About solutions of an initial equation adjoint to its known solutions

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A concept of solution of initial equation adjoint to its known solutions are developed and used for searching new symmetries of differential equations.

Suppose a given nonlocal transformation

$$\begin{array}{lll} \mathcal{T}: & x^i = h^i(y, v_{(k)}), & u^K = H^K(y, v_{(k)}), \\ & i = 1, \ldots, n, & K = 1, \ldots, m. \end{array} \tag{1}$$

maps an *initial* equation

$$F_0(x, u_{(n)}) = 0 \tag{2}$$

into the equation $\Phi(y, v_{(q)}) = 0$ of order $q = n+k$, which admits a factorization to another, we call them a *target*, equation

$$F_1(y, v_{(s)}) = 0, \tag{3}$$

i.e.,

$$\Phi(y, v_{(q)}) = \lambda F_1(y, v_{(s)}). \tag{4}$$

λ is a differential operator of order $n + k - s$. Above results in algorithms for finding solutions of (2) from known ones of (3). An existence of the factorization equation (4) give rise to technique of finding a special solution to the initial equation (2). Further we name it *adjoint*.

Let's assume, that a given function $v = f(y)$ *is not a solution* of equation (3), that is, substituting this function into (3), we get the equation

$$F_1(y, v_{(s)}) = w(y). \tag{5}$$

Suppose, nevertheless, that equation (4) holds and the PDE

$$\lambda(y, v_{(s)})w(y) = 0. \tag{6}$$

appears true. Here w runs through the solution set of a *linear* equation with variable coefficients of spatial form. Solving this equation with respect to unknown function w, one can find its solution as a function depending on $y, v_{(k)}(y)$

$$w = W(y, v_{(k)}(y)). \tag{7}$$

After substitution of this expression $w = W(y, v(y))$ into the equation (5) we obtain an *inhomogeneous* equation for dependent variable v in the form

$$F_1(y, v_{(s)}) = W(y, v_{(k)}(y)). \tag{8}$$

A function $v(y)$ determined by this equation is a solution of equation (4). Substituting obtained $v(y)$ into the formulae of a nonlocal transformation $\mathcal{T}$ one can find appropriate solution of a given equation (2). Moreover, having the information on point symmetries of the inhomogeneous equation (8), one can construct r-parametrical family of solutions to it and, consequently, find the corresponding parametrical sets of solutions to Eq. (2). We illustrate this construction by some examples.