the heat semi-group. We say that $P_x(M) = \cup L_{x,y}(M)$ by using the following formula:

$$dP_1^x = p_1(x, y) dy \otimes dP_{1,x,y}. \tag{1.10}$$

By repeating the same considerations, $\tau(\omega)$ is an element of $H^2(L_{x,y}, Z)$ if we consider smooth loops. In particular, we can consider a formal line bundle $\Lambda_{x,y}$ over $L_{x,y}(M)$. The problem is to glue together all these formal line bundles $\Lambda_{x,y}$ into a formal line bundle Λ_x over $P_x(M)$. If we consider smooth paths, the obstruction is measured in [5]. It is $d\beta = -\omega$. We can perturb $\tau(\omega)$ by

$$\tilde{\tau}(\omega) = \beta(\gamma_1) + \tau(\omega). \tag{1.11}$$

such that $\tilde{\tau}(\omega) = \tau(\omega)$ over $L_{x,y}(M)$ and such that $\tilde{\tau}(\omega)$ is closed Z-valued over $P_x(M)$. So, we can construct the formal global line bundle Λ_x over $P_x(M)$ by its sections. It remains to show that a section of Λ_x restricts into a section of $\Lambda_{x,y}$. We meet the problem that a section of Λ_x is only almost surely defined. We will proceed as in the quasi-sure analysis [10, 1]: a smooth functional over the flat Wiener space restricts into a functional over a finite codimensional manifold by using integration by parts formulas. We would like to state the analogous result for a smooth section of Λ_x.

In order to speak of Sobolev spaces over $P_x(M)$, we consider the tangent space (1.1) with only the condition $H_0 = 0$. This allows us to perform integration by parts [8, 3]. We can speak of the connection one form A_i of Λ_x over $B(\gamma_i, \delta)$. We get over $P_x(M)$

$$\nabla^{\Lambda_x}\alpha_i = d\alpha_i + A_i(\gamma)\alpha_i \tag{1.12}$$

which is almost surely consistent with the change of charts. Following the convention of differential geometry, we can iterate the operation of covariant differentiation and we get the operation ∇^{k,Λ_x} which is a k Hilbert-Schmidt cotensor in the tangent space connection, if we add the trivial connection in the tangent space (the tangent space of $P_x(M)$ is trivial modulo the parallel transport τ_t).

Definition 1.6. *The space $W_{k,p}(\Lambda_x)$ is the space of sections of the formal line bundle Λ_x such that $\nabla^{k',\Lambda_x}\phi$ belongs to L^p for $k' \leq k$. The space of smooth sections $W_{\infty,\infty-}(\Lambda_x)$ is the intersection of all Sobolev spaces $W_{k,p}(\Lambda)$.*

Theorem 1.7. *A smooth section ϕ of Λ_x restricts to a section Φ_y of $\Lambda_{x,y}$.*

2. String structure over the Brownian bridge

Let us consider the finite energy based path space $P_f(G)$ and the finite energy loop group $L_f(G)$. Let us consider the two form over $P_f(G)$, which on the level of a Lie algebra satisfies to:

$$c(X_\cdot, Y_\cdot) = \frac{1}{8\pi^2}\int_0^1 (\langle X_s, dY_s\rangle - \langle Y_s, dX_s\rangle)\,. \tag{2.1}$$

Its restriction over $L_f(G)$ gives a central extension $\tilde{L}_f(G)$ of $L_f(G)$ if G is supposed simple simply laced [23]: in particular, $Spin_{2n}$ is simply laced.

Let us introduce the Bismut bundle over $L_x(M)$ if Q is a principal bundle over M with structure group G: q_s is a loop over γ_s such that $q_s = \tau_s^Q g_s$, where τ_s^Q is the parallel transport over Q for any connection over Q. We suppose that $g_\cdot$ is of finite energy. Moreover, $g_1 = (\tau_1^Q)^{-1}$. Let f be the map from $L_x(M)$ to G:

$$\gamma_\cdot \to (\tau_1^Q)^{-1}. \tag{2.2}$$

Let π be the projection from $L_e(Q)$ over $L_x(M)$ and π the projection from $P_f(G)$ over G: $g_\cdot \to g_1$. We get a commutative diagram of bundles (see [6]):

$$\begin{array}{ccc} L_e(Q) & \rightarrow & P_f(G) \\ \downarrow & & \downarrow \\ L_x(M) & \rightarrow & G \end{array} \tag{2.3}$$

Let ω be the 3 form:

$$\omega(X, Y, Z) = \frac{1}{8\pi^2}\langle X, [Y, Z]\rangle. \tag{2.4}$$

We get [6, 17]: $\pi^*\omega = dc$. If the first Pontryagin class of Q is equal to 0, $(f^*)^*c - \pi = A8*\nu$ is a closed form over $L_e(Q)$: in such a case namely $f^*\omega = d\nu$, where ν is a nice iterated integral over $L_x(M)$. We can perturb $(f^*)^*c - \pi^*\nu$ by a closed iterated integral in the basis such that we get a closed Z-valued 2 form F_Q over $L_e(Q)$ (in the smooth loop context). We do the following hypothesis, for smooth loop:

Hypothesis. *$L_e(Q)$ is simply connected. $L_f(G)$ is simply connected. $L_x(M)$ is simply connected.*

The obstruction to trivialize $L_e(Q)$ is the holonomy: we can restrict the transition functions $P_f(G) \to G$ to be an element of smooth loop in G. We can introduce a connection over this bundle ∇^∞ and we can pullback this connection into a connection ∇^∞ over the bundle $L_e(Q) \to L_x(M)$. This allows us to lift the distinguished paths over $L_x(M)$ (we are now in the stochastic context) into distinguished paths over $L_e(Q)$, which will allow us to produce a system of transition functions with values in $\tilde{L}_f(Q)$, because the stochastic part of F_Q is a sum of iterated integrals. But the obstruction to trivialize $L_e(Q)$ is τ_1^Q which is almost surely defined. We cannot work over open neighborhoods in order to trivialize our lift.

Let us resume: we can find a set of subset O_i of $L_x(M)$ such that:

i) $\cup O_i = L_x(M)$ almost surely.

ii) There exists a sequence of smooth functionals G_i^n such that $G_i^n > 0$ is included into O_i and such that G_i^n tends increasingly almost surely to the indicatrix function of O_i.

iii) Over $O_i \cap O_j$, there exists a map $\tilde{\rho}_{i,j}(\gamma)$ with values in $\tilde{L}_f(G)$ such that

$$\tilde{\rho}_{i,j}(\gamma)\tilde{\rho}_{j,i}(\gamma) = \tilde{e} \tag{2.5}$$

($\tilde{e}$ is the unique element of $\tilde{L}_f(G)$) and such that, over $O_i \cup O_j \cup O_k$,

$$\tilde{\rho}_{i,j}(\gamma)\tilde{\rho}_{j,k}(\gamma)\tilde{\rho}_{k,i}(\gamma)\tilde{e} \tag{2.6}$$

iv) Moreover, $\tilde{\rho}_{i,j}(\gamma)$ is smooth in the following way: $\tilde{\rho}_{i,j}(\gamma) = (l_{i,j,\cdot}(\gamma), \alpha_{i,j})$, where $l_{i,j,\cdot}(\gamma)$ is a path in $L_f(G)$ starting from $e_\cdot$ which depends smoothly on γ and $\alpha_{i,j}$ a functional in S^1 which depends smoothly on γ. (Let us recall that the central extension $\tilde{L}_f(G)$ can be seen as a set of couple of paths in $L_f(G)$ and an element of the circle, submitted to an equivalence relation as was discussed in the first part).

Let us suppose that there exists a unitary representation $Spin_\infty$ of $\tilde{L}_f(G)$.

Definition 2.1. *A measurable section of Spin is a family of random variables from O_i into $Spin_\infty$ submitted to the relation:*

$$\psi_j = \tilde{\rho}_{j,i}\psi_i. \tag{2.7}$$

The modulus $\|\psi\|$ is intrinsically defined, because the representation is unitary.

Definition 2.2. *A L^p section of Spin is a measurable section ψ such that*

$$\|\psi\|_{L^p} = \| \, \|\psi\| \, \|_{L^p} < \infty. \tag{2.8}$$

We would like to speak of a bundle whose fiber should be $\tilde{L}_f(G)$, by doing as in the first part: the problem is that there is no Haar measure over $\tilde{L}_f(G)$. We will begin to construct an S^1 bundle over $L_e(Q)$, and for that, we need to construct a measure over $L_e(Q)$. This requires to construct a measure over $L_f(G)$.

For that, we consider the stochastic differential equation:

$$dg_s = g_s(C + B_s), \qquad g_0 = e, \tag{2.9}$$

where B_s is a Brownian motion starting from 0 in the Lie algebra of G and C is an independent Gaussian variable over the Lie algebra of G with average 0 and covariance I_d. The law of g_1 has a density $q(g) > 0$. If we consider a C^2 deterministic path in G starting from e, $k_s g_s$ and $g_s k_s$ have a law which is absolutely continuous with respect to the original law, but the density is only in L^1 (unlike the traditional case of [2] for continuous loops, where the density is in all the L^p). We can get infinitesimal quasi-invariance formulas, that is integration by parts formulas: we take as tangent vector fields $X_s = g_s K_s$ or $X_s = K_s g_s$ ($K_0 = 0$), where K_s has values in the Lie algebra of G. We take as the Hilbert norm $\int_0^1 \|K"_s\|^2 ds$ (instead of $\int_0^1 \|K'_s\|^2 ds$ for the continuous case). If $K_.$ is deterministic, we get an integration by parts formula:

$$E_{P_f(G)}\left[\langle dF, X\rangle\right] = E_{P_f(G)}[F \text{ div } X], \tag{2.10}$$

where div X belongs to all the L^p.

This allows us, since $q(g) > 0$, to desintegrate the measure over the set of finite energy paths going from e to g. We get a measure dP_g, and we get a measure over $L_e(Q)$:

$$d\mu_{tot} = dP_{1,x} \otimes dP_{(\tau_1^Q)^{-1}}. \tag{2.11}$$

Let us suppose that $(\tau_1^Q)^{-1}$ belongs to a small open neighborhood G_i of G, where the bundle $P_f(G) \to G$ is trivial. Let $K_{i,.}$ be the connection one form of this bundle in this trivialization. We split the tangent space of $L_e(Q)$ into the orthonormal sum of a vertical one and a horizontal one:

— The vertical one is constituted of vector fields of the type $q_s K_s$ (K_s is a loop in the Lie algebra of G with two derivatives) with the Hilbert norm $\int_0^1 \|K"_s\|^2 ds$).

— The horizontal one is a set of vectors of the type

$$X_s^h = \tau_s H_s - K_{i,s}\left(\langle d(\tau_1^Q)^{-1}), X\rangle\right) g_s, \tag{2.12}$$

where $X_s = \tau_s H_s$ and H_s checks (1.1). The Hilbert norm is $\int_0^1 \|H'_s\|^2 ds$.

We get:

Proposition 2.3. *Let F be a cylindrical functional over $L_e(Q)$. We have the integration by parts formula:*

$$\mu_{tot}[< dF, X >] = \mu_{tot}[F \text{ div } X] \tag{2.13}$$

if X corresponds to a vertical or a horizontal vector fields which are associated to deterministic K_s or deterministic H_s. Moreover, $\operatorname{div} X$ *belongs to all the L^p.*

This integration by parts formula allows us to get Sobolev spaces over $L_e(Q)$.

We repeat the considerations which lead to the transition functions $\tilde{\rho}_{i,j}(\gamma)$ but over $L_e(Q)$. We get a system of charts $(O_i, \rho_{i,j}(q_.)$ with the difference that $O_i \subset L_e(Q)$ and $\rho_{i,j}(q_;)$ belong to S^1. It is easier to say in this context that $\rho_{i,j}(q_.)$ belong locally to all the Sobolev spaces.

Definition 2.4. *A measurable functional $\tilde{F}(\tilde{q}_.)$ associated to the formal circle bundle over $L_e(Q)$ constructed from the system of $\rho_{i,j}$ is a family of measurable functionals $F_i : O_i \times S^1 \to R$ such that almost surely in $q_.$ and u (we choose the Haar measure over the circle)*

$$F_i(q_., u_i) = F_j(q_., u_j), \tag{2.14}$$

where $u_j = u_i \rho_{i,j}(q_.)$ almost surely.

Since there is the Haar measure over the circle, we can do as in the first part in order to speak of functionals $\tilde{F}(\tilde{q}_.)$ which belong to $L^p(\tilde{\mu}_{tot})$, by integrating in the fiber.

An "element" $\tilde{q}_.$ of $\tilde{L}_e(Q)$ can be seen formally as the couple of a distinguished path in $L_e(Q)$ arriving in $q_. = \pi(\tilde{q}_.)$ and an element of the circle. An element $\tilde{g}_.$ of $\tilde{L}_2(G)$ can be seen as the couple of a distinguished path in $L_2(G)$, the group of based loops in G with two derivatives, and an element of the circle. $\tilde{g}_.$ acts over $\tilde{q}_.$ by multiplying vertically the end loop of the distinguished path associated to $\tilde{q}_.$ and multiplying the two element of the circle. Moreover, the action of $L_2(G)$ leads to quasi-invariance formulas over $P_f(G)$. This motivates the following definition:

Definition 2.5. *A stochastic gauge transform Ψ of the formal bundle $\tilde{L}_e(Q)$ is a measurable application from $L_x(M)$ into $\tilde{L}_2(G)$.*

A gauge transform induces a transformation Ψ of functionals over $\tilde{L}_2(Q)$:

$$\Psi(\tilde{F}(\tilde{q}_.) = \tilde{F}(\tilde{q}_. \Psi(q_.)). \tag{2.15}$$

This definition has a rigorous sense at the level of functionals.

Theorem 2.6. *The group of gauge transforms acts naturally by isometries over $L^\infty(\tilde{\mu}_{tot})$.*

3. The case of the free loop space

Let us suppose that we consider the free loop space of continuous applications from the circle S^1 into M with the measure

$$dP = p_1(x,x)dx \otimes dP_{1,x}. \tag{3.1}$$

The tangent space is as in (1.1), but we have to take the periodicity condition $X_0 = X_1$. We take as Hilbert structure the Hilbert structure $\int_0^1 \|H'_s\|^2 ds + \int_0^1 \|X_s\|^2 ds$ which is invariant by rotation.

Let A_Q be the connection 1-form over the principal bundle $Q \to M$: it is a one form from the tangent space of Q into the Lie algebra of Q. We associate following [7] the 2 form over $L(Q)$ to the free loop space of Q for smooth loops:

$$\omega_Q = \frac{1}{4\pi^2}\int_0^1 1/2\langle A_q, d/dt A_Q\rangle. \tag{3.2}$$

This two form gives the central extension of $L(G)$ in the fiber of $L(Q)$. We have to perturb it by a form in order to get a form which is Z-valued closed over $L(Q)$ and which gives ω_Q in the fiber.

Following [7], we perturb ω_Q into a form ω'_Q over the smooth free loop space of Q:

$$\omega'_Q = \frac{1}{4\pi^2}\int_0^1 1/2\langle A_Q, d/dt A_Q\rangle - \langle R_Q, A_Q(dq_s)\rangle, \tag{3.3}$$

where R_Q is the curvature tensor of A_Q. Let σ_Q be the 3 form over Q:

$$\sigma_Q = \frac{1}{8\pi^2}\langle A_Q, R_Q - 1/6[A_Q, A_Q]\rangle. \tag{3.4}$$

We get

$$d\sigma_Q = \pi^* p_1(Q), \tag{3.5}$$

where $p_1(Q)$ is the first Pontryiagin class of Q. If $p_1(Q) = 0$, we can choose a form over M such that $p_1(Q) = -d\nu$ and such that $\sigma_Q - \pi^*\nu$ is a Z-valued closed 3 form over Q. Let τ be the operation of transgression over $L(Q)$. Following [7], we choose as curvature term in order to define the S^1 bundle over $L(Q)$ the expression $F_Q = \omega'_Q - \tau(\pi^*\nu)$.

We repeat the considerations of the second part with the main difference that we don't have globally a commutative diagram as (2.2). This leads to some difficulties, but since we have such a diagram locally at the starting point, we can produce a system of charts $O_i, \tilde\rho_{i,j}(\gamma)$ with the difference that $O_i \subset L(M)$ and that $\tilde\rho_{i,j}(\gamma)$ takes its values in $\tilde L_f(G)$, the basical central extension of the free loop space of G (we consider loops with finite energy).

If $G = Spin_{2n}$, the central extension of the free loop group has got a unitary representation by using the fermionic Fock space. This allows us to repeat Definition 2.1 and Definition 2.2.

Moreover, F_q is invariant under rotation. The natural circle action lifts to the section of the spin bundle over $L(M)$.

References

[1] Airault H. and Malliavin P., Quasi sure analysis, Publication Paris VI, 1991.

[2] Albeverio S. and Hoegh-Krohn R., The energy representation of Sobolev Lie groups, *Compositio Math.*, 1978, V.36, 37–52.

[3] Bismut J.M., Large deviations and the Malliavin calculus, *Progress in Math.*, 1984, V.45. Birkhauser.

[4] Borthwick D., The Pfaffian line bundle, *C.M.P.*, 1992, V.149, 463–493.

[5] Brylinski J.L. and Mac Laughlin D., The geometry of degree-four characteristic classes and of line bundles on loop spaces I., *Duke Math. J.*, 1994, V.75, 603–638.

[6] Carey A.L. and Murray M.K., String structure and the path fibration of a group, *C.M.P.*, 1991, V.141, 441–452.

[7] Coquereaux R. and Pilch K., String structure on loop bundles, *C.M.P.*, 1989, V.120, 353–378.

[8] Driver B., A Cameron-Martin type quasi-invariance for Brownian motion on compact manifolds, *J.F.A.*, 1992, V.110, 272–376.

[9] Elworthy D., Stochastic differential equations on manifold, L.M.S. Lectures Notes Serie 20, Cambridge University Press. 1982.

[10] Getzler E., Dirichlet form on loop spaces, *Bull. Sci. Maths.*, 1989, V.113, 155–174.

[11] Ikeda N. and Watanabe S., Stochastic Differential Equations and Diffusion Processes, North-Holland, 1981.

[12] Jones J. and Léandre R., L^p Chen forms over loop spaces, in: Stochastic Analysis, Barlow M. and Bingham N. edit., Cambridge University Press, 1991, 104–162.

[13] Jones J.D.S. and Léandre R., A stochastic approach to the Dirac operator over the free loop space, To be published in "Loop Space", A. Sergeev edit.

[14] Killingback T., World-sheet anomalies and loop geometry, *Nucl. Phys. B*, 1987, V.288, 578–588.

[15] Léandre R., Integration by parts formulas and rotationally invariant Sobolev calculus on the free loop space, XXVII Winter School of theoretical physics, Gielerak R. and Borowiec A. edit., *J. of Geometry and Phys.*, 1993, V.11, 517–528.

[16] Léandre R., Invariant Sobolev calculus on the free loop space, To be publish an *Acta Applicandae Mathematicae.*

[17] Léandre R., String structure over the Brownian bridge, To be published in *C.M.P.*

[18] Léandre R., Hilbert space of spinors fields over the free loop space, To be published in *Reviews in Math. Phys.*

[19] Léandre R., Stochastic gauge transform of the string bundle, Preprint.

[20] MacLaughlin D., Orientation and string structures on loop spaces, *Pac. J. Math.*, 1992, V.155, 143–156.

[21] Malliavin M.P. and Malliavin P., Integration on loop group III. Asymptotic Peter-Weyl orthogonality, *J.F.A.*, 1992, V.108, 13–46.

[22] Pickrell D., Invariant measures for unitary forms of Kac-Moody groups, Preprint.

[23] Pressley A. and Segal G., Loop Groups, Oxford University Press, 1986.

[24] Segal G., Elliptic cohomology, Séminaire Bourbaki, Exposé 695, Astérisque 161–162, 1988, 187–201.

[25] Taubes C., S^1 actions and elliptic genera, *C.M.P.*, 1989, V.122, 455–526.

[26] Witten Ed., The index of the Dirac operator in loop space, in: Elliptic curves and modular forms in algebraic topology, Landweber edit., L.N.M. 1326, Springer, 1988, 161–181.