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To Separation of Variables in a (1+2)-Dimensional Fokker-Planck Equation

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Abstract

The problem of separation of variables in a (1+2)-dimensional Fokker-Planck equation is considered. For the case of constant coefficients of second derivatives, the classification of coordinate systems, where the variables in the Fokker-Planck equation are separable, is made and new coordinate systems for one class of this equation, which provide a separation of variables, are obtained.

Let us consider the Fokker-Planck equation

$$u_t = (d_{ij}u_{x_j})_{x_i} - (a_i u)_{x_i} \qquad i,j = \overline{1,n}.$$

Here, $u_t = \dfrac{\partial u}{\partial t}$ and $u_{x_i} = \dfrac{\partial u}{\partial x_i}$. Coefficients d_{ij} and a_i depend on $t, \vec{x}$, and d_{ij} are non-negative definite quadratic form coefficients. The summation is done over indeices i and j. If $d_{ij} = \text{const}$, then the two-dimensional Fokker-Planck equation is reduced to

$$u_t = u_{x_1x_1} + u_{x_2x_2} - (a_1 u)_{x_1} - (a_2 u)_{x_2} \tag{1}$$

by a linear change of variables.

To separate variables in the given equation means to find new variables $\omega_0, \omega_1, \omega_2$ and a function $Q(t, x_1, x_2)$ such that equation (1) splits after the substitution of

$$u = Q(t, x_1, x_2)\varphi_0(\omega_0, \lambda_1, \lambda_2)\varphi_1(\omega_1, \lambda_1, \lambda_2)\varphi_2(\omega_2, \lambda_1, \lambda_2) \tag{2}$$

into three ordinary differential equations for the functions $\varphi_0, \varphi_1, \varphi_2$. Here, λ_1, λ_2 are separation constants.

A method of classification of coordinate systems, which allow a separation of variables for the Schrödinger equation, is realized in [1, 2]. It allows one also to obtain explicitly potentials for which a separation of variables is possible. Let us use this method for the classification of coordinate systems, which allow us to separate variables in equation (1). Let

$$\omega_0 = t, \quad \omega_i(t, x_1, x_2), \quad i = 1, 2,$$

and suppose that solution (2) and

$$u' = Q'\varphi_0'(t, \lambda_1', \lambda_2')\varphi_1'(\omega_1', \lambda_1', \lambda_2')\varphi_2'(\omega_2', \lambda_1', \lambda_2')$$

are equivalent if

$$Q' = Q\Phi_1(\omega_1)\Phi_2(\omega_2), \quad \omega_i' = \Omega_i(\omega_i), \quad \lambda_i' = \Lambda_i(\lambda_1, \lambda_2), \qquad i = 1, 2. \tag{3}$$

Substituting (2) to (1), splitting the obtained equation with respect to φ_0, φ_1, φ_2, $\dfrac{d\varphi_1}{d\omega_1}$, $\dfrac{d\varphi_2}{d\omega_2}$, λ_1, λ_2, and considering the relations of equivalency (3), we have that (1) splits into the system of three linear ordinary differential equations

$$\begin{aligned} &\frac{d\varphi_0}{dt} = (\lambda_1 r_1(t) + \lambda_2 r_2(t) + r_0(t))\varphi_0,\\ &\frac{d^2\varphi_1}{d\omega_1^2} = (\lambda_1 b_{11}(\omega_1) + \lambda_2 b_{21}(\omega_1) + b_{01}(\omega_1))\varphi_1\\ &\frac{d^2\varphi_2}{d\omega_2^2} = (\lambda_1 b_{12}(\omega_2) + \lambda_2 b_{22}(\omega_2) + b_{02}(\omega_2))\varphi_2, \end{aligned} \tag{4}$$

where

$$\operatorname{rank}\begin{pmatrix} r_1 & r_2 \\ b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} = 2$$

and the functions ω_1, ω_2, Q, a_1, a_2 satisfy the system of equations

$$\omega_{1x_1}\omega_{2x_1} + \omega_{1x_2}\omega_{2x_2} = 0, \tag{5}$$

$$b_{i1}(\omega_1)(\omega_{1x_1}^2 + \omega_{1x_2}^2) + b_{i2}(\omega_2)(\omega_{2x_1}^2 + \omega_{2x_2}^2) + r_1(t) = 0, \quad i = 1, 2, \tag{6}$$

$$Q\omega_{it} - Q\triangle\omega_i - 2Q_{x_1}\omega_{ix_1} - 2Q_{x_2}\omega_{ix_2} + a_1 Q\omega_{ix_1} + a_2 Q\omega_{ix_2} = 0, \quad i = 1, 2, \tag{7}$$

$$\begin{aligned} &Q_t - \triangle Q - Qb_{01}(\omega_1)(\omega_{1x_1}^2 + \omega_{1x_2}^2) - Qb_{02}(\omega_2)(\omega_{2x_1}^2 + \omega_{2x_2}^2) -\\ &\quad -Qr_0(t) + Q_{x_1}a_1 + Q_{x_2}a_2 + Q(a_{1x_1} + a_{2x_2}) = 0. \end{aligned} \tag{8}$$

While solving equations (5) and (6), we define coordinate systems providing a separation of variables in equation (1), determined up to the equivalency relation (3). Substituting the obtained expressions for ω_1, ω_2 into equation (7), we have functions $Q(t, x_1, x_2)$ explicitly for each of the found coordinate systems and also the condition, imposed on a_1, a_2 for a providing of system compatibility. The substitution of the obtained results into (8) gives the equation for coefficients a_1, a_2. If we succeed in solution of this equation, we obtain the functions a_1, a_2, with which the variables in equation (1) separate in the corresponding coordinate systems. Here, we give the list of coordinate systems, which provide the separation of variables in equation (1) and the functions $Q(t, x_1, x_2)$ explicitly, which correspond to these variables.

1) $$\begin{aligned} &\omega_1 = A(t)(x_1\cos\alpha t + x_2\sin\alpha t) + W_1(t),\\ &\omega_2 = B(t)(x_1\sin\alpha t - x_2\cos\alpha t) + W_2(t),\\ &Q(t, x_1, x_2) = \exp\left(\frac{\dot{A}}{4A}(x_1^2\cos^2\alpha t + x_2^2\sin^2\alpha t) + \frac{\dot{B}}{4B}(x_1^2\sin^2\alpha t + x_2^2\cos^2\alpha t)\right)\times\\ &\times\exp\left(\frac{1}{2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)x_1x_2\sin\alpha t\cos\alpha t + \frac{\dot{W}_1}{2A}(x_1\cos\alpha t + x_2\sin\alpha t)\right)\times\\ &\times\exp\left(\frac{\dot{W}_2}{2B}(x_1\sin\alpha t - x_2\cos\alpha t) + \frac{F}{2}\right). \end{aligned}$$

2) $x_1 = W(t)e^{\omega_1}\cos(\omega_2 + \alpha t) + W_1(t), \quad x_2 = W(t)e^{\omega_1}\sin(\omega_2 + \alpha t) + W_2(t),$

$Q(t, x_1, x_2) = R(t, x_1, x_2)$

3) $x_1 = W(t)\left(\omega_1\omega_2\cos\alpha t + \frac{1}{2}(\omega_2^2 - \omega_1^2)\sin\alpha t\right) + W_1(t),$

$x_2 = W(t)\left(\omega_1\omega_2\sin\alpha t + \frac{1}{2}(\omega_2^2 - \omega_1^2)\cos\alpha t\right) + W_2(t),$

$Q(t, x_1, x_2) = R(t, x_1, x_2).$

4) $x_1 = W(t)(\cosh\omega_1\cos\omega_2\cos\alpha t + \sinh\omega_1\sin\omega_2\sin\alpha t) + W_1(t),$

$x_2 = W(t)(\sinh\omega_1\sin\omega_2\cos\alpha t - \cosh\omega_1\cos\omega_2\sin\alpha t) + W_2(t),$

$Q(t, x_1, x_2) = R(t, x_1, x_2).$

Here, $A(t)$, $B(t)$, $W(t)$, $W_1(t)$, and $W_2(t)$ are arbitrary sufficiently smooth functions,

$$\frac{\partial F}{\partial x_1} = \alpha x_2 + a_1, \qquad \frac{\partial F}{\partial x_2} = -\alpha x_1 + a_2, \qquad a_{1x_2} - a_{2x_1} = -2\alpha,$$

$$R(t, x_1, x_2) = \exp\left(-\frac{\dot{W}}{4W}((x_1 - W_1)^2 + (x_2 - W_2)^2) - \frac{1}{2}\dot{W}_1x_1 - \frac{1}{2}\dot{W}_2x_2\right) \times$$
$$\times \exp\left(-\frac{\alpha}{2}W_2x_1 + \frac{\alpha}{2}W_1x_2 + \frac{F}{2}\right).$$

We note that equation (1) reduces for $a_{2x_1} - a_{1x_2} = 0$ to the heat equation with potential (see [2]).

As an example , let us consider the equation:

$$u_t = u_{x_1x_1} + u_{x_2x_2} - ((a_{11}x_1 + a_{12}x_2)u)_{x_1} - ((a_{21}x_1 + a_{22}x_2)u)_{x_2}. \qquad (9)$$

If $a_{12} = a_{21}$ ($\alpha = 0$), it is reduced [2] to heat equation and provides the separation of variables in five coordinate systems. Equation (9) proves to allow the separation of variables also when $a_{12} = -a_{21}$, $a_{11} = a_{22}$. In this case, four coordinate systems are obtained, in which equation (9) is separated:

$$\omega_1 = \begin{cases} (\sin(a_{11}(t + C_1)))^{-1}(x_1\cos\alpha t + x_2\sin\alpha t) + C_2(\sin(a_{11}(t + C_1)))^{-2}, \\ (C_3 + \sin(2a_{11}(t + C_1)))^{-1/2}(x_1\cos\alpha t + x_2\sin\alpha t), \end{cases}$$

$$\omega_2 = \begin{cases} (\sin(a_{11}(t + C_4)))^{-1}(x_1\sin\alpha t - x_2\cos\alpha t) + C_5(\sin(a_{11}(t + C_4)))^{-2}, \\ (C_6 + \sin(2a_{11}(t + C_4)))^{-1/2}(x_1\sin\alpha t - x_2\cos\alpha t), \end{cases}$$

$C_1, \ldots, C_6$ are arbitrary real constants.

References

[1] Zhdanov R.Z., Revenko I.V. and Fushchych W.I., On the new approach to variable separation in the time-dependent Schrödinger equation with two space dimensions, *J. Math. Phys.*, 1995, V.36, 5506–5521.

[2] Zhdanov R.Z., Separation of variables in (1+2)-dimensional Schrödinger equation, *J. Math. Phys*, 1997, V.38, 1197–1217.