Ukrainian Mathematical Journal, Vol. 57, No. 9, 2005

MEASURE-VALUED DIFFUSIONS AND CONTINUAL SYSTEMS OF INTERACTING PARTICLES IN A RANDOM MEDIUM

A. Yu. Pilipenko UDC 519.21

We consider continual systems of stochastic equations describing the motion of a family of interacting particles whose mass can vary in time in a random medium. It is assumed that the motion of every particle depends not only on its location at given time but also on the distribution of the total mass of particles. We prove a theorem on unique existence, continuous dependence on the distribution of the initial mass, and the Markov property. Moreover, under certain technical conditions, one can obtain the measure-valued diffusions introduced by Skorokhod as the distributions of the mass of such systems of particles.

In the last decades, the theory of Markov measure-valued processes has been extensively developed. This is explained, first of all, by the variety of areas of its application in numerous fields of science: biology, genetics, chemistry, physics, etc. (see, e.g., [1]).

The most interesting examples of measure-valued processes, such as, e.g., the Fleming–Viot, Dawson–Watanabe, and McKean – Vlasov processes, are usually obtained in the following way: First, one considers a prelimit system consisting of finitely many particles. Further, certain assumptions are made concerning the character of the evolution of particles (interaction or independence of motion, possibility of birth or death, etc.). Then one lets the initial number of particles tend to infinity, whereas the mass of every particle tends to zero. Introducing, if necessary, a certain normalization of time or space variables, one proves the weak relative compactness of a sequence of measure-valued processes and writes the problem of martingales for the limits. It should be noted that the proof of the uniqueness of the limit is usually a much more complicated problem than the proof of its existence (see, e.g., [1, 2]).

Also note that the processes obtained as solutions of the problem of martingales are defined, generally speaking, on "a certain" probability space. Furthermore, investigating only the evolution of measures, one does not always succeed in establishing a relationship between mass changes and the motion of particles on the phase space, the presence of a flow, etc. As an illustration, consider the following deterministic example [3]:

Assume that the phase space is $X = \mathbb{R}^2$. Consider the following two dynamical systems: $\varphi_t^1 = id$ is the identical mapping and φ_t^2 is the rotation by an angle t about the origin of coordinates. Let $\mu_t^j = \mu \circ \left(\varphi_t^j\right)^{-1}$, $j = 1, 2,$ be a measure-valued process that can be obtained as a result of the transfer of the initial mass μ by the flow φ_t^j. In this case, if $\mu = N(0,1)$ is a Gaussian measure with mean-value zero and unit covariance operator, then $\mu_t^1 = \mu_t^2 = \mu$, i.e., the distribution of the total mass remains constant, though it is generated by absolutely different mass-transfer processes.

The approach to studying mass-transfer processes together with the investigation of the motion of interacting particles in a random medium was considered in detail in [3, 4].

In the present work, we analyze measure-valued diffusions for which the motion of interacting particles whose mass may vary in time serves as a prelimit model. Skorokhod [5, 6] proposed the following model:

Institute of Mathematics, Ukrainian Academy of Sciences, Kiev.

Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 57, No. 9, pp. 1289–1301, September, 2005. Original article submitted June 17, 2005.

Assume that, at the initial time $t = 0$, particles of masses $c_1, \ldots, c_n$ are located at points $u_1, \ldots, u_n \in \mathbb{R}^d$, respectively. Let x_t^j and c_t^j denote the location and the mass of the jth particle at certain time $t \geq 0$. Then $\mu_t = \sum_j c_t^j \delta_{x_t^j}$ is the distribution of the total mass at time t. Further, we assume that the variation in mass and weight is described by the system of interacting stochastic equations

$$dx_t^j = a_0\left(x_t^j, \mu_t\right)dt + \sum_k a_k\left(x_t^j, \mu_t\right)dw_k(t), \tag{1}$$

$$dc_t^j = \left(b_0\left(x_t^j, \mu_t\right)dt + \sum_k b_k\left(x_t^j, \mu_t\right)dw_k(t)\right)c_t^j, \tag{2}$$

$$\mu_t = \sum_j c_t^j \delta_{x_t^j} \tag{3}$$

with the initial conditions

$$x_0^j = u_j \in \mathbb{R}^d, \quad c_0^j = c_j \geq 0, \tag{4}$$

where $\{w_k(t)\}$ are independent Wiener processes.

Under certain natural conditions of smoothness and boundedness of the coefficients of Eqs. (1)–(4), the weak relative compactness of the measure-valued random processes μ_t^n was proved in [5, 6] in the case where the initial distributions $\{\mu_0^n\}_{n\geq 1}$ converge weakly to a certain finite measure, and the problem of martingales was written for possible limits.

Remark 1. The case $b_k \equiv 0$, i.e., the case where the mass of every particle remains constant, was considered in [3].

The main subject studied in the present paper is the continual system of stochastic equations

$$dx_t(u) = a_0(x_t(u), \mu_t)dt + \sum_{k=1}^{n} a_k(x_t(u), \mu_t)dw_k(t), \tag{5}$$

$$d\rho_t(u) = \left(b_0(x_t(u), \mu_t)dt + \sum_{k=1}^{n} b_k(x_t(u), \mu_t)dw_k(t)\right)\rho_t(u), \tag{6}$$

$$\mu_t = (\rho_t \mu) \circ x_t^{-1}, \tag{7}$$

$$x_0(u) = u, \qquad \rho_0(u) = 1, \tag{8}$$

where μ is an arbitrary finite measure. Here, $\rho_t \mu$ is interpreted as a measure having the Radon–Nikodym density ρ_t with respect to μ, and $(\rho_t \mu) \circ x_t^{-1}$ is the image of $\rho_t \mu$ under the mapping x_t.

In the special case where the initial distribution is a discrete measure, i.e., $\mu = \sum_{j=1}^{m} c_j \delta_{u_j}$, the processes $x_t^j := x_t(u_j)$ and $c_t^j := \rho_t(u_j)$ satisfy Eqs. (1)–(4) and $\sum_{j=1}^{m} c_t^j(t)\delta_{x_t^j} = (\rho_t \mu) \circ x_t^{-1}$.

In the present work, we prove the existence and uniqueness of a strong solution of system (5)–(8) as well as its continuous dependence on the initial measure and Markov property. As a result, the process μ_t obtained in the course of the solution of system (5)–(8) is a measure-valued diffusion in the sense of definition from [5, 6].

1. Theorem on Existence and Uniqueness

Let $\mathfrak{M}$ be the space of finite measures on $\mathbb{R}^d$ with the topology of weak convergence, i.e., a sequence $\{\mu_n, n \geq 1\} \subset \mathfrak{M}$ converges to $\mu \in \mathfrak{M}$ if, for any bounded continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$, the following convergence takes place:

$$\langle \mu_n, f \rangle := \int f\, d\mu_n \to \int f\, d\mu, \quad n \to \infty.$$

Assume that $a_k, b_k\colon \mathbb{R}^d \times \mathfrak{M} \to \mathbb{R}^d$, $u \in \mathbb{R}^d$, $t \in [0, T]$, $\mu \in \mathfrak{M}$, and $\{ w_k(t), k = \overline{1, n} \}$ are independent Wiener processes.

Definition 1. *A pair* (x, ρ) *is called a solution of system (1)–(4) if, for any* $t \in [0, T]$, *the process* $(x, \rho) = (x_s(u, w), \rho_s(u, w))$, $s \in [0, t]$, $u \in \mathbb{R}^d$, $w \in \Omega$, *is measurable with respect to the* σ*-algebra* $\mathcal{B}_{[0,t]} \times \mathcal{B}_{\mathbb{R}^d} \times \mathcal{F}$ *and an integral analog of (1)–(4) is satisfied. Here,* $\mathcal{B}_{[0,t]}$ *and* $\mathcal{B}_{\mathbb{R}^d}$ *are the Borel* σ*-algebras on* $[0, t]$ *and* $\mathbb{R}^d$, *respectively, and* $\mathcal{F}_t = \sigma(w_k(s), s \leq t, k = \overline{1, n})$.

We denote by $\mathfrak{K}$ the set of functions from $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ bounded by unity and satisfying the Lipschitz condition with a constant not greater than 1.

Theorem 1. *Suppose that functions* a_k *and* b_k *satisfy the following conditions:*

(i) a_k *and* b_k, $k = \overline{0, m}$, *are bounded;*

(ii) there exists $L > 0$ *such that*

$$\forall k = \overline{0, m} \quad \forall u_1, u_2 \in \mathbb{R}^d \quad \forall \mu_1, \mu_2 \in \mathfrak{M}:$$

$$\left| a_k(u_1, \mu_1) - a_k(u_2, \mu_2) \right| + \left| b_k(u_1, \mu_1) - b_k(u_2, \mu_2) \right|$$

$$\leq L \left(\sup_{k \in \mathfrak{K}} \left| \int K(u_1, v)\mu_1(dv) - \int K(u_2, v)\mu_2(dv) \right| + \left| u_1 - u_2 \right| \right). \tag{9}$$

Then there exists a unique solution of system (5)–(8).

Remark 2. Generally speaking, inequality (9) is more general than the inequality

$$\left|a_k(u_1,\mu_1)-a_k(u_2,\mu_2)\right| + \left|b_k(u_1,\mu_1)-b_k(u_2,\mu_2)\right|$$

$$\le L\left(\sup_{f\in\mathfrak{F}}\left|\int f(v)\mu_1(dv)-\int f(v)\mu_2(dv)\right|+\left|u_1-u_2\right|\right), \tag{9'}$$

where $\mathfrak{F}$ is the set of functions from $\mathbb{R}^d \to \mathbb{R}$ bounded by unity and satisfying the Lipschitz condition with a constant not greater than 1. Integral functionals of the form $a_k(u,\mu)=\int K(u,v)\mu(dv)$ with $K\in\mathfrak{K}$ not necessarily satisfy (9') but satisfy inequality (9).

In Sec. 2, we show that if the processes $x_t(u)$ and $\rho_t(u)$ are a solution of system (4)–(6), then they have a continuous modification with respect to t and u. Therefore, in what follows (Secs. 3–6), we assume that $x_t(\cdot)$ and $\rho_t(\cdot)$, $t\in[0;T]$, are random processes with values in the spaces $C\left(\mathbb{R}^d;\mathbb{R}^d\right)$ and $C\left(\mathbb{R}^d\right)$, respectively, with the topology of uniform convergence on compact sets.

We prove the theorem in two steps: trajectorywise uniqueness (Sec. 3) and weak existence (Sec. 4). Then the existence of a strong solution follows from the Yamada–Watanabe theorem [7] [the proof of the corresponding theorem was given for stochastic differential equations in $\mathbb{R}^n$, but it can applied to Eqs. (5)–(8) almost word for word].

The trajectorywise uniqueness of a solution of system (5)–(8) is understood as follows:

Definition 2. *System (5)–(8) possesses the property of trajectorywise uniqueness if the fact that* (x,ρ) *and* $(\bar{x},\bar{\rho})$ *are solutions of system (5)–(8) implies that*

$$\mathrm{P}\left(\forall t\in[0,T]\ \ \forall u\in\mathbb{R}^d:\ x_t(u)=\bar{x}_t(u),\ \rho_t(u)=\bar{\rho}_t(u)\right)=1.$$

Note that, as follows from the continuity of (x,ρ) in t and u, this definition is equivalent to the condition that, for any $t\in[0,T]$ and $u\in\mathbb{R}^d$, the following equality is true:

$$x_t(u)=\bar{x}_t(u),\qquad \rho_t(u)=\bar{\rho}_t(u)\quad \text{a.s.}$$

2. Auxiliary Statements

In the present paper, we use numerous constants $C_1, C_2, \ldots$. For simplicity, we omit subscripts and write simply $C.$, meaning, generally speaking, different constants.

Assume that $u_1, u_2\in\mathbb{R}^d$, $\rho_1,\rho_2\colon \mathbb{R}^d\mapsto[0,\infty)$, $x_1, x_2\colon \mathbb{R}^d\to\mathbb{R}^d$, $\nu_1=(\rho_1\mu)\circ x_1^{-1}$, and $\nu_2=(\rho_2\mu)\circ x_2^{-1}$. In this case, inequality (9) yields

$$\left|a_k(u_1,\nu_1)-a_k(u_2,\nu_2)\right| \le L\left(\left|u_1-u_2\right|+\sup_{K\in\mathfrak{K}}\left|\int (K(u_1,x_1(v))\rho_1(v)-K(u_2,x_2(v))\rho_2(v))\mu(dv)\right|\right)$$

$$\le C.\left(\left(\left|u_1-u_2\right|+\left(\int|x_1-x_2|^2d\mu\right)^{1/2}\right)\left(1+\left(\int\rho_1^2d\mu\right)^{1/2}\right)+\left(\int(\rho_1-\rho_2)^2d\mu\right)^{1/2}\right). \tag{10}$$

In particular, if $\rho_1 = \rho_2 = \rho$ and $x_1 = x_2$, then

$$\left|a_k(u_1, v) - a_k(u_2, v)\right| \le C.\left|u_1 - u_2\right|\left(1 + \left(\int \rho^2 d\mu\right)^{1/2}\right). \tag{11}$$

Using the boundedness of the functions a_k and b_k, we easily obtain the following *a priori* moment estimates for the processes $\rho_t(u)$ and $x_t(u)$ (see [8, Chap. 4.5]):

Lemma 1. *Let* $\rho_t(u)$ *and* $x_t(u)$ *be solutions of Eqs. (1)–(4). Then, for any* $p > 1$, *there exists* $K_p > 0$ *independent of the initial measure* μ *and such that*

$$\sup_{u\in\mathbb{R}^d} \mathrm{E} \sup_{t\in[0,T]} \left(\rho_t(u)^p + |x_t(u) - u|^p\right) \le K_p .$$

Remark 3. The random process $\rho_t(u)$, $t \ge 0$, satisfies a linear stochastic differential equation with positive initial condition. Hence, $\rho_t(u) > 0$ a.s.

Denote

$$\tau_m = \inf\left\{t \ge 0 \colon \left(\int \left(1 + \rho_t^2(v)\right)\mu(dv)\right)^{1/2} \ge m\right\},$$

$$x_t^m(u) = x_{t\wedge\tau_m}(u), \quad \rho_t^m(u) = \rho_{t\wedge\tau_m}(u).$$

Remark 4. As follows from Lemma 1, $\tau_m \uparrow \infty$ as $m \to \infty$ a.s.

Relations (7) and (8) imply that x_t^m and ρ_t^m satisfy the Itô system of equations

$$dx_t^m(u) = a_0^m\left(x_t^m(u), t\right)dt + \sum_{k=1}^n a_k^m\left(x_t^m(u), t\right)dw_k(t),$$

$$d\rho_t^m(u) = \left(b_0^m\left(x_t^m(u), t\right)dt + \sum_{k=1}^n b_k^m\left(x_t^m(u), t\right)dw_k(t)\right)\rho_t^m(u)$$

with bounded coefficients $a_k^m(u,t) = a_k(u, \mu_t)\, \mathbb{1}_{\{t\le\tau_m\}}$ and $b_k^m(u,t) = b_k(u,\mu_t)\, \mathbb{1}_{\{t\le\tau_m\}}$ satisfying the Lipschitz condition (with a constant depending only on m).

Lemma 2. $\forall p > 1\ \forall m \in \mathbb{N}\ \exists L = L(m, p)\ \forall t_1, t_2 \in [0, T]\ \forall u_1, u_2 \in \mathbb{R}^d$:

$$\mathrm{E}\left|x_{t_1}^m(u_1) - x_{t_2}^m(u_2)\right|^p \le L\left(|t_1 - t_2|^{p/2} + \left|u_1 - u_2\right|^p\right).$$

Lemma 2 can be proved in the standard way (see, e.g., [8, Chap. 4]).

As follows from the Kolmogorov theorem, the processes $x_t^m(u)$ and $\rho_t^m(u)$ are continuous in (t, u). Remark 4 implies that there also exists a continuous modification for the processes $x_t(u)$ and $\rho_t(u)$. Furthermore, it is easy to verify that if $\bar{x}_t(u)$, $\bar{\rho}_t(u)$ is such a modification, then it also satisfies system (5)–(8), and the equality $(\rho_t \mu) \circ x_t^{-1} = (\bar{\rho}_t \mu) \circ \bar{x}_t^{-1}$, $t \in [0, T]$, holds with probability 1. Therefore, we assume in what follows that, for the pair (x, ρ), a modification continuous in (t, u) is already chosen.

Lemma 3. $\forall \varepsilon > 0$:

$$\mathrm{P}\left(\sup_{t \in [0,T]} \sup_u \frac{|x_t(u) - u| + \rho_t(u)}{1 + |u|^{\varepsilon}} < \infty \right) = 1. \tag{12}$$

Equality (12) can be obtained by analogy with the corresponding statement for the derivative of a solution of the stochastic equation with respect to the initial data (see [8, Chap. 4]).

3. Proof of the Trajectorywise Uniqueness of a Solution

Let (x_t, ρ_t) and $(\bar{x}_t, \bar{\rho}_t)$ be two solutions of (1). We introduce the stopping time

$$\tau_n = \inf\left\{ t \geq 0 \colon \left(\int \left(1 + \rho_t^2(v)\right) \mu(dv) \right)^{1/2} \geq n \right\}.$$

As follows from Lemma 1, $\tau_n \uparrow \infty$ as $n \to \infty$ a.s. In this case, using the Itô formula and estimates (10) and (11), we establish that, for any $t \geq 0$,

$$\begin{aligned} &\left| x_{t \wedge \tau_n}(u) - \bar{x}_{t \wedge \tau_n}(u) \right|^2 \left(1 + \rho_{t \wedge \tau_n}^2(u)\right) \\ &\quad \leq L. \int_0^{t \wedge \tau_n} \left(\left(|x_s(u) - \bar{x}_s(u)|^2 + \int |x_s(v) - \bar{x}_s(v)|^2 \mu(dv) \right) \left(1 + \int \rho_s^2(v) \mu(dv)\right) \right. \\ &\quad\quad \left. + \int |\rho_s(v) - \bar{\rho}_s(v)|^2 \mu(dv) \right) \left(1 + \rho_s^2(u)\right) ds + M_t(u), \end{aligned} \tag{13}$$

where $M_t(u)$ is a certain martingale (with respect to t) and

$$\sup_u \sup_{t \in [0,T]} \mathrm{E}|M_t(u)|^2 < \infty,$$

$$\mathrm{E} M_t(u) = 0.$$

By analogy, taking the definition of τ_n into account, we obtain the following inequality with $\mathrm{E} N_t(u) = 0$:

$$\begin{aligned} \left| \rho_{t \wedge \tau_n}(u) - \bar{\rho}_{t \wedge \tau_n}(u) \right|^2 &\leq L. \int_0^{t \wedge \tau_n} \left(|\rho_s(u) - \bar{\rho}_s(u)|^2 + \left(1 + \rho_s^2(u)\right) \right. \\ &\quad \left. \times \left(|x_s(u) - \bar{x}_s(u)|^2 + \int |x_s(v) - \bar{x}_s(v)|^2 \left(1 + \rho_s^2(v)\right) \mu(dv) \right) \right) ds + N_t(u). \end{aligned} \tag{14}$$

We take the mathematical expectation of both sides of (13) and integrate with respect to $\mu(du)$. Recall that $\int\left(1+\rho^2_{s\wedge\tau_n}(u)\right)\mu(du)\le n$. Then, replacing u by v in the integrals, we obtain the following inequality [generally speaking, with a different constant $L_\cdot = L_\cdot(n)$]:

$$\int \mathrm{E}\left|x_{t\wedge\tau_n}(v)-\bar{x}_{t\wedge\tau_n}(v)\right|^2\rho^2_{t\wedge\tau_n}(v)\mu(dv)$$

$$\le L_\cdot\left(2\int \mathrm{E}\int_0^{t\wedge\tau_n}|x_s(v)-\bar{x}_s(v)|^2\rho_s^2(v)\mu(dv)ds+\int \mathrm{E}\int_0^{t\wedge\tau_n}|\rho_s(v)-\bar{\rho}_s(v)|^2\mu(dv)ds\right).$$

$$\le L_\cdot\left(2\int_0^t\int \mathrm{E}\left|x_{s\wedge\tau_n}(v)-\bar{x}_{s\wedge\tau_n}(v)\right|^2\rho^2_{s\wedge\tau_n}(v)\mu(dv)ds+\int_0^t\int \mathrm{E}\left|\rho_{s\wedge\tau_n}(v)-\bar{\rho}_{s\wedge\tau_n}(v)\right|^2\mu(dv)ds\right). \quad (15)$$

The existence of the integrals on both sides of this inequality follows from Lemma 1.

Taking the mathematical expectation in (14) and integrating with respect to $\mu(du)$, we arrive at the following inequality by analogy with (15):

$$\int \mathrm{E}\left|\rho_{t\wedge\tau_n}(v)-\bar{\rho}_{t\wedge\tau_n}(v)\right|^2\mu(dv)$$

$$\le L_\cdot\left(\int_0^t\int \mathrm{E}\left|\rho_{s\wedge\tau_n}(v)-\bar{\rho}_{s\wedge\tau_n}(v)\right|^2\mu(dv)ds+\int_0^t\int \mathrm{E}\left|x_{s\wedge\tau_n}(v)-\bar{x}_{s\wedge\tau_n}(v)\right|^2\left(1+\rho_{s\wedge\tau_n}(v)^2\right)\mu(dv)ds\right). \quad (16)$$

Applying the Gronwall lemma to (15) and (16) for any $t\in[0,T]$, we get $x_{t\wedge\tau_n}(v)=\bar{x}_{t\wedge\tau_n}(v)$ and $\rho_{t\wedge\sigma_n}(v)=\bar{\rho}_{t\wedge\tau_n}(v)$ for μ-almost all $v\in\mathbb{R}^d$ and almost all $\omega\in\Omega$. Since $\tau_n\to+\infty$ as $n\to\infty$ a.s., for any $t\in[0,T]$ we have

$$x_t(v)=\bar{x}_t(v),\qquad \rho_t(v)=\bar{\rho}_t(v)\quad \mu\text{-a.s.}$$

Substituting these equalities into (13) and (14), taking the mathematical expectation, and using the Gronwall lemma, we obtain

$$\forall t\in[0,T]\quad \forall u\in\mathbb{R}^d:\ x_t(u)=\bar{x}_t(u),\quad \rho_t(u)=\bar{\rho}_t(u)\quad \text{P-a.s.}$$

4. Weak Existence

Let $\mu^m=\sum_{k=1}^m c_{k,m}\delta_{u_{k,m}}$, $m\ge 1$, be a sequence of discrete measures that converge weakly to the measure μ.

We denote by x_t^m, ρ_t^m, μ_t^m the solution of system (5)–(8) with the initial condition μ^m instead of μ.

First, note that solutions of the corresponding equations exist and are unique. Indeed, it is easy to deduce from the assumptions concerning the coefficients a_i and b_i that the functions

$$(u_1, \dots, u_m, \rho_1, \dots, \rho_m) \mapsto a_i\left(u_k, \sum_{j=1}^m c_j \rho_j \delta_{u_j}\right),$$

$$(u_1, \dots, u_m, \rho_1, \dots, \rho_m) \mapsto b_i\left(u_k, \sum_{j=1}^m c_j \rho_j \delta_{u_j}\right)$$

satisfy the condition of linear growth and the local Lipschitz condition.

Remark 5. For simplicity, $c_{k,m}$ and $u_{k,m}$ are replaced here by c_k and u_k, respectively.

Thus, the system

$$dx_t^m(u_k) = a_0\left(x_t^m(u_k), \sum_{j=1}^m c_j \rho_t^m(u_j)\delta_{x_t^m(u_j)}\right)dt + \sum_{i=1}^n a_i\left(x_t^m(u_k), \sum_{j=1}^m c_j \rho_t^m(u_j)\delta_{x_t^m(u_j)}\right)dw_i(t),$$

$$d\rho_t^m(u_k) = \left(b_0\left(x_t^m(u_k), \sum_{j=1}^m c_j \rho_t^m(u_j)\delta_{x_t^m(u_j)}\right)dt + \sum_{i=1}^n b_i\left(x_t^m(u_k), \sum_{j=1}^m c_j \rho_t^m(u_j)\delta_{x_t^m(u_j)}\right)dw_i(t)\right)\rho_t^m(u_k),$$

$$x_0^m(u_k) = u_k, \qquad \rho_0^m(u_k) = 1$$

has a unique solution.

For $u \neq u_k$, we determine $x_t^m(u)$ from the equation

$$dx_t^m(u) = a_0\left(x_t^m(u), \sum_{j=1}^m c_j \rho_t^m(u_j)\delta_{x_t^m(u_j)}\right)dt + \sum_{i=1}^n a_i\left(x_t^m(u), \sum_{j=1}^m c_j \rho_t^m(u_j)\delta_{x_t^m(u_j)}\right)dw_i(t).$$

The corresponding equation for $\rho_t^m(u)$ can be written by analogy.

Note that x^m and ρ^m thus constructed form the unique solution of system (5)–(8) with initial condition $\mu_0^m = \sum_{k=1}^m c_{k,m}\delta_{u_{k,m}}$.

By analogy with Lemmas 1 and 2, we have the following inequalities

$$\forall p > 1 \quad \forall n \geq 1 \quad \exists L_n \quad \forall m \quad \forall t_1, t_2 \in [0, T] \quad \forall u_1, u_2 \in \mathbb{R}^d:$$

$$\mathrm{E}\left|x^m_{t_1 \wedge \tau_{n,m}}(u_1) - x^m_{t_2 \wedge \tau_{n,m}}(u_2)\right|^p + \left|\rho^m_{t_1 \wedge \tau_{n,m}}(u_1) - \rho^m_{t_2 \wedge \tau_{n,m}}(u_2)\right|^p \leq L_n\left(|t_1 - t_2|^{p/2} + |u_1 - u_2|^p\right),$$

$$\forall p > 1 \quad \exists C: \quad \sup_m \sup_{u \in \mathbb{R}^d} \mathrm{E} \sup_{t \in [0,T]}\left(\left|x_t^m(u) - u\right|^p + \left|\rho_t^m(u)\right|^p\right) < C, \tag{17}$$

where

$$\tau_{n,m} = \inf\left\{t \geq 0: \left(\int\left(1 + \left(\rho_t^m(v)\right)^2 \mu^m(dv)\right)\right)^{1/2} \geq n\right\}.$$

As follows from Theorem 1.4.7 in [8], for any $n \in \mathbb{N}$ the sequence of random fields $\left\{\left(x^m_{\cdot\wedge\tau_{n,m}}(\cdot),\ \rho^m_{\cdot\wedge\tau_{n,m}}(\cdot)\right),\ m \ge 1\right\}$ is weakly relatively compact in the space $C\left([0,T];\ C\left(\mathbb{R}^d;\mathbb{R}^d\right)\right) \times C\left([0,T],\ C\left(\mathbb{R}^d\right)\right)$, where the spaces $C\left(\mathbb{R}^d;\mathbb{R}^d\right)$ and $C\left(\mathbb{R}^d\right)$ are equipped with the topology of uniform convergence on compact sets.

Inequalities (17) yield

$$\forall c > 0: \quad \lim_{n\to\infty} \sup_m \mathrm{P}\left\{\tau_{n,m} < c\right\} = 0,$$

and, hence, the sequence $\left\{\left(x^m_\cdot(\cdot),\ \rho^m_\cdot(\cdot)\right), m \ge 1\right\}$, is also weakly relatively compact.

Let us show that the sequence of measure-valued processes $\left\{\mu^m_t,\ m \ge 1\right\}$ is weakly relatively compact in $C([0,T],\ \mathfrak{M})$. For this purpose, it suffices to verify that the following conditions are satisfied [1]:

(i) for any bounded Lipschitzian function f, the sequence of random processes $\left\langle \mu^m_t, f\right\rangle = \int f\, d\mu^m_t$, $m \ge 1$, is weakly relatively compact in $C([0,T])$;

(ii) for any $\varepsilon > 0$, one has

$$\sup_m \mathrm{P}\left(\sup_{t\in[0,T]} \mu^m_t\{\|u\| \ge n\} \ge \varepsilon\right) \to 0, \quad n \to \infty. \tag{18}$$

Taking into account that $\left\langle \mu^m_t, f\right\rangle = \int f\left(x^m_t(u)\right) \times \rho^m_t(u)\,\mu^m(du)$ and applying the arguments used in the proof of the weak compactness of the flows $\left(x^m, \rho^m\right)$, we easily verify condition (i).

Let us verify (18). We have

$$\mathrm{P}\left(\sup_t \mu^m_t\{|u| \ge n\} \ge \varepsilon\right) \le \mathrm{P}\left(\sup_{t\in[0,T]} \sup_{|u|\le\sqrt{n}} |x_t(u)| \ge \frac{n}{2}\right) + \mathrm{P}\left(\sup_{t\in[0,T]} \int_{|u|>\sqrt{n}} \rho^m_t(u)\mu^m(du) \ge \varepsilon\right).$$

According to Lemma 3, the first term tends to zero as $n \to \infty$. To estimate the second term, we use the Chebyshev inequality and Lemma 1. As a result, we get

$$\mathrm{P}\left(\sup_{t\in[0,T]} \int_{|u|>\sqrt{n}} \rho^m_t(u)\mu^m(du) \ge \varepsilon\right) \le \varepsilon^{-1}\mu_m\left(u: |u| > \sqrt{n}\right) \sup_u \mathrm{E} \sup_{t\in[0,T]} \rho^m_t(u) \to 0, \quad n \to \infty.$$

Thus, the weak relative compactness is proved.

Passing, if necessary, to a subsequence, we assume that $(x^m_\cdot,\ \rho^m_\cdot,\ \mu^m_\cdot)$ converges weakly as $m \to \infty$ to the limit process $(x_\cdot, \rho_\cdot, \mu_\cdot)$ (defined, generally speaking, on a different probability space).

Let us prove that

$$\mathrm{P}\left(\forall t \in [0,T]:\ \mu_t = (\rho_t\mu)\circ x_t^{-1}\right) = 1. \tag{19}$$

It is easy to see that $(\rho_t\mu) \circ x_t^{-1}$ is a continuous measure-valued process. Hence, to verify (19), it suffices to prove that

$$\forall t \in [0, T]: \quad \mathrm{P}\left(\mu_t = (\rho_t\mu) \circ x_t^{-1}\right) = 1.$$

In turn, for this relation to be true, it is sufficient that, for any $k \geq 1$ and any continuous finite function $f\colon \mathbb{R}^{kd} \to \mathbb{R}$, the following mathematical expectations be equal:

$$\mathrm{E} \int_{\mathbb{R}^{kd}} f(x_t(u_1), \ldots, x_t(u_k))\rho_t(u_1) \ldots \rho_t(u_k)\mu(du_1) \ldots \mu(du_k) = \mathrm{E} \int_{\mathbb{R}^{kd}} f(u_1, \ldots, u_k)\mu_t(du_1) \ldots \mu_t(du_k).$$

Without loss of generality [9], we may assume that the processes $(x_t^m, \rho_t^m, \mu_t^m)$ and (x_t, ρ_t, μ_t) are defined on the same probability space and the convergence takes place almost surely. For simplicity, we consider only the case $k = 1$.

For any $m \geq 1$, we have $\mu_t^m = \left(\rho_t^m \mu^m\right) \circ \left(x_t^m\right)^{-1}$, whence

$$\mathrm{E}\int f\left(x_t^m(u)\right)\rho_t^m(u)\mu^m(du) = \mathrm{E}\int f(u)\mu_t^m(du). \tag{20}$$

For almost all ω, the following convergence takes place:

$$\int f(u)\mu_t^m(du) \to \int f(u)\mu_t(du).$$

Furthermore, the sequence of random variables $\left\{\int f\, d\mu_t^m, m \geq 1\right\}$ is uniformly integrable because

$$\sup_m \mathrm{E}\left(\int f\, d\mu_t^m\right)^2 \leq \sup_u |f(u)|^2 \sup_m \mathrm{E}\left(\int \rho_t^m\, d\mu_t^m\right)^2 \leq \sup_u |f(u)|^2 \sup_m \sup_u \mathrm{E}\left|\rho_t^m(u)\right|^2 \sup_m \mu^m\left(\mathbb{R}^d\right) < \infty.$$

Therefore, the right-hand side of (20) converges to $\mathrm{E}\int f\, d\mu_t$. Consider its left-hand side. It is easy to verify that the sequence $\left\{\mathrm{E}\int f\left(x_t^m\right)\rho_t^m d\mu^m, m \geq 1\right\}$ is uniformly integrable. Therefore, it suffices to verify the convergence of the following integrals in probability:

$$\int f\left(x_t^m\right)\rho_t^m d\mu_t^m \to \int f(x_t)\rho_t d\mu, \qquad m \to \infty. \tag{21}$$

Recall that the following uniform convergence on compact sets takes place almost surely:

$$x_t^m \to x_t \quad \text{and} \quad \rho_t^m \to \rho_t \quad \text{as} \quad m \to \infty. \tag{22}$$

Let $U \subset \mathbb{R}^d$ be a certain compact set. Then

$$\left| \int_{\mathbb{R}^d} f\left(x_t^m\right)\rho_t^m \, d\mu^m - \int_{\mathbb{R}^d} f(x_t)\rho_t \, d\mu \right|$$

$$\leq \left| \int_{\mathbb{R}^d} f(x_t)\rho_t \, d\mu - \int_{\mathbb{R}^d} f(x_t)\rho_t \, d\mu^m \right| + \int_U \left| f(x_t)\rho_t - f\left(x_t^m\right)\rho_t^m \right| d\mu^m$$

$$+ \int_{\mathbb{R}^d \setminus U} \left(|f(x_t)\rho_t| + \left| f\left(x_t^m\right)\rho_t^m \right| \right) d\mu^m. \tag{23}$$

The first term on the right-hand side of (23) converges to zero with probability 1 by virtue of the weak convergence of measures, the continuity of x_t and ρ_t in the parameter u, and Lemma 1. The second term does not exceed $\sup_{u \in U}\left(\left| f(x_t(u))\rho_t(u) - f\left(x_t^m(u)\right)\rho_t^m(u)\right|\right) \sup_m \mu^m(U)$, and, hence, it converges to zero almost surely for any fixed compact set U.

By the choice of U, we can also make the expression

$$\sup_m \mathrm{E} \int_{\mathbb{R}^d \setminus U} \left(|f(x_t)\rho_t| + \left| f\left(x_t^m\right)\rho_t^m \right| \right) d\mu^m \leq C \sup_m \mu^m\left(\mathbb{R}^d \setminus U\right)$$

arbitrarily small. These results show that (21) converges, and, therefore, relation (19) is true.

By analogy with the arguments presented above, one can easily prove the weak relative compactness of the sequence

$$\zeta_m = \left(x_\cdot^m(\cdot), \rho_\cdot^m(\cdot), \mu_\cdot^m(\cdot), \int_0^\cdot a_k\left(x_s^m(\cdot), \mu_s^m\right) ds, \right.$$

$$\left. \int_0^\cdot b_k\left(x_s^m(\cdot), \mu_s^m\right)\rho_s^m \, dw_k(s), \; w_k(\cdot), \; k = \overline{0, n}, \; m \geq 1 \right),$$

where $w_0(t) \equiv t$.

Passing to subsequences, if necessary, and using the Skorokhod theorem [9], we choose a common probability space and the sequence

$$\tilde{\zeta}_m = \left(\tilde{x}_\cdot^m, \tilde{\rho}_\cdot^m, \tilde{\mu}_\cdot^m, \int_0^\cdot a_k\left(\tilde{x}_s^m(\cdot), \tilde{\mu}_s^m\right) ds, \; \int_0^\cdot b_k\left(\tilde{x}_s^m(\cdot), \tilde{\mu}_s^m\right) dw_k(s), \; \tilde{w}_k(\cdot), \; k = \overline{0, n}, \; m \geq 1 \right)$$

on it such that $\tilde{\zeta}_m \overset{\mathfrak{D}}{=} \zeta_m$, and, furthermore, this sequence converges almost surely to a certain element

$$\tilde{\zeta} = \left(\tilde{x}, \tilde{\rho}, \tilde{\mu}, \alpha_k, \beta_k, \tilde{w}_k, \; k = \overline{0, n} \right),$$

where $\tilde{w}_k$, $k = \overline{1, n}$, are Wiener processes and, as proved above, $\tilde{\mu} = \left(\tilde{\rho}\mu^0\right) \circ \tilde{x}^{-1}$.

By analogy with Theorem 1 in [10, Chap. 5, Sec. 2], one can easily verify that

$$\alpha_t^k(u) = \int_0^t a_k\big(\tilde{x}_s(u), \tilde{\mu}_s\big)\, d\tilde{w}_k(s),$$

$$\beta_t^k(u) = \int_0^t b_k\big(\tilde{x}_s(u), \tilde{\mu}_s\big)\rho_s(u)\, d\tilde{w}_k(s),$$

and $\tilde{x}$, $\tilde{\rho}$, $\tilde{\mu}$, and $\tilde{w}_k$ are connected by relations (5)–(8). Thus, the weak existence and, hence, Theorem 1 are proved.

Remark 6. For the existence of a weak solution, one can weaken condition (9) of Theorem 1. Namely, instead of this condition, it suffices to require that the functions a_k and b_k be continuous in (u, μ) and have the Lipschitz property with respect to the first argument:

$$\exists K \quad \exists L \quad \forall k = \overline{0, m} \quad \forall u_1,\ u_2 \in \mathbb{R}^d \quad \forall \mu \in \mathfrak{M}:$$

$$\big|a_k(u_1, \mu) - a_k(u_2, \mu)\big| + \big|b_k(u_1, \mu) - b_k(u_2, \mu)\big| \le L|u_1 - u_2|.$$

Under similar conditions, a theorem on the existence of a weak solution for equations with interaction but without variation in the mass of particles (i.e., for $b_k \equiv 0$) was proved in [11].

5. Continuous Dependence on the Initial Measure

In this section, we establish the continuous dependence of a solution of system (5)–(8) on the initial measure μ.

Theorem 2. *Suppose that $\mu^m \to \mu$ as $m \to \infty$ in $\mathfrak{M}$. Let $\left(x^m, \rho^m\right)$ denote a solution of system (1)–(4) with initial condition μ^m. Then, for any compact set $U \subset \mathbb{R}^d$, the convergence in probability*

$$\sup_{t\in[0,T]} \sup_{u\in U} \Big(\big|x_t^m(u) - x_t(u)\big| + \big|\rho_t^m(u) - \rho_t(u)\big|\Big) \xrightarrow{\text{P}} 0, \quad m \to \infty, \tag{24}$$

and the convergence in probability of the processes

$$\mu_t^m \to \mu_t, \quad m \to \infty \qquad in \quad C([0, T], \mathfrak{M}) \tag{25}$$

take place.

Proof. By analogy with the proof of the weak existence of a solution of system (5)–(8), one can easily verify that the sequence of processes $\eta^m = \left(x^m, \rho^m, x, \rho, w_k, k = \overline{1, n}\right)$ is weakly compact in

$$\Big(C\big([0,T], C\big(\mathbb{R}^d; \mathbb{R}^d\big)\big) \times C\big([0,T], C\big(\mathbb{R}^d\big)\big) \times C\big([0,T], C\big(\mathbb{R}^d, \mathbb{R}^d\big)\big) \times C\big([0,T], C\big(\mathbb{R}^d\big)\big) \times C([0,T])\Big).$$

Using the Skorokhod theorem [9], we choose a common probability space and a subsequence $\tilde{\eta}^m \overset{\mathcal{D}}{=} \eta^m$ that converges almost surely to a certain limit process $\tilde{\eta} = \left(\bar{x},\ \bar{\rho},\ \tilde{x},\ \tilde{\rho},\ \tilde{w}_k,\ k = \overline{1, n}\right)$. As in Sec. 3, we can establish that $(\bar{x}, \bar{\rho})$ and $(\tilde{x}, \tilde{\rho})$ are solutions of system (5)–(8) with $\tilde{w}_k$ instead of w_k. Theorem 1 yields

$$\forall u \in \mathbb{R}^d \quad \forall t \in [0, T]: \quad \tilde{x}_t(u) = \bar{x}_t(u), \quad \tilde{\rho}_t(u) = \bar{\rho}_t(u) \quad \text{a.s.},$$

and, hence, the uniform convergence on compact sets $\tilde{x}^m \to \tilde{x}$ and $\tilde{\rho}^m \to \tilde{\rho}$ as $m \to \infty$ takes place almost surely. Since the distributions $\left(\tilde{x}^m,\ \tilde{\rho}^m,\ \tilde{x},\ \tilde{\rho}\right)$ and $\left(x^m,\ \rho^m,\ x,\ \rho\right)$ coincide, this yields (24).

Convergence (25) is a consequence of (24) and the moment estimates in Lemma 1.

6. Markov Property of the Measure-Valued Process μ_t

The proof of the Markov property of the process μ_t can be carried out by analogy with the classical scheme of the proof of the Markov property for solutions of (ordinary) stochastic differential equations (see [10, Chap. 6, Sec. 1]).

Assume that $s > 0$, $t \ge s$, $\varepsilon > 0$, and $\mu \in \mathfrak{M}$. Denote

$$\mathcal{F}_s = \sigma\left(w_k(\tau),\ \tau \in [0, s],\ k = \overline{1, n}\right),$$

$$\mathcal{F}_{s,t} = \sigma\left(w_k(\tau) - w_k(s),\ \tau \in [s, t],\ k = \overline{1, n}\right).$$

Note that the σ-algebras $\mathcal{F}_s$ and $\mathcal{F}_{s,t}$ are independent.

Denote by $x_{s,t,\nu}(u)$, $\rho_{s,t,\nu}(u)$, $\mu_{s,t,\nu}$ a solution of the system

$$dx_{s,t,\nu}(u) = a_0\left(x_{s,t,\nu}(u), \mu_{s,t,\nu}\right)dt + \sum_{k=1}^{n} a_k\left(x_{s,t,\nu}(u), \mu_{s,t,\nu}\right)dw_k(t), \tag{26}$$

$$d\rho_{s,t,\nu}(u) = \left(b_0\left(x_{s,t,\nu}(u), \mu_{s,t,\nu}\right)dt + \sum_{k=1}^{n} b_k\left(x_{s,t,\nu}(u), \mu_{s,t,\nu}\right)dw_k(t)\right)\rho_{s,t,\nu}(u), \quad t \ge s, \quad u \in \mathbb{R}^d, \tag{27}$$

$$x_{s,s,\nu}(u) = u, \quad \rho_{s,s,\nu}(u) = 1, \quad \mu_{s,t,\nu} = \nu \circ x_{s,t,\nu}^{-1}. \tag{28}$$

As follows from Theorem 2, there exists a measurable version of the mapping

$$\Omega \times \mathfrak{M} \ni (\omega, \nu) \mapsto \left(x_{s,t,\nu}, \rho_{s,t,\nu}, \mu_{s,t,\nu}\right) \in C\left(\mathbb{R}^d, \mathbb{R}^d\right) \times C\left(\mathbb{R}^d\right) \times \mathfrak{M}.$$

Using a slightly modified version of Theorem 1 for nonanticipating initial conditions, one can easily verify that system (26)–(28) with initial condition μ_s has a unique solution.

Consider the random processes

$$y_t(u) = x_t\left(x_s^{-1}(u)\right), \quad r_t(u) = \frac{\rho_t\left(x_s^{-1}(u)\right)}{\rho_s\left(x_s^{-1}(u)\right)}, \quad t \ge s.$$

Remark 7. For any $s > 0$, the mapping $u \mapsto x_s(u)$ is a homeomorphism of $\mathbb{R}^d$ [8], and, therefore, the inverse mapping x_s^{-1} is well defined.

It is easy to see that the pair $y_t,\ r_t$ satisfies the system

$$dy_t(u) = a_0(y_t(u), \mu_t)\,dt + \sum_{k=1}^{n} a_k(y_t(u), \mu_t)\,dw_k(t), \quad t \ge s,$$

$$dr_t(u) = \left(b_0(y_t(u), \mu_t)\,dt + \sum_{k=1}^{n} b_k(y_t(u), \mu_t)\,dw_k(t)\right) r_t(u),$$

$$y_s(u) = u, \qquad r_s(u) = 1.$$

It should also be noted that

$$\mu_t = (\rho_t \mu) \circ x_t^{-1} = \left(\left(\frac{\rho_t}{\rho_s}\rho_s \mu\right) \circ x_s^{-1}\right) \circ \left(x_t \circ x_s^{-1}\right)^{-1}$$

$$= \left(\frac{\rho_t \circ x_s^{-1}}{\rho_s \circ x_s^{-1}}(\rho_s \mu) \circ x_s^{-1}\right) \circ \left(x_t \circ x_s^{-1}\right)^{-1} = (r_t \mu_s) \circ y_t^{-1}.$$

Since a solution of system (26)–(28) with initial condition $v = \mu_s$ is unique, we have

$$y_t = x_{s,t,\mu_s}, \qquad r_t = \rho_{s,t,\mu_s}, \qquad \mu_t = \mu_{s,t,\mu_s}.$$

The independence of the $\mathcal{F}_{s,t}$-measurable mapping $\mu_{s,t,\cdot}$ and the $\mathcal{F}_s$-measurable random measure μ_s implies that the process μ_t possesses the Markov property.

By analogy with the arguments presented above and the results of [3, Sec. 3.3], one can verify a more general statement.

Theorem 3. *1. For any* $k, l \ge 0,\ k \ge l,$ *and* $u_1, \ldots, u_k \in \mathbb{R}^d$, *the random process*

$$(\mu_t, x_t(u_1), \ldots, x_t(u_k),\ \rho_t(u_1), \ldots, \rho_t(u_l)), \quad t \ge 0,$$

is a Markov process.

2. The random process

$$(\mu_t,\ x_t(\cdot),\ \rho_t(\cdot)), \quad t \ge 0,$$

is a Markov process in $\mathfrak{M} \times C(\mathbb{R}^d; \mathbb{R}^d) \times C(\mathbb{R}^d)$, *where the spaces* $C(\mathbb{R}^d; \mathbb{R}^d)$ *and* $C(\mathbb{R}^d)$ *are considered with the topology of uniform convergence on compact sets.*

This work was partially supported by the Ministry of Education and Science of Ukraine (project No. GP/F8/0086).

REFERENCES

1. D. A. Dawson, *Measure-Valued Markov Processes,* Springer, Berlin (1993).
2. S. Cerrai and P. Clement, "Well-posedness of the martingale problem for some degenerate diffusion processes occurring in dynamics of populations," *Bull. Sci. Math.,* **128**, No. 5, 355–370 (2004).
3. A. A. Dorogovtsev and P. Kotelenez, *Stochastic Flows with Interaction and Random Measures,* Kluwer, Dordrecht (2004).
4. A. A. Dorogovtsev, "Stochastic flows with interaction and measure-valued processes," *Int. J. Math. Sci.,* No. 63, 3963–3977 (2003).
5. A. V. Skorokhod, "Measure-valued diffusions," *Theory Stochast. Proc.,* **3**(19), No. 1-2, 7–12 (1997).
6. A. V. Skorokhod, "Measure-valued diffusions," *Ukr. Mat. Zh.,* **49**, No. 3, 458–464 (1997).
7. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes,* North-Holland, Amsterdam (1981).
8. H. Kunita, *Stochastic Flows and Stochastic Differential Equations,* Cambridge University Press, Cambridge (1990).
9. O. Kallenberg, *Foundations of Modern Probability,* Springer, New York (2002).
10. I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations and Their Applications* [in Russian], Naukova Dumka, Kiev (1982).
11. M. P. Karlikova, "On a weak solution for an equation for an evolution flow with interaction," *Ukr. Mat. Zh.,* **57**, No. 7, 895–903 (2005).