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NONLINEAR TRANSFORMATIONS OF SMOOTH MEASURES ON INFINITE-DIMENSIONAL SPACES

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We investigate the properties of the image of a differentiable measure on an infinitely-dimensional Banach space under nonlinear transformations of the space. We prove a general result concerning the absolute continuity of this image with respect to the initial measure and obtain a formula for density similar to the Ramer – Kusuoka formula for the transformations of the Gaussian measure. We prove the absolute continuity of the image for classes of transformations that possess additional structural properties, namely, for adapted and monotone transformations, as well as for transformations generated by a differential flow. The latter are used for the realization of the method of characteristics for the solution of infinite-dimensional first-order partial differential equations and linear equations with an extended stochastic integral with respect to the given measure.

Introduction

The problem of absolute continuity of the image of a measure on an infinite-dimensional space under nonlinear transformations has a rich history. One of the first results in this direction was obtained by Gikhman and Skorokhod for transformations of measures on a Hilbert space (see [1]; see also Theorem 3 in [2, Chap. 3]). The density formula obtained in the works cited above in the special case of a measure on a finite-dimensional space is an immediate consequence of an ordinary formula for a change of variables. The application of this formula for obtaining infinite-dimensional results requires the approximation of the original transformation by a finite-dimensional one and the proof of the convergence of the sequence of the corresponding densities.

Conditions that guarantee this convergence in a more general and compact form were obtained for the transformations of Gaussian measures (see [3, 4] and the survey [5]). The available results for non-Gaussian measures are considerably weaker and, as a rule, they require additional strong assumptions concerning the structure of a measure or a transformation [2, 6, 7]. Significant difficulties arise even if one tries to formulate conditions of quasiinvariance of a differentiable measure, which are close to necessary conditions.

In the present paper, we obtain general conditions for a transformation of a space that are sufficient for the absolute continuity of the image of a non-Gaussian differentiable measure. The basic assumption concerning the original measure (Condition **B**, Sec. 2) is somewhat restrictive. However, on the one hand, this condition can easily be verified and, on the other hand, it guarantees the realization of moment estimates, which allows us to pass to the limit. Note that the direct verification of estimates of this type often meets serious difficulties.

Parallel with the general results, we obtain results concerning the absolute continuity of the image of a measure for transformations that possess additional structural properties, namely, for nonanticipative (a non-Gaussian analog of the Girsanov theorem) and monotone transformations, as well as for transformations generated by a differential flow. The latter transformations are used for the realization of the method of characteristics for the solution of the first-order partial differential equations in an infinite-dimensional space and linear equations with a generalized stochastic integral with respect to a given measure.

1. Smooth Measures and Sobolev Classes of Transformations on Infinite-Dimensional Spaces

Assume that X is a real separable Banach space, H is a separable Hilbert space imbedded into X by a continuous operator $j\colon H \hookrightarrow X$, and μ is a probability measure on the Borel σ-algebra $\mathcal{B}(X)$ differentiable along the

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directions from jH [8]. We assume that the logarithmic derivative $\{(\rho, h), h \in H\}$ of the measure μ has a weak order p for any $p < +\infty$, i.e.,

$$\forall p \in R^+ \ \exists C_p < +\infty : \int |(\rho, h)|^p d\mu \le C_p \|h\|_H^p, \qquad h \in H. \tag{1}$$

Definition 1. *The stochastic derivative* $D_{p,h}$, $p \in (1, +\infty)$, *along the direction* $h \in H$ *is defined as the closure in the space* $L_p(X, \mu)$ *of the operator*

$$\nabla_h : f \mapsto (\nabla f, jh)$$

defined on the set of functions f *continuous and bounded together with their Fréchet derivatives* ∇f. *By virtue of the differentiability of the measure* μ, *this closure is well defined [5].*

Definition 2. *A function* $f \in L_p(X, \mu)$ *belongs to the Sobolev space* $W_p^1(X, \mu)$ *if the following conditions are satisfied:*

(i) for any $h \in H$, *the derivative* $D_{p,h} f$ *is defined;*

(ii) there exists a random element $g_f \in L_p(X, H, \mu)$ *such that*

$$D_{p,h} f = (g_f, h)_H \quad \text{a.e.}, \qquad h \in H.$$

The element g_f *is called the stochastic derivative of the function* f *and is denoted by* $D_p f$. *If this does not lead to misunderstanding, we omit the index* p *in the expressions* $D_{p,h} f$ *and* $D_p f$.

The stochastic derivatives

$$D_{h,p}^E : L_p(X, E, \mu) \to L_p(X, E, \mu) \qquad \text{and} \qquad D_{h,p}^E : L_p(X, E, \mu) \to L_p(X, E \otimes H, \mu)$$

and the Sobolev spaces $W_p^1(X, E, \mu)$ for elements that take values in the separable Hilbert space E are defined by analogy. Multiple stochastic derivatives and Sobolev spaces of higher order $W_p^n(X, E, \mu)$ are defined iteratively.

Definition 3. *The adjoint operator*

$$I_p^E = \left(D_p^E\right)^* : L_q(X, E \otimes H, \mu) \to L_p(X, E, \mu), \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

is called the operator of stochastic integration or the generalized stochastic integral by analogy with the generalized stochastic Skorokhod integral with respect to a Wiener process.

The domain of definition of the operator I_p^E contains, in particular, elements of the form

$$g = \sum_{k=1}^{n} g_k \otimes h_k, \quad g_k \in W_{q'}^1(X, E, \mu), \quad h_k \in H, \quad k = \overline{1, n}, \quad q' > q. \tag{2}$$

For these elements, we have

$$I_p^E(g) = -\sum_{k=1}^{n}(\rho, h_k)g_k - \sum_{k=1}^{n}(D_{q'}^E g_k, h_k)_H. \tag{3}$$

In the case $E = \mathbb{R}$, we write $I_p^E = I_p$.

Definition 4. *A function* $f \in L_\infty(X, E, \mu)$ *belongs to the class* $W_\infty^k(X, E, \mu)$, $k \geq 1$, *if the following conditions are satisfied:*

(i) $f \in \bigcap_p W_p^k(X, E, \mu)$;

(ii) *the functions* $f, Df, \ldots, D^k f$ *are essentially bounded random elements with values in* E, $E \otimes H, \ldots,$ $E \otimes H^{\otimes k}$, *respectively.*

Definition 5. *A function* f *belongs to the class* $WL_\infty^k(X, E, \mu)$ *if* $f \in W_\infty^k(X, E, \mu)$ *and, for* $l = 0, \ldots, k-1$, *there exists a modification* $\widetilde{D^l f}$ *of the derivative* $D^l f$ *that satisfies the Lipschitz condition along* H, *i.e., is such that*

$$\left\| \widetilde{D^l f}(x) - \widetilde{D^l f}(y) \right\|_{E \otimes H^{\otimes l}} \leq L_l \|x - y\|_H, \quad x, y \in X. \tag{4}$$

Remark 1. A function that satisfies condition (4), belongs to the class $W_\infty^k(X, E, \mu)$. Let us verify this fact for $k = 1$. We fix $p \in \mathbb{R}^+$, $h \in H$, and show that the derivative $D_{p,h} f$ is defined. We decompose the space X into the sum $X = X_0 + \langle jh \rangle$. This relation induces the decomposition $\mu = \pi_0 \otimes \{\mu_x, x \in X_0\}$, where π_0 is the projection of the measure μ onto X_0 and $\{\mu_x, x \in X_0\}$ is a family of conditional measures concentrated on the straight lines $\{x + tjh, t \in \mathbb{R}\}$. By virtue of the one-dimensional Rademacher theorem [9], for π_0-almost all $x \in X_0$ the derivative $D_{p,h}^x f$ of the section $f|_{x+\langle jh \rangle}$ is defined and μ_x-almost everywhere bounded in the norm $\|\cdot\|_E$ by the constant $L_0 \|h\|_H$. This implies that the derivative $D_{p,h} f$ is defined and bounded almost everywhere in the norm by the constant $L_0 \|h\|_H$. Then [10] condition (ii) in Definition 2 is satisfied, i.e., the stochastic derivative $D_p^E f$ is defined and $\left\| D_p^E f \right\|_{E \otimes H} \leq L_0$ almost everywhere. Repeating this argument, we establish that the required statement is true for $k \in \mathbb{N}$.

The authors do not know whether the converse implication is true, i.e., whether the classes W_∞^k and WL_∞^k coincide. In certain special cases, we can use the following reasoning: for $f \in W_\infty^k$, we can construct a sequence $\{f_n\}$ of smooth cylindrical functions whose derivatives are bounded almost everywhere by the same constants as the corresponding derivatives of f and, furthermore, $f_n \xrightarrow[n \to \infty]{} f$ in W_p^k, $p \geq 1$, and then we can set $\widetilde{D^l f}(x) = \lim_{n \to \infty} D^l f_n(x)$.

This method of the proof of the equality $W_\infty^k = WL_\infty^k$ was realized in [11] in the important case of a Gaussian measure μ; in this case, $\{f_n\}$ can be chosen as a sequence of projections on finite-dimensional linear σ-algebras. A similar result is true for measures that are close, in a certain sense, to product measures [12]. For logarithmically convex measures, the equality $W_\infty^k = WL_\infty^k$ can be proved by analogy with the proof of Lemma 5 in [13].

Remark 2. Arguments similar to the proof of Lemma 3 in [2, Chap. 2] show that, for the derivative of a function $f \in W_\infty^k(X, E, \mu)$ and any basis $\{e_n\}$ in H, there exist modifications $\{\widetilde{\widetilde{D^l f}}, l = 0, \ldots, k-1\}$ that satisfy the Lipschitz condition along $\tilde{H}$, which is the linear span of $\{e_n\}$, with the constants $\{\operatorname{ess\,sup} \|D^{l+1} f\|, l = 0, \ldots, k-1\}$, respectively. If condition (4) is satisfied, then the modifications $\widetilde{D^l f}$ can be chosen so that

$$L_l = \operatorname{ess\,sup} \left\| D^{l+1} f \right\|, \quad l = 0, \ldots, k-1.$$

2. General Formula for a Change of Variables for Smooth Measures

In what follows, we assume that the measure μ satisfies the following condition:

B. For any $h \in H$, the logarithmic derivative (ρ, h) belongs to $\bigcap_p W_p^1(X, \mu)$ and there exists a bounded random operator [10] that acts in H and is such that $\beta_\mu = \operatorname{ess\,sup} \|B\|_{\mathfrak{L}(H)} < +\infty$ and $Bh = -D(\rho, h)$ almost everywhere, $h \in H$. Condition **B**, in particular, is sufficient for the realization of estimates (1).

The main result of the present section is the following theorem:

Theorem 1. *Suppose that a mapping* $F : X \to X$ *has the form*

$$F(x) = x + j\Phi(x), \quad x \in X,$$

and the function Φ *belongs to the class* $HC^1(X, H, \mu)$ *of functions from* $W_\infty^1(X, H, \mu)$ *such that, for any* $x \in X$, *the functions*

$$H \ni h \mapsto \Phi(x+h) \in H, \qquad H \ni h \mapsto D\Phi(x+h) \in H^{\otimes 2}$$

are continuous. Also assume that $\mu(\{x \mid \text{the operator } \mathbb{1}_H + D\Phi(x) \text{ is noninvertible}\}) = 0$.

Then the image μ_F *of the measure* μ *under the mapping* F *is absolutely continuous with respect to* μ *and*

$$p_F(x) \equiv \frac{d\mu_F}{d\mu}(x) = \sum_{y \in F^{-1}(\{x\})} \left[J_F^\mu(y) \right]^{-1} \tag{5}$$

almost everywhere with respect to the completion of the measure μ; *here,*

$$J_F^\mu(x) = \left|\det{}_2(\mathbb{1}_H + D\Phi(x))\right| \exp\left[-I(\Phi)(x) - \int_0^1 (1-\tau)(B(x+\tau\Phi(x))\Phi(x), \Phi(x))_H \, d\tau\right], \quad x \in X, \tag{6}$$

and the symbol $\det_2$ *denotes the Carleman–Fredholm determinant [14].*

Remark 3. The stochastic integral $I(\Phi)$ for $\Phi \in W_\infty^1(X, H, \mu)$ is well defined [15].

Remark 4. For measures for which the classes W_∞^k and WL_∞^k coincide [e.g., for product measures or logarithmically convex (in particular, Gaussian) measures], the condition $\Phi \in W_\infty^2(X, H, \mu)$ is a sufficient condition for the smoothness of the function Φ.

Prior to the proof of Theorem 1, we present several auxiliary results. First of all, we recall certain properties of the Carleman–Fredholm determinant, which are necessary for what follows.

Lemma 1 [14].

1. *The function*

$$T \mapsto \det(\mathbb{I}_H + T)\exp(-\operatorname{tr} T)$$

defined on the set $\mathcal{L}_{fin}(H)$ *of linear continuous operators in* H *that have finite-dimensional images is continuous on this set in the norm of the space* $\mathfrak{L}_2(H)$ *and can be continuously extended to the entire space* $\mathfrak{L}_2(H)$.

This extension is denoted by $\det_2(\mathbb{I}_H + \cdot)$.

2. *For* $T \in \mathfrak{L}_2(H)$, *we have* $\det_2(\mathbb{I}_H + T) = 0 \Leftrightarrow 0 \in \sigma(\mathbb{I}_H + T)$.

3. *The function* $\det_2(I_H + \cdot)$ *is uniformly continuous and bounded on every set of the form* $\{\|\cdot\|_{\mathfrak{L}_2} \le K\}$.

4. *For any* $\delta < 1$, *the function* $[\det_2(I_H + \cdot)]^{-1}$ *is uniformly continuous and bounded on* $\{\|\cdot\|_{\mathfrak{L}_2} \le \delta\}$.

The general scheme of the proof of Theorem 1 coincides with scheme of the proof of the corresponding statement for Gaussian measures proposed by Kusuoka, which reduces [4, 16] to a local representation of the mapping F in the form of a composition of an affine transformation of the space X and a mapping that slightly differs from the identity transformation. The first step in the realization of this scheme is the following result:

Theorem 2. *Let* $\Phi \in WL_\infty^1(X, H, \mu)$ *and let*

$$\beta_1 \equiv \operatorname{ess\,sup} \|D\Phi\|_{\mathfrak{L}_2(H)} < 1.$$

Then the following assertions are true:

(i) $\mu_F \sim \mu$;

(ii) *there exists a measurable mapping* $F^{-1}: X \to X$ *such that*

$$F \circ F^{-1} = F^{-1} \circ F = id_X \quad \mu\text{-almost everywhere;}$$

in this case,

$$p_F(x) = \left[J_F^\mu(F^{-1}(x))\right]^{-1}, \qquad x \in X;$$

(iii) for any ε, $|\varepsilon| < \beta_1^{-1}$, *there exists a constant* C *dependent only on* ε, β_μ, β_1, *and* $\beta_0 = \operatorname{ess\,sup} \|\Phi\|_H$ *and such that*

$$\mathrm{E}\exp \varepsilon I(\Phi) \le C.$$

Remark 5. Assertion (i) of Theorem 2 guarantees that the value of $\int_0^1 (1-\tau)(B(\cdot + \tau\Phi)\Phi, \Phi)\,d\tau$ and, hence, J_F^μ do not depend on the choice of a modification of the random operator B.

Proof of Theorem 2. First, we consider the case $X = H = \mathbb{R}^n$. In this case, we have $\mu \sim \lambda^n$, and the standard theorem on the change of variables implies the equivalence of the measures μ_F, λ^n, and $\mu_{F^{-1}}$ (the mapping F^{-1} exists by virtue of the theorem on contracting mappings). We have

$$\frac{d\mu_{F^{-1}}}{d\mu}(x) = \left|\det(\mathbb{I}_H + D\Phi)\right| \exp[\ln p(F(x)) - \ln p(x)]. \tag{7}$$

Taking into account that, in this case, $\rho = (\ln p)'$, $B = -(\ln p)''$, writing the Taylor formula with the remainder term in the Lagrange form for the difference of logarithms in (7), and then applying equality (3), we get

$$\frac{d\mu_{F^{-1}}}{d\mu}(x) = J_F^\mu(x),$$

which completes the proof of assertion (ii) in Theorem 2.

The proof of assertion (iii) follows from the relation

$$\mathrm{E}\exp(\varepsilon I(\Phi)) = \mathrm{E}\, J^\mu_{id_X - \varepsilon\Phi}[\det_2(I - \varepsilon\nabla\Phi)]^{-1} \exp\{\varepsilon^2 \int_0^1 (1-\tau)\,(B(\cdot - \tau\varepsilon\Phi)\Phi, \Phi)\,d\tau\}$$

$$\le \mathrm{E} \sup_{\|T\|_{\mathfrak{L}_2(H)} \le |\varepsilon/\beta_1|} [\det_2(\mathbb{I}_H + T)]^{-1} \exp\left\{\frac{\varepsilon^2}{2}\beta_0 \cdot \beta_\mu\right\} = C(\varepsilon, \beta_0, \beta_1, \beta_\mu) < +\infty.$$

Now let X be infinite-dimensional. We prove assertion (i) under the additional assumption that there exists a finite-dimensional space $H_0 \subset j^* X^*$ such that $\Phi(x) \subset H_0$. We decompose the space X into the direct sum $X = X_0 + \left[(j^*)^{-1} H_0\right]^\perp$, $\dim X_0 < +\infty$. This relation generates the following decomposition of the measure μ:

$$\mu(d(x,y)) = \mu_y(dx)\,\pi(dy), \qquad (x,y) \in X,$$

where $\{\mu_y,\ y \in [(j^*)^{-1} H_0]^\perp\}$ and π are the conditional measures and the image of the measure μ under the projection onto $[(j^*)^{-1} H_0]^\perp$, respectively. On the other hand, since $\Phi(X) \subset H_0$, the mapping F can be represented in the form

$$F(x, y) = (x + j\varphi_y(x),\ y), \qquad (x, y) \in X.$$

Furthermore, for any y, the mapping φ_y satisfies the conditions of Theorem 2. By virtue of the principle of contracting mappings, for all y there exist the inverse mappings $(id_{X_0} + j\varphi_y)^{-1}$. Since π-almost all conditional measures $\{\mu_y\}$ satisfy condition **B**, for π-almost all y we have

$$p_y(x) \equiv \frac{d\mu_y \circ (id_{x_0} + j\varphi_y)^{-1}}{d\mu_y}(x) = \left[J^{\mu_y}_{(id_{X_0} + j\varphi_y)}\big((id_{x_0} + j\varphi_y)^{-1}(x)\big)\right]^{-1}, \qquad x \in X_0,$$

which proves the required statement.

Finally, we consider the general case. In H, we choose a basis $\{h_k\} \subset j^*X^*$ and set $P_n = \mathrm{Pr}_{\langle h_1, \ldots, h_n \rangle}$ and $F_n = id_X + jP_n\Phi$. The statement proved above implies that, for any $n \geq 1$, there exists the inverse mapping F_n^{-1}, $\mu_n \ll \mu$, $\mu_{F_n^{-1}} \ll \mu$, and

$$p_{F_n}(x) = \left[J^{\mu}_{F_n}(F_n^{-1}(x))\right]^{-1}, \qquad p_{F_n^{-1}}(x) = J^{\mu}_{F_n}(x), \qquad x \in X.$$

By construction, $\mu_n \Rightarrow \mu$, $n \to \infty$. Furthermore, in view of assertion (iii) proved above for finite-dimensional mappings, we have

$$\mathrm{E}\left[p_{F_n}(x)\right]^2 = \mathrm{E}\left[J^{\mu}_{F_n}(x)\right]^{-2} J^{\mu}_{F_n}(x) = \mathrm{E}\left[J^{\mu}_{F_n}(x)\right]^{-1} \leq C_1(\beta_0, \beta_1, \beta_\mu),$$

i.e., the family $\{\, p_{F_n},\ n \geq 1 \,\}$ is uniformly integrable. Assertion (i) now follows from the general result in [1].

Theorem 3 [1]. *Suppose that* X *is a separable metric space,* μ *is a probability measure defined on a Borel* σ*-algebra, and* $\{f_n\colon X \to X \mid n \geq 0\}$ *is a sequence of measurable mappings. Assume that the following conditions are satisfied:*

(i) for any $n \geq 1$, *the measure* $\mu \circ (f_n)^{-1}$ *is absolutely continuous with respect to the measure* μ;

(ii) the sequence of densities $\left\{d\mu \circ (f_n)^{-1}/d\mu\colon n \geq 1\right\}$ *is uniformly integrable;*

(iii) f_n *converges to* f_0 *in the measure* μ *as* $n \to \infty$.

Then $\mu \circ (f_0)^{-1} \ll \mu$, *and if the sequence* $\left\{d\mu \circ (f_n)^{-1}/d\mu\right\}$ *converges to a certain function* p, *then*

$$p = \left\{\frac{d\mu \circ (f_0)^{-1}}{d\mu}\right\}.$$

By construction, each function F_n^{-1} is a H-Lipschitz function with the constant $1/(1 - \beta_1)$ for all $n \geq 1$, $\|F_n - F\| \to 0$ as $n \to \infty$ almost everywhere, and the limit $\lim_{n\to\infty} F_n^{-1} \equiv F^{-1}$ exists almost everywhere in H. Then, by virtue of the absolute continuity of $\mu_{F^{-1}} \ll \mu$, we have

$$\left\| F \circ F^{-1} - id_X \right\|_H \le \limsup_{n\to\infty} \left\| F \circ F^{-1} - F_n \circ F^{-1} \right\|_H + \limsup_{n\to\infty} \left\| F_n \circ F^{-1} - F_n \circ F_n^{-1} \right\|_H$$

$$\le \limsup_{n\to\infty} \left\| F \circ F^{-1} - F_n \circ F^{-1} \right\|_H + \beta_1 \limsup_{n\to\infty} \left\| F^{-1} - F_n^{-1} \right\|_H = 0 \quad \text{a.e.}$$

By analogy, we can prove that $F^{-1} \circ F = id_X$ almost everywhere. To complete the proof, it remains to verify that

$$\int_0^1 (1-\tau)(B(\cdot + \tau\Phi_n)\Phi_n, \Phi_n)\, d\tau \to \int_0^1 (1-\tau)(B(\cdot+\tau\Phi)\Phi, \Phi)\, d\tau \quad \text{a.e.}$$

This convergence is guaranteed by the following auxiliary statement:

Lemma 2. *Suppose that* X *and* Y *are complete separable metric spaces with metrics* ρ_X *and* ρ_Y, *respectively,* μ *is a probability measure defined on the Borel* σ*-algebra* $\mathcal{B}(X)$, *and* $\varphi_n \colon X \to X$ *and* $f_n \colon X \to Y$, $n \ge 0$, *are measurable mappings. Assume that the following conditions are satisfied:*

(i) $\varphi_n \xrightarrow{\mu} \varphi_0$, $n \to \infty$, $f_n \xrightarrow{\mu} f_0$, $n \to \infty$;

(ii) for any $n \ge 1$, *the measure* $\mu \circ \varphi_n^{-1}$ *is absolutely continuous with respect to* μ;

(iii) the sequence of densities $\left\{ d\mu \circ (\varphi_n)^{-1} / d\mu \colon n \ge 1 \right\}$ *is uniformly integrable.*

Then $f_n \circ \varphi_n \xrightarrow{\mu} f_0 \circ \varphi_0$, $n \to \infty$.

Proof. Without loss of generality, we can assume that the convergences

$$f_n \to f_0, \quad n \to \infty, \qquad \varphi_n \to \varphi_0, \quad n \to \infty,$$

hold μ-almost everywhere. In addition, according to Theorem 3, $\mu \circ \varphi_0^{-1} \ll \mu$. Let $\varepsilon > 0$ be given. By using the Ulam theorem and the uniform integrability of the densities $\left\{ d\mu \circ (\varphi_n)^{-1} / d\mu \colon n \ge 0 \right\}$, we choose a compact set $\tilde{K} \subset .X$ such that $\mu(X \setminus \tilde{K}) < \varepsilon$ and $\mu(\varphi_n \notin \tilde{K}) < \varepsilon$ for any $n \ge 0$. It follows from the theorems of Ulam, Luzin, and Egorov that there exists a compact set $K \subset \tilde{K}$, $\mu(X \setminus K) < 2\varepsilon$, such that the following assertions are true:

(a) the function f_0 is continuous on K;

(b) the following uniform convergence holds on K:

$$f_n \overset{K}{\rightrightarrows} f_0, \quad n \to \infty, \qquad \varphi_n \overset{K}{\rightrightarrows} \varphi_0, \quad n \to \infty.$$

We choose $\delta > 0$ and $n_0 \in \mathbb{N}$ so that

$$\forall x, y \in K, \quad \rho_X(x, y) < \delta \colon \rho_Y(f_0(x), f_0(y)) < \varepsilon;$$

$$\forall x \in K \quad \forall n \ge n_0: \quad \rho_X(\varphi_n(x), \varphi_0(x)) < \delta, \quad \rho_Y(f_n(x), f_0(x)) < \varepsilon;$$

$$\forall n \ge n_0: \quad \mu(\rho_X(\varphi_n, \varphi_0) \ge \delta) < \varepsilon.$$

Then, for $n \ge n_0$, we have

$$\begin{aligned} \mu(\rho_Y(f_n \circ \varphi_n, f_0 \circ \varphi_0) \ge 2\varepsilon) \;\le\;& \mu(f_n \notin K) + \mu(\varphi_0 \notin K) + \mu(\rho_X(\varphi_n, \varphi_0) \ge \delta) \\ &+ \mu(\varphi_n \in K,\ \varphi_0 \in K,\ \rho_X(\varphi_n, \varphi_0) < \delta,\ \rho_Y(f_n \circ \varphi_n, f_0 \circ \varphi_n) \ge \varepsilon) \\ &+ \mu(\varphi_n \in K,\ \varphi_0 \in K,\ \rho_X(\varphi_n, \varphi_0) < \delta,\ \rho_Y(f_0 \circ \varphi_n, f_0 \circ \varphi_0) \ge \varepsilon) \\ \le\;& 5\varepsilon, \end{aligned}$$

which proves Lemma 2.

For the next step in the proof of Theorem 1, we need an analog of the theorem on inverse functions.

We fix an orthonormal basis $\{e_n, n \ge 1\} \subset j^* X^*$ in H and denote $H_n \equiv \langle e_1, \ldots, e_n \rangle$.

Theorem 4. *Suppose that* Φ *satisfies the conditions of Theorem 1. Then there exist countable families of measurable sets* $\{V_i^\alpha\}$ *and functions* $\{\Psi_i^\alpha \colon V_i^\alpha \to H\}$, $\alpha = 1, 2, 3$, *such that* $V_i^{\alpha+1} = T_i^\alpha V_i^\alpha$, $\alpha = 1, 2$, $T_i^\alpha = id_{V_i^\alpha} + j\Psi_i^\alpha$ *and, furthermore, the following assertions are true:*

(i) Ψ_i^3 *are deterministic mappings with values in* H_{n_i}, $i \ge 1$;

(ii) for every $i \ge 1$, *there exists a deterministic element* $D\Psi_i^1$ *in* $\mathfrak{L}(H_{n_i}) \subset \mathfrak{L}_2(H)$ *and* $\det_2(\mathbb{I}_H + D\Psi_i^1) \ne 0$;

(iii) the mapping T_i^2 *satisfies the conditions of Theorem 2 on the set* V_i^2;

(iv) $\bigcup_i V_i^1 = \{x \mid \det_2(\mathbb{I}_H + D\Phi(x)) \ne 0\}$ *and* $F_i \equiv F|_{V_i^1} = T_i^3 \circ T_i^2 \circ T_i^1$;

(v) the functions $(id_H + jP_{H_n}\Psi_i^2)$ *are one-to-one functions on* V_i^2, *the sets* $V_i^{3,n} = (id_H + jP_{H_n}\Psi_i^2)$ *satisfy the conditions* $V_i^{3,\infty} \equiv \liminf\limits_{n\to\infty} V_i^{3,n} \supset V_i^3$, *and* $\|\kappa_i(x) - \kappa_i(x)\|_H \to 0$, $n \to \infty$, $x \in V_i^3$, *where*

$$(id_X + j\Psi_i^2)^{-1}(x) = id_X + j\kappa_i(x), \quad x \in V_i^3,$$

and

$$(id_X + jP_{H_n}\Psi_i^2)^{-1}(x) = id_X + j\kappa_i^n(x), \quad x \in V_i^{3,n}.$$

Assertions (i)–(iv) were proved by Kusuoka in [4] (see also [16]). In this case, the properties of the measure (the Gaussian measure in the cited work) were not used. We obtain the proof of assertion (v) for the families $\{V_i^\alpha\}$ and $\{\Psi_i^\alpha\}$ constructed in [4] by direct application of the principle of contracting mappings.

The proof of Theorem 2 consists of the local application (to the functions $F_i = F|_{V_i^1}$) of the arguments of Theorem 1. For any fixed $n \geq 1$ and $i \geq 1$, the images of the measure $\mu|_{V_i^1}$ and $\mu|_{T_i^3 V_i^{3,n}}$ under the mappings $F_i^n \equiv T_i^3 \circ (id_X + jP_{H_n}\Psi_i^2) \circ T_i^1$ and $(F_i^n)^{-1}$, respectively, are absolutely continuous with respect to the measure μ, and the corresponding densities are equal to

$$\left[J^{\mu}_{F_i^n}((F_i^n)^{-1}(x))\right]^{-1}, \quad x \in T_i^3 V_i^{3,n}, \qquad \text{and} \qquad J^{\mu}_{F_i^n}(x), \quad x \in V_i.$$

To verify this assertion, it suffices to decompose the measure μ into a family of conditional measures concentrated on layers parallel to $H_{n_i \vee n}$ and use the finite-dimensional formula of the change of variables. Passing to the limit as $n \to +\infty$ and using assertion (v) of Theorem 4 and Lemma 2, we obtain

$$\mu|_{V_i^1} \circ F^{-1} \ll \mu, \quad p_i(x) = \frac{d\mu|_{V_i^1} \circ F^{-1}}{d\mu}(x) = [J_F^{\mu}((F_i)^{-1}(x))]^{-1}, \quad x \in F(V_i^1).$$

Taking into account that $\mu(\bigcup_i V_i^1) = 1$ and, hence, $\mu \circ F^{-1} = \mu|_{\bigcup_i V_i^1} \circ F^{-1}$ and assuming that $V_i^1 \cap V_j^1 = \varnothing$, $i \neq j$ (which can be done without loss of generality), we get $\mu \circ F^{-1} \ll \mu$ and

$$p(x) = \frac{d\mu \circ F^{-1}}{d\mu}(x) = \sum_{i: x \in F(V_i^1)} J_F^{\mu}((F_i^{-1})(x)) = \sum_{\substack{y \in \bigcup_i V_i^1 \\ f(y) = x}} J_F^{\mu}(y), \quad x \in X.$$

The proof of Theorem 1 is completed by the following analog of the Sard theorem:

Theorem 5. *Suppose that* $\Phi \in HC^1(X, H, \mu)$. *Then*

$$\mu^*(F(\{x \mid \det_2(\mathbb{1}_H + D\Phi(x)) = 0\})) = 0.$$

Proof. An approach to the proof of finite-dimensional analogs of the Sard theorem, which consists of using the implicit-function theorem for the local reduction of the transformation F to a finite-dimensional transformation, was proposed in [17]. In the present theorem, this approach can be realized in the following way:

Let $n \geq 1$. We set $X_n = \{x \mid \det_2(\mathbb{1}_H + (\mathbb{1}_H - P_{H_n})D\Phi(x)) \neq 0\}$. By virtue of Theorem 4, there exists a family of sets $\{V_i^n\}$ such that

$$\mu|_{V_i^n} \circ \left(id_X + j(\mathbb{1}_H - P_{H_n})\Phi\right)^{-1} \ll \mu, \qquad \bigcup_i V_i^n = X_n.$$

Taking into account that the image of the mapping $\Phi \circ (id_X + j(\mathbb{1}_H - P_{H_n})\Phi)^{-1}$ takes values in H_n, we conclude that, on V_i^n, the mapping F has the form $F|_{V_i^n} = F_{i,n}^2 \circ F_{i,n}^1$, where $F_{i,n}^1(x) = x + j\Psi_n(x)$, $\Psi_n(x) \in H_n$, $x \in V_i$, and $F_{i,n}^2$ transforms the measure μ into an absolutely continuous measure. Applying the finite-dimensional Sard theorem to $F_{i,n}^1$ and carrying out the summation over i, we obtain

$$\mu^*(F\{x \in X_n \mid \det_2(\mathbb{1}_H + D\Phi) = 0\}) = 0.$$

Taking into account that $\bigcup_n X_n = X$, we obtain the required statement.

Theorem 5 is proved.

3. Several Applications. Girsanov Theorem for Smooth Measures

We present several special cases of formula (5), which are of independent interest.

We assume that an isomorphism identified the space H with $L_2([0,1])$ is fixed. We set $H_t = L_2([0,t]) \subset H$ and define a flow of σ-algebras $\{\mathcal{F}_t = \sigma(\varphi \in W^1_\infty(X,\mu)\colon D\varphi \in H_t \text{ almost everywhere}),\ t \in [0,1]\}$.

The following statement is true:

Theorem 6 [18]. *Suppose that a process* $\{g_t,\ t \in [0,1]\}$ *is adapted with the flow* $\{\mathcal{F}_t\}$ *and* $\mathrm{E}\int_0^1 g_t^2 dt < +\infty$.

Then g, *as a random element in* $L_2([0,1])$, *is stochastically integrable with* $p = 2$. *Furthermore, there exists a sequence of nonanticipative step processes*

$$g_t^n = \sum_{k=1}^{N} \alpha_k^n \mathbb{1}_{t \in [t_n^{k-1}, t_n^k]}, \quad t \in [0,1], \quad n \geq 1, \tag{8}$$

such that $\mathrm{E}\|g - g^n\|^2_{L_2([0,1])} \to 0,\ n \to \infty$, *and*

$$I_2(g) = L_2 - \lim_{n\to\infty} \sum_{k=1}^{N_n} \alpha_k^n \big[m(t_n^k) - m(t_n^{k-1})\big],$$

where $\{m(t) = I(\mathbb{1}_{[0,t]}),\ t \in [0,1]\}$. *The process* $m(\cdot)$ *is called the logarithmic process of the measure* μ.

Remark 6. The process m is not necessary adapted with the flow $\{\mathcal{F}_t\}$ [15].

Theorem 7. *Let* g *be a process adapted with* $\{\mathcal{F}_t\}$ *and let* $\|g\|_{H,\infty} \equiv \operatorname{ess\,sup} \|g\|_H < +\infty$.

Then the distribution μ_F *in* X *of a random element* F,

$$F(x) = x + jg(x), \qquad x \in X, \tag{9}$$

is equivalent to the measure μ. *Furthermore,*

$$p^F(F(x)) \equiv \frac{d\mu}{d\mu_F}(F(x)) = \exp\left\{-I(g)(x) - \int_0^1 (1-\tau)\big(B(x + \tau jg)g(x),\ g(x)\big)_H\, d\tau\right\}. \tag{10}$$

Remark 7. Assume that $X = C_0([0,1])$, $H = L_2([0,1])$, $\mu = \mu_W$ is the Wiener measure, and $j\colon h(\cdot) \mapsto \int_0^\cdot h(s)ds$. In this case, Theorem 7 establishes the absolute continuity of the distribution of the Itô process $\{\xi_t,\ t \in [0,1]\}$ with the differential $d\xi_t(x) = g_t dt + dx_t,\ t \in [0,1]$, with respect to the Wiener measure. The stochastic integral of the nonanticipative process g with respect to a Wiener process coincides with the ordinary Itô integral and the operator B is identically equal to $\mathbb{1}_H$. Hence, in this case, formula (10) takes the form

$$\frac{d\mu_w}{d\mu_\xi}(\xi) = \exp\left(-\int_0^1 g_t\,dw_t - \frac{1}{2}\int_0^1 g_t^2\,dt\right) = \exp\left(-\int_0^1 g_t\,d\xi_t + \frac{1}{2}\int_0^1 g_t^2\,dt\right),$$

which coincides with the classical Girsanov formula. Thus, Theorem 7 can be regarded as an analog of the Girsanov formula for smooth measures.

Proof of Theorem 7. First, assume that the element g has the form (8) and every of random variables α_n^k is an $\mathcal{F}_{t_n^{k-1}}$-measurable random variable from WL_∞^2. Then mapping (9) satisfies the conditions of Theorem 1. Since

$$\left(Dg(x),\ \mathbb{1}_{[t_n^{k-1},t_n^k]} \otimes \mathbb{1}_{[t_n^{j-1},t_n^j]}\right) = 0$$

for all $x \in X$ and $k \geq j$, we have $\det_2(\mathbb{I}_H + Dg(x)) = 1$. It is easy to verify that the mapping $F = id_X + jg$ has the inverse mapping $F^{-1} = id_X + jq$, where the element q has the form (8). Equality (10) now immediately follows from formula (5).

By analogy with the proof of assertion (iii) in Theorem 2, we obtain

$$\mathrm{E}\exp CI(g) \leq \exp\left\{\frac{C^2}{2}\beta_\mu \|g\|_{H,\infty}^2\right\}, \qquad C \in \mathbb{R}. \tag{11}$$

Passing to the limit and using the result of Lemma 2, we obtain the statement for an arbitrary process g that satisfies the conditions of Theorem 7. In this case, estimate (11) guarantees the uniform integrability of the family of densities. Theorem 7 is proved.

Corollary. *Under the conditions of Theorem 7, for any* $\alpha < (\beta_\mu \|g\|_{H,\infty})^{-1}$ *we have*

$$\mathrm{E}\exp\frac{\alpha}{2}I^2(g) < +\infty.$$

An analogous result is true for monotone mappings.

A mapping $F = id_X + j\Phi$ is called monotone if $D\Phi(x) \geq 0,\ x \in X$.

The following result is proved by analogy with Theorem 7:

Theorem 8. *Let* $F = id_X + j\Phi$ *be a monotone mapping and let* $\Phi \in WL_\infty^1(X, H, \mu)$. *Then* F *is injective and the following assertions are true:*

(i) $\mu_F \ll \mu$ *and*

$$\frac{d\mu_F}{d\mu_F}(x) = [J_F^\mu(y)]^{-1}, \qquad F(y) = x, \quad x \in X;$$

(ii) $\mathrm{E}\exp CI(g) \leq \mathrm{exr}\ C^2/2\beta_\mu \|\Phi\|_{H,\infty},\ C < 0;$

(iii) for $\alpha < (\beta_\mu \|\Phi\|_{H,\infty})^{-1}$, *we have*

$$\mathrm{E}\exp\frac{\alpha}{2}(I(\Phi)\wedge 0)^2 < +\infty.$$

4. Transformations Generated by Evolution Flow

We consider the absolute continuity of the original measure μ under the action of a flow generated by the integral equation

$$u_t(x) = x + j\left(\int_0^1 b_s(u_s(x))\,ds\right), \quad x \in X, \quad t \in [0, 1]. \tag{12}$$

Note that Eq. (12) possesses a special property. It is natural to require that a solution of Eq. (12) do not change if we replace the random process b_t, $t \in [0, 1]$, by a stochastically equivalent process. However, generally speaking, this is not true if the measures $\mu \circ (u_s)^{-1}$ and μ are singular. Hence, the problem of absolute continuity of the measures $\mu \circ (u_t)^{-1}$ with respect to μ and the problem of existence of a solution of (12) must be considered simultaneously.

Definition 6. *A measurable random process* u_t, $t \in [0, 1]$, *is a solution of Eq. (12) if the following conditions are satisfied:*

(a) there exists a set $X_0 \subset X$ *of complete measure such that relation (12) is true for all* $x \in X_0$, $t \in [0, 1]$;

(b) the measures $\mu \circ (u_t)^{-1}$, $t \in [0, 1]$, *are absolutely continuous with respect to* μ.

The main result of the present section is the following theorem:

Theorem 9. *Suppose that a measure* μ *satisfies condition* ***B***, *a function* $b = b_t(x) : [0, 1] \times X \to H$ *is measurable with respect to the pair of variables,* $b_t \in W^1_\infty(X, H, \mu)$ *for almost all* $t \in [0, 1]$, *and* $\operatorname{ess\,sup}_t \|b_t\|_{W^1_\infty} < \infty$.

Then the following assertions are true:

(i) Eq. (12) has a solution;

(ii) for any $t \in [0, 1]$, *the equation*

$$v_{s,t}(x) = x - j\left(\int_s^t b_z(v_{z,t}(x))\,dz\right), \quad 0 \le s \le t, \tag{13}$$

has a solution; furthermore, the function $v_t := v_{0,t}$ *is an almost-everywhere inverse mapping of* u_t, *i.e.,*

$$v_t(u_t(x)) = u_t(v_t(x)) = x \quad \text{for } \mu\text{-almost all } x \quad \text{and} \quad v_{s,t} = u_s \circ v_t \quad \mu\text{-almost everywhere;}$$

(iii) *the measures* μ, $\mu \circ (u_t)^{-1}$, *and* $\mu \circ (v_{s,t})^{-1}$, $0 \le s \le t \le 1$, *are equivalent and the Radon–Nikodym densities have the form*

$$\frac{d\mu \circ (u_t)^{-1}}{d\mu}(x) = \exp\left(\int_0^t (Ib_s)(v_{s,t}(x))\,ds\right), \tag{14}$$

$$\frac{d\mu \circ (v_{s,t}(x))^{-1}}{d\mu}(x) = \exp\left(-\int_s^t (Ib_z)(u_z(x))\,dz\right); \tag{15}$$

(iv) *if, for almost all* $t \in [0, 1]$, *the mapping* b_t *belongs to the class* $WL^1_\infty(X, H, \mu)$, *then a solution of Eq. (12) is unique to within stochastic equivalence.*

Remark 8. For the first time, the problem of transformation of a measure by a flow generated by a differential equation in an infinite-dimensional space was considered by Cruzeiro [19] for Gaussian measures. A strong generalization of this result in the case of an arbitrary differentiable measure was obtained by Bogachev and Mayer-Wolf in [20]. It was proved that if $b : X \to H$ is a measurable mapping, μ is a probability measure on a Banach space X differentiable along a Hilbert space $H \hookrightarrow X$, and

(α) $\forall c \in \mathbb{R}\ \forall h \in H$: $\exp\{c\rho_h\} \in L_1(X, \mu)$,

(β) $\forall c > 0$: $\exp(c\|b\|_H)$, $\exp(c\|Db\|_{\text{op}}) \in L_1(X, \mu)$, where $\|\cdot\|_{\text{op}}$ is an operator norm,

(γ) there exists a sequence of finite-dimensional orthogonal projectors $\{p_n, n \ge 1\}$ such that $P_n b \in \mathcal{D}(I)$, $n \ge 1$, and $\forall c \in \mathbb{R}$: $\sup_n \mathrm{E}\exp(cI(P_n b)) < \infty$,

then there exists a unique solution of the equation

$$u_t(x) = x + j\left(\int_0^t b(u_s(x))\,ds\right).$$

Note that the conditions on the measure and the function b in Theorem 9 are much stronger than conditions (a) and (b). On the other hand, as a rule, it is difficult to verify condition (c).

We prove the existence of a solution by analogy with Sec. 2 by using the approximation of a solution of the required equation by solutions of an equation of the form (12) with coefficients $P_n b$, where P_n is a finite-dimensional projector. For this purpose, we need the following statement on the properties of solutions of an ordinary differential equation with Lipschitz coefficients:

Lemma 3. *Suppose that a measurable function* $b\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfies the following conditions:*

(i) b *is bounded;*

(ii) b *satisfies the Lipschitz condition with respect to* x, *i.e.,*

$$L := \sup_{t \in [0,1]} \sup_{x \neq y} \frac{\|b(t,x) - b(t,y)\|}{\|x - y\|} < \infty.$$

Then the following assertions are true:

(a) *the integral equations*

$$u_t(x) = x + \int_0^t b(s, u_s(x))ds, \qquad t \in [0, 1], \quad x \in \mathbb{R}^n, \tag{16}$$

$$v_{s,t}(x) = x - \int_s^t b(z, v_{z,t}(x))ds, \qquad s \in [0, 1], \quad x \in \mathbb{R}^n, \tag{17}$$

have unique solutions; furthermore, the mappings u_t, $v_{s,t} : \mathbb{R}^n \to \mathbb{R}^n$ *are bijections and* $v_{0,t}(u_t(x)) = u_t(v_{0,t}(x)) = x$ *and* $v_{s,t}(x) = u_s \circ v_t(x)$ *for all* $x \in \mathbb{R}^n$;

(b) *for all* $s, t, 0 \le s \le t \le 1$, *the mappings* u_t *and* $v_{s,t}$ *are differentiable* λ^n*-almost everywhere and the relations*

$$\nabla u_t(x) = id_{\mathbb{R}^n} + \int_0^t \nabla b(s, u_s(x)) \nabla u_s(x)\, ds, \qquad t \in [0, 1], \tag{18}$$

$$v_{s,t}(x) = id_{\mathbb{R}^n} - \int_s^t \nabla b(z, v_{z,t}(x)) \nabla v_{z,t}(x)\, dz, \qquad s \in [0, t], \tag{19}$$

$$\det \nabla u_t(x) = \exp\left(\int_0^t tr\ \nabla b(s_s, u\ (x))ds \right) \qquad t \in [0, 1], \tag{20}$$

$$\det \nabla v_{s,t}(x) = \exp\left(-\int_s^t tr\ \nabla b(z, v_{z,t}(x))dz \right), \qquad s \in [0, t], \tag{21}$$

hold on a certain set $X_0 \subset \mathbb{R}^n$ *of complete Lebesgue measure;*

(c) *the measures* $\lambda^n \circ (u_t)^{-1}$, $\lambda^n \circ (v_{s,t})^{-1}$, *and* λ^n *are equivalent and the following relations hold* λ^n*-almost everywhere:*

$$\frac{d\lambda^n \circ (u_t)^{-1}}{d\lambda^n}(x) = \det \nabla v_{0,t}(x), \tag{22}$$

$$\frac{d\lambda^n \circ (v_{s,t})^{-1}}{d\lambda^n}(x) = \exp\left(\int_s^t tr\ \nabla b(z, u_z(x))dz \right). \tag{23}$$

Proof of Lemma 3. Assertion (a) of Lemma 3 is the standard statement from the theory of differential equations [21]. In the case where the function $b(t,\cdot)$ is continuously differentiable for any $t\in[0,1]$, assertions (b) and (c) are also well known.

Assume that $\psi_r\colon \mathbb{R}^n\to\mathbb{R}$ is an infinitely differentiable function, $\psi_r(x)=1$ for $\|x\|\le r$, and $\psi_r(x)=0$ for $\|x\|\ge r+1$. We set $b_r(t,x):=b(t,x)\,\psi_r(x)$. Let $u_{r,t}$ be a solution of the equation

$$u_{r,t}(x) = x + \int_0^t b_r(s,u_{r,s}(x))\,ds.$$

We fix $c>0$. Note that if $\|x\|\le c$ and r is sufficiently large, then $u_{r,t}(x)=u_t(x)$, $t\in[0,1]$. Hence, without loss of generality, we can assume that the function b has a compact support, i.e.,

$$\exists r_0>0 \quad \forall t\in[0,1] \quad \forall x\in\mathbb{R}^n, \qquad \|x\|\ge r_0 : \|b(t,x)\| = 0.$$

Note that, for any $t\in[0,1]$ and $x_1,x_2\in\mathbb{R}^n$, we get $\|x_1\|\le r_0$, $\|x_2\|\ge r_0$: $\|u_t(x_1)\|\le r_0$, and $u_t(x_2)=x_2$. Therefore, it suffices to verify that the restrictions of the functions u_t and $v_{s,t}$ to the set $\{\|x\|\le r_0+1\}$ possess the properties indicated in Lemma 3, the measures $\lambda^n|_{\{\|x\|\le r_0+1\}}$, $\lambda^n|_{\{\|x\|\le r_0+1\}}\circ(u_t)^{-1}$ are equivalent, and the Radon–Nikodym density is defined by relation (22).

The function u_t satisfies the Lipschitz condition. Indeed, for any $x,y\in\mathbb{R}^n$ and $t\in[0,1]$, we have

$$\|u_t(x)-u_t(y)\| \le \|x-y\| + \int_0^t \|b(s,u_s(x)) - b(s,u_s(y))\|ds \le \|x-y\| + L\int_0^t \|u_s(x)-u_s(y)\|ds.$$

By virtue of the Gronwall–Bellman lemma, this yields

$$\sup_{x\ne y}\frac{\|u_t(x)-u_t(y)\|}{\|x-y\|} \le e^{Lt}, \qquad t\in[0,1].$$

We consider a sequence of functions $\{b_k,\ k\ge 1\}$:

$$b_k(t,x) = \int_{\mathbb{R}^n} b(t,y)\,\varphi_k(x-y)\,dy,$$

where $\varphi_k\colon\mathbb{R}^n\to\mathbb{R}$, $\varphi_k(x)=k^n\varphi(kx)$, φ is a nonnegative finite infinitely differentiable function, and $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$.

For any $t\in[0,1]$, the functions $b_k(t,\cdot)$ are continuously differentiable and

$$\sup_{k,t,x}\|b_k(t,x)\| \le \sup_{t,x}\|b(t,x)\| < \infty,$$

$$\sup_{k,t,x}\|\nabla b_k(t,x)\| \le \sup_t \operatorname*{ess\,sup}_x \|\nabla b(t,x)\| < \infty,$$

$$\sup_{t,x}\|b_k(t,x)-b(t,x)\| \to 0, \quad k\to\infty,$$

$$\forall t \in [0,1] : \nabla b_k(t,x) \xrightarrow{\lambda^n} \nabla b(t,x), \quad k \to \infty.$$

Remark 9. The function $b(t,\cdot)$ satisfies the Lipschitz condition with respect to x. Hence, by virtue of the Rademacher theorem [9], it is differentiable λ^n-almost everywhere.

We denote by $u_t^{(k)}$ and $v_{s,t}^{(k)}$ solutions of Eqs. (16) and (17) with the function b replaced by b_k. For any $t \in [0, 1]$, the functions $u_t^{(k)}$ and $v_{s,t}^{(k)}$ are continuously differentiable and, for these functions, relations (a)–(c) are true.

It is easy to see that

$$\sup_{t,x} \left\| u_t^{(k)}(x) - u_t(x) \right\| \to 0, \quad k \to \infty,$$

$$\sup_{s,t,x} \left\| v_{s,t}^{(k)}(x) - v_{s,t}(x) \right\| \to 0, \quad k \to \infty.$$

We also note that, for sufficiently large k,

$$\left\| v_{s,t}^{k}(x) \right\|, \left\| u_t^{(k)}(x) \right\| \le r_0 + 1 \quad \text{for} \quad \|x\| \le r_0 + 1,$$

$$u_t^{(k)}(x) = v_{s,t}^{(k)}(x) = x \quad \text{for} \quad \|x\| > r_0 + 1.$$

The sequence of densities

$$\left\{ \frac{d\lambda^n|_{\{\|x\|\le r_0+1\}} \circ \left(u_t^{(k)}\right)^{-1}}{d\lambda^n|_{\{\|x\|\le r_0+1\}}} = \exp\left(\int_0^t \operatorname{tr} \nabla v_{s,t}^{(k)} \, ds \right) : k \ge 1 \right\}$$

is uniformly integrable with respect to the Lebesgue measure. Hence, by virtue of Theorem 3, we have the absolute continuity $\lambda^n|_{\{\|x\|\le r_0+1\}} \circ (u_t)^{-1} \ll \lambda^n|_{\{\|x\|\le r_0+1\}}$ and, consequently, $\lambda^n \circ (u_t)^{-1} \ll \lambda^n$. By analogy, one can verify that $\lambda^n \circ (v_{s,t})^{-1} \ll \lambda^n$. Hence, the measures λ^n, $\lambda^n \circ (u_t)^{-1}$, and $\lambda^n \circ (v_{s,t})^{-1}$ are equivalent.

We verify equality (18). Note that the functions $b(t,\cdot)$, $b(t, u_t(\cdot))$, and $u_t(\cdot)$ satisfy the Lipschitz condition. Hence, by virtue of the Rademacher theorem, they are differentiable λ^n-almost everywhere. It follows from the equivalence of the measures λ^n and $\lambda^n \circ (u_t)^{-1}$ that, for any $t \in [0, 1]$, the relation

$$\nabla(b(t, u_t(x))) = \nabla b(t, u_t(x)) \, \nabla u_t(x) \tag{24}$$

holds for λ^n-almost all x.

Hence, relation (24) is true on a certain set $X_0 \subset \mathbb{R}^n$ of complete Lebesgue measure for almost all $t \in [0, 1]$. The validity of (18) [and, by analogy, (19)] for all $x \in X_0$ follows from the theorem on the differentiation under the sign of a Lebesgue integral. The other assertions are proved by passing to the limit and using Lemma 2 and Theorem 3.

Lemma 3 is proved.

Lemma 4. *Let* $\mu(dx) = e^{-V(x)}dx$ *be a probability measure in* $\mathbb{R}^n$ *and let* $b : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ *be a measurable mapping. Assume that the following conditions are satisfied:*

(i) *V is continuously differentiable, the derivative V' satisfies the Lipschitz condition, and the second derivative V'' is essentially bounded:*

$$\beta_\mu := \operatorname*{ess\,sup}_x \|V''(x)\| < \infty, \quad \textit{where } \|\cdot\| \textit{ is the operator norm;}$$

(ii) *the function b is bounded by a constant β_0;*

(iii) *the function b satisfies the Lipschitz condition with respect to x and*

$$\beta_1 := \sup_t \operatorname*{ess\,sup}_x \|\nabla b(t, x)\|_{\mathfrak{L}_2} < \infty,$$

where $\|\cdot\|_{\mathfrak{L}_2}$ is the Hilbert–Schmidt norm.

Denote by u_t and $v_{s,t}$ solutions of Eqs. (16) and (17). Then the following assertions are true:

(a) *the measures μ, $\mu \circ (u_t)^{-1}$, and $\mu \circ (v_{s,t})^{-1}$ are equivalent and*

$$p_t := \frac{d\mu \circ (u_t)^{-1}}{d\mu} = \exp\left\{ \int_0^t (Ib(s, \cdot))(v_{s,t})\,ds \right\}, \tag{25}$$

$$p_{-t} := \frac{d\mu \circ (v_t)^{-1}}{d\mu} = \exp\left\{ -\int_0^t (Ib(s, \cdot))(u_s)\,ds \right\}, \tag{26}$$

where $I = \langle V', \cdot\rangle - \operatorname{tr}\nabla$ is a generalized stochastic integral corresponding to the measure μ (3);

(b) *there exists a constant $C = C(\beta_0, \beta_1, \beta_\mu) < +\infty$ that depends only on β_0, β_1, and β_μ (but does not depend on the dimension of $\mathbb{R}^n$) and is such that*

$$\sup_{t \in [0,1]} \int_{\mathbb{R}^n} p_t |\ln p_t|\, d\mu \le C. \tag{27}$$

Proof. To prove relations (25) and (26), it suffices to note that

$$\frac{d\mu \circ (u_t)^{-1}}{d\mu}(x) = \frac{d\lambda^n \circ (u_t)^{-1}}{d\lambda^n} e^{V(v_t(x)) - V(x)},$$

use formula (22) for $d\lambda^n \circ (u_t)^{-1} / d\lambda^n$, and use the Newton–Leibnitz formula for $(V(v_t(x)) - V(x))$.

By using Lemma 2 and the Taylor formula with the remainder in the Lagrange form, approximating the function V by twice continuously differentiable functions, and approximating the mapping u_t by the sequence $\{u_t^{(k)}\}$, we obtain from Lemma 3 the following equality for μ-almost all x:

$$V(v_t(x)) - V(x) = \int_0^t \langle V'(x), v_s(x) - x\rangle ds + \int_0^t \int_s^t \langle V''(v_{z,t}(x))b(z, v_{z,t}(x)),\ b(s, v_{s,t}(x))\rangle dz ds.$$

We also note that

$$\frac{d\lambda^n \circ (u_t)^{-1}}{d\lambda^n} = \exp\left\{-\int_0^t \mathrm{tr}\left[\nabla(b(s, v_{s,t})) - \int_s^t \nabla b(s, v_{s,t})\nabla b(z, v_{z,t})\nabla v_{z,t}\, dz\right] ds\right\}.$$

Hence,

$$p_t = \exp\left\{-\int_0^t \int_s^t \mathrm{tr}\big[\nabla(b(s, v_{s,t}))\nabla b(z, v_{z,t})\nabla v_{z,t}\big] dz ds \right.$$
$$\left. + \ I(v_t(x) - x) - \int_0^t \int_s^t \langle V''(v_{z,t})b(z, v_{z,t}),\ b(s, v_{s,t})\rangle dz ds\right\}.$$

Similarly,

$$p_{-t} = \exp\left\{-\int_0^t \int_0^s \mathrm{tr}\,[\nabla b(s, u_s)\nabla b(z, u_z)\nabla u_z\,] dz ds + I(u_t(x) - x) - \int_0^t \int_0^s \langle V''(u_z)b(z, u_z),\ b(s, u_s)\rangle dz ds\right\}.$$

By using the change of variables $x = v_t(y)$, we obtain

$$\int_{\mathbb{R}^n} p_t(y)|\ln p_t(y)|\mu(dy) = \int_{\mathbb{R}^n} |\ln p_{-t}(x)|\mu(dx)$$
$$\le \int_{\mathbb{R}^n} \int_0^t \int_0^s \big|\mathrm{tr}[\nabla b(s, u_s)\nabla b(z, u_z)]\nabla u_z\big| dz ds d\mu$$
$$+ \int_{\mathbb{R}^n} |I(u_t(x) - x)|\mu(dx) + \beta_\mu \int_0^t \int_0^s \|b(z, u_z)\|\,\|b(s, u_s)\|\, dz ds. \tag{28}$$

For any t, x, we have $\|u_t(x) - x\| \le \beta_0 t$. Applying the Gronwall–Bellman lemma to Eq. (18), we obtain the inequality

$$\|\nabla(u_t(x) - x)\|_{\mathfrak{L}_2} \le e^{\beta_1 t}\beta_1, \qquad t \in [0, 1], \quad x \in \mathbb{R}^n.$$

Lemma 5 ([22], Chap. 1, Sec. 3). *Suppose that a measure* μ *on* X *satisfies condition* ***B*** *and a function* f: $X \to H$ *belongs to the Sobolev space* $W_2^1(X, H, \mu)$. *Then* f *belongs to the domain of definition of the operator* $I = I_2$ *and*

$$\mathrm{E}(If)^2 \le \mathrm{E}\|Df\|^2_{\mathfrak{L}_2} + \beta_1 \mathrm{E}\|f\|^2_H.$$

Applying the indicated estimates and Lemma 5 to the second term in (28), we obtain

$$\int p_t |\ln p_t| d\mu \;\le\; \int_0^t \int_0^s \beta_1^2 (1 + e^{\beta_1 z}) dz ds + \sqrt{\beta_\mu \beta_0^2 t^2 + e^{2\beta_1 t}} + \frac{1}{2}\beta_\mu \beta_0^2 .$$

Lemma 3 is proved.

Proof of Theorem 9. Assume that $\{e_n\colon n \ge 1\}$ is an orthonormal basis of the space H, $H_n = \langle e_1, \ldots, e_n \rangle$, and $P_n\colon H \to H_n$ is an orthoprojector onto H_n. We decompose X into the direct sum $X = Y_n \oplus j(H_n)$, where $Y_n \subset X$ is a certain closed space. We identify this decomposition and the product $Y_n \times \mathbb{R}^n$ so that the vectors $j(e_1), \ldots, j(e_n)$ transform into the natural basis of $\mathbb{R}^n$. We denote the point that is the image of $x \in X$ under this identification by $(y_n, x_n) \in Y_n \times \mathbb{R}^n$.

Assume that $\pi_n = \pi_n(dy_n)$ is the projection of the measure μ onto Y_n and $\{\mu_{y_n}^{(n)}(dx_n)\colon y_n \in Y_n\}$ is a family of conditional measures with respect to the bundle $X = Y_n \times \mathbb{R}^n$ on a plane of the form $\{y_n\} \times \mathbb{R}^n = \mathbb{R}^n$. As noted in Sec. 2, for π^n-almost all y_n these conditional measures are differentiable and satisfy condition **B** (on $\mathbb{R}^n$).

It follows from the conditions of Theorem 9 (see Remark 2 in Sec. 1) that there exists a function $\tilde{b}\colon [0,1] \times X \to H$ such that $\tilde{b}_t(x) = b_t(x)$ for almost all (t, x) and

$$\forall m \ge 1,\ h \in H_m,\ t \in [0,1]\colon \left\| \tilde{b}_t(x + j(h)) - \tilde{b}_t(x) \right\|_H \;\le\; \beta_1 \|h\|_H, \tag{29}$$

where $\beta_1 := \operatorname{ess\,sup}_{t,x} \|Db(t,x)\|_{\mathfrak{L}_2}$.

We prove Theorem 9 under the assumption that the function b satisfies relation (29).

Consider the auxiliary sequence of equations

$$u_t^{(n)}(x) \;=\; x + j\left(\int_0^t b_s^{(n)}(u_s^{(n)}(x)) ds \right), \qquad x \in X, \quad t \in [0,1], \tag{30}$$

where $b_s^{(n)} = P_n b$.

We note that, for any $x = (y_n, x_n)$, the function $u_t^{(n)}$ does not change the coordinate y_n, i.e., $u_t^{(n)}(y_n, x_n) = (y_n, \tilde{u}_t^{(n)}(y_n, x_n))$, where $\tilde{u}_t^{(n)}$ is a solution of the following integral equation in $\mathbb{R}^n$:

$$\tilde{u}_t^{(n)}(y_n, x_n) \;=\; x_n + \int_0^t b_s^{(n)}(y_n, \tilde{u}_s^{(n)}(y_n, x_n)) ds .$$

Lemma 3 implies that, for π^n-almost all points $y_n \in Y_n$, the measures $\mu_{y_n}^{(n)}$ and $\mu_{y_n}^{(n)} \circ (\tilde{u}_t^{(n)}(y_n, \cdot))^{-1}$ are equivalent. Hence [2], $\mu \sim \mu \circ (u_t^{(n)})^{-1}$ and, for μ-almost all (y_n, x_n), the following equality is true:

$$\frac{d\mu \circ (u_t^{(n)})^{-1}}{d\mu}(y_n, x_n) \;=\; \frac{d\mu_{y_n}^{(n)} \circ (u_t^{(n)}(y_n, \cdot))^{-1}}{d\mu_{y_n}^{(n)}}(x_n). \tag{31}$$

Furthermore, it follows from (27) that

$$\sup_{n,t}\int p_t^{(n)}\left|\ln p_t^{(n)}\right| d\mu < \infty,$$

where $p_t^{(n)} = d\mu\circ(u_t^{(n)})^{-1}/d\mu$ and, hence, the sequence $\{p_t^{(n)}\}$ is uniformly integrable.

We estimate the deviation

$$\begin{aligned}\mathrm{E}\sup_{s\le t}\left\|u_s^{(n)} - u_s^{(m)}\right\|_H &\le \mathrm{E}\int_0^t \left\|b_s^{(n)}(u_s^{(n)}) - b_s^{(m)}(u_s^{(m)})\right\|_H ds \\ &\le \mathrm{E}\int_0^t \left(\left\|P_n b_s(u_s^{(n)}) - P_n b_s(u_s^{(m)})\right\|_H + \left\|(P_n - P_m) b_s(u_s^{(m)})\right\|_H\right) ds \\ &\le \beta_1 \mathrm{E}\int_0^t \left\|u_s^{(n)} - u_s^{(m)}\right\|_H ds + \int_0^t\int_X \|(P_n - P_m) b_s(x)\|_H\, p_s^{(m)}(x)\,\mu(dx)\,ds. \qquad (32)\end{aligned}$$

Here, to estimate the first term in the last inequality, we have used inequality (29). Note that, by virtue of the uniform integrability of $\{p_s^{(m)}\}$ and the Lebesgue theorem on dominated convergence, the second term on the right-hand side of (32) converges to zero as $n, m \to \infty$.

It follows from the Gronwall–Bellman lemma that the sequence $\{u^{(n)} : n \ge 1\}$ is fundamental and there exists a random function u_t, $t \in [0, 1]$, with continuous trajectories such that

$$\lim_{n\to\infty} \mathrm{E}\sup_t \left\|u_t^{(n)} - u_t\right\|_H = 0.$$

Theorem 3 yields the absolute continuity $\mu\circ(u_t)^{-1} \ll \mu$ and Lemma 2 implies that equality (12) holds for almost all (t, x). By virtue of the continuity of the left-hand and right-hand sides of (12) in t, equality (12) is true for all $t \in [0, 1]$ on a certain set $X_0 \subset X$ of μ-complete measure.

The existence of a solution of (13) can be proved by analogy. The equality $v_t(u_t(x)) = u_t(v_t(x)) = x$ for μ-almost all x is obtained from Lemma 2 and, hence, $\mu\circ(u_t)^{-1} \sim \mu$.

We verify formula (14).

By virtue of Lemma 4 and relation (31), for μ-almost all $x = (y_n, x_n) \in X$ we have

$$p_t^{(n)}(y_n, x_n) = \exp\left\{\int_0^t (I^{y_n} b_s^{(n)})(v_{s,t}^{(n)}(x))\,ds\right\},$$

where $I^{y_n} : L_2(\mathbb{R}^n, \mathbb{R}^n, \mu_{y_n}^{(n)}) \to L_2(\mathbb{R}^n, \mu_{y_n}^{(n)})$ is a generalized stochastic integral.

Note that [see (3)], for any function $f \in W_\infty^1(X, h, \mu)$, we have

$$(If)(y_n, x_n) = (I^{y_n} f)(y_n, x_n) \quad \text{for } \mu\text{-almost all } x = (y_n, x_n) \in X.$$

Hence,

$$p_t^{(n)}(x) = \exp\left\{\int_0^t (Ib_s^{(n)})(v_{s,t}^{(n)}(x))ds\right\}.$$

It follows from Lemma 5 that, for any $s \in [0, 1]$,

$$\lim_{n\to\infty}\int_X \left(Ib_s^{(n)}(x) - Ib_s(x)\right)^2 \mu(dx) = 0.$$

Applying Theorem 3 and Lemma 2, we obtain (14). Formula (15) is proved by analogy. Thus, Theorem 9 is proved in the case where the function b satisfies (29). Let us consider the general case. Let $\tilde{b}$ be a modification of the mapping b that satisfies relation (29). We denote by u_t, $t \in [0, 1]$, a solution of the equation

$$u_t(x) = x + j\left(\int_0^t \tilde{b}_s(u_s(x))ds\right), \qquad t \in [0, 1].$$

Note that,in this case, by virtue of the equivalence of the measures μ and $\mu \circ (u_t)^{-1}$, $t \in [0, 1]$, u_t satisfies Eq. (12). Equation (13) is considered by analogy.

We now prove the uniqueness of a solution of (12) under the assumption $b_t \in WL_\infty^1(X, H, \mu)$ for almost all $t \in [0, 1]$. Assume that $\tilde{u}_t$, $t \in [0, 1]$, is a solution of (12). By analogy with (12), we get the following estimate:

$$\mathrm{E}\sup_{s\le t}\left\|\tilde{u}_s - u_s^{(n)}\right\|_H \le \beta_1 \mathrm{E}\int_0^t \left\|\tilde{u}_s - u_s^{(n)}\right\|_H ds + \int_0^t\int_X \|(id_H - P_n)b_s(x)\|_H\, p_s^{(n)}(x)\,\mu(dx)ds.$$

This implies the convergence

$$\lim_{n\to\infty}\mathrm{E}\sup_{t\in[0,1]}\left\|\tilde{u}_t - u_t^{(n)}\right\| = 0.$$

Hence, for μ-almost all x, we have $\tilde{u}_t(x) = u_t(x)$, $t \in [0, 1]$. Theorem 9 is proved.

5. Linear Equations with Generalized Stochastic Integrals

In this section, we consider linear equations with generalized stochastic integrals. Since a generalized stochastic integral is an analog of a first-order differential operator [see (3)], equations with the operator I can be regarded as infinite-dimensional partial differential equations. First, we consider the following simple equation:

$$\begin{gathered}\frac{\partial \xi_t(x)}{\partial t} = -\langle D\xi_t(x), b_t(x)\rangle_H + a_t(x)\xi_t(x), \qquad t \in [0, 1], \quad x \in X, \\ \xi_0(x) = f(x),\end{gathered} \tag{33}$$

where $b\colon [0,1]\times X \to H$, $a\colon [0,1]\times X \to \mathbb{R}$, and $f\colon X \to \mathbb{R}$ are measurable functions and D is the stochastic derivative.

Let us solve Eq. (33) by the method of characteristics proposed by Dorogovtsev in [23, 24] for the solution of equations that contain stochastic derivatives or generalized stochastic integrals.

The equation for characteristics has the form (12) and, hence,

$$\xi_t(x) = f(v_t(x)) \exp\left(\int_0^t a_s(v_{s,t}(x))\, ds \right), \tag{34}$$

where u_t and $v_{s,t}$ are solutions of (12) and (14), respectively, and $v_t = v_{0,t}$.

Note that if the function b satisfies the conditions of Theorem 9, then the process is well defined in (34). By using the approximation of the mappings u_t and $v_{s,t}$ proposed in Theorem 9, we can establish the stochastic differentiability of the mappings $(u_t(x) - x)$ and $(v_{s,t}(x) - x)$. By analogy with the well-known finite-dimensional argument [25], we obtain the following theorem:

Theorem 10. *Suppose that a measure* μ *and a function* b *satisfy the conditions of Theorem 9,* $a \in L_\infty([0,1] \times X, dt \times d\mu)$, $f \in HC^1$, $a_t \in HC^1$ *for any* $t \in [0,1]$, *and* $\sup_t \| a_t \|_{W^1_\infty} < \infty$.

Then the random process ξ_t *defined in (34) satisfies relation (33) for all* $t \in [0,1]$ μ*-almost everywhere.*

We describe more exactly equations with the operator I considered in what follows.

We set $H = L_2([0,1])$. Let $a = a_t(x)$ and $b = b_t(x)$ be measurable mappings from $[0,1] \times X$ into $\mathbb{R}$ and let ξ_0 be a random variable.

We consider the equation

$$\xi_t = \xi_0 + \int_0^t a_s \xi_s\, ds + I(\mathbb{1}_{[0,t]}(\cdot)\, b.\xi.), \quad t \in [0,1]. \tag{35}$$

Remark 10. If μ is the distribution of a Wiener process $\omega(t)$, $t \in [0,1]$, in the space $X = C_0([0,1])$ of continuous functions originating at zero and $H = L_2([0,1])$ is imbedded into X by the operator $j : h \mapsto \int_0^{\cdot} h(s)\, ds$, then Eq. (35) takes the form

$$\xi_t = \xi_0 + \int_0^t a_s \xi_s\, ds + \int_0^t b_s \xi_s\, dw_s, \tag{36}$$

where the integral with respect to the Wiener process is a generalized Skorokhod integral.

Formally, we have the equality

$$If = -\langle \rho, f \rangle - \operatorname{tr} Df.$$

However, generally speaking, for a function f from $W^1_p(X, H, \mu)$, the trace of the operator $\operatorname{tr} Df$ is not defined and an H-valued random element ρ such that $\langle \rho, h \rangle_H = \rho_h$, μ-almost everywhere also does not exist.

Let us apply the method of characteristics to the solution of (35) (first, we do this without mathematical justification).

The equation for the characteristic takes the form

$$u_t(x) = x + j(\mathbb{1}_{[0,t]}(\cdot)\, b.(u.(x))), \tag{37}$$

where $t \in [0, 1]$ and $x \in X$, and a solution of (35) has the form

$$\xi_t(x) = \xi_0(v_t(x)) \exp\left(\int_0^t a_s(v_{s,t}(x))\,ds\right) \exp\left(-\int_0^t \sum_{k=1}^{\infty} \left[D_{e_k} b_s(v_{s,t}(x)) + b_s(v_{s,t}(x))\, \rho_{e_k}(v_{s,t}(x))\, e_k(s)\right] ds\right), \quad (38)$$

where $v_t \colon X \to X$ is the mapping inverse to u_t, $v_{s,\,t} = u_s \circ v_t$, and $\{e_k \colon k \geq 1\}$ is an orthonormal basis in $H = L_2([0,1])$.

The third factor in the expression is similar to the Radon–Nikodym density $L_t := \dfrac{d\mu \circ (v_t)^{-1}}{d\mu}$ [cf. (14)], and formula (38) takes the form

$$\xi_t = \xi_0(v_t) \exp\left(\int_0^t a_s(v_{s,t})\,ds\right) L_t. \quad (39)$$

Note that, in fact, the heuristic formula (39) determines a solution of Eq. (35) in the case of Eq. (36) with a generalized stochastic Skorokhod integral with respect to a Wiener process [26].

Further, we prove an analogous fact for smooth measures that satisfy condition **B**. For this purpose, we present conditions for the existence and uniqueness of a solution of (37), establish the invertibility of u_t and the absolute continuity $\mu \circ \left(v_{s,\,t}\right)^{-1} \ll \mu$, and prove that a random process defined by (39) is a solution of (35).

Theorem 11. *Suppose that a measure* μ *satisfies condition* ***B*** *and a measurable function* $b = b_t(x) \colon [0, 1] \times X \to \mathbb{R}$ *satisfies the following conditions:*

(i) for almost all $t \in [0, 1]$, *the function* $b_t \colon X \to \mathbb{R}$ *belongs to* WL^1_∞;

(ii) $\operatorname{ess\,sup}_{t \in [0,1]} \| b_t \|_{\infty,1} < \infty$.

Then the following assertions are true:

(a) there exist measurable functions $u = u_t(x) \colon [0, 1] \times X \to \mathbb{R}$ *and* $v = v_{s,\,t}(x) \colon \{(s, t, x) \,|\, 0 \leq s \leq t \leq 1,\ x \in X\} \to \mathbb{R}$ *and a set* $X_0 \subset X$ *of complete measure such that relation (37) and the equality*

$$v_{s,t}(x) = x - j(\mathbb{1}_{[s,t]}(\cdot)\, b_{\cdot}(v_{\cdot,t}(x))), \qquad s \in [0, t], \quad (40)$$

are true for all $x, s, t \colon x \in X_0,\ 0 \leq s \leq t \leq 1$;

(b) for all $s, t, 0 \leq s \leq t \leq 1$, *the measures* μ, $\mu \circ (u_t)^{-1}$, *and* $\mu \circ \left(v_{s,\,t}\right)^{-1}$ *are equivalent;*

(c) the mapping $v_t := v_{0,t}$ *is inverse almost everywhere to* u_t: $v_t(u_t(x)) = u_t(v_t(x)) = x$ *for* μ*-almost all* x; $v_{s,t} = u_s \circ v_t$ μ*-almost everywhere;*

(d) solutions of Eqs. (37) and (40) are unique to within stochastic equivalence.

Remark 11. Equations of the form (37), (40) are sometimes called equations with singular drift. In the Gaussian case, they were investigated in [26–28], where results similar to Theorem 11 were obtained. The method of conditional mathematical expectation, which was efficiently applied to Gaussian measures in the works cited above, cannot be directly used in the case under consideration.

Proof of Theorem 11. Without loss of generality, we can assume that

$$\exists c>0 \ \ \forall t\in[0,1],\ x\in X,\ h\in H:\quad |b_t(x+j(h))-b_t(x)| \ \le\ c\,\|h\|_H, \tag{41}$$

$$\exists \beta_0>0 \ \ \forall t\in[0,1],\ x\in X:\quad \sup_{t,x}|b(t,x)| \ \le\ \beta_0. \tag{42}$$

In this case, for any $x \in X$, Eqs. (37) and (40) have unique solutions, which can be obtained by the method of successive approximations. We verify this for Eq. (37).

We define a sequence $\{u_n = u_{n,t}(x)\colon n \ge 0\}$ in the following way:

$$u_{0,t}(x) \ := x, \qquad t\in[0,1], \quad x\in X;$$

$$u_{n+1,t}(x) \ := \ x \ + \ j(\mathbb{1}_{[0,t]}(\cdot)\, b_\cdot(u_{n,\cdot}(x))), \quad n \ \ge \ 0.$$

Then

$$\|u_{n+1,t}(x) - u_{n,t}(x)\|_H^2 \ = \ \int_0^t (b_s(u_{n,s}(x)) - b_s(u_{n-1,s}(x)))^2 ds \ \le \ c\int_0^t \|u_{n,s}(x) - u_{n-1,s}(x)\|_H^2 ds.$$

Using the inequality obtained and the argument standard for the method of successive approximations, we establish the existence of the required solution. The uniqueness is established in the standard way.

Let $\{e_n\colon n\ge 1\}$ be an orthonormal basis in $L_2([0,1])$ that consists of Haar functions [29] and let P_n be the orthoprojector onto $H_n := \langle e_1,\dots,e_n\rangle$. We consider the auxiliary sequences of equations

$$u_t^{(n)}(x) \ = \ x \ + \ j\big[P_n(\mathbb{1}_{[0,t]}(\cdot)\, b_\cdot(u_\cdot^{(n)}(x)))\big], \quad t\in[0,1], \tag{43}$$

$$v_{s,t}^{(n)}(x) \ = \ x \ - \ j\big[P_n(\mathbb{1}_{[s,t]}(\cdot)\, b_\cdot(v_{\cdot,t}^{(n)}(x)))\big], \quad s\in[0,t]. \tag{44}$$

By analogy with Sec. 4, we decompose the space X into the sum $X = Y_n \oplus j(H_n) \ = \ Y_n \times R^n$. Then Eqs. (43) and (44) take the form

$$u_t^{(n)}(x) \ = \ \left(y_n,\ x_n + \int_0^t b_s(u_s^{(n)}(x))e^{(n)}(s)\,ds \right), \tag{45}$$

$$v_{s,t}^{(n)}(x) \ = \ \left(y_n,\ x_n - \int_s^t b_z(v_{z,t}^{(n)}(x))e^{(n)}(z)\,dz \right), \tag{46}$$

where $x = (y_n, x_n) \in X$, $e^{(n)} = (e_1,\dots,e_n) \in L_2\big([0,1],\mathbb{R}^n\big)$.

By analogy with the proof of Theorem 9, we establish that the measures μ, $\mu \circ \left(u_t^{(n)}\right)^{-1}$, and $\mu \circ \left(v_{s,t}^{(n)}\right)^{-1}$ are equivalent, $v_t^{(n)}\left(u_t^{(n)}(x)\right) = u_t^{(n)}\left(v_t^{(n)}(x)\right) = x$ for μ-almost all x, and $v_s \circ u_t = v_{s,\,t}$ μ-almost everywhere. We prove the uniform integrability of the family of densities

$$\left\{ \frac{d\mu \circ (u_t^{(n)})^{-1}}{d\mu} : \; n \geq 1, \; t \in [0,1] \right\}.$$

For this purpose, it suffices to verify that

$$\sup_{t,n,x} \left\| u_t^{(n)}(x) - x \right\|_H < \infty, \tag{47}$$

$$\sup_{n,t} \operatorname*{ess\,sup}_{x} \left\| \nabla_{\mathbb{R}^n} (u_t^{(n)}(x) - x) \right\|_{\mathbb{R}^n \otimes \mathbb{R}^n} < \infty.$$

Relation (47) follows from the estimate

$$\left\| u_t^{(n)}(x) - x \right\|_H^2 \leq \int_0^t b^2(s, u_s^{(n)}(x)) \leq \beta_0^2 t,$$

where β_0 is the constant from (42).

By using Lemma 3, we establish that $u_t^{(n)}$ satisfies the Lipschitz condition along the space $\mathbb{R}^n$ and, for μ-almost all $x \in X$, the following relations are true:

$$\nabla_{\mathbb{R}^n} u_t^{(n)}(x) = \left(0, \; id_{\mathbb{R}^n} + \int_0^t e^{(n)}(s) \nabla_{\mathbb{R}^n} b\left(s, u_s^{(n)}(x)\right) ds \right) = P_n\left(\mathbb{I}_{[0,t]}(\cdot)\, \nabla_{\mathbb{R}^n} b\left(\cdot, u_\cdot^{(n)}(x)\right) \nabla_{\mathbb{R}^n} u_\cdot^{(n)}(x) \right).$$

Hence, for almost all $(t, x) \in [0,1] \times X$, we have

$$\left\| \nabla_{\mathbb{R}^n} (u_t^{(n)}(x) - x) \right\|_{\mathbb{R}^n \otimes \mathbb{R}^n} \leq 2\beta_1^2 \int_0^t \left\| \nabla_{\mathbb{R}^n} (u_s^{(n)}(x) - x) \right\|_{\mathbb{R}^n \otimes \mathbb{R}^n} ds + 2\beta_1^2 t,$$

where $\beta_1 = \operatorname*{ess\,sup}_{t,x} \| Db_t(x) \|_H$. Therefore,

$$\sup_{n,t} \operatorname*{ess\,sup}_{x} \left\| \nabla_{\mathbb{R}^n} (u_t^{(n)}(x) - x) \right\|_{\mathbb{R}^n \otimes \mathbb{R}^n}^2 \leq 2\beta_1^2 e^{2\beta_1^2}$$

and the uniform integrability of the densities

$$\left\{ \frac{d\mu \circ (u_t^{(n)})^{-1}}{d\mu} : \; n \geq 1, \; t \in [0,1] \right\}$$

is proved.

To prove the absolute continuity $\mu \circ (u_t)^{-1} \ll \mu$, it suffices to show (Theorem 3) that, for any t and x, we have $u_t^{(n)}(x) \to u_t(x)$, $n \to \infty$.

Indeed,

$$\begin{aligned}\left\| u_t(x) - u_t^{(n)}(x) \right\|^2_{L_2([0,1])} &= \left\| \mathbb{1}_{[0,t]}(\cdot)\, b(\cdot, u_\cdot(x)) - P_n\big(\mathbb{1}_{[0,t]}(\cdot)\, b(\cdot, u_\cdot^{(n)}(x))\big) \right\|^2_{L_2([0,1])} \\ &\le 2\left\| (id_H - P_n)\big(\mathbb{1}_{[0,t]}(\cdot)\, b(\cdot, u_\cdot(x))\big) \right\|^2_{L_2([0,1])} \\ &\quad + 2\left\| P_n\left[\mathbb{1}_{[0,t]}(\cdot)\big(b(\cdot, u_\cdot(x)) - b(\cdot, u_\cdot^{(n)}(x))\big) \right] \right\|^2_{L_2([0,1])} \\ &\le o_x(1) + 2\beta_1^2 \int\limits_0^t \left\| u_s(x) - u_s^{(n)}(x) \right\|^2_{L_2([0,1])} ds,\end{aligned}$$

where, for any $x \in X$, we have $o_x(1) \to 0$, $n \to \infty$.

Combining this and the Gronwall–Bellman lemma, we obtain the required convergence.

Similarly, for any s and t, $s \le t$, we have $\mu \circ (v_{s,t})^{-1} \ll \mu$, and the existence is proved. The uniqueness is proved by analogy with the proof of Theorem 9.

Theorem 11 is proved.

For simplicity, we present the subsequent argument for the special case where $X = C_0([0,1])$, $H = L_2([0,1])$, and $j(h) = \int_0^{\cdot} h(s)\, ds$.

Definition 7. *A random process* ξ_t, $t \in [0, 1]$, *is called an* L_1*-solution of (35) if the following conditions are satisfied:*

(i) $\xi_t \in L_1(X, \mu)$, $t \in [0, 1]$; $a_\cdot \xi_\cdot, b_\cdot \xi_\cdot \in L_1([0,1] \times X, dt \times d\mu)$;

(ii) for any f from the class

$$\mathcal{F}C_b^\infty = \left\{ \varphi(\langle x_1^*, \cdot\rangle, \ldots, \langle x_n^*, \cdot\rangle) \colon n \in \mathbb{N},\ x_i^* \in X^*,\ \varphi \in C_b^\infty(\mathbb{R}^n) \right\},$$

the equality

$$\mathrm{E} f\left(\xi_t - \xi_0 - \int\limits_0^t b_s \xi_s\, ds \right) = \mathrm{E} \int\limits_0^t b_s \xi_s D_s f\, ds \tag{48}$$

is true.

Note that the right-hand side of (48) is well defined because $Df \in L_\infty([0, 1] \times X, dt \times d\mu)$. If the random function $f_{b\xi}(s) = b_s \xi_s$, $s \in [0, 1]$, is such that

$$\mathrm{E}\left(\int_0^1 (b_s\xi_s)^2\,ds\right)^{p/2} < \infty$$

for certain $p \geq 1$, then the right-hand side of (48) can be written as $\mathrm{E}(f_{b\xi}, Df)_H$ and condition (ii) of Definition 7 is equivalent to the following: $f_{b\xi}\,\mathbb{1}_{[0,t]} \in \mathcal{D}(I_p)$ for any $t \in [0, 1]$, and equality (35) holds for any t μ-almost everywhere.

Thus, Definition 7 is consistent with the definition of generalized stochastic integrals.

Theorem 12. *Suppose that a measure* μ *and a random function* b_t, $t \in [0, 1]$, *satisfy the conditions of Theorem 11 and* $a \in L_\infty([0,1]\times X,\, dt\times d\mu)$, $\xi_0 \in L_\infty(X, \mu)$. *Then the random function* ξ_t *defined by relation (39) is an* L_1*-solution of Eq. (35). Here,* $v_{s,t}$ *is a solution of (40),* $v_t = v_{0,t}$, *and* $L_t = d\mu\circ(v_t)^{-1}/d\mu$.

Proof. Assume that $f \in \mathcal{F}C_b^\infty$ and ξ_t is defined by relation (39). Recall that $v_{s,t} = u_s \circ v_t$ almost everywhere (see Theorem 11). Then, by using the change of variables $y = v_t(x)$, we obtain

$$\mathrm{E}f\left(\xi_t - \xi_0 - \int_0^t a_s\xi_s\,ds\right) = \mathrm{E}\left[f(u_t)\xi_0\exp\left(\int_0^t a_s(u_s)\,ds\right) - f\xi_0 - \xi_0\int_0^t a_s(u_s)f(u_s)\exp\left(\int_0^s a_z(u_z)\,dz\right)ds\right]. \tag{49}$$

By virtue of (37), for $f \in \mathcal{F}C_b^\infty$ and all $x \in X$ the following equality is true:

$$f(u_t(x)) = f(x) + \int_0^t [D_s f](u_s(x))\,b_s(u_s(x))\,ds, \qquad t \in [0, 1]. \tag{50}$$

By using (50), we integrate the third term in (49) by parts. As a result, we get

$$\begin{aligned}
\mathrm{E}f\left(\xi_t - \xi_0 - \int_0^t a_s\xi_s\,ds\right) &= \mathrm{E}\left[f(u_t)\xi_0\exp\left\{\int_0^t a_s(u_s)\,ds\right\} - \xi_0 f - \xi_0\left\{f(u_s)\exp\left(\int_0^s a_z(u_z)\,dz\right)\right\}\Bigg|_{s=0}^{s=t}\right.\\
&\qquad \left. + \,\xi_0\int_0^t [D_s f](u_s)\,b_s(u_s)\exp\left\{\int_0^s a_z(u_z)\,dz\right\}ds\right]\\
&= \mathrm{E}\int_0^t \xi_0 [D_s f](u_s)\,b_s(u_s)\exp\left\{\int_0^s a_z(u_z)\,dz\right\}ds\\
&= \mathrm{E}\int_0^t \xi_0(v_s)\exp\left\{\int_0^s a_z(v_{z,s})\,dz\right\}L_s b_s D_s f\,ds,
\end{aligned}$$

whence we conclude that the random process ξ_t, $t \in [0, 1]$, defined by formula (39) is an L_1-solution of (35). Theorem 12 is proved.

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