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Differential Form Symmetry Analysis of Two Equations Cited by Fushchych

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Abstract

In Wilhelm Fushchych's address, "Ansatz '95", given to the first conference "Symmetry in Nonlinear Mathematical Physics" [1], he listed many differential equations on which he and others had done some symmetry analysis. In this talk, the present author treats two of these equations rather extensively, using differential forms to find the symmetries, based on a method by F. B. Estabrook and himself [2]. A short introduction to the differential form method will be presented.

1 Introduction

Most calculations for symmetries of differential equations are done with the classical method. However, in 1971 Frank B. Estabrook and the author published a method [2] for finding the symmetries of differential equations, using a differential form technique, with a geometrical flavor. We refer to that paper as paper I. Since that technique has not been used widely in the literature, I would like to review it, and then to apply it to two equations cited by Fushchych in his talk "Ansatz 95", which he gave here at Kyiv at the first conference on nonlinear mathematical analysis [1].

2 Differential forms

I give a brief review of differential forms here. A simple, clever definition of differential forms, due to H. Flanders [3], is that differential forms are the things found under integral signs. That gives an immediate picture of differential forms, but we need to look at their foundation.

We begin by writing out a general tensor field, in terms of components with a basis formed of tensor products of basis tangent vectors e_i and 1-forms ω_i, as shown:

$$T = T^{ik\ldots}_{\ldots mn\ldots} e_i \otimes e_k \otimes \ldots \omega^m \otimes \omega^n \otimes \ldots . \tag{1}$$

The components may be functions of position.

One may work in the "natural bases" for these spaces, written, for coordinates x^i,

$$e_i = \partial/\partial x^i, \quad \omega^i = dx^i. \tag{2}$$

Differential forms are now defined as totally antisymmetric covariant tensor fields, that is, fields in which only the ω^i appear and in which the components are totally antisymmetric. It is usual to use an antisymmetric basis written as

$$\omega^i \wedge \omega^j \wedge \omega^k \ldots = \sum_\pi (-1)^\pi \pi[\omega^i \otimes \omega^j \otimes \omega^k \ldots], \tag{3}$$

composed of antisymmetric tensor products of the ω^i. π represents a permutation of the ω^i, and the sum is over all possible permutations. The symbol $\wedge$ is called a hook or wedge product. Then a form α, say, of rank p, or p-form, may be written as

$$\alpha = \alpha_{ijk\ldots} \omega^i \wedge \omega^j \wedge \omega^k \ldots \tag{4}$$

(p factors), with sums over $i, j, k, \ldots$ (Typically the sum is written for $i < j < k \ldots$ to avoid repetition.) A 0-form is simply a function. It is common to use the natural basis and to write p-forms as sums of hook products of p of the dx^i, as

$$\beta = \beta_{ijk\ldots} dx^i \wedge dx^j \wedge dx^k \ldots \tag{5}$$

We now may work with the set of differential forms on a manifold by itself. I give a brief summary of the rules. They may be found in many references, such as paper I; a good one for mathematical physicists is Misner, Thorne, and Wheeler [4]. Vectors will be needed in defining the operations of contraction and Lie derivative.

2.1 Algebra of forms

Forms of the same rank comprise a vector space and may be added and subtracted, with coefficients as functions on the manifold. Forms may be multiplied in terms of the hook product. Multiplication satisfies a distribution rule

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma \tag{6}$$

and a commutation rule

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha, \tag{7}$$

where $p = \text{rank}(\alpha)$ and $q = \text{rank}(\beta)$. Thus, products of 1-forms, in particular, are antisymmetric. This implies, if the base manifold is n-dimensional, that all forms of rank greater than n vanish, since the terms would include multiples of the same 1-form, which would be zero by antisymmetry. The number of independent p-forms in n-dimensional space is $\binom{n}{p}$ and the total number of independent forms, from rank 0 to n, is 2^n.

2.2 Calculus of forms

We define the exterior derivative d as a map from p-forms to (p+1)-forms. If

$$\gamma = f dx^i \wedge dx^j \wedge \ldots \tag{8}$$

then d is defined by (sum on k)

$$d\gamma = (\partial f / \partial x^k)\, dx^k \wedge dx^i \wedge dx^j \wedge \ldots \tag{9}$$

(one simply writes $d\gamma = df \wedge dx^i \wedge dx^j \ldots$ and expands df by the chain rule.) The exterior derivative satisfies these postulates:
Linearity:

$$d(\alpha + \beta) = d\alpha + d\beta \tag{10}$$

Leibnitz rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{11}$$

($p = \text{rank}(\alpha)$). In particular, if f is a function,

$$d(f\alpha) = df \wedge \alpha + f d\alpha.$$

Closure rule:

$$dd\alpha = 0 \tag{12}$$

for any form α. A form β satisfying $d\beta = 0$ is said to be "closed".

In three-dimensional space, these various rules are equivalent to many familiar vector identities.

2.3 Contraction with a vector

Contraction with a vector $(\cdot)$ is a map from p-forms to (p-1)-forms, defined by, if $v = v^i(\partial/\partial x^i)$,

$$v \cdot (\alpha_k dx^k) = v^i \alpha_i, \tag{13}$$

$$v \cdot (\alpha \wedge \beta) = (v \cdot \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \cdot \beta), \tag{14}$$

where $p = \text{rank}(\alpha)$. Thus,

$$v \cdot dx^i = v^i. \tag{15}$$

Contractions may be added or multiplied linearly by a scalar.

2.4 Lie derivative

The Lie derivative $\pounds_v$ is a generalization of the directional derivative and requires a vector v for definition. It may be defined on tensors and geometrical objects in general, but we consider only its definition on forms here. The Lie derivative of a p-form is another p-form. If f is a function, then

$$\pounds_v\, f = v \cdot df = v^i(\partial f/\partial x^i), \tag{16}$$

$$\pounds_v\, d\alpha = d(\pounds_v\, \alpha). \tag{17}$$

Thus, $\pounds_v(x^i) = v^i$ and $\pounds_v(dx^i) = dv^i$ (dv^i is to be expanded by the chain rule.) Further identities are

$$\pounds_v(\alpha \wedge \beta) = (\pounds_v\, \alpha) \wedge \beta + \alpha \wedge (\pounds_v\, \beta) \tag{18}$$

and

$$\pounds_v\, \alpha = v \cdot d\alpha + d(v \cdot \alpha). \tag{19}$$

Applying the Lie derivative is often called *dragging* a form. If a Lie derivative is zero, the form may be said to be invariant under the transformation represented by the dragging.

3 Writing differential equations as forms

Any differential equation or a set of differential equations, ordinary or partial, may be written in terms of differential forms. The method is straightforward (see paper I). One first reduces the equation(s) to a set of first-order differential equations by introducing new variables as necessary. As an example, consider the heat equation, where subscripts indicate differentiation:

$$u_{xx} = u_t. \tag{20}$$

We define new dependent variables $z = u_x$, $w = u_t$, so that $z_t = w_x$, $z_x = w$. We consider a differentiable manifold in the five variables x, t, u, z, and w. On this manifold we introduce a set of forms by inspection. These forms are chosen such that if we (a) consider the dependent variables to be functions of the independent variables (a process called sectioning) – so that we can write their exterior derivatives in terms of the independent variables – and then (b) set the forms equal to zero (a process called annulling), we recover the original differential equations. Thus, we simply restrict the forms to the solution manifold of the differential equation(s).

For the heat equation, we first define

$$\alpha = -du + z\,dx + w\,dt. \tag{21}$$

Sectioning gives

$$\alpha = -(u_x\,dx + u_t\,dt) + z\,dx + w\,dt$$

and annulling gives $z = u_x$, $w = u_t$ back again. Now

$$d\alpha = dz\,dx + dw\,dt \tag{22}$$

where the hook product $\wedge$ is to be understood. Sectioning gives $d\alpha = (z_x\,dx + z_t\,dt)\,dx + (w_x\,dx + w_t\,dt)\,dt = z_t\,dt\,dx + w_x\,dx\,dt = (w_x - z_t)\,dx\,dt$, since $dx\,dx = dt\,dt = 0$ and $dt\,dx = -dx\,dt$, and annulling gives $w_x = z_t$ again. Finally, we write

$$\beta = dz\,dt - w\,dx\,dt = (z_x\,dx + z_t\,dt)\,dt - w\,dx\,dt, \tag{23}$$

giving $w = z_x$ when β is annulled. Thus, the set $\{\alpha, d\alpha, \beta\}$ – which we call the ideal I of forms – represents the original equation(s) when sectioned and annulled.

The forms as given are not unique. For example, we may construct an alternate set simply from $z = u_x$ and $z_x = u_t$, yielding the forms $\gamma = du\,dt - z\,dx\,dt$ and $\delta = dz\,dt + du\,dx$ and giving an alternate ideal $I' = \{\gamma, \delta\}$. These ideals should be closed, and they are: $dI \subset I$ and $dI' \subset I'$, since $d\beta = d\alpha \wedge dx$, $d\delta = 0$, and $d\gamma = \delta \wedge dx$.

4 Invariance of the differential equations

It is now simple to treat the invariance of a set of differential equations. A set of equations is invariant if a transformation leaves the equations still satisfied, provided that the original equations are satisfied. In the formalism we have introduced, this is easily stated: the Lie derivative of forms in the ideal must lie in the ideal:

$$\pounds_v I \subset I. \tag{24}$$

Then if the basis forms in the ideal are annulled, the transformed equations are also annulled. In practice, this means simply that the Lie derivative of each of the (basis) forms in I is a linear combination of the forms in I. For the heat equation, by using the ideal I, we get the equations

$$\pounds_v \, \alpha = \lambda_1 \alpha, \tag{25}$$

which gives

$$\pounds_v \, d\alpha = d\lambda_1 \wedge \alpha + \lambda_1 d\alpha, \tag{26}$$

so that $\pounds_v \, d\alpha$ is automatically in the ideal, and

$$\pounds_v \, \beta = \lambda_2 \beta + \lambda_3 d\alpha + \sigma \wedge \alpha, \tag{27}$$

where the λ_i and σ are multipliers to be eliminated. The λ_i are functions (0-forms) and σ is an arbitrary 1-form. The resulting equations, after this elimination, are simply the usual determining equations for the symmetry generators, the components v^i of v – which was called the *isovector* in paper I.

For the example considered, we consider first Eq. (25). It is simplified by putting

$$H = v \cdot \alpha = -v^u + z v^x + w v^t, \tag{28}$$

where the superscripts indicate components of v. H is to be considered a function of all variables and is as yet unspecified. Then, by Eq. (19),

$$\pounds_v \alpha = v \cdot d\alpha + d(v \cdot \alpha) = \lambda_1 \alpha \tag{29}$$

or

$$v^z dx - v^x dz + v^w dt - v^t dw + dH = \lambda_1(-du + z\, dx + w\, dt). \tag{30}$$

dH is to be expanded by the chain rule. We now set the coefficients of dx, dt, etc., to zero. From the coefficient of du, we get $\lambda_1 = -H_u$. The other coefficients give, after substitution for λ_1,

$$\begin{aligned} v^x &= H_z, \quad v^z = -H_x - zH_u \\ v^t &= H_w, \quad v^w = -H_t - wH_u. \end{aligned} \tag{31}$$

We note from Eq. (28) and Eq. (31) that

$$v^u = -H + zH_z + wH_w. \tag{32}$$

We do not need to consider $d\alpha$ separately, as noted above.

We now expand Eq. (27), using the rules for Lie differentiation given in Eqs. (16) through (18), and also substitute for the forms on the right-hand side. We get

$$\begin{aligned} dv^z\, dt &+ dz\, dv^t - v^w\, dx\, dt - w\, dv^x\, dt - w\, dx\, dv^t \\ &= \lambda_2(dz\, dt - w\, dx\, dt) + \lambda_3(dz\, dx + dw\, dt) \\ &+ (\sigma_1\, dx + \sigma_2\, dt + \sigma_3\, dz + \sigma_4\, dw) \wedge (-du + z\, dx + w\, dt), \end{aligned} \tag{33}$$

in which the σ_i (which are the components of σ), along with λ_2 and λ_3, are multipliers to be eliminated. We do not include a du in σ because it can be replaced by α.

We now expand the dv^i by the chain rule and set the coefficients of all ten possible basis 2-forms $(dx\,dt, dx\,du, \ldots, dz\,dw)$ equal to zero. After elimination of the multipliers, we get the determining equations, where commas indicate differentiation,

$$\begin{aligned} {v^t}_{,w} &= 0 \\ {v^t}_{,x} + w{v^t}_{,z} + z{v^t}_{,w} &= {v^z}_{,w} - w{v^x}_{,w} \\ {v^z}_{,x} - v^w - w{v^x}_{,x} &= -w{v^z}_{,z} + w^2{v^x}_{,z} - z{v^z}_{,u} + wz{v^x}_{,u}\,. \end{aligned} \tag{34}$$

Solution of these equations together with Eqs. (31) and (32) now gives the usual symmetry group, or *isogroup*, for the one-dimensional heat equation.

Invariant variables are now found in the usual way by solving the equation(s)

$$\frac{dt}{v^t} = \frac{dx}{v^x} = \frac{du}{v^u}. \tag{35}$$

Geometrically, this gives the characteristics for the first-order differential equation $v \cdot \alpha = 0$, in which we restore $z = u_x$, $w = u_t$.

Mathematically, this is equivalent to the traditional method of finding the invariances of differential equations. So why spend time learning a new method?

(1) It is easy to apply. One reduces the set of differential equations to first-order equations by defining appropriate variables, writes them by inspection as differential forms, writes out the Lie derivative equations and sets the coefficients of the various basis forms to zero, and eliminates the multipliers. Calculations may be long because there may be many equations and many multipliers to eliminate, but they are very straightforward. In some cases many terms in the expansion drop out because of the antisymmetry of 1-forms, simplifying the treatment. This happens in the second example discussed below.

(2) One may get some geometrical insight into the process because of the inherently geometrical nature of forms. As an example, it leads immediately to the invariant surface condition used in finding nonclassical symmetries(see paper I, also [5]). For treatments that stress this geometrical nature, see Ref. [4] and [6]. (Forms may also be used in other, related contexts, such as searching for Bäcklund transformations, conservation laws, etc. [7])

(3) It allows the possibility for the independent variable components of the isovector v to be functions of the dependent variables. Usually these components of v are automatically assumed to be functions of only the independent variables, usually without loss of generality. However, in the case of a hodograph transformation, for example, those components do depend on the dependent variables.

(4) This method is nicely adaptable to computer algebraic calculations. As an example, Paul Kersten [8] developed a very nice treatment many years ago, for use in REDUCE. It enables one to set up the forms, find the determining equations, and then interactively work on their solution. Ben T. Langton [9], a Ph.D. student of Edward Fackerell's at the University of Sydney, is just finishing work on a modification of that technique which will improve its usefulness. There is also a MAPLE code which uses this technique [10]. See a brief discussion of these by Ibragimov [11].

(5) It is easy to make Ansatzen in the variable dependence of the isovector components, simply by specifying it when one writes out their exterior derivatives in the expansion of the Lie derivatives.

5 Short wave gas dynamic equation; symmetry reduction

This equation was cited by Fushchych in his Ansatz '95 talk [1] at the last Kyiv conference. It is his Eq. (4.4),

$$2u_{tx} - 2(2x + u_x)u_{xx} + u_{yy} + 2\lambda u_x = 0. \tag{36}$$

Only a brief mention of it, with a simple Ansatz, was given at that time. We give a longer treatment here.

We define new variables

$$w = u_x,\ z = u_y; \tag{37}$$

then Eq. (36) takes the form

$$2w_t - (4x + 2w)w_x + z_y + 2\lambda w = 0. \tag{38}$$

We could write a 1-form here, to be annulled:

$$-du + w\,dx + u_t\,dt + z\,dy,$$

but it involves u_t, which is not one of our variables. So we write a 2-form by multiplying this 1-form by dt in order to remove the unwanted term:

$$\alpha = (-du + w\,dx + z\,dy)\,dt. \tag{39}$$

Then $d\alpha$ is a 3-form:

$$d\alpha = (dw\,dx + dz\,dy)\,dt \tag{40}$$

and the equation itself is expressed as the 3-form

$$\beta = 2dw\,dx\,dy + (4x + 2w)dw\,dt\,dy + dz\,dt\,dx + 2\lambda w\,dt\,dx\,dy. \tag{41}$$

We note that $d\beta$ is proportional to $d\alpha$, so that the ideal $\{\alpha, d\alpha, \beta\}$ is closed.

We now consider

$$\pounds_v\,\alpha = \lambda\alpha \tag{42}$$

(β and $d\alpha$ are 3-forms and so are not included on the right-hand side.) In expansion of this equation, most terms involve only the t-component of the isovector, v^t, and they show simply that $v^t = K(t)$, a function of t only. The other terms involving dt then provide the only other useful information; elimination of the multiplier λ leaves four equations. These are conveniently written by defining $H = v^u - wv^x - zv^y$. Then they become

$$\begin{aligned} v^x &= -H_w, \quad v^z = H_y + zH^u, \\ v^y &= -H_z, \quad v^w = H_x + wH_u. \end{aligned} \tag{43}$$

v^u then can be written as

$$v^u = H - wH_w - zH_z. \tag{44}$$

We now note that $\pounds_v\,d\alpha = d\lambda \wedge \alpha + \lambda d\alpha$, thus being in the ideal, so we need not write a separate equation for it.

The remaining equation is

$$\pounds_v \beta = \nu\beta + \sigma d\alpha + \omega \wedge \alpha, \tag{45}$$

where ν and σ are 0-form multipliers and ω is a 1-form multiplier. We now write out the terms involving all 20 possible basis 3-forms and eliminate the multipliers. We see immediately that v^w and v^y are independent of u and z and that v^x is independent of u, z and w. Use of the equations shows that

$$H = -wA(x,y,t) - zB(y,t) + C(x,y,t,u) \tag{46}$$

with $v^x = A$ and $v^y = B$; A, B, and C are as yet undetermined. Eventually we find expressions for the generators, with two remaining equations, which are polynomials in z and w. We equate the polynomial coefficients to zero and get these expressions for the generators:

$$\begin{aligned}
v^t &= K, \\
v^x &= xJ - (y^2/4)(K'' + J') - yL' - N', \\
v^y &= (y/2)(K' + J) + L, \\
v^u &= u(2J - K') + G, \\
v^w &= w(J - K') + G_x, \\
v^z &= (3z/2)(J - K') + (wy/2)(K'' + J') + wL' + G_y,
\end{aligned} \tag{47}$$

where J, K, L and N are functions of t and G is a function of x, y, and t. Primes indicate d/dt. We also have

$$\begin{aligned}
J' &= (1/3)K'' - (4/3)(\lambda + 2)K', \\
0 &= 2G_{xt} - 4xG_{xx} + G_{yy} + 2\lambda G_x, \\
G_x &= N'' + 2N' - x(2K' + J') + y(L'' + 2L') + (y^2/4)(K''' + J'' + 2K'' + 2J'),
\end{aligned} \tag{48}$$

From these equations, we may solve for an explicit expression for G_{yy}. Then the consistency of $(G_{yy})_{,x}$ and $(G_x)_{,yy}$ yields the equation

$$(\lambda - 1)(\lambda - 4)K' = 0. \tag{49}$$

Thus, if $\lambda = 1$ or $\lambda = 4$, we may take K to be an arbitrary function of t.

We integrate the first of Eqs. (48) to get

$$J = (1/3)K' - (4/9)(\lambda + 2)K + 4a, \tag{50}$$

where a is a constant. We also define a new function, $\alpha(t)$, by

$$\frac{\alpha'}{\alpha} = -\frac{J + K'}{2K} = -\frac{2K'}{3K} + \frac{2\lambda + 4}{9} - \frac{2a}{K}. \tag{51}$$

We can also integrate for G, but the expression is long and we do not write it here.

We now find the invariant independent variables for the system. We have, as in the manner of Eq. (35),

$$\frac{dy}{dt} = \frac{v^y}{v^t} = -\frac{y\alpha'}{\alpha} + \frac{L}{K}. \tag{52}$$

Solution gives an invariant variable

$$\xi = y\alpha + \beta \tag{53}$$

where

$$\beta = -\int K^{-1} L\alpha \, dt. \tag{54}$$

From

$$\frac{dx}{dt} = \frac{v^x}{v^t} = -(\frac{2\alpha'}{\alpha} + \frac{K'}{K})x + \frac{y^2}{2K}(\frac{K\alpha'}{\alpha})' - \frac{yL' + N'}{K}, \tag{55}$$

we get the invariant variable

$$\eta = K\alpha^2 x - (1/2)K\alpha\alpha' y^2 - K\alpha\beta' y - \gamma, \tag{56}$$

where

$$\gamma = \int (-N'\alpha^2 + K\beta'^2) \, dt. \tag{57}$$

We now can use ξ and η as given in terms of K, α, β, and γ without ever referring back to L, N, etc.

From

$$\frac{du}{dt} = \frac{v^u}{v^t} = -(\frac{3K'}{K} + \frac{4\alpha'}{\alpha})u + \frac{G}{K}, \tag{58}$$

we get the invariant variable

$$F = K^3\alpha^4 u - \int K^2\alpha^4 G \, dt. \tag{59}$$

To get a solution of the original differential equation Eq. (36), we assume $F = F(\xi, \eta)$. To evaluate the integral in Eq. (59), we need to write out G, substitute for x and y in terms of ξ and η, do the t integrals while keeping ξ and η constant, and then reexpress ξ and η in terms of x and y. This is an extremely complicated procedure. One may simplify it, however, by noting, by inspection, that the completed integral will be a polynomial in ξ up to ξ^4, also with terms $\eta, \eta\xi, \eta\xi^2$, and η^2. If these are replaced by their expressions in x and y, these will yield terms in y up to y^4, also x, xy, xy^2, and x^2. So we write

$$\begin{aligned} u &= p_1 + p_2 y + p_3 y^2 + p_4 y^3 + p_5 y^4 + p_6 x \\ &\quad + p_7 xy + p_8 xy^2 + p_9 x^2 + K^{-3}\alpha^{-4} F(\xi, \eta), \end{aligned} \tag{60}$$

where the p_i are as yet undetermined functions of t, and substitute into Eq. (36). By using the expressions for ξ and η (Eqs. (53) and (56)), identifying coefficients, redefining F to include some polynomial terms in order to simplify the equation, we finally get–in which b is a new constant and a was defined in Eq. (50):

$$F_{\xi\xi} - 2F_\eta F_{\eta\eta} + 18aF_\eta + 4b\eta = 0, \tag{61}$$

$$(\lambda - 1)(\lambda - 4)K^2 = 9b + 81a^2, \tag{62}$$

(consistent with Eq. (49)), and

$$\begin{aligned}
p_9 &= -1 + \alpha'/\alpha + K'/(2K), \\
p_8 &= -(K\alpha)'/(2K\alpha), \\
p_7 &= -(K\beta')'/(K\alpha), \\
p_6 &= (\beta'^2/2 - \gamma'/K)/\alpha^2, \\
p_5 &= -(1/6)(p_8' + \lambda p_8 - 2p_8p_9 + b\alpha'/(K^2\alpha)), \\
p_4 &= -(1/3)(p_7' + \lambda p_7 - 2p_7p_9 + 2b\beta'/(K^2\alpha)), \\
p_3 &= -(p_6' + \lambda p_6 - 2p_6p_9 + 2b\gamma/(K^3\alpha^2)),
\end{aligned} \tag{63}$$

and p_1 and p_2 are arbitrary. These are expressed in terms of the functions K, α, β, and γ that occur in the invariant variables ξ and η.

6 Nonlinear heat equation with additional condition

This equation also was cited by Fushchych in Ansatz '95 [1], Eqs. (3.29) and (3.30), which are given here:

$$u_t + \nabla \cdot [f(u)\nabla u] = 0, \tag{64}$$

$$u_t + (2M(u))^{-1}(\nabla u)^2 = 0. \tag{65}$$

In Theorem 5 in that treatment, he showed that the first equation (here, Eq. (64)) is conditionally invariant under Galilei operators if the second equation (Eq. (65)) holds. Here we start with Eqs. (64) and (65) and study their joint invariance, a different problem.

If we define variables

$$q = u_x,\ r = u_y,\ s = u_z, \tag{66}$$

we can write Eqs. (64) and (65) as

$$u_t = -(2M)^{-1}(q^2 + r^2 + s^2) \tag{67}$$

and

$$-(2M)^{-1}(q^2 + r^2 + s^2) + (fq)_x + (fr)_y + (fs)_z = 0. \tag{68}$$

Now we can write forms as follows:

$$\alpha = -du - (2M)^{-1}(q^2 + r^2 + s^2)\,dt + q\,dx + r\,dy + s\,dz, \tag{69}$$

$$\begin{aligned}
\beta &= d\alpha + (M'/2M^2)(q^2 + r^2 + s^2)\alpha\,dt \\
&= (M'/2M^2)(q^2 + r^2 + s^2)(q\,dx + r\,dy + s\,dz)\,dt \\
&\quad -(1/M)(qdq + rdr + sds)\,dt + dq\,dx + dr\,dy + ds\,dz
\end{aligned} \tag{70}$$

which combination gets rid of the du terms, and

$$\gamma = g(u)(q^2 + r^2 + s^2)dx\,dy\,dz\,dt + (dq\,dy\,dz + dr\,dz\,dx + ds\,dx\,dy)\,dt, \tag{71}$$

where

$$g(u) = f^{-1}(f' - (2M)^{-1}) \tag{72}$$

and in all of which a prime indicates d/du. Now we assume that v^x, v^y, v^z, and v^t are functions of x, y, z, and t; v^u is a function of u, x, y, z, and t; and that v^q, v^r, and v^s are linear in q, r, and s (with additional terms independent of those variables), and with coefficients which are functions of u, x, y, z, and t.

We now consider

$$\pounds_v\, \alpha = \lambda\alpha. \tag{73}$$

Expansion of this equation, using the conditions stated above, shows that $\lambda = v^u{}_{,u}$; it has terms up to the quadratic in q, r, and s. Setting the coefficients of the terms in these variables equal to zero and solving yields the expressions

$$\begin{aligned} v^t &= \eta(t), \\ v^x &= \xi x + \sigma_3 y - \sigma_2 z + \zeta_1 t + \nu_1, \\ v^y &= \xi y + \sigma_1 z - \sigma_3 x + \zeta_2 t + \nu_2, \\ v^z &= \xi z + \sigma_2 x - \sigma_1 y + \zeta_3 t + \nu_3, \end{aligned} \tag{74}$$

where the σ_i, ζ_i, and ν_i are constants, ξ is linear in t, and $\eta(t)$ is quadratic in t. v^q, v^r, v^s, and v^u are quadratic in x, y, and z. The u dependence is not yet entirely determined because $M(u)$ and $f(u)$ have not been specified.

From Eq. (73) we have, as before, $\pounds_v\, d\alpha = \lambda d\alpha + d\lambda \wedge \alpha$, automatically in the ideal.

The remaining calculation is the determination of $\pounds_v\, \gamma$ and setting it equal to a linear combination of $\alpha, d\alpha$ (or β), and γ, and to eliminate the multipliers. This appears to be a formidable task; however, by use of the assumptions and information we already have, it turns out to be surprisingly easy.

We expand $\pounds_v\, \gamma$ and substitute the values for the v^i that we already have. After this calculation, we find that there are terms proportional to $dx\, dy\, dz\, dt$, $du\, dy\, dz\, dt$, $du\, dz\, dx\, dt$, $du\, dx\, dy\, dt$, and $(dq\, dy\, dz + dr\, dz\, dx + ds\, dx\, dy)\, dt$. We substitute for the latter sum of three terms from Eq. (71), thus giving a term proportional to γ and one proportional to $dx\, dy\, dz\, dt$. In the three terms involving du, we substitute for du from Eq. (69). It is now seen that $\pounds_v\, \gamma$ is the sum of three terms: one proportional to γ, one proportional to α, and one proportional to $dx\, dy\, dz\, dt$. Thus, $\pounds_v\, \gamma$ is already in the desired form except for the last term! But this last term cannot be represented as a sum of terms in α, β, and γ, as is seen by inspection. Hence its coefficient must vanish, and that condition provides the remaining equations for the generators v^i.

The coefficient has terms proportional to $q^2 + r^2 + s^2$, q, r, s, and a term independent of those variables. Setting this last term equal to zero shows that $\xi_t = 0$. The q, r, and s terms give equations which can be written collectively as

$$\zeta_i(M' + gM) = 0, \quad i = 1, 2, 3. \tag{75}$$

The coefficient of $q^2 + r^2 + s^2$ gives a relation among the various functions of u. We may now summarize the results.

In addition to Eqs. (74), with ξ now constant, we get

$$\eta(t) = at + b, \tag{76}$$

where a and b are constant, and

$$\begin{aligned} v^u &= M(\zeta_1 x + \zeta_2 y + \zeta_3 z) + h(u), \\ v^q &= qJ + \sigma_3 r - \sigma_2 s + \zeta_1 M, \\ v^r &= rJ + \sigma_1 s - \sigma_3 q + \zeta_2 M, \\ v^s &= sJ + \sigma_2 q - \sigma_1 r + \zeta_3 M, \end{aligned} \tag{77}$$

where

$$J = M'(\zeta_1 x + \zeta_2 y + \zeta_3 z) + h' - \xi. \tag{78}$$

The quadratic dependence on x, y, and z has dropped out because of the condition $\xi_t = 0$. We also have these relations among the functions of u: Eqs. (72), (75) and the further equations

$$h' = hM'/M + 2\xi - a, \tag{79}$$

$$[hM^{-1}(M' + gM)]' = 0. \tag{80}$$

There are now two cases.

Case I. If

$$M' + gM = 0, \tag{81}$$

then

$$fM = u/2 + c, \tag{82}$$

where c is a constant. Integration for h from Eq. (79) now gives

$$h = kM + (2\xi - a)M \int M^{-1} du, \tag{83}$$

where k is a constant.

Case II. If $M' + gM \neq 0$, then all $\zeta_i = 0$. We may write h by Eq. (83), but now h must satisfy the additional condition Eq. (80).

Fushchych's Theorem 5 now follows: if $M = u/(2f)$, the original equations are invariant under a Galilean transformation

$$G_i = t\partial_i + Mx^i\partial_u. \tag{84}$$

The general operator is then $G = \sum_i \zeta_i G_i$, giving $v^i = \zeta_i t$ and $v^u = M(\zeta_1 x + \zeta_2 y + \zeta_3 z)$, a possible choice of generators from the above calculation. From Eq. (82) we see that his conclusion thus holds in a slightly more general case, when the constant c is not zero.

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