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Non-local Symmetry of the 3-Dimensional Burgers-Type Equation

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Abstract

Non-local transformation, which connects the 3-dimensional Burgers-type equation with a linear heat equation, is constructed. Via this transformation, nonlinear superposition formulae for solutions are obtained and the conditional non-local symmetry of this equation is studied.

The multidimensional generalization of the Burgers equation

$$L_1(u) = u_0 - u|\nabla u| - \triangle u = 0, \tag{1}$$

is called further the Burgers-type equation. This equation was suggested by W. Fushchych in [1]. We use here such notations:

$$\partial_\mu u = \frac{\partial u}{\partial x_\mu}, \qquad \{x_\mu\} = (x_0, x_1, ..., x_{n-1}),$$
$$\nabla u = \|u_1, u_2, ..., u_n\|^T, \quad (\mu = 0, n-1),$$
$$|\nabla u| = \sqrt{(\nabla u)^2}, \qquad \triangle = (\nabla)^2 = \partial_1^2 + \partial_2^2 + \cdots + \partial_{n-1}^2.$$

In the present paper, we construct the non-local transformation, which connects the 3-dimensional equation (1) with a linear heat equation. Via this transformation, we obtain nonlinear superposition formulae for solutions of equation (1). Also we investigate the conditional non-local symmetry of equation (1) and obtain formulae generating solutions of this equation.

1. Conditional non-local superposition

Let us consider the 3-dimensional scalar heat equation

$$L_2(w) = w_0 - \triangle w = 0. \tag{2}$$

For the vector-function H, such equations are fulfiled:

$$H = 2\nabla \ln w, \qquad H = \|h^1, h^2, h^3\|^T, \tag{3}$$

$$\frac{1}{2}(H)^2 + (\nabla \cdot H) = \partial_0 \ln w, \qquad \nabla \times H = 0. \tag{4}$$

From the integrability condition for equations (3), (4), it follows that equation (2) is connected with the vector equations

$$H_0 - \frac{1}{2}(H)^2 - \nabla(\nabla \cdot H) = 0, \qquad \nabla \times H = 0. \tag{5}$$

Let $|H|^2 = (h^1)^2 + (h^2)^2 + (h^3)^2 = u^2$ for $H = \theta \cdot u$, where $|\theta| = 1$, $\theta = |\theta^1, \theta^2, \theta^3|^T$. Then we obtain from (5) that $\nabla \times \theta = 0$ and the equality

$$\theta\left[u_0 - u|\nabla u| - \triangle u\right] = u\left[\theta_0 - 2\nabla \ln u(\nabla \cdot \theta) - \nabla(\nabla \cdot \theta)\right].$$

Let relations

$$L_1(u) = u_0 - u|\nabla u| - \triangle u = 0, \tag{6}$$

$$\begin{aligned} &\theta_0 - 2\nabla \ln u(\nabla \cdot \theta) - \nabla(\nabla \cdot \theta) = 0, \\ &\nabla \times \theta = 0. \end{aligned} \tag{7}$$

be fulfiled on the some subset of solutions of equation (5). System (3), (4) in new variables, which connect equations (2) and (6), has the form

$$\nabla \ln w = \frac{1}{2}\theta u, \tag{8}$$

$$\partial_0 \ln w = \frac{1}{4}u^2 + \frac{1}{2}(\nabla \cdot \theta u). \tag{9}$$

So via transformations (8), (9), PDE (6) is reduced to the linear equation (2). The corresponding generalization of the Cole-Hopf transformation (substitution) is obtained in the form

$$u = 2\sqrt{(\nabla \ln w)^2}. \tag{10}$$

Theorem 1. *The non-local substitution (10) is a linearization of the 3-dimensional Burgers-type equation (1), and the operator equality*

$$\partial_0(\nabla \ln w) - 2(\nabla \ln w) \cdot \triangle \ln w - \nabla(\triangle \ln w) = [w^{-1}\nabla - w^{-2} \cdot \nabla w] \cdot (w_0 - \triangle w) = 0$$

is fulfiled.

Example 1. The function

$$u = 2\sqrt{[\nabla \ln \varphi(x_0, \omega)]^2}, \qquad \omega = \alpha \cdot x = \alpha_a x_a, \quad (a = 1, 2, 3), \tag{11}$$

is a non-Lie solution of the nonlinear equation (1). Let us substitute this non-local ansatz into equation (1). We find the condition on φ:

$$[\varphi^{-1} \cdot \partial_\omega - \varphi^{-2} \cdot \varphi_\omega] \cdot [\varphi_0 - \varphi_{\omega\omega}] = 0. \tag{12}$$

So, we obtain the non-local reduction of equation (1) to the form (12). Analogously, with the ansatz

$$u = 2\sqrt{[\nabla \ln\{2(n-1)x_0 + x\}]^2}, \quad x \equiv (x_1, x_2, x_3), \quad (n = 4), \tag{13}$$

one obtains

$$[w^{-1}\nabla - w^{-2}\nabla w] \cdot [\partial_0 - \triangle] \cdot \{2(n-1)x_0 + x\}. \tag{14}$$

Here, $w \equiv 2(n-1)x_0 + x$.

The linearization of the differential equation (1) makes it possible via a linear superposition of solutions of equation (2) and the non-local transformation (8), (9) to construct the principle of nonlinear superposition for solutions of equation (1).

Theorem 2. *The superposition formula for a subset of solutions* $\overset{(k)}{u}(x)$, $(k = 1,2)$ *of equation (1) has the form*

$$\begin{aligned}
&\overset{(3)}{u}(x) = 2\left[\frac{\overset{(1)}{\tau}\overset{(1)}{u}\overset{(1)}{\theta} + \overset{(2)}{\tau}\overset{(2)}{u}\overset{(2)}{\theta}}{\overset{(1)}{\tau} + \overset{(2)}{\tau}}\right],\\
&\overset{(3)}{\theta}\overset{(3)}{u}(x) = 2\left(\overset{(1)}{\tau} + \overset{(2)}{\tau}\right)^{-1}\left(\overset{(1)}{\tau}\overset{(1)}{u}\overset{(1)}{\theta} + \overset{(2)}{\tau}\overset{(2)}{u}\overset{(2)}{\theta}\right),\\
&\overset{(k)}{\tau_0} = \frac{1}{2}\overset{(k)}{\tau}\left[\overset{(k)}{u}{}^2 + (\nabla\cdot\overset{(k)}{\theta}\overset{(k)}{u})\right],\\
&\nabla\overset{(k)}{\tau} = \frac{1}{2}\overset{(k)}{\tau}\overset{(k)}{u}\overset{(k)}{\theta}, \nabla\times\overset{(k)}{\theta},\\
&\partial_0\ln(\overset{(1)}{\tau} + \overset{(2)}{\tau}) = \left[\frac{\overset{(1)}{\tau}\overset{(1)}{u}\overset{(1)}{\theta} + \overset{(2)}{\tau}\overset{(2)}{u}\overset{(2)}{\theta}}{\overset{(1)}{\tau} + \overset{(2)}{\tau}}\right] + \left[\nabla\frac{\overset{(1)}{\tau}\overset{(1)}{u}\overset{(1)}{\theta} + \overset{(2)}{\tau}\overset{(2)}{u}\overset{(2)}{\theta}}{\overset{(1)}{\tau} + \overset{(2)}{\tau}}\right].
\end{aligned} \tag{15}$$

Here, $\overset{(k)}{u}$, $(k = 1,2)$ *are known solutions of equation (1),* $\overset{(s)}{u}$, $(s = 1,2,3)$ *and* $\overset{(k)}{\tau}, (k = 1,2)$ *are functional parameters, and* $\overset{(k)}{\tau_0} - \triangle\overset{(k)}{\tau} = 0$.

2. Conditional non-local invariance

Let us consider the potential hydrodynamic-type system in $R(1,3)$ of independent variables

$$H_0 + (H,\nabla)H - \triangle H = 0, \qquad \nabla\times H = 0, \tag{16}$$

or, in the form (5),

$$H_0 + \frac{1}{2}\nabla(H)^2 - \nabla(\nabla\cdot H) = 0, \qquad \nabla\times H = 0. \tag{17}$$

Let $|H| = u$ or, in other words

$$|H|^2 = (H^1)^2 + (H^2)^2 + (H^3)^2 = u^2.$$

Then we have such equalities:

$$H = \theta\cdot|H| = \theta\cdot u, \qquad (\theta = H\cdot|H|^{-1}). \tag{18}$$

So, θ is a unit vector collinear with H. Let $\nabla \times \theta = 0$. It makes one possible to obtain

$$\begin{aligned}
&[\nabla \times \theta u] = u \cdot [\nabla \times \theta] + [\nabla u \times \theta] \longrightarrow \nabla u \times \theta = 0, \\
&\nabla u = \theta \cdot |\nabla u|, \qquad (\nabla \cdot \theta u) = u(\nabla \cdot \theta) + \theta \cdot \nabla u, \\
&\nabla(\nabla \cdot \theta u) = u \cdot \nabla(\nabla \cdot \theta) + 2\nabla u \cdot (\nabla \cdot \theta) + \theta \triangle u.
\end{aligned} \tag{19}$$

Substituting H of the form (18) into equation (17), we find the operator equation

$$\theta[u_{0+}u|\nabla u| - \triangle u] = -u[\theta_0 - 2\nabla \ln u(\nabla \cdot \theta) - \nabla(\nabla \cdot \theta)].$$

Let now pick out the subset of solutions of this equation, which consists of solutions of the system

$$u_{0+}u|\nabla u| - \triangle u = 0, \tag{20}$$

$$\theta_0 - 2\nabla \ln u(\nabla \cdot \theta) - \nabla(\nabla \cdot \theta) = 0. \tag{21}$$

Variables $\theta = \| \theta^1, \theta^2, \theta^3 \|^T$ are playing the role of supplementary parameters. Notice that, for a known solution u of equation (20), equation (21) is linear.

Let us take another copy of equation (17):

$$Q_0 + \frac{1}{2}\nabla(Q)^2 - \nabla(\nabla \cdot Q) = 0, \qquad \nabla \times Q = 0. \tag{22}$$

Let $Q = \tau \cdot w$, where $w = |Q|$ and $\nabla \times \tau = 0$. The differential equation (22) can be substituted now by the system

$$w_0 + w|\nabla w| - \triangle w = 0, \tag{23}$$

$$\tau_0 - 2\nabla \ln w \cdot (\nabla \cdot \tau) - \nabla(\nabla \cdot \tau) = 0. \tag{24}$$

Assume now such equalities:

$$\begin{aligned}
&-2\nabla \ln w = H - Q, \\
&-\partial_0 \ln w = \frac{1}{2}(H \cdot Q) - \left[\frac{1}{2}(H)^2 - (\nabla \cdot H)\right].
\end{aligned} \tag{25}$$

Excluding w from this system by cross-differentiation, we obtain

$$H_0 + \frac{1}{2}\nabla(H)^2 - \nabla(\nabla \cdot H) = Q_0 + \frac{1}{2}\nabla(H \cdot Q). \tag{26}$$

So, with the condition

$$Q_0 + \frac{1}{2}\nabla(H \cdot Q) = 0, \tag{27}$$

we get equation (17). Let

$$\frac{1}{2}(H \cdot Q) = \frac{1}{2}(Q)^2 - (\nabla \cdot Q). \tag{28}$$

Then (27) becomes equation (22). As one can see, condition (28) is the necessary condition for reducing equations (20), (21) via the non-local transformation (25), (28) to system (23), (24). Substituting

$$H = \tau \cdot w - 2\nabla \ln w$$

into (28), one obtains

$$w_0 + w \cdot |\nabla w| - \triangle w = \frac{1}{2} w\{|\nabla w| - (\nabla \cdot Q)\}.$$

Let us put the condition

$$|\nabla w| = (\nabla \cdot Q). \tag{29}$$

Since

$$(\nabla \cdot \tau w) = w(\nabla \cdot \tau) + |\nabla w|,$$

it follows from condition (29) that

$$(\nabla \tau) = -4|\nabla \ln w|.$$

Introducing notations $u \equiv \overset{(2)}{u}$, $w \equiv \overset{(1)}{u}$, we can formulate such a theorem:

Theorem 3. *Let* $\overset{(1)}{u}$ *be a known solution of equation (1)*

$$u_0 + u \cdot |\nabla u| - \triangle u = 0.$$

Then its new solution $\overset{(2)}{u}$ *is defined by the formulae*

$$\overset{(2)}{u} = (\theta \cdot \tau) \overset{(1)}{u} - 2 \cdot \theta \cdot \nabla \ln \overset{(1)}{u},$$

$$\frac{1}{2}(\theta \cdot \tau) \overset{(1)}{u}\overset{(2)}{u} + 2\partial_0 \ln \overset{(1)}{u} = \frac{1}{2} \overset{(2)}{u}{}^2 - (\nabla \cdot \theta \overset{(2)}{u}),$$

$$\theta \overset{(2)}{u} = \tau \overset{(1)}{u} - 2 \cdot \nabla \ln \overset{(1)}{u},$$

$$\theta_0 - 2 \cdot \nabla \ln \overset{(2)}{u} (\nabla \cdot \theta) - \nabla(\nabla \cdot \theta) = 0,$$

$$\tau_0 - 2 \cdot \nabla \ln \overset{(1)}{u} (\nabla \cdot \tau) - \nabla(\nabla \cdot \tau) = 0,$$

$$\nabla \times \theta = 0, \quad \nabla \times \tau = 0, \quad (\nabla \cdot \tau) = -4|\nabla \ln w|.$$

The last equalities are additional conditions for the non-local invariance of equation (1) with respect to the non-local transformation (25), (26).

References

[1] Fushchych W.I., Symmetry in mathematical physics problems, in: Algebraic-Theoretical Studies in Mathematical Physics, Inst. of Math. Acad. of Sci. Ukraine, Kyiv, 1981, 6–28.