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Transformation of Gaussian measure by infinite-dimensional stochastic flow

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Abstract — We study the propagation of absolute continuity of Gaussian measure in infinite-dimensional space by a flow generated by stochastic equation.

INTRODUCTION

Let (X, μ, H) be an abstract Wiener space, $W(t), t \geq 0$ be a cylindric Wiener process in a Hilbert space E [Kuo].

Consider a stochastic flow φ_{st} generated by the following stochastic equation:

$$\begin{cases} d\varphi_{st}(x) = a(\varphi_{st}(x))dt + \sigma(\varphi_{st}(x))dW(t), \ t \geq s \\ \varphi_{ss}(x), \ x \in X, \end{cases} \tag{0.1}$$

where $a : X \to H, \ \sigma : X \to \mathcal{L}_2(E, H)$.

If ω, s, t are fixed, then a function $\varphi_{st}(\cdot, \omega)$ generates a transformation from X to X.

We consider a question on absolute continuity of the image-measure $\mu \circ \varphi_{st}^{-1}$ with respect to measure μ.

Note that the case when $\sigma \equiv 0$, i.e. the equation (0.1) is non-random differential equation, is well studied [2,3,4]. Some results on propagation of absolute continuity in stochastic case ($\sigma \neq 0$) were obtained in [5] (additive noise), [6].

The mapping φ_{st} in papers above was such that $\varphi_{st}(x) - x \in H$. It worth to mention a class of results on absolute continuity of the image-measure (deterministic case) where $\varphi_{st}(x) - x$ does not belong to Cameron–Martin space, but φ_{st} corresponds to some random notation of Cameron–Martin space [4, 7, 8].

We will consider coefficients a, σ from the more general class than smooth H-valued (or $\mathcal{L}_2(E, H)$-valued) functions, such that a solution of (1.1) could be, for example, a semigroup of random linear operators that corresponds in some sense to a unitary stochastic semigroup in H.

In Section 1 we consider a linear stochastic equation of the form

$$\begin{cases} dU_{st} = (Adt + BdW_t)U_{st}, \ t \geq s, \\ U_{ss} = 1\!\!1_H, \end{cases} \tag{0.2}$$

where $A \in \mathcal{L}(H), \ B \in \mathcal{L}(E, \mathcal{L}(H)), 1\!\!1_H$ is identity operator in H.

Sufficient conditions which ensure that U_{st} is a stochastic unitary semigroup in H are given. Note that there are a lot of difficulties concerned with stochastic integration of operator-valued stochastic processes [9, 10], especially if we do not demand a Hilbert–Schmidt structure.

In Section 2 we identify some measure-preserving transformation of X with unitary operator U_{st} and study the existence and uniqueness of mild solution for the equation

$$\begin{cases} d\varphi_t(x) = (Adt + BdW_t)\varphi_t(x) + a(\varphi_t(x))dt + b(\varphi_t(x))dW_t, \\ \varphi_0(x) = x, \end{cases} \tag{0.3}$$

where $a : X \to H,\ b : X \to \mathcal{L}_2(E, H)$.

In Section 3 we give some sufficient conditions for the absolutely continuity of measures $\mu \circ \varphi_t^{-1} \ll \mu$, where φ_t gives a solution to (0.3).

1. UNITARY STOCHASTIC SEMIGROUPS

Let H be a real separable Hilbert space. The following definitions on random operator and stochastic semigroups are due Skorokhod A.V. [10]

Definition 1.1. Let A be a linear mapping from a space H to a space of H-valued random elements $L_0(\Omega, H)$. Then A is called a strong random operator if a mapping $x \to Ax$ it is continuous in probability, that is

$$\forall \varepsilon > 0\ \forall x \in H\colon \lim_{x_n \to x} P(|Ax_n - Ax| > \varepsilon) = 0.$$

It should be noted, that not every strong random operator is bounded, i.e. an object $A = A(\omega)$ does not exists for a.a. ω, in general.

Definition 1.2. [10] **Ch.15.3** A family of (strong or bounded) random operators $\{U_{st}, s \le t\}$ in H is said to be a (strong or bounded) stochastic semigroup if the following conditions hold:

1) for all $t_1 \le t_2 \le t_3 :\ U_{t_1t_3} = U_{t_2t_3}U_{t_1t_2}$ a.s., $U_{t_1t_1} = \mathbb{I}$;

2) for all $t_1 \le t_2 \le t_3$ σ-algebras $\sigma\{U_{\alpha\beta},\ t_1 \le \alpha \le \beta \le t_2\}$ and $\{U_{\delta\gamma},\ t_2 \le \delta \le \gamma \le t_3\}$ are independent.

Remark. See [10] for details concerning the definition of a composition of strong random operators. But in the articles only bounded linear operators are studied, however some general results that proved for strong linear operators are applied.

A stochastic semigroup U_{st} is unitary if for all s, t an operator U_{st} is unitary a.s. It is said to be homogeneous if for all $s \le t$ and $\delta > 0$ the distribution of $U_{t_1t_2}$ coincides with the distribution of $U_{t_1+\delta, t_2+\delta}$.

In some cases [10] an operator semigroup can be obtained as a solution of operator stochastic differential equation. The main result of this section is the following theorem.

Theorem 1.1. *Assume that continuous linear operators $A_n \in \mathcal{L}(H), n \ge 0$ are such that:*

1) for each $n \ge 0$ we have $A_n = -A_n^*$, where A_n^* is the adjoint operator for A_n;

2) the series $\sum_{n=1}^{\infty} A_n^2$ converges in a strong sense.

Let $w_n(t), n \ge 1$ be a sequence of independent one–dimensional Wiener processes.

Then the operator stochastic equation

$$dU_{st} = (A_0 + \frac{1}{2}\sum_{n\geq 1} A_n^2)U_{st}dt + \sum_{n\geq 1} A_n U_{st} dw_n(t), \tag{1.1}$$

$$U_{ss} = \mathbf{1}.$$

has the unique strong solution and the collection of operators $U_{st},\ s \leq t$ *generates a homogeneous unitary semigroup.*

Moreover, conjugate operators $V_{st} = U_{st}^*$ *satisfy the following equation:*

$$dV_{st} = V_{st}(-A_0 + \frac{1}{2}\sum_{n\geq 1} A_n^2)dt - \sum_{n\geq 1} V_{st} A_n dw_n(t), \tag{1.2}$$

$$V_{ss} = \mathbf{1}.$$

Remark. Here by the solution of (1.1) we mean a process U_{st} such that for each $x \in H$ the processes

$$\int_s^t (A_0 + \frac{1}{2}\sum_{n\geq 1} A_n^2)U_{sz}xdz \quad \text{and} \quad \int_s^t A_n U_{sz} x dw_n(z)$$

are well-defined (as integrals of H-valued stochastic processes) and the following equality holds:

$$U_{st}x = \int_s^t (A_0 + \frac{1}{2}\sum_{n\geq 1} A_n^2)U_{sz}xdz + \sum_{n\geq 1}\int_s^t A_n U_{sz} x dw_n(z). \tag{1.3}$$

Eq. (1.2) is understood in the same sense.

Remark. If $\{U_{st}\}$ is a homogeneous stochastic unitary semigroup in finite-dimensional space $H = \mathbb{R}^d$ then under very general assumptions the existence of Wiener processes $w_n(t)$ and operators $A_n = -A_n^*, n \geq 0$, such that U_{st} satisfies (1.1), can be proved.

Proof of Theorem 1.1. The existence and uniqueness of strong operator processes $U_{st}, V_{st}, s \leq t$ satisfying (1.1), (1.2) is proved in [10], §14. That is, for each $x \in H$ there exists H-valued random element $U_{st}x$ such that a mapping $x \to U_{st}x$ is continuous in probability, linear in x, all integrals in (1.3) are well-defined and (1.3) holds true (analogously for $V_{st}x$).

To prove the Theorem it is sufficient to verify that:

a) U_{st} and V_{st} are bounded operators;

b) U_{st} is an isometry, i.e. for all $x, y \in H$:

$$\langle U_{st}x, U_{st}y\rangle = \langle x, y\rangle \ \text{ a.e.},$$

where $\langle \cdot, \cdot \rangle$ is a scalar product in H;

c) $U_{st}V_{st} = V_{st}U_{st} = \mathbb{1}$;

d) verify the third condition of the definition 1.1.

Remark. The item d) follows from a) and reasoning of [10], Ch. 5.

Let us check a),b). Due to Ito's formula

$$
\begin{aligned}
&d\langle U_{st}x, U_{st}y\rangle \\
&= \int_s^t \left(\langle (A_0 + \frac{1}{2}\sum_{n\geq 1} A_n^2)U_{sz}x, U_{sz}y\rangle + \langle U_{sz}x, (A_0 + \frac{1}{2}\sum_{n\geq 1} A_n^2)U_{sz}y\rangle \right) dz \\
&+ \sum_{n\geq 1}\int_s^t \left(\langle U_{sz}x, A_nU_{sz}y\rangle + \langle A_nU_{sz}x, U_{sz}y\rangle\right) dw_n(z) \\
&+ \int_s^t \sum_{n\geq 1}\langle A_nU_{sz}x, A_nU_{sz}u\rangle dz = 0,
\end{aligned}
$$

because of condition 1) of the Theorem.

So, for all $x, y \in H$ we have $\langle U_{st}x, U_{st}y\rangle = \langle x, y\rangle$ a.e. By Theorem 2 §1.2 of [10] the operator U_{st} is a bounded random operator, $\|U_{st}x\| = \|x\|$ a.e. for each x, so $\|U_{st}\|_{op} = 1$ a.e. Here $\|\cdot\|_{op}$ is an operator norm.

Let $\{e_k, k \geq 1\}$ be an orthonormal basis of H, P_N be orthoprojection on $\{e_1, \ldots, e_N\}$. Consider auxiliary equations

$$
dU_{st}^N = \left(P_NA_0P_N + \frac{1}{2}\sum_{n\geq 1}(P_NA_nP_N)^2 \right) U_{st}^N dt + \sum_{n\geq 1} P_NA_nP_NU_{st}^N dw_n(t), \tag{1.4}
$$

$$
dV_{st}^N = V_{st}^N \left(-P_NA_0P_N + \frac{1}{2}\sum_{n\geq 1}(P_NA_nP_N)^2 \right) dt - \sum_{n\geq 1} V_{st}^N P_NA_nP_N dw_n(t), \tag{1.5}
$$

$$
U_{st}^N = V_{st}^N = \mathbb{1}.
$$

It is easy to check that operators $\{P_NA_nP_N\}_{n\geq 1}$ satisfy conditions 1), 2) of the Theorem.

Note that for each $x \in H$:

$$
U_{st}^N x = P_NU_{st}^N P_Nx + (\mathbb{1} - P_N)x, \tag{1.7}
$$

$$
V_{st}^N x = P_NV_{st}^N P_Nx + (\mathbb{1} - P_N)x, \tag{1.8}
$$

and therefore, processes $P_NU_{st}^N P_N$, $P_NV_{st}^N P_N$ are solutions of (1.4), (1.5) in finite-dimensional space P_NH. Thus, $P_NU_{st}P_N$ is unitary operator in P_NH (as isometry in finite-dimensional space), and hence $V_{st}^N = (U_{st}^N)^{-1}$ in H.

The main step of the Theorem proof is the following Lemma.

Lemma 1.1. *For all $s \leq t$, $x \in H$ the following convergence*

$$
U_{st}^N x \to U_{st}x, \ N \to \infty; \tag{1.9}
$$

$$
V_{st}^N x \to V_{st}x, \ N \to \infty \tag{1.10}
$$

holds in $L_2(\Omega, \mathcal{F}, \mathbf{P};\ H)$.

The proof of the Lemma will be presented after the proof of the Theorem.

Let us choose a subsequence $\{N_k\}$ such that, for each basis vector $e_n,\ n \geq 1$ the following convergence holds:

$$U_{st}^{N_k} e_n \to U_{st} e_n,\ k \to \infty \text{ a.e;} \tag{1.11}$$

$$V_{st}^{N_k} e_n \to V_{st} e_n,\ k \to \infty \text{ a.e.} \tag{1.12}$$

The fact that $\|U_{st}^{N_k}\|_{op} = 1, \|V_{st}^{N_k}\|_{op} = 1$ a.e. and (1.11), (1.12) imply the strong operator convergence

$$U_{st}^{N_k} \longrightarrow U_{st},\ k \to \infty;$$

$$V_{st}^{N_k} \longrightarrow V_{st},\ k \to \infty.$$

The limit of the product of strongly convergent sequences of operators is equal to the product of limits. So,

$$\mathbb{1} = V_{st}^{N_k} U_{st}^{N_k} \longrightarrow V_{st} U_{st}, k \to \infty,$$

$$\mathbb{1} = U_{st}^{N_k} V_{st}^{N_k} \longrightarrow U_{st} V_{st}, k \to \infty,$$

which means that $U_{st}V_{st} = V_{st}U_{st} = \mathbb{1}$. The Theorem is proved.

Proof of Lemma 1.1. Without loss of generality we can assume that $s = 0$. Put $V_t^N := V_{0t}^N,\ U_t^N := U_{0t}^N$. Let us verify (1.9):

$$\begin{aligned}
&\mathbf{E}\,\|(U_t^N - U_t)x\|^2 \\
&\leq t\mathbf{E} \int_0^t \| \left(P_N A_0 P_N + \sum_n (P_N A_n P_N)^2 \right) (U_z^N - U_z)x\|^2 dz \\
&\quad + t\mathbf{E} \int_0^t \| \left(P_N A_0 P_N - A_0 + \sum_n ((P_N A_n P_N)^2 - A_n^2) \right) U_z x\|^2 dz \\
&\quad + \int_0^t \sum_n \mathbf{E}\,\|(P_N A_n P_N - A_n) U_z x\|^2 dz \\
&\quad + \int_0^t \mathbf{E} \sum_n \|P_N A_n P_N (U_z^N - U_z)x\|^2 dz.
\end{aligned} \tag{1.13}$$

Observe that

$$\begin{aligned}
\|\sum_n (P_N A_n P_N)^2\|_{op} &= \sup_{\|x\|=1} |\langle \sum_n (P_N A_n P_N)^2 x, x\rangle| \\
&= \sup_{\|x\|=1} \sum_n \|P_N A_n P_N x\|^2 \\
&\leq \sup_{\|y\|=1} \sum_n \|P_N A_n y\|^2 \\
&\leq \sup_{\|y\|=1} \sum_n \|A_n y\|^2 \\
&= \sup_{\|y\|=1} \sum_n \langle A_n^2 y, y\rangle = \|\sum_n A_n^2\|_{op},
\end{aligned} \tag{1.6}$$

because $\sum_n A_n^2$ and $\sum_n (P_N A_n P_N)^2$ are self-adjoint operators.

Due to (1.6) the right hand side of (1.13) is not greater than the following expression:

$$\left(2t(\|A_0\|_{op}+\|\sum_n A_n^2\|_{op}^2)+\|\sum_n A_n^2\|_{op}\right)\int_0^t \mathbf{E}\,\|(U_z^N-U_z)x\|^2dz$$
$$+\,t\int_0^t \mathbf{E}\,\|\Big((P_NA_0P_N-A_0)+\sum\big((P_NA_nP_N)^2-A_n^2)\big)\Big)\,U_zx\|^2dz. \tag{1.14}$$

The convergence in (1.9) will follow from (1.14) and Gronwall's lemma, if we verify that

$$\forall n\ge 0\ :\ \ P_NA_nP_N\xrightarrow{s}A_n,\ N\to\infty; \tag{1.15}$$

$$\sum_n(P_NA_nP_N)^2\xrightarrow{s}\sum_m A_n^2,\ N\to\infty. \tag{1.16}$$

The convergence in (1.15) is obvious. Let us check (1.16). Let $x\in H$ be fixed. Then

$$\begin{aligned}&\|\sum_n\big((P_NA_nP_N)^2-A_n^2\big)\,x\|\\&\le\|P_N\sum_n(A_nP_N)(P_NA_n)(P_Nx-x)\|\\&+\|P_N\sum_n\big((A_nP_N)(P_NA_n)-A_n^2\big)\,x\|+\|(P_N-1\!\!1)\sum_n A_n^2x\|.\end{aligned}$$

The series $\sum_n(A_nP_N)(P_NA_n)$ are strongly convergent, the sum is non-positive self-adjoint operator, and (see (1.6))

$$\sup_N\|\sum_n(A_nP_N)(P_NA_n)\|_{op}\le\|\sum_n A_n^2\|_{op}.$$

Let us prove weak convergence

$$\sum_n(A_nP_N)(P_NA_n)\xrightarrow{w}\sum_n A_n^2,\ N\to\infty. \tag{1.17}$$

Indeed, let $x,y\in H$. Choose m and N_0 such that

$$\Big|\sum_{n=m+1}^{\infty}\langle A_n^2x,y\rangle\Big|<\varepsilon,\ \sum_{n=m+1}^{\infty}\|A_nx\|^2=-\sum_{n=m+1}^{\infty}\langle A_n^2x,x\rangle<\varepsilon,\ \sum_{n=m+1}^{\infty}\|A_ny\|^2<\varepsilon$$

and

$$\Big|\sum_{n=1}^{m}\langle\big((A_nP_N)(P_NA_n)-A_n^2\big)\,x,y\rangle\Big|<\varepsilon$$

for all $N\ge N_0$.

Then

$$\begin{aligned}&\Big|\sum_n\langle\left(\sum_n(A_nP_N)(P_NA_n)-A_n^2\right)x,y\rangle\Big|\\&\le 2\varepsilon+|\sum_{n=m+1}^{\infty}\langle P_NA_nx,P_NA_ny\rangle|\\&\le 2\varepsilon+\sqrt{\sum_{n=m+1}^{\infty}\|P_NA_nx\|^2\sum_{n=m+1}^{\infty}\|P_NA_ny\|^2}\\&\le 2\varepsilon+\sqrt{\sum_{n=m+1}^{\infty}\|A_nx\|^2\sum_{n=m+1}^{\infty}\|A_ny\|^2}\le 3\varepsilon.\end{aligned}$$

So, (1.17) is proved. The application of the following lemma completes the proof of convergence (1.16) and, therefore, (1.9).

Lemma 1.2. *Let* $\{B_n\}_{n\geq 1}$ *be a sequence of non-negative operators, which converges weakly to zero-operator. Then the sequence* $\{B_n\}_{n\geq 1}$ *converges strongly.*

Proof. Observe that $\sqrt{B_n} \xrightarrow{s} 0, n \to \infty$, because

$$\forall x \in H : \|\sqrt{B_n}x\|^2 = \langle \sqrt{B_n}x, \sqrt{B_n}x\rangle = \langle B_n x, x\rangle \to 0, n \to \infty.$$

Thus, $B_n = \sqrt{B_n}\sqrt{B_n}$ also converges (to zero-operator) as a product of two strongly convergent sequences.

Lemma 1.2 is proved.

Let us check (1.10):

$$\begin{aligned}\mathbf{E}\,\|V_t x - V_t^N x\|^2 \leq {} & 2t\int_0^t \mathbf{E}\,\|(V_s - V_s^N)\sum_n A_n^2 x\|^2 ds \\ & + 2t\int_0^t \mathbf{E}\,\|V_s^N \sum_n \left(A_n^2 - (P_N A_n P_N)^2\right) x\|^2 ds \\ & + \sum_n \int_0^t \mathbf{E}\,\|(V_s - V_s^N) A_n x\|^2 ds \\ & + \sum_n \int_0^t \mathbf{E}\,\|V_s^N (A_n - P_N A_n P_N) x\|^2 ds.\end{aligned}$$

Iterating similar estimates, taking into account that $\|V_s^N\|_{op} = 1$ a.e. and convergence (1.15), (1.16), we obtain

$$\begin{aligned}\mathbf{E}\,\|V_t x - V_t^N x\|^2 \leq {} & \varepsilon_{m,N} \\ & + \sum_{n_1 \ldots n_m} \sum_{\alpha_{n_k} \in \{0;1\}} (2t \vee 1)^m \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} \|(V_{t_m} - V_{t_m}^N) \\ & \times \prod_{k=1}^m (\sum_l A_l^2)^{\alpha_{n_k}} A_{n_k}^{1-\alpha_{n_k}} x\|^2 dt_m \ldots dt_2 dt_1,\end{aligned} \tag{1.18}$$

where $\varepsilon_{m,N} \to 0$ as $N \to 0$ for each fixed m.

The inequality (1.6) implies that right hand side of (1.18) is not greater than $\varepsilon_{m,N} + \frac{((2t\vee 1)tc)^m}{m!}$, where $c > 0$ is some constant. Letting $m \to \infty$ and then $N \to \infty$ we obtain the proof of (1.10). Lemma 1.1 is proved.

2. STOCHASTIC EQUATIONS IN THE INFINITE-DIMENSIONAL SPACE

Let (X, μ, H) be an abstract Wiener space. Assume that the sequence of operators $\{A_k, k \geq 0\} \subset \mathcal{L}(H)$ satisfies the assumptions of §1; U_{st}, V_{st} are unitary stochastic semigroups in H corresponding to the equations (1.1),(1.2).

Observe that every unitary operator U in H can be naturally extended to a linear measurable (w.r.t. the measure μ) transformation of the space X, which preserves a measure μ. This can be done in the following way. Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis of H, $\widehat{e}_n$ be a linear measurable functional corresponding to e_n. Then [11]Th.3.5.1. series

$$\widehat{U}x := \sum_n \widehat{e}_n(x)Ue_n,\ x\in X \tag{2.1}$$

are convergent for μ-almost all $x\in X$ and its distribution is equal to μ.

Remark. If $\{f_n\}_{n\geq 1}$ is another orthonormal basis of H, then

$$\sum_n \widehat{f}_n(x)Uf_n = \sum_n \widehat{e}_n(x)Ue_n$$

for μ-a.a. $x\in X$, though the sets of convergence in general are different.

Remark. For each $h\in H$ we have the equality $\widehat{U}h = Uh$ a.s.

There are measurable in s,t,ω,x version of the processes $\widehat{U}_{st}x$, $\widehat{V}_{st}x$. Indeed, let $e_n\in H$ be fixed. Is it not difficult to verify that a process $U_{st}x$ is continuous in probability in (s,t), so there exists a measurable in (s,t,ω) modification of $U_{st}e_n$. The existence of measurable modification of $\widehat{U}_{st}x$ follows from (2.1) and [12]. The case of $\widehat{V}_{st}x$ is similar.

We will say that $\widehat{U}_{st}$, $\widehat{V}_{st}$ are mild solutions of the equations

$$\begin{cases} d\widehat{U}_{st}x = \left(A_0 + \frac{1}{2}\sum_{n=1}^{\infty}A_n^2\right)\widehat{V}_{st}xdt + \sum_{n=1}^{\infty}A_n\widehat{U}_{st}xdw_n(t),\\ \widehat{U}_{ss}x = x,\end{cases}$$

$$\begin{cases} d\widehat{V}_{st}x = \widehat{V}_{st}\left(-A_0 + \frac{1}{2}\sum_{n=1}^{\infty}A_n^2\right)xdt - \sum_{n=1}^{\infty}V_{st}A_nxdw_n(t),\\ \widehat{V}_{ss}x = x.\end{cases}$$

Put $\widehat{U}_t := \widehat{U}_{0,t},\ \widehat{V}_t := \widehat{V}_{0,t}$.

Assume that $a : X\to H, \sigma_n : X\to H$ are measurable functions.

Definition 2.1. Stochastic process φ_t with values in X is called a mild solution of the equation

$$\begin{cases} d\varphi_t(x) = & \left(\left(A_0 + \frac{1}{2}\sum_{n=1}^{\infty}A_n^2\right)\varphi_t(x) + a(\varphi_t(x))\right)dt+\\ & +\sum_n(A_n\varphi_t(x) + \sigma_n(\varphi_t(x)))dw_n(t),\\ \varphi_0(x) = x,\end{cases} \tag{2.2}$$

if $\varphi_t(x) = \widehat{U}_t\psi_t(x)$, where $\psi_t(x)$ satisfies the equation

$$\psi_t(x) = x + \int_0^t V_s a(\widehat{U}_s\psi_s(x))ds - \sum_n\int_0^t V_sA_n\sigma_n(\widehat{U}_s\psi_s(x))ds+$$
$$+\sum_n\int_0^t V_s\sigma_n(\widehat{U}_s\psi_s(x))dw_n(s). \tag{2.3}$$

Definition 2.2. Let E be a separable Hilbert space. A mapping $f : X\to H$ belongs to a class $\mathcal{H}C^k(E)$, $k\in\mathbb{N}$ if for each $x\in X$ a mapping

$$H\ni h\mapsto f(x+h)\in E$$

is k-times continuously Frechet differentiable and for each $j = 0, \dots, k$ a derivative $D^j f(x)$ belongs to a space $H^{\otimes j} \otimes E$.

The set of functions $f \in \mathcal{H}C^k(E)$ with finite norm

$$\|f\|_{\infty,k} := \sup_{x\in X} \sum_{j=0}^{k} \|D^j f(x)\|_{H^{\otimes j}\otimes E}$$

is denoted by $\mathcal{H}C^k_b(E)$.

Lemma 2.1. *Assume that functions $a, \sigma_n : X \to H$ belong to a space $\mathcal{H}C^1(H)$ and*

$$\|a\|_{\infty,1} + \sum_n \|\sigma_n\|^2_{\infty,1} < \infty,$$

where $\|\cdot\|_\infty$ is a supremum norm.

Then for each $x \in X$ there exists the unique solution of the equation (2.3).

Remark. If $X = H = \mathbb{R}^n$ is finite-dimensional space then under assumptions of Lemma 2.1 there exists the unique solution for (2.2) and $\varphi_t(x) = U_t\psi_t(x)$, where $\psi_t(x)$ is a solution of (2.3).

Proof of Lemma 2.1. Set $\widetilde{\psi}_t(x) = \psi_t(x) - x$. The random element $\psi_t(x)$ satisfies the equation (2.3) if and only if H-valued random element $\widetilde{\psi}_t(x)$ satisfies the following Ito equation:

$$\begin{aligned}\widetilde{\psi}_t(x) = &\int_0^t \left[V_s a(\widehat{U}_s x + U_s\widetilde{\psi}_s(x)) + \sum_n A_n U_s \sigma_n(\widehat{U}_s x + U_s\widetilde{\psi}_s(x))\right] ds \\ &+ \sum_n \int_0^t V_s\sigma_n(\widehat{U}_s x + U_s\widetilde{\psi}_s(x)) dw_k(s).\end{aligned} \tag{2.4}$$

In order to verify that coefficients of the equation (2.4) satisfy conditions of uniqueness and existence it is sufficient to show that the function

$$f : H \ni h \to \sum_n V_s A_n \sigma(\widehat{U}_s x + U_s h) \in H$$

is well-defined and satisfies Lipschitz condition uniformly in (w, s, h).

The proof of this fact follows from the next Lemma and inequality:

$$\begin{aligned}&\sum_n \|\sigma_n(\widehat{U}_s x + U_s h_2) - \sigma_n(\widehat{U}_s x + U_s h_1)\|^2_H \\ &\le \|h_2 - h_1\|^2_H \sup_{y\in X} \sum_n \|D\sigma_n(y)\|^2_{HS},\end{aligned}$$

where $\|\cdot\|_{HS}$ is Hilbert-Schmidt norm, $D\sigma_n = D^1\sigma_n$.

Lemma 2.2. *Suppose that $\{f_n\}_{n\ge 1} \subset H$ is such that $\sum_n \|f_n\|^2 < \infty$.*

Then the series $\sum_n A_n f_n$ are weakly convergent and

$$\|\sum_n A_n f_n\|^2_H \le \|\sum_n A_n^2\|_{\text{op}} \sum_n \|f_n\|^2_H,$$

where $\|\cdot\|_{op}$ *is operator norm.*

The proof of the convergence follows from the estimate:

$$\begin{aligned}\forall x \in H: \quad \left(\sum_n \langle A_n f_n, x\rangle\right)^2 &= \left(\sum_n \langle f_n, A_n x\rangle\right)^2 \\ &\leq \sum_n \|f_n\|^2 \sum_n \|A_n x\|^2 \\ &= -\sum_n \|f_n\|^2 \sum_n \langle A_n^2 x, x\rangle.\end{aligned}$$

Remark. A modification of the process $\psi_t(x), t \in [0,T], x \in X$, which is measurable in t, x, ω and is continuous in t (for all x, ω), can be chosen. Only this modification is considered further.

Remark. It was mentioned at the beginning of the chapter that a process $\widehat{U}_t(\omega)x$ is defined up to measure μ. If $\widetilde{U}_t(\omega)x$ is another version and $\overline{\psi}_t(x)$ is a solution of (2.3) with $\widetilde{U}_t$ instead of $\widehat{U}_t$ then for μ-almost all $x \in X$ and P-almost all $\omega \in \Omega$ we have the equality

$$\overline{\psi}_t(x) = \psi_t(x), \ t \in [0,T].$$

3. PROPAGATION OF ABSOLUTE CONTINUITY

Let $\varphi_t(x)$ be a mild solution of (2.2). In this section we find conditions that guarantee the absolute continuity of measures $\mu_t := \mu \circ \varphi_t^{-1}$ with respect to $\mu, \ t \in [0,T]$.

The main result of the article is the following theorem.

Theorem 3.1. *Assume that*

1) a sequence of operators $\{A_n : n \geq 0\} \subset \mathcal{L}(H)$ *satisfies conditions of Theorem 1.1;*

2) $a \in \mathcal{H}C_b^1(H), \ \sigma_n \in \mathcal{H}C_b^2(H)$,

$$\|a\|_{\infty,1} + \sum_{n=1}^{\infty} \|\sigma_n\|_{\infty,2}^2 < \infty.$$

Then there exists a set $\Omega_0 \subset \Omega$ *of full probability such that*

$$\mu_t(\omega) \ll \mu \text{ for all } \omega \in \Omega_0, t \in [0,T].$$

Remark. Recall that we consider continuous in t versions of processes U_t and ψ_t.

Proof of Theorem 3.1. The proof of the theorem will be done by the finite-dimensional approximations.

Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis in H, P_N be the orthoprojection on $\mathcal{L}(e_1, \dots, e_N)$.

Denote by $\widehat{e}_n = \widehat{e}_n(x)$ a measurable linear functional that corresponds to the element $e_n \in H$.

Let $\mathcal{B}_N$ be a σ-algebra in X generated by $\{\widehat{e}_1, \dots, \widehat{e}_N\}$. Put $a^N = P_N E_\mu(a/\mathcal{B}_N)$, $\sigma_n^N = P_N E_\mu(\sigma_n/\mathcal{B}_N)$, $A_n^N = P_N A_n P_N$, where $E_\mu(\cdot/\mathcal{B}_N)$ is conditional expectation.

Observe [11] that a^N, σ_n^N, A_n^N also satisfy conditions of the Theorem and

$$\|a^N\|_{\infty,2} \le \|a\|_{\infty,2},\ \|\sigma_n^N\|_{\infty,2} \le \|\sigma_n\|_{\infty,2},$$

$$\|\sum_n (A_n^N)^2\|_{\mathrm{op}} \le \|\sum_n A_n^2\|_{\mathrm{op}}.$$

Denote by $\varphi_t^N, \psi_t^N, U_t^N, V_t^N$ the solutions for the equations (2.2), (2.3), (1.1), (1.2) respectively.

Lemma 3.1. *For P-a.a ω measures $\mu_t^N := \mu \circ (\varphi_t^N)^{-1}$ and μ are equivalent for all $t \in [0,T]$.*

Moreover

$$\sup_{t\in[0,T]} \sup_N \mathbf{E}_P \mathbf{E}_\mu |\rho_t^N \ln \rho_t^N| < \infty, \tag{3.1}$$

where $\rho_t^N = \frac{d\mu_t^N}{d\mu}$.

Proof of Lemma 3.1. Let us represent X as a direct sum $X = X_N \oplus \mathcal{L}(e_1, \dots, e_N)$, where $X_N \subset X$ is a subspace of X. Identify linear span $\mathcal{L}(e_1, \dots, e_N)$ with $\mathbb{R}^N$ in such a way that vectors $\{e_1, \dots, e_N\}$ corresponds to a natural basis of $\mathbb{R}^N$. So, we can assume that $X = X_N \times \mathbb{R}^N$ and μ is a product of Gaussian measures $\mu_N \times \mu^N$, besides $\mu^N = N(0, \mathrm{Id}_{\mathbb{R}^N})$.

Observe that

$$\varphi_t^N(x^N, x_N) - (x^N, 0) = \varphi_t^N(y^N, x_N) - (y^N, 0)$$

for all $x^N, y^N \in X_N, x_N \in \mathbb{R}^N, t \in [0,T]$. I.e. φ_t^N depends only on the second coordinate. Denote by the $\widetilde{\varphi}_t^N(x_N) = P_N \varphi_t^N(0, x_N)$ the restriction of φ_t^N to the space $\mathbb{R}^N$.

It is well-known [13] that a solution of s.d.e. in finite-dimensional space with twice differentiable local characteristics generates a flow of diffeomorphisms, so it maps each measure with positive density to equivalent. Thats why

$$\mu^N \circ (\widetilde{\varphi}_t^N)^{-1} \sim \mu^N.$$

Note that

$$\frac{d\mu_t}{d\mu} = \frac{d\mu^N \circ (\widetilde{\varphi}_t^N)^{-1}}{d\mu^N}.$$

Denote by $\widetilde{\varphi}_{-t}^N$ the inverse mapping to $\widetilde{\varphi}_t^N$,

$$\widetilde{\varphi}_{-t}^N(\widetilde{\varphi}_t^N(x_N))) = \widetilde{\varphi}_t^N(\widetilde{\varphi}_{-t}^N(x_N))) = \overline{x}_N,$$

and by $\rho_t^N,\ t \in \mathbb{R}$ the density $\frac{d\mu \circ (\widetilde{\varphi}_t^N)^{-1}}{d\mu}$.

The change of variables formula implies:

$$\mathbf{E}_\mu |\rho_t^N \ln \rho_t^N| = \mathbf{E}_\mu |\ln \rho_{-t}^N|. \tag{3.2}$$

In order to estimate the density ρ_{-t}^N we need the following statement.

Lemma 3.2. *Let a stochastic flow* $\xi_t = \xi_t(x)$ *be generated by the following s.d.e.*

$$\begin{cases} d\xi_t(x) = \alpha_t(\xi_t(x))dt + \sum_{n=1}^{\infty} \beta_t^n(\xi_t(x))dw_n(t), \ t \in [0,T] \\ \xi_0(x) = x, \ x \in \mathbb{R}^d, \end{cases} \tag{3.3}$$

where $\alpha = \alpha_t(x,\omega)$, $\beta^n = \beta_t^n(x,\omega)$ *are adapted measurable processes.*

Assume that local characteristics of the flow $\alpha_t(x)$ *and*

$$b_t(x,y) = \sum_{n=1}^{\infty} \beta_t^n(x)(\beta_t^n(y))^*$$

are twice continuously differentiable and bounded with their derivatives.

1) Then there exists a modification of ξ_t *such that for a.a.* ω *the mapping* $\xi_t(\cdot,w) : \mathbb{R}^d \to \mathbb{R}^d$ *is a diffeomorphism for all* $t \in [0,T]$;

2) Let $m = N(0, Id_{\mathbb{R}^d})$ *be mean-value Gaussian measure with identity covariation operator in* $\mathbb{R}^d$.

Then measures m *and* $m \circ (\xi_{-t})^{-1}$ *are equivalent and*

$$\begin{aligned} \frac{dm \circ (\xi_{-t})^{-1}}{dm} = \exp\bigg\{ &\int_0^t (\delta\alpha_s)(\xi_s)ds + \sum_{n=1}^{\infty} \int_0^t (\delta\beta_s^n)(\xi_s)dw_n(s) \\ &+ \frac{1}{2}\int_0^t \sum_{n=1}^{\infty} (\mathrm{Tr}(\nabla\beta_s^n)^2(\xi_s) + |\beta_s^n(\xi_s)|^2)ds \bigg\}, \end{aligned} \tag{3.4}$$

where the action of the operator δ *on a smooth function* $f = (f_1, \dots, f_d)$ *is*

$$\delta f = \sum_{k=1}^{d} \Big(-\frac{\partial f_k}{\partial x_k} + x_k f_k \Big).$$

The proof of the first statement of the lemma follows from [13]. If we write down the equation (3.3) in Stratonovich form, then [13] we obtain the following expression for the density:

$$\frac{dm \circ (\xi_{-t})^{-1}}{dm} = \exp\bigg\{ \int_0^t (\delta\widetilde{\alpha}_s)(\xi_s)ds + \sum_{n=1}^{\infty} \int_0^t (\delta\beta_s^n)(\xi_s) \circ dw_n(s) \bigg\}, \tag{3.5}$$

where $\widetilde{\alpha}_t = \alpha_t - \frac{1}{2}\sum_{n=1}^{\infty} \nabla\beta_t^n\beta_t$ is a corrected coefficient.

Writing (3.5) back to the Ito form we get the formula

$$\begin{aligned} &\ln \frac{dm \circ (\xi_{-t})^{-1}}{dm} - \int_0^t (\delta\alpha_s)(\xi_s)ds - \sum_{n=1}^{\infty} \int_0^t (\delta\beta_s^n)(\xi_s)dw_n(s) \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \int_0^t \Big(\langle (\nabla\delta\beta_s^n)(\xi_s), \beta_s^n(\xi_s) \rangle - \delta(\nabla\beta_s^n\beta_s^n)(\xi_s) \Big) ds. \end{aligned} \tag{3.6}$$

Let $f = (f_1, \dots, f_d)$ be a smooth function. Then

$$\begin{aligned} &\langle (\nabla\delta f), f \rangle - \delta(\nabla f f) \\ &= \sum_{k,j=1}^{d} \bigg[\frac{\partial}{\partial x_k}\Big(-\frac{\partial f_j}{\partial x_j} + x_j f_j \Big) f_k - \Big(-\frac{\partial}{\partial x_j} + x_j \Big)\Big(\frac{\partial f_j}{\partial x_k} f_k \Big) \bigg] \\ &= \sum_{k,j=1}^{d} \frac{\partial f_j}{\partial x_k}\frac{\partial f_k}{\partial x_j} + \sum_{k=1}^{d} f_k^2. \end{aligned} \tag{3.7}$$

The substitution (3.7) into (3.6) finishes the proof of Lemma 3.2.

The mapping U_t^N preserves measure μ. So for each measurable function f we have the equality

$$\mathbf{E}_\mu f\left(\frac{d\mu \circ (\psi_t^N)^{-1}}{d\mu}\right) = \frac{d\mu \circ (\varphi_t^N)^{-1}}{d\mu}.$$

Thus, to prove (3.1) it is sufficient to verify (see (3.2)) that

$$\sup_{t\in[0,T]} \sup_N \mathbf{E}_p \mathbf{E}_\mu \left| \ln\left(\frac{d\mu \circ (\psi_{-t}^N)^{-1}}{d\mu}\right)\right| < \infty.$$

Applying Lemma 3.2 to a process ψ_t^N we get the formula:

$$\begin{aligned}
&\ln \frac{d\mu_N \circ (\psi_{-t}^N)^{-1}}{d\mu_N} \\
&= \int_0^t (\delta \widetilde{a}_s^N)(\psi_s^N) ds + \sum_{n=1}^\infty \int_0^t (\delta \widetilde{\sigma}_s^{N,n})(\psi_s^N) dw_n(s) \\
&+ \frac{1}{2} \int_0^t \sum_{n=1}^\infty \big(\operatorname{Tr}(\nabla \widetilde{\sigma}_s^{N,n})^2(\psi_s^N) + \|\widetilde{\sigma}_s^{N,n})(\psi_s^N)\|^2\big) ds,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{a}_t^N(\overline{x}_N) &= V_t^N a^N(U_t^N \overline{x}_N) - V_t^N \sum_{n=1}^\infty A_n^N \sigma_n^N(U_t^N \overline{x}_N), \\
\widetilde{\sigma}_t^{N,n}(\overline{x}_N) &= V_t^N \sigma_n^N(U_t^N \overline{x}_N),\ \overline{x}_N = (x_1, \dots, x_N).
\end{aligned}$$

The rest of the proof of Lemma 3.1 can be done similarly to [6] §4.

Lemma 3.3. $\forall T > 0 : \sup_{t\in[0,T]} \mathbf{E}_{\mu\times P} \|\psi_t - \psi_t^N\|_H^2 \to 0,\ n \to \infty.$

Proof.

$$\begin{aligned}
&\mathbf{E}_{\mu\times P} \|\psi_t - \psi_t^N\|_H^2 \\
&\le 3\Big(t \sup_{t\le T} \int_0^t \|V_s a(\widehat{U}_s \psi_s) - V_s^N a^N(\widehat{U}_s^N \psi_s^N)\|^2 ds \\
&\quad + \mathbf{E}_{\mu\times P} \int_0^t \sum_n \|V_s A_n \sigma_n(\widehat{U}_s \psi_s) - V_s^N A_n^N \sigma_n^N(\widehat{U}_s^N \psi_s^N)\|^2 ds \\
&\quad + \int_0^t \sum_n \|V_s \sigma_n(\widehat{U}_s \psi_s) - V_s^N \sigma_n^N(\widehat{U}_s^N \psi_s^N)\|^2 ds\Big) \\
&= 3(tI_1 + I_2 + I_3).
\end{aligned} \tag{3.8}$$

In order to shorten exposition, let us estimate the first term of (3.8) only. The considerations for I_2, I_3 are similar.

Taking into account that $\|V_t^N h\|_H = \|V_t h\|_H = \|h\|_H$ for all $h \in H$ we get:

$$\begin{aligned}
I_1 \le 4\Big(&\mathbf{E}_{\mu\times P} \int_0^t \|a^N(\widehat{U}_s \psi_s^N) - a(\widehat{U}_s^N \psi_s^N)\|^2 ds \\
&+ \mathbf{E}_{\mu\times P} \int_0^t \|a(\widehat{U}_s \psi_s^N) - a(\widehat{U}_s \psi_s)\|^2 ds \\
&+ \mathbf{E}_{\mu\times P} \int_0^t \|a(\widehat{U}_s^N \psi_s^N) - a(\widehat{U}_s^N \psi_s^N)\|^2 ds + \mathbf{E}_{\mu\times P} \int_0^t \|(V_s^N - V_s) a(\widehat{U}_s \psi_s)\|^2 ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&= 4\Big(\mathbf{E}_{\mu\times P}\int_0^t \|a^N - a\|_H^2 \frac{d\mu\circ(\varphi_s^N)^{-1}}{d\mu}\,ds \\
&\quad + \mathbf{E}_{\mu\times P}\int_0^t \|a(\widehat{U}_s^N) - a(\widehat{U}_s)\|^2 \frac{d\mu\circ(\psi_s^N)^{-1}}{d\mu}\,ds \\
&\quad + \mathbf{E}_{\mu\times P}\int_0^t \sup_y \|Da(y)\|^2\|\psi_s^N - \psi_s\|^2 ds + \mathbf{E}_{\mu\times P}\int_0^t \|(V_s^N - V_s)a(\widehat{U}_s\psi_s)\|^2 ds\Big) \\
&= J_1^N + J_2^N + J_3^N + J_4^N.
\end{aligned}
$$

The item J_4^N tends to zero as $N\to\infty$ due to Lemma 1.1 and Lebesgue dominated theorem.

The function a is bounded, so a sequence $a^N = P_N E_\mu(a/\mathcal{B}_N)$ is bounded and converges to a in measure μ. Because of Lemma 3.1 we have uniform integrability of $\int_0^t \|a^N - a\|_H^2 \frac{d\mu\circ(\varphi_s^N)^{-1}}{d\mu} ds$ and so the convergence $J_1^N \to 0, N\to\infty$.

Similarly, we get convergence $J_2^N\to 0, N\to\infty$ if we prove that $a(U_s^N)\to a(\widehat{U}_s), N\to\infty$ is measure $\mu\times P$. This follows from the next lemma.

Lemma 3.4. *Let $\{B_N, N\ge 0\}$ be a sequence of unitary operators in H, B_N converges strongly to B_0 as $N\to\infty$.*

Then

1) $\widehat{B}_N x \xrightarrow{\mu} \widehat{B}_0 x, N\to\infty$;

2) for each measurable function $a : X\to H$ we have the following convergence

$$a(\widehat{B}_N x) \xrightarrow{\mu} a(\widehat{B}_0 x), N\to\infty.$$

Proof. 1) All random elements $\{\widehat{B}_N x\}_{N\ge 0}$ have the same distribution in X, which is equal to μ. Let $\widehat{P}_n x = \sum_{k=1}^n \widehat{e}_k(x)e_k, x\in X$ (see the definition of $\widehat{e}_k$ at the beginning of §2). By Ito-Nisio theorem we have convergence $P_n x\to x, n\to\infty$ in measure m. Chose a number n such that

$$\mu\{x : \|\widehat{B}_N x - \widehat{P}_n\widehat{B}_N x\|_X \ge \varepsilon\} \le \varepsilon.$$

for all N.

Then

$$
\begin{aligned}
\mu\{x : \|\widehat{B}_N x - \widehat{B}_0 x\|_X \ge 3\varepsilon\} &\le \mu\{x : \|\widehat{B}_0 x - \widehat{P}_n\widehat{B}_0 x\|_X \ge \varepsilon\} \\
&\quad + \mu\{x : \|\widehat{B}_N x - \widehat{P}_n\widehat{B}_N x\|_X \ge \varepsilon\} \\
&\quad + \mu\{x : \|\widehat{P}_n(\widehat{B}_0 x - \widehat{B}_N x)\|_X \ge \varepsilon\}.
\end{aligned}
$$

The first two terms are not greater then ε, the third item converges to zero as $N\to\infty$. Indeed

$$
\begin{aligned}
\|\widehat{P}_n(\widehat{B}_N x - \widehat{B}_0 x)\|_X &= \|\sum_{k=1}^n\sum_{j=1}^\infty \widehat{e}_j(x)\langle(B_N - B_0)e_j, e_k\rangle_H e_k\|_X \\
&\le \sum_{k=1}^n \|e_k\|_X\Big|\sum_{j=1}^\infty \widehat{e}_j(x)\langle(B_N - B_0)^* e_k, e_j\rangle_H\Big|.
\end{aligned}
$$

The expression inside the modulus is mean-zero a Gaussian random variable with variance

$$\sum_j \langle (B_N - B_0)^* e_k, e_j\rangle^2 = \|(B_N - B_0)^* e_k\|_H \to 0, N \to \infty.$$

So, the first part of Lemma 3.4 is proved.

2) By Egorov's theorem for each $\varepsilon > 0$ there exists a compact set $K \subset X, \mu(K) > 1-\varepsilon$ such that a function a is continuous on K. Hence a is uniformly continuous on K. Chose $\delta > 0$ such that $\|a(x) - a(y)\| < \varepsilon$ for all $x, y,\ \|x - y\| < \delta$.

A mapping $\widehat{B}_N : X \to X$ preserves a measure μ, so $\mu\{x \in K : \widehat{B}_N x \notin K\} \le 2\varepsilon$.

Thus

$$\begin{aligned}
&\overline{\lim_{N\to\infty}}\ \mu(\|a(\widehat{B}_N x) - a(\widehat{B}_0 x)\| \le \varepsilon)\\
&\ge \overline{\lim_{N\to\infty}}\ \mu(x \in K : \widehat{B}_N x \in K, \widehat{B}_0 x \in K, \|\widehat{B}_N x - \widehat{B}_0 x\| < \delta)\\
&\ge 1 - 3\varepsilon.
\end{aligned} \tag{3.9}$$

The number $\varepsilon > 0$ is arbitrary, so (3.9) implies the proof of Lemma 3.4.

Considering other items in (3.8) similarly to I_1 we obtain the inequality

$$\mathbf{E}_{\mu\times P}\|\psi_t - \psi_t^N\|_H^2 \le C(T)\bigg(\int_0^t \mathbf{E}_{\mu\times P}\|\psi_s - \psi_s^N\|_H^2 + o_N(1)\bigg),\ t \in [0, T],$$

where $o_N(1) \to 0, N \to \infty$ uniformly in t, $C(T) = const.$

The application of Gronwall's lemma proves the inequality (3.8).

Corollary 3.1. $\forall T > 0 :\ \mathbf{E}\sup_{t\in[0,T]} \|\psi_t^N - \psi_t\|_H^2 \to 0, N \to \infty.$

The proof of corollary is standard. It uses Burkholder inequality for the estimation of stochastic integrals and Gronwall lemma.

Corollary 3.2. $\forall t :\ \varphi_t^N \xrightarrow{P\times\mu} \varphi_t, N \to \infty.$

The proof follows from Lemmas 3.3, 3.4.

In order to prove Theorem 3.1 we need the following limit statement on absolute continuity.

Lemma 3.5. [14]. *Let Y be complete separable metric space, ν be a probability on Y. Assume that a sequence of measurable mappings $f_n : Y \to Y, n \ge 0$ is such that*

1) $\forall n \ge 1 :\ \nu \circ f_n^{-1} \ll \nu$;

2) $f_n \xrightarrow{\nu} f_0, n \to \infty$;

3) the sequence of Radon–Nykodym densities $\left\{\frac{d\nu\circ f_n}{d\nu}, n \ge 1\right\}$ is uniformly integrable w.r.t. ν.

Then the image measure $\nu \circ f_n^{-1}$ is absolutely continuous w.r.t. ν.

By Lemmas 3.1, 3.5 and Corollary 3.2 we get that:

for all $t \ge 0$ for P-almost all $\omega \in \Omega$ the image measure $\mu \circ \varphi_t^{-1}$ is absolute continuous with respect to μ.

Observe that the corresponding set of P-null measure depends on t, generally. Let us prove that it can be chosen independently on t.

The processes ψ_t and $\widehat{U}_t$ are continuous in t with probability 1. So, by Lemma 3.4, with a probability 1 we have the following continuity:

$$\exists \Omega_0,\ \mathbf{P}(\Omega_0) = 1\ \forall \omega \in \Omega_0\ \forall t_0 \in [0, T] :\ \varphi_t \xrightarrow{\mu} \varphi_{t_0},\ t \to t_0. \tag{3.10}$$

Similarly to reasoning of §6 [6] it can be verified that a process

$$\mathbf{E}_{\mu}\frac{d\mu\circ\varphi_t^{-1}}{d\mu}\left|\ln\frac{d\mu\circ\varphi_t^{-1}}{d\mu}\right| = \mathbf{E}_{\mu}\frac{d\mu\circ\psi_t^{-1}}{d\mu}\left|\ln\frac{d\mu\circ\psi_t^{-1}}{d\mu}\right|$$

has continuous in $t\in[0,T]$ modification.

So, for P-a.a. $\omega\in\Omega$ the sequence $\{\rho_t(\omega,\cdot), t\in\mathbb{Q}\cap[0,T]\}$ is uniformly integrable with respect to μ.

The application of Lemma 3.5 and (3.10) finishes the proof of Theorem 3.1.

REFERENCES

1. H.-H. Kuo. *Gaussian measures in Banach spaces.* Berlin, New York, Springer-Verlag (1975).
2. Cruzeiro A.-B. Equations differentielles sur l'espace de Wiener et formues de Cameron-Martin non-lineaires, *J. Func. Anal.*, **54**, 206–227 (1983).
3. Cruzeiro, Ana Bela. Unicite de solutions d'equations differentielles sur l'espace de Wiener, *J. Funct. Anal.* **58**, 3, 335–347 (1984).
4. Ustunel A. S., Zakai M. Absolute continuity on the Wiener space and some applications. *Stochastic analysis* (Ithaca, NY, 1993), 487–494, Proc. Sympos. Pure Math., 57, Amer. Math. Soc., Providence, RI, 1995.
5. Tolmachev N. A., Khitruk F. A. Transformations of Gaussian measures by stochastic flows. *Dokl. Akad. Nauk* 399, 4, 454–459 (2004). (Russian).
6. Pilipenko A. Yu. Transformation of measures in infinite-dimensional spaces by the flow induced by a stochastic differential equation, Sbornik: Mathematics **194**, 4, 551-573 (2003).
7. Peters G. Anticipating flows on the Wiener space generated by vector fields of low regularity. *J. Funct. Anal.* **142**, 1, 129–192 (1996).
8. Kusuoka S. Nonlinear transformation containing rotation and Gaussian measure. *J. Math. Sci. Univ.*, (Tokyo) **10**, 1, 1–40 (2003).
9. Da Prato G., Zabczyk J. *Stochastic equations in infinite dimensions.* Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 454 pp. (1992).
10. Skorohod A. V. *Random linear operators.* Translated from the Russian. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, xvi+199 pp. (1984).
11. Bogachev V. I. Gaussian measures. Mathematical Surveys and Monographs, 62. American Mathematical Society, Providence, RI, xii+433 pp. (1998).
12. C. Stricker and M. Yor. Calcul stochastique dependant dun param'etre, *Z. Wahrsch. Verw. Gebiete* 45:2, 109-133 (1978).
13. Kunita H. *Stochastic flows and stochastic differential equations*, Cambridge studies in advanced mathematics (1990).
14. I. I. Gikhman, A. V. Skorokhod: On densities of probability measures in functional spaces. *Uspehi Mat. Nauk*, **21**, 6, 83–152 (1966).