SMOOTHNESS OF DISTRIBUTION FOR SOLUTIONS OF SDE'S WITH INTERACTION*

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Abstract. Consider the following system of stochastic differential equation with interaction

$$
\begin{aligned}
dx(u,t) = &\int b_0(x(u,t),x(v,t))\mu_0(dv)dt+ \\
&+\sum_{k=1}^{m} b_k(x(u,t),x(v,t))\mu_k(dv)d\omega_k(t),\ t\in[0,T]
\end{aligned}
\tag{1}
$$

$$x(u,t)=\varphi(u),\ u\in\mathbb{R}^d.$$

The conditions that guarantee the smoothness of $x(u,t)$ distribution are studied.

Consider the equation (1), where $b_k : \mathbb{R}^{2d} \to \mathbb{R}^d, k = \overline{1,m},\ \varphi : \mathbb{R}^d \to \mathbb{R}^d,\ \mu_k$ are probability measures on $\mathbb{R}^d$, $\omega_1(t),\dots,\omega_m(t)$ are independent one-dimensional Wiener processes. The aim of this article is to find some conditions that ensure smoothness of $x(u,t)$ distribution for fixed $u\in\mathbb{R}^d, t>0$.

The process $x(u,t), u\in\mathbb{R}$ can be considered as an random element in some infinite-dimensional space ($C(\mathbb{R},\mathbb{R})$ roughly). So our investigation can be studied as a part of the following problem. Let $\xi(t)$ be a solution of SDE

$$d\xi(t)=A(\xi(t))dt+B(\xi(t))dW(t)$$

in some multidimensional space E. Let π be a projection in E to some finite dimensional subspace of E. The problem is to prove the smoothness of $\pi\xi(t)$ distribution w.r.t. the Lebesgue measure on πE. There are some results that ensure differentiability of the $\xi(t)$ distribution w.r.t. the directions of some infinitedimensional subspace $H\subset E$ (cf. [1,2]). This property is more strong than differentiability of $\pi\xi(t)$. But reasoning of [1,2] cannot be applied in a case of the equation (1). Really, if we have only finite number of Wiener processes then (infinitedimensional) Malliavin's matrix cannot be invertible.

Let us introduce the following notations. Set

$$(B\psi)(u)=\|B_k\psi(u)\|_{k=\overline{1,m}}=\|\int_{\mathbb{R}^d} b_k(\psi(u),\psi(v))\mu_k(dv)\|_{k=\overline{1,m}} \tag{2}$$

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for each measurable function $\psi : \mathbb{R}^d \to \mathbb{R}^d$.

For fixed $u \in \mathbb{R}^d, \psi$ a map $B\psi(u)$ will be considered as a linear operator from $\mathbb{R}^d$ to $\mathbb{R}^m$ or as $d \times m$ matrix.

The main result of the article is the following theorem

Theorem 1. *Assume that*

1) *for each $k = 0, \dots, m$ a function $b_k : \mathbb{R}^{2d} \to \mathbb{R}^d$ is infinitely differentiable and bounded with all its derivatives;*
2) *for each $k = 0, \dots, m$ a probability measure μ_k has all positive moments;*
3) *a function $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is infinitely differentiable and $\sup_x \|\nabla^k \varphi(x)\| < \infty$ for each $k \in \mathbb{N}$.*

Suppose that for some orthoprojection $\pi : \mathbb{R}^d \to \mathbb{R}^d$ and a point $u \in \mathbb{R}^d$ the following condition is satisfied:
$\exists c > 0 \ \forall \psi : \mathbb{R}^d \to \mathbb{R}^d$

$$\pi B\psi(u)(B\psi(u))^* \pi^* \geq c\pi.$$

Then the distribution of $\pi x(u,t)$ in $\pi\mathbb{R}^d$ has C^∞-density with respect to the Lebesgue measure in $\pi\mathbb{R}^d$.

Remark 1. The result of a Theorem 1 can easily be generalized to a case of multiple interaction, i.e. if we consider terms

$$\int b_k(x(u,t), x(v_1,t), \dots, x(v_n,t))\mu_k(dv_1, \dots, dv_n)$$

instead of $\int b_k(x(u,t), x(v,t)\mu_k(dv)$. This can be done by extension of a space.

Remark 2. The existence and uniqueness of solution (1) is proved in [3]. Besides there exists a modification of $x(u,t)$ which is continuous in (u,t). Namely this modification is considered further.

Remark 3. There is one difference between research of smoothness of a solution of SDE or its projection. At first case it is natural to demand some non-degenerativity of coefficients in the starting point. But in the second case it is natural to require a non-degenerativity of coefficients in the terminal point (cf.[4]). The same effect appears in our situation also.

We need the following well-known theorem of the Malliavin calculus.

Theorem 2 [5]**.** *Let (X, μ, H) be an abstract Wiener space, $F = (F_1, \dots, F_d) : X \to \mathbb{R}^d$. Assume that*

1) *$F_1, \dots, F_d$ lie in $\cap_{p,k\geq 1} W_{p,k}$;*
2) *the Malliavin matrix $(DF_i, DF_j)_{i,j=1}^d$ is invertible a.e.;*
3) $(\det \|(DF_i, DF_j)\|)^{-1} \in \cap_{p\geq 1} L_p$.

Then the distribution of F has C^∞-density in $\mathbb{R}^d$.

Further, we consider $X = C_0([0,T], \mathbb{R}^m), X$ is a space of continuous $\mathbb{R}^m$-valued functions starting at zero. Let μ be the distribution of Wiener process $(\omega_1(t), \dots, \omega_m(t))$ in $X, H = L_2([0,T], \mathbb{R}^m)$. A Hilbert space H is imbedded to X by the inclusion map

$$H \ni h \mapsto \int_0^\cdot h(s)ds \in X.$$

Lemma 1. *The random elements $x(u,t)$ belongs to $\cap_{p,k\geq 1} W_{p,k}(\mathbb{R}^d)$ and their stochastic derivatives*

$$y_s(u,t) := D_s x(u,t)$$

satisfy the following SDE with interaction

(3)

$$y_s(u,t) = B(x(\cdot,s))(u)+$$

$$\int_s^t \int_{\mathbb{R}^d} ((b_0)'_1(x(u,z),x(v,z))y_s(u,z) + (b_0)'_2(x(u,z),x(v,z))y_s(v,z))\mu_0(dv)dz+$$

$$+\sum_{k=1}^m \int_s^t \int_{\mathbb{R}^d} ((b_k)'_1(x(u,z),x(v,z))y_s(u,z) + (b_k)'_2(x(u,z),x(v,z))y_s(v,z))\mu_k(dv)dw_k(z)$$

where B is defined in (2), $s \leq t$.

Besides

$\forall p \geq 1\ \exists C_p > 0\ \forall s_1, s_2 \in [0,T]\ \forall t \in [0,T], t \geq s_1 \vee s_2\ \forall u \in \mathbb{R}^d :$

$$E\|y_{s_1}(u,t)\|^p \leq C_p; \tag{4}$$

$$E\|y_{s_1}(u,t) - y_{s_2}(u,t)\|^p \leq C_p|t_1 - t_2|^{p/2}. \tag{5}$$

The proof of the Lemma 1 is quite standard. So let us only outline the main steps.

Step 1. $\forall p \geq 1\ \exists C_p\ \forall s_1, s_2 \in [0,T]\ \forall u_1, u_2 \in \mathbb{R}^d :$

$$E\|x(u,s_1)\|^p \leq C_p, \tag{6}$$

$$E\|x(u_1,s_1) - x(u_2,s_2)\|^p \leq C_p(|s_1 - s_2|^{p/2} + \|u_1 - u_2\|^{p/2}). \tag{7}$$

To prove this the technique of [6] for stochastic flows is used again with estimations of [3].

Step 2. There exists a solution of (3) that satisfies (4), (5).

The proof can be done using (6), (7), analogously to a proof of Step 1.

Step 3. Let $\varphi_n(t) = \frac{k}{2^n}$, where $t \in \left[\frac{k}{2^n}; \frac{k+1}{2^n}\right)$. Denote by $x_n(u,t)$ the solution of

$$dx_n(u,t) = B_0 x_n(\cdot, \varphi_n(t))(u)dt + \sum_{k=1}^n (B_k x_n(\cdot, \varphi_n(t)))(u)dw_k(t)$$

$$x_n(u,0) = \varphi(u),$$

where B_k are defined in (2).

Analogously to the case of diffusion equations (without interaction) [5] it can be proved that

1) $x_n(u,t)$ lies in $\cap_{p,k\geq 1} W_{p,k}(\mathbb{R}^d)$;
2) for all u,t a sequence $x_n(u,t)$ converges to $x(u,t)$ in all $L_p, p \geq 1$ uniformly in $u \in \mathbb{R}^d$;
3) for all $m \in \mathbb{N}$ $s_1, \ldots, s_n \in [0,t], u \in \mathbb{R}^d$ there exists a limit of $D_{s_1} \ldots D_{s_m} x_n(u,t)$ in each $L_p, p \geq 1$ uniformly in $s_1, \ldots, s_n, t, u$.
4) $D_s x_n(u,t)$ converges to $y_s(u,t)$ which is defined in (3).

Combining results of Steps $1-3$ we get the proof of the Lemma 1.

The application of the following proposition to the Theorem 2 and Lemma 1 completes a proof of the Theorem 1.

Lemma 2. *Assume that a random vector* $\xi = (\xi_1, \dots, \xi_j) \in \cap_{p,k\geq 1} W_{p,k}(\mathbb{R}^j)$ *satisfies the following conditions*

1. $\forall \alpha > 1 \exists p = p(\alpha) > 0 \exists c = c(\alpha) > 0 \forall s, t \in [0,T]$:

$$E\|D_s\xi - D_t\xi\|^p \leq c|t-s|^\alpha; \tag{8}$$

2. $\exists t_0 \in [0,T] \exists A > 0$:

$$(D_{t_0}\xi)(D_{t_0}\xi)^* \geq A id_{\mathbb{R}^j} \ \ a.s.,$$

where $id_{\mathbb{R}^j}$ *is identity matrix in* $\mathbb{R}^j$.

Then the distribution of ξ *in* $\mathbb{R}^j$ *has* C^∞*-density.*

Proof. By a Theorem 2 we have only to verify that

$$\det\left(\int_0^T D_s\xi(D_s\xi)^* ds\right) \in \cap_{p\geq 1} L_p. \tag{9}$$

The inequality (8) implies the existence of continuous in s version of $D_s\xi$. Moreover [7]:
$\forall \alpha > 1 \ \exists k = k(\alpha) \ \forall \varepsilon > 0$:

$$E \sup_{|s-t_0|<\varepsilon} \|D_s\xi - D_{t_0\xi}\|^{p(\alpha)} \leq K\varepsilon^\alpha.$$

Observe that

$$\begin{aligned} D_s\xi(D_s\xi)^* &\geq \frac{1}{2} D_{t_0}\xi(D_{t_0\xi})^* - (D_s\xi - D_{t_0}\xi)(D_s\xi - D_{t_0}\xi)^* \geq \\ &\geq \frac{A}{2} id_{\mathbb{R}^j} - (D_s\xi - D_{t_0\xi})(D_s\xi - D_{t_0\xi})^*. \end{aligned}$$

So for each $\alpha > 0$:

$$\begin{aligned} &P\left(\det\|D\xi(D\xi)^*\| < \left(\frac{A}{4}\right)^j \varepsilon\right) \leq \\ &\leq P\left(\exists s \in [0,T], |s-t_0| < \varepsilon : D_s\xi(D_s\xi)^* \geq \frac{A}{4} id_{\mathbb{R}^j}\right) \leq \\ &\leq P\left(\sup_{|s-t_0|<\varepsilon} \|D_s\xi - D_{t_0}\xi\| > \frac{A^{1/2}}{2}\right) \leq \\ &\leq \frac{E\sup_{|s-t_0|<\varepsilon} \|D_s\xi - D_{t_0}\xi\|^{p(\alpha)}}{(A^{1/2}/2)^{p(\alpha)}} \leq \text{const} \cdot \varepsilon^\alpha. \end{aligned}$$

Thus

$$P(\det D_\xi(D\xi)^* < \varepsilon) = o(\varepsilon^\alpha), \varepsilon \to 0+$$

for each $\alpha > 0$ and therefore (9) is satisfied.

Lemma 2 is proved.

By (3) we have

$$D_t\pi x(u,t) = \pi B(x(\cdot,t))(u).$$

So all conditions of the Lemma 2 are satisfied with $\xi = \pi x(u,t), t_0 = t, j = \dim \pi\mathbb{R}^d$, $\mathbb{R}^j = \pi\mathbb{R}^d, p(\alpha) = 2\alpha$ (see (5)).

Theorem 1 is proved.

References

1. Bell D.R., *Degenerate stochastic differential equations and hypoellipticity*, Pitman Monographs and Surveys in Pure and Applied Mathematics **79** (1995).
2. Belopolskaya Ya.I., Daletsky Yu.L., *Smoothness of transition probabilities of Markov processes described by stochastic differential equations in Hilbert space*, Reports of Ukrainian Academy of Sciences **9** (1994), 40-45.
3. A.A.Dorogovtsev, P.Kotelenez, *Smooth stationary solutions of quasilinear stochastic partial differential equations: 1.Finite mass*, Probability theory and its applications (to appear) (2002).
4. Takeuchi A., *The Malliavin calculus for SDE with jumps and the partially hypoelliptic problem*, Osaka journal of mathematics(to appear) (2002).
5. Watanabe S., *Lectures on stochastic differential equations and the Malliavin calculus*, TATA Inst. of Fundamental Research, 1984.
6. H.Kunita, *Stochastic flows and stochastic differential equations*, vol. 24, Cambridge studies in advanced mathematics, 1990.
7. Buldygin V.V., Kozachenko Yu.V., *Metric characterization of random variables and random processes*, vol. 188, Amer. Math. Soc, Providence, 1998.

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