Lie symmetries of fundamental solutions of the Kramers equation

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In our talk the equation

$$u_t = u_{yy} - yu_x + (\omega^2 x + y)u_y + u \tag{1}$$

is considered. In this equation we use the following designations: $u = u(t,x,y)$ is the unknown function to be found, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_{yy} = \frac{\partial^2 u}{\partial y^2}$; ω is an arbitrary real constant fulfilling the condition $\omega^2 < \frac{1}{4}$.

Note that equation (1) is the particular case of the known Kramers equation [1]

$$u_t = u_{yy} - (yu)_x + ((V'(x) + y)u)_y,$$

describing the motion of a particle in the fluctuating environments with the external potential $V(x)$.

By using the Berest–Aksenov algorithm [2, 3], the Lie invariance algebra of fundamental solutions of equation (1) was found, namely, it was proved the statement.

Theorem 1. *The equation*

$$u_t - u_{yy} + yu_x - (\omega^2 x + y)u_y - u = \delta(t,x,y), \tag{2}$$

defining the fundamental solutions of equation (1) ($\delta = \delta(t,x,y)$ is the delta-function) admits a non-trivial two-dimensional Lie algebra of invariance, which is spanned by the generators:

$$\begin{aligned}
Y_1 &= \left\{(\mu_1 - \mu_2)\, e^{\mu_1 t} - e^{-\mu_1 t} - 2\mu_1 e^{-\mu_2 t}\right\}\partial_x + \left\{\left(\mu_1^2 - \omega^2\right) e^{\mu_1 t} + \mu_1 e^{-\mu_1 t} + 2\omega^2 e^{-\mu_2 t}\right\}\partial_y + \\
&\quad + \left\{\left(\omega^2 e^{-\mu_1 t} + 2\mu_1\omega^2 e^{-\mu_2 t}\right)x - \left(\mu_1 e^{-\mu_1 t} + 2\omega^2 e^{-\mu_2 t}\right)y\right\}u\partial_u,\\
Y_2 &= \left\{(\mu_1 - \mu_2)\, e^{\mu_2 t} + 2\mu_2 e^{-\mu_1 t} + e^{-\mu_2 t}\right\}\partial_x + \left\{\left(\omega^2 - \mu_2^2\right) e^{\mu_2 t} - 2\omega^2 e^{-\mu_1 t} - \mu_2 e^{-\mu_2 t}\right\}\partial_y + \\
&\quad + \left\{-\left(2\mu_2\omega^2 e^{-\mu_1 t} + \omega^2 e^{-\mu_2 t}\right)x + \left(2\omega^2 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right)y\right\}u\partial_u,
\end{aligned}$$

where μ_i $(i = 1, 2)$ are the roots of the equation $\mu^2 + \mu + \omega^2 = 0$.

The fundamental solution of equation (1) was found in explicit form by Chandrasekhar in his famous article [4]:

$$u = \frac{\theta(t)\, e^t}{2\pi\sqrt{\Delta}} \cdot \exp\left\{-\frac{A(t)\, x^2 + B(t)\, xy + C(t)\, y^2}{2\Delta}\right\}, \tag{3}$$

where $\theta = \theta(t)$ is the Heaviside function; the functions $A(t)$, $B(t)$, $C(t)$, and Δ are equal respectively

$$A(t) = 4\omega^2 e^t + (1 - 4\omega^2)\, e^{2t} + \mu_1 e^{-2\mu_2 t} + \mu_2 e^{-2\mu_1 t},\ B(t) = 4e^t - 2e^{-2\mu_2 t} - 2e^{-2\mu_1 t},$$

$$C(t) = 4e^t + \frac{1 - 4\omega^2}{\omega^2}\, e^{2t} + \frac{1}{\mu_1}\, e^{-2\mu_2 t} + \frac{1}{\mu_2}\, e^{-2\mu_1 t},\ \Delta = 8e^t + \frac{1 - 4\omega^2}{\omega^2}(e^{2t} + 1) - \frac{1}{\omega^2}\left(e^{-2\mu_1 t} + e^{-2\mu_2 t}\right).$$

It is easy to prove that the fundamental solution (3) is invariant with respect to the generators Y_1 and Y_2. Hence, the following statement takes place.

Theorem 2. *The fundamental solution (3) of equation (1) is invariant with respect to the two-parametric group of point transformations corresponding to the Lie algebra $\langle Y_1, Y_2\rangle$ of the symmetry generators of equation (2).*

This theorem gives us the possibility to construct the fundamental solution (3) of equation (1) as the weak invariant solution of equation (2).

References

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