STROOCK-VARADHAN THEOREM FOR FLOWS GENERATED BY STOCHASTIC DIFFERENTIAL EQUATIONS WITH INTERACTION

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We prove a theorem that characterizes the support of a flow generated by a system of stochastic differential equations with interaction.

Consider the system of stochastic differential equations with interaction

$$\begin{cases} dx(u,t) &= \int_{\mathbb{R}^d} a(x(u,t),x(v,t))\mu(dv)dt+ \\ &+\sum_{k=1}^m \int_{\mathbb{R}^d} b_k(x(u,t),x(v,t))\mu_k(dv))\circ dw^k(t),\ t\in[0,T] \\ x(u,0) &= \varphi(u),\ u\in\mathbb{R}^d, \end{cases} \tag{1}$$

where $a:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$, $b_k:\mathbb{R}^d\to\mathbb{R}^d$, $\varphi:\mathbb{R}^d\to\mathbb{R}^d$, μ,μ_k are probability measures on $\mathcal{B}(\mathbb{R}^d)$ and $w^1(t),\ldots,w^m(t)$ are independent one-dimensional Wiener processes.

Under some natural conditions on the smoothness and growth of the coefficients a and b_k of the equation and the function φ [1], there exists a unique solution of Eq.(1). In this case, a certain modification of $x(u,t)$ has derivatives with respect to u, which are continuous in (u,t) almost surely. Thus, the flow $\{x(u,t),u\in\mathbb{R}^d\}_{t\in[0,T]}$ can be regarded as a random element in the space $C([0,T]\to C^n(\mathbb{R}^d;\mathbb{R}^d))$, , where the space $C^n(\mathbb{R}^d,\mathbb{R}^d)$ of functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ that are n times continuously differentiable is equipped with the topology of uniform convergence of derivatives on compacta.

As proved by Stroock and Varadhan (see, e.g., [2]), the support of a diffusion process $x(t),t\in[0,T]$, that satisfies the equation

$$\begin{cases} dx(t) &= a(x(t))dt+\sum_{k=1}^m b_k(x(t))\circ dw^k(t),\ t\in[0,T] \\ x(0) &= x_0 \end{cases}$$

coincides with the closure of the set of functions $\Big\{x^\psi \mid \psi = (\psi_1, \dots, \psi_m) \in C^2([0,T];\mathbb{R}^m)\Big\}$ in the space $C([0,T],\mathbb{R}^d)$; here, x^ψ is defined by the ordinary differential equation

$$\begin{cases} \frac{dx^\psi(t)}{dt} & = a(x^\psi(t)) + \sum_{k=1}^m b_k(x^\psi(t)),\ t \in [0,T] \\ x^\psi(0) & = x_0 \end{cases}$$

An analogous statement on a flow of diffeomorphisms generated by a stochastic differential equation was proved in [3]. The main result of the present paper is the proof of an analog of the Stroock-Varadhan theorem for flows generated by the stochastic equation with interaction (1).

We introduce the following notation: $\|f\|_T := \sup_{t\in[0,T]} \|f(t)\|$, where the function f acts from $[0,T]$ into the Euclidean space $\mathbb{R}^n$;

$\|g\|_U := \sup_{u\in U} \|g(u)\|$, for $g : U \subset \mathbb{R}^d \to \mathbb{R}^d$;

$\|h\|_{U,T} := \sup_{u\in U} \sup_{t\in[0,T]} \|h(u,t)\|$, for $h : U \times [0,T] \to \mathbb{R}^d$.

In what follows, as a rule, we omit the indices T and U. For example, $\|w\|$ is understood as $\|w\|_T = \sup_t \|(w(t), \dots, w_m(t))\|_{\mathbb{R}^m}$.

Let $\psi = (\psi_1, \dots, \psi_m) :\ [0,T] \to \mathbb{R}^m$ be a continuously differentiable function.

We denote by x^ψ a solution of the following system of differential equations:

$$\begin{cases} \frac{\partial x^\psi(u,t)}{\partial t} & = \int_{\mathbb{R}^d} a(x^\psi(u,t), x^\psi(v,t))\mu(dv) + \\ & + \sum_{k=1}^m \int_{\mathbb{R}^d} b_k(x^\psi(u,t), x^\psi(v,t))\mu_k(dv))(\psi^k)'(t),\ t \in [0,T] \\ x^\psi(u,0) & = \varphi(u). \end{cases} \tag{2}$$

Theorem 1. *Suppose that the following conditions are satisfied:*

(i) $\varphi \in C^n(\mathbb{R}^d, \mathbb{R}^d)$;

(ii) the functions a and $b_k, k = 1, \dots, m$ are continuously differentiable with respect to the first variable $(n+2)$ and $(n+4)$ times, respectively, and are bounded together with their derivatives;

(iii) the functions a and b_k and all their derivatives satisfy the global Lipschitz condition with respect to the second variable, i.e.,

$$\exists L > 0\ \forall u, v_1, v_2 \in \mathbb{R}^d\ \forall l = 0, \dots, n+2,\ p = 0, \dots, n+4 :$$

$$\|\frac{\partial^l a(u,v_1)}{\partial u^l}-\frac{\partial^l a(u,v_2)}{\partial u^l}\|\le L\|v_1-v_2\|,$$
$$\|\frac{\partial^p b_k(u,v_1)}{\partial u^p}-\frac{\partial^p b_k(u,v_2)}{\partial u^p}\|\le L\|v_1-v_2\|;$$

(iv) μ and μ_k are probability measures on $\mathbb{R}^d$.

Then, for any function $\psi\in C^2([0,T],\mathbb{R}^m)$, a bounded set $U\subset\mathbb{R}^d$ and $\theta>0$ one has

$$\lim_{\delta\to 0}P\left(\sup_{u\in U}\sup_{t\in[0,T]}\sup_{0\le l\le n}\|\frac{\partial^l x(u,t)}{\partial u^l}-\frac{\partial^l x^\psi(u,t)}{\partial u^l}\|\ge\theta\ /\ \|w-\psi\|<\delta\right)=0. \tag{3}$$

The main result of the present paper is the theorem below, which is a direct consequence of relation (3).

Theorem 2. *The support of the distribution $x(u,t), u\in\mathbb{R}^d,\ t\in[0,T]$ contains the closure of the set $\{x^\psi|\psi\in C^2([0,T];\mathbb{R}^m)\}$ in the space $C([0,T]\to C^n(\mathbb{R}^d;\mathbb{R}^d))$.*

Remark 1. The author knows the proof of the inverse inclusion. The proof is based, as in [2, 3], on the approximation theorem and will be presented in a separate paper.

Proof of Theorem 1. In [1], it was proved that if the conditions of Theorem 1 are satisfied, then the process $x(u,t)$ has an n-times continuously differentiable modification. In what follows, we consider exactly this modification. It is also known that the derivatives $\frac{\partial^k x(u,t)}{\partial u^k}$ satisfy the equations obtained by formally differentiating (1) k times.

First, consider the case $n=0$. Assume that $\psi=0$. The case of arbitrary ψ can be proved with the use of the Girsanov theorem by analogy with the theorem on the support of diffusion processes (Theorem 8.2 in [2, Chap. 6]). We restrict ourselves to the case where $U\subset\mathbb{R}^d$ is a closed ball.

Let us verify that, for any $\varepsilon>0$ we have

$$P\left(\left\|\int_0^\cdot\int b_k(x(u,t),x(v,t))\mu_k(dv)\circ dw^k(t)\right\|_{U,T}\ge\varepsilon/\|w\|<\delta\right)\to 0,\delta\to 0+. \tag{4}$$

It is known [4], that, for any $p > d$, the Sobolev space $W_p^1(U)$ is continuously imbedded into $C(U)$. Hence, to prove (4), it suffices to verify that

$$E\left(\sup_t \int_U \left(\| \int_0^t \int b_k(x(u,t),x(v,t))\mu_k(dv) \circ dw^k(s)\|^p + \right.\right. \tag{5}$$

$$\left.\left.\| \int_0^t \int (b_k)'_1(x(u,s),x(v,s))\mu_k(dv)\frac{\partial x(u,s)}{\partial u} \circ dw^k(s)\|^p\right) du / \|w\| < \delta\right) \to 0, \delta \to 0+.$$

Estimate (5) follows from the relations

$$\sup_{u\in U} E\left(\left\|\int_0^\cdot \int b_k(x(u,t),x(v,t))\mu_k(dv) \circ dw^k(s)\right\|_T^p / \|w\| < \delta\right) \to 0, \delta \to 0+, \tag{6}$$

$$\sup_{u\in U} E\left(\left\|\int_0^\cdot \int (b_k)'_1(x(u,s),x(v,s))\mu_k(dv)\frac{\partial x(u,s)}{\partial u} \circ dw^k(s)\right\|_T^p / \|w\| < \delta\right) \to 0, \delta \to 0+. \tag{7}$$

We need the following definition:

Definition. *A semimartingale $b(t), t \in [0,T]$,*

$$db(t) = b_0(t)dt + b_k(t)dw^k(t)$$

belongs to the class B_0, if there exists a constant β, such that, for any $k \in \{0,\dots,m\}$, $t \in [0,T]$ we have

$$|b_k(t)| \le \beta_0 \quad a.s.$$

If each diffusion characteristic $b_k(t), 1 \le k \le$, is a semimartingale from B_0, then we say that $b(t)$ belongs to the class B_1.

Similarly, if

$$db_k(t) = b_{k,0}(t)dt + b_{k,j}(t)dw^j(t),$$

where $b_{k,j}(t)$ belongs to $B_0, 1 \le k,j \le m$, then $b(t), t \in [0,T]$ belongs to the class B_2, and so on.

Lemma 1. *Suppose that a semimartingale $b(t), t \in [0,T]$ belongs to the class B_3. Then there exist positive constants K_1, $K_2 = K_2(b)$, $K_3 = K_3(B)$, and a positive non-decreasing function $M = M(\delta)$ such that $M(\delta) \to 0$ as $\delta \to 0+$ and*

$$\forall \varepsilon > 0\ \exists \delta_0 > 0\ \forall \delta \in (0,\delta_0)\ \forall M > M(\delta) \forall k \in \{1,\dots,m\} :$$

$$P\left(\|\int_0^{\cdot} b(s)\circ dw^k(s)\|_T \geq \varepsilon/\|w\|<\delta\right)\leq$$

$$\leq K_1\exp\left(-\frac{\min\left\{\frac{\varepsilon^2}{K_2};K_1M^2;\frac{\varepsilon^2}{K_3M^2}-1\right\}}{\delta^2}\right). \tag{8}$$

Remark 2. It follows from the Stroock–Varadhan theorem that the probability in (8) tends to zero as $\delta\to 0+$. However, we need more exact asymptotics.

Proof of Lemma 1. We apply the formula of integration by parts to the stochastic integral

$$\int_0^t b(s)\circ dw^k(s)=b(t)w^k(t)-\int_0^t w^k(s)\tilde{b}_0(s)ds-$$

$$-\int_0^t b_j(s)w^k(s)\circ dw^j(s)=I_1(t)+I_2(t)+I_3(t). \tag{9}$$

Here,

$$db(t)=b_0(t)dt+b_j(t)dw^j(t)=\tilde{b}_0(t)dt+b_j(t)\circ dw^j(t),$$

and the process $\tilde{b}_0(t), t\in[0,T]$ is obtained by using the Ito formula.

In the proof, we use many constants. To avoid the use of numerous different indices, we write a dot instead of them. For example, $K.$ and $K.(b)$, are understood, respectively, as a universal constant (possibly, dependent on d or T) and a constant dependent only on the properties of the semimartingale b. Note that

$$\|I_1\|\leq K.(b)\delta \text{ and } \|I_2\|\leq K.(b)\delta \tag{10}$$

on the set $\{\|w\|<\delta\}$.

Denote $\xi^{kj}(t):=\int_0^t w^k(s)\circ dw^j(s)$.

Then

$$I_3(t)=-b_j(t)\xi^{kj}(t)+\int_0^t\xi^{kj}(s)b_{j,0}(s)ds+\int_0^t\xi^{kj}(s)b_{j,l}(s)\circ dw^l(s)=$$

$$=\mathfrak{L}_1(t)+\mathfrak{L}_2(t)+\mathfrak{L}_3(t). \tag{11}$$

In the integral $\mathfrak{L}_3$ we pass from the Stratonovich integral to the Ito integral and apply the Ito formula once again. As a result, we get

$$\mathfrak{L}_3(t) = \alpha(\xi^{kj}, b)(t) - \int_0^t \xi^{kj}(s) w^l(s) b_{j,l,i}(s) dw^i(s) -$$

$$- \int_0^t b_{j,l}(s) w^l(s) d\xi^{kj}(s), \tag{12}$$

where $\|\alpha(\xi^{kj}, b)(\cdot)\|_T \leq K.(b)\|\xi^{kj}\|_T$.

Let $M > 0$ be a certain number, let $\beta(t), t \in [0, T]$ be a non-anticipating process, and let $|\beta(t)| \leq B$ a.s., where $B = \text{const} > 0$.

Then

$$P\left(\| \int_0^{\cdot} \xi^{kj}(s) w^l(s) \beta(s) dw^i(s) \geq \varepsilon / \|w\| < \delta\right) \leq$$

$$\leq P\left(\|\xi^{kj}\| \geq M / \|w\| < \delta\right) +$$

$$+ P\left(\|\xi^{kj}\| < M, \| \int_0^{\cdot} \xi^{kj}(s) w^l(s) \beta(s) dw^i(s)\| \geq \varepsilon / \|w\| < \delta\right). \tag{13}$$

Lemma 2. *There exist positive constants c_1 and c_2 such that, for any $M > 0, \delta > 0$ and $M > \delta^2$, the following inequality is true:*

$$P\left(\|\xi^{kj}\| \geq M / \|w\| < \delta\right) \leq \frac{c_1 \delta}{M} exp\left\{-\frac{c_2 M^2}{\delta^2}\right\}. \tag{14}$$

Proof. Assume that $k \neq j$ and

$$\eta^{kj}(t) = \frac{1}{2} \int_0^t [w^k(s) dw^j(s) - w^j(s) dw^k(s)],$$

$$a(t) = \frac{1}{4} \int_0^t [(w^k)^2(s) + (w^j)^2(s)] ds.$$

It is proved in [2, Chap. 6, Sec. 6], that $B(t) \equiv \eta^{kj}(a^{-1}(t))$ is a Brownian motion independent of $\{(w^k)^2(t) + (w^j)^2(t)\}$. Therefore, by analogy with Lemma 8.2 in [2], we obtain

$$P\{\|\eta^{kj}\|_T \geq M / \|w\| < \delta\} = P\{\|B(a(t))\|_T \geq M / \|w\| < \delta\} \leq$$

$$\leq P\{\max_{0 \leq s \leq T\delta^2/4} |B(s)| > M\} <$$

$$< 4P\{B(T\delta^2/4) > M\} = \frac{4}{\sqrt{2\pi}} \int_{\frac{2M}{\sqrt{T}\delta}}^{\infty} e^{-\frac{x^2}{2}} dx \le \frac{2\sqrt{T}\delta}{M} \exp\left\{-\frac{2M^2}{T\delta^2}\right\}. \tag{15}$$

Inequality (14) follows directly from relation (15) in view of the fact that

$$\xi^{kj}(t) = -\eta^{kj}(t) + \frac{1}{2} w^k(t) w^j(t).$$

If $k = j$, then the statement of Lemma 2 is obvious because $\xi^{kk}(t) = \frac{1}{2}(w^k)^2(t)$.

Lemma 2 is proved.

Lemma 3 [2, Lemma 8.1 in Chap. 6]. *There exist constants c_3 and c_4, such that*

$$P(\|w\| < \delta) \sim c_3 \exp\left(\frac{-c_4}{\delta^2}\right), \delta \to +0.$$

To estimate the right-hand side of (13), we use the following known fact: If $M(t)$ is a square-integrable martingale with characteristic $\langle M(t)\rangle$, and continuous trajectories and $M(0) = 0$, then there exists a Brownian motion W (possibly, on an extension of the probability space) such that $M(t) = W(\langle M\rangle(t))$. Hence, the second term on the right-hand side of (13) does not exceed the following value:

$$P\{\sup_{t\in[0,T]} \|W(tB^2\delta^2M^2)\| \ge \varepsilon\}/P\{\|w\| < \delta\} \le$$

$$\le P\left\{\sup_{s\in[0,T]} \|W(s)\| \ge \frac{\varepsilon}{B\delta M}\right\} \Big/ P\{\|w\| < \delta\} \le$$

$$\le K. \exp\left\{\frac{\left(\frac{\varepsilon^2}{B^2M^2} - 1\right) K.}{\delta^2}\right\}.$$

The last term in (12) can be examined by analogy with regard for the fact that

$$\int_0^t b_{j,l}(s) w^l(s) d\xi^{kj}(s) = \int_0^t b_{j,l}(s) w^l(s) w^k(s) dw^j(s) +$$

$$+\frac{\delta_{k,j}}{2}\int_0^t b_{j,l}w^l(s)ds.$$

The statement of Lemma 1 now follows from estimates (9)–(13).

We continue the proof of Theorem 1. Let us verify the convergence of (6). Let $u \in U$. Note that the random process $\int b_k(x(u,t), x(v,t))\mu_k(dv), t \in [0,T]$, satisfies the conditions of Lemma 1. Set $\xi(u) = \| \int_0^{\cdot} \int b_k(x(u,s), x(v,s))\mu_k(dv) \circ dw^k(t)\|$. Hence,

$$E\left(\xi(u)^p / \|w\| < \delta\right) \leq$$

$$\leq p\int_0^1 y^{p-1} P(\xi(u) > y/\|w\| < \delta)dy +$$

$$+\frac{E\left(\xi(u)^p \mathbb{1}_{\{\xi(u)\geq 1\}}\mathbb{1}_{\{\|w\|<\delta\}}\right)}{P(\|w\| < \delta)} = I_1(\delta) + I_2(\delta). \qquad (16)$$

It is easy to see [see estimate (8)] that, for any $y > 0$, we have

$$\sup_{u\in U} P\left(\left\|\int_0^{\cdot}\int b_k(x(u,s), x(v,s))\mu_k(dv)\circ dw^k(t)\right\| > y/\|w\| < \delta\right) \to 0, \delta \to 0+.$$

Hence, by virtue of the Lebesgue dominated-convergence theorem, we have convergence to zero in (16) uniform with respect to $u \in U$.

Note that $\sup_u E\xi(u)^p < \infty$ for each $p > 0$.

Therefore, due to Lemmas 1 and 3:

$$I_2(\delta) \leq \left(E\xi(u)^\alpha\right)^{1/\alpha}\left[P(\xi(u) > 1, \|w\| < \delta)\right]^{1/\beta} / P(\|w\| < \delta) =$$

$$= c.\left(P(\xi(u) > 1/\|w\| < \delta)\right)^{1/\beta}\left(P(\|w\| < delta)\right)^{-1/\alpha} \leq$$

$$\leq \exp\left\{\frac{-\frac{1}{\beta}\left(\min\{\frac{1}{K_2}, K_1 M, \frac{1}{K_3 M^2} - 1\}\right) + \frac{c.}{\alpha}}{\delta^2}\right\}, \qquad (16.5)$$

where $1/\alpha + 1/\beta = 1$.

Choose M and α so that the expression in the numerator of (16.5) be negative and let $\delta \to 0+$. Then $I_2(\delta) \to 0$.

Note that the integrand in (7) does not satisfy the conditions of Lemma 1 because, generally speaking, $\frac{\partial x(u,t)}{\partial u}$ is unbounded. Therefore, we need some additional arguments.

Assume that $c > 0$ and $f_c : \mathbb{R}^d \to \mathbb{R}^d$ is a finite, nonnegative, infinitely differentiable function such that $f_c(x) = x$ for $|x| \leq c$.

Denote by $y(u,t)$ a solution of the equation

$$\begin{cases} dy(u,t) & = \int_{\mathbb{R}^d} a_1'(x(u,t),x(v,t))\mu(dv)f_c(y_c(u,t))dt + \\ & \sum_{k=1}^m \int (b_k)_1'(x(u,s),x(v,s))\mu_k(dv)f_c(y_c(u,t)) \circ dw^k(t) \\ y_c(u,0) & = \nabla\varphi(u). \end{cases} \tag{17}$$

Note that

$$P\left(\{\omega : \|y_c(u,t,\omega)\| \le c, u \in U, t \in [0,T]\}\triangle\right.$$

$$\left.\triangle\{\omega : \|\frac{\partial x(u,t,\omega)}{\partial u}\| \le c, u \in U, t \in [0,T]\}\right) = 0$$

and, on the set $\left\{\sup_{u\in U, t\in[0,T]} \|\frac{\partial x(u,t)}{\partial u}\| \le c\right\}$ the following equality holds almost surely:

$y_c(u,t) = \frac{\partial x(u,t)}{\partial u}$ for all $t \in [0,T], u \in U$.

Assume that ω and u are such that

$$\|\sum_{k=1}^m \int_0^{\cdot} \int (b_k)_1'(x(u,s),x(v,s))\mu_k(dv)\frac{\partial x(u,t)}{\partial u} \circ dw^k(t)\|_T \le \varepsilon.$$

Then, applying the Gronwall-Bellman lemma to the equations for the derivative

$$\begin{cases} d\frac{\partial x(u,t)}{\partial u} & = \int_{\mathbb{R}^d} a_1'(x(u,t),x(v,t))\mu(dv)\frac{\partial x(u,t)}{\partial u}dt + \\ & \sum_{k=1}^m \int (b_k)_1'(x(u,s),x(v,s))\mu_k(dv)\frac{\partial x(u,t)}{\partial u} \circ dw^k(t) \\ \frac{\partial x(u,t)}{\partial u} & = \nabla\varphi(u). \end{cases}$$

we obtain

$$\|\frac{\partial x(u,t)}{\partial u}\| \le (\|\nabla\varphi(u)\| + \varepsilon)\exp(\|a_1'\|_\infty T). \tag{18}$$

Let $\varepsilon > 0$ be fixed. We choose $c > 0$, so that the right-hand side of (18) does not exceed $c$for all $u \in U$. Then $y_c(u,t) = \frac{\partial x(u,t)}{\partial u}$, $u \in U, t \in [0,T]$ for almost all ω from the set

$\{\| \sum_{k=1}^m \int_0^{\cdot} \int (b_k)_1'(x(u,s),x(v,s))\mu_k(dv)f_c(y_c(u,t)) \circ d\omega^k(t)\|_T \ge \varepsilon\}$. Denote

$$\xi_t(u) := \sum_{k=1}^m \int_0^t \int (b_k)_1'(x(u,s),x(v,s))\mu_k(dv)\frac{\partial x(u,s)}{\partial u} \circ d\omega^k(s).$$

Then

$$E(\|\xi_\cdot(u)\|_T^p/\|w\|<\delta)=E(\|\xi_\cdot(u)\|_T^p \mathbb{1}_{\{\|\xi_\cdot(u)\|_T<\varepsilon\}}/\|w\|<\delta)+$$
$$+E(\|\xi_\cdot(u)\|_T^p \mathbb{1}_{\{\|\xi_\cdot(u)\|_T\geq\varepsilon\}}/\|w\|<\delta)=I_1(\delta)+I_2(\delta).$$

The proof of the convergence of $I_1(\delta)$ to zero as $\delta\to 0$ is similar to the proof of (6) because

$$I_1(\delta)\leq E(\|\sum_{k=1}^m\int_0^\cdot\int b_k(x(u,s),x(v,s))\mu_k(dv)f_c(y_c(u,s))\circ dw^k(s)\|_T^p/\|w\|<\delta).$$

To estimate $I_2(\delta)$, we use the Holder inequality

$$I_2(\delta)\leq\frac{(E\|\xi_\cdot(u)\|_T^{p\alpha})^{1/\alpha}(P(\|w\|<\delta,\|\xi_\cdot(u)\|_T\geq\varepsilon))^{1/\beta}}{P(\|w\|<\delta)},\tag{19}$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

It is easy to verify that $K(U,\alpha,p)=(E\|\xi_\cdot(u)\|_T^{p\alpha})^{1/\alpha}<\infty$. Hence, the expression in (19) does not exceed the following value (see Lemmas 1 and 3):

$$K(U,\alpha,p)\sup_{u\in U}\frac{[P(\|\sum_{k=1}^m\int_0^\cdot\int b_k(x(u,s),x(v,s))\mu_k(dv)f_c(y_c(u,s))\circ d\omega^k(s)\|_T\geq\varepsilon/\|w\|<\delta)]}{[P(\|w\|<\delta)]^{1/\alpha}}\leq$$

$$\leq K(U,\alpha,p)\exp\left[\left\{-\frac{1}{\beta}(\frac{\theta^2}{K(b_k,U,\varepsilon,f_c)})\wedge K.M^2\wedge\right.\right.$$
$$\left.\left.\wedge\left(\frac{\theta^2}{K(b_k,U,\varepsilon,f_c)M^2}-1\right)+\frac{c_2}{\alpha}\right\}\frac{1}{\delta^2}\right].\tag{20}$$

We choose M and then α so that the quantity in the numerator in (20) is negative. Then $I_2(\delta)\to 0$ as $\delta\to 0+$. Thus, we have proved relation (7) and, consequently, relation (4).

We now pass to the proof of (3) for $\psi=0$. Let $\tilde U\subset\mathbb{R}^d$ be a closed ball and let $u\in\tilde U$. Denote $o(\tilde U):=\|\int_0^\cdot b_k(x(u,s))\circ dw^k(s)\|_{\tilde U,T}$. Then

$$\sup_{s\in[0,T]}\|x(u,s)-x^0(u,s)\|\leq$$

$$\leq\int_0^t\int_{\tilde U}\|a(x(u,s),x(v,s))-a(x^0(u,s),x^0(v,s))\|\mu(dv)ds+$$

$$+2\sup_{x,y}\|a(x,y)\|\mu(\mathbb{R}^d\setminus\tilde{U})+o(\tilde{U})\le$$

$$\le K.(a)(\int_0^t \sup_{z\in[0,s]}\|x(u,s)-x^0(u,s)\|ds+$$

$$+\int_0^t\int_{\tilde{U}}\sup_{z\in[0,s]}\|x(v,s)-x^0(v,s)\|\mu(dv)ds+\mu(\mathbb{R}^d\setminus\tilde{U}))+o(\tilde{U}). \quad (21)$$

Integrating (21) with respect to the parameter $u\in\tilde{U}$ and replacing u by v, we obtain

$$\int_{\tilde{U}}\sup_{s\in[0,t]}\|x(v,s)-x^0(v,s)\|\mu(dv)\le$$

$$\le K.(a)\left(\int_0^t\int_{\tilde{U}}\sup_{z\in[0,s]}\|x(v,s)-x^0(v,s)\|\mu(dv)ds+\mu(\mathbb{R}^d\setminus\tilde{U})\right)+o(\tilde{U}). \quad (22)$$

Applying the Gronwall-Bellman lemma to (22), substituting the expression obtained into (21), and applying the Gronwall–Bellman lemma once again, we get

$$\sup_{s\in[0,t]}\|x(u,s)-x^0(u,s)\|\le K.(a,T)(\mu(\mathbb{R}^d\setminus\tilde{U})+o(\tilde{U})), t\in[0,T].$$

Let $\theta>0$ and $\varepsilon>0$ be fixed. We choose a set $\tilde{U}$ that contains U so that

$$K.(a,T)\mu(\mathbb{R}^d\setminus\tilde{U})<\frac{\theta}{2}.$$

Then [see (4)]

$$\lim_{\delta\to0+}P\left(\sup_{t\imath[0,T]}\sup_{u\in U}\|x(u,s)-x^0(u,s)\|\ge\theta/\|w\|<\delta\right)\le$$

$$\le\lim_{\delta\to0+}P\left(K.(a,T)o(\tilde{U})\ge\frac{\theta}{2}/\|w\|<\delta\right)=0.$$

Theorem 1 is proved in the case $n=0$. The general case differs from the case $n=1$ insignificantly, and, therefore, we restrict ourselves to the question about the support of the random process $x(u,t)$ in the space

$C([0,T] \to C^1(\mathbb{R}^d,\mathbb{R}^d))$. We consider Eq. (1) and the equations for the derivative $y(u,t) := \frac{\partial x(u,t)}{\partial u}$,

$$\begin{cases} dy(u,t) & = [\int_{\mathbb{R}^d} a_1'(x(u,t),x(v,t))\mu(dv)]y(u,t)dt + \\ & \sum_{k=1}^m \int_{\mathbb{R}^d} (b_k)_1'(x(u,t),x(v,t))\mu_k(du)y(u,t) \circ d\omega^k(t) \\ y(u,0) & = \frac{\partial\varphi(u)}{\partial u}, \end{cases} \tag{23}$$

as a system of stochastic equations with interaction. Unfortunately, Eqs. (1) and (23) do not satisfy the conditions of Theorem 1 for $n=0$ because the coefficients in the equation for $y(u,t)$ are unbounded. To prove the theorem, we use the localization principle. Let $y_c(u,t)$ and f_c be the same as in (17).

Let

$$\tau_c(U) = \inf\{t|\exists u \in U, \|y(u,t)\| \geq c\},$$
$$\tilde{\tau}_c(U) = \inf\{t|\exists u \in U, \|y_c(u,t)\| \geq c\} \ -$$

be the exit times of $y(u,t)$ and $y_c(u,t), u \in U$, respectively, from the ball of radius c centered at the origin. Note that $\tau_c(U) = \tau_c(\tilde{U})$ a.s. and $y(u,t) = y_c(u,t)$ a.s. for all $u \in U, t \in [0,\tau_c(U)]$.

Let $\theta > 0$ be given. As in the beginning of the proof of Theorem 1, we assume that $\psi = 0$. Denote by $y^0(u,t)$ and $y_c^0(u,t)$ solutions of the following equations:

$$\frac{\partial y^0(u,t)}{\partial t} = \int_{\mathbb{R}^d} a_1'(x^0(u,t),x^0(v,t))\mu(dv)y^0(u,t)$$

$$\frac{\partial y_c^0(u,t)}{\partial t} = \int_{\mathbb{R}^d} a_1'(x^0(u,t),x^0(v,t))\mu(dv)f_c(y_c^0(u,t))$$

$$y^0(u,0) = y_c(u,0) = \frac{\partial\varphi(u)}{\partial u},$$

where x^0 is a solution of (2) for $\psi = 0$.

We choose $c > 0$ so that

$$\sup_{u\in U, t\in[0,T]} \|y^0(u,t)\| < c - \theta.$$

Then $y^0(u,t) = y_c^0(u,t)$ for all $u \in U$ and $t \in [0,T]$, and

$$\lim_{\delta\to 0} P(\|y - y^0\|_{U,T} \leq \theta / \|w\| < \delta) =$$

$$= \lim_{\delta\to 0} P(\|y - y^0\|_{U,T} \leq \theta, \tau_c(U) > T / \|w\| < \delta) =$$

$$= \lim_{\delta\to 0} P(\|y_c - y_c^0\|_{U,T} \le \theta / \|w\| < \delta) = 1,$$

because relations (1) and (17), regarded as a system of stochastic equations with values in $\mathbb{R}^{d+d^2}$, satisfy the conditions of Theorem 1 for $n = 0$.

Theorem 1 is proved.

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