Flows generated by stochastic equations with reflection[1 2]

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Abstract— The properties of stochastic flow generated by stochastic equation with reflection are studied. We investigate an instant of time and a place where particles carrying by such a flow can coalesce.

Introduction. Consider the following stochastic differential equation

$$\begin{cases} d\varphi_{st}(x) = a_0(\varphi_{st}(x))dt + \sum_{k=1}^{m} a_k(\varphi_{st}(x))dw_k(t), t \geq s, \\ \varphi_{ss}(x) = x, x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $a_k : \mathbb{R}^d \to \mathbb{R}^d, k = \overline{0,m}, w_k(t)$ are Wiener processes. It is well known (cf. [1,2]) that under some assumptions on smoothness and boundedness of coefficients the solution of (1) generates a flow of diffeomorphisms of a space $\mathbb{R}^d$.

More precisely: there exists a modification of the solution such that, for each ω the mapping $\mathbb{R}^d \ni x \mapsto \varphi_{st}(x,\omega) \in \mathbb{R}^d$ is a diffeomorphism and it is a flow, i.e.

$$\forall t_1 \leq t_2 \leq t_3 \; \forall \omega : \; \varphi_{t_2 t_3} \circ \varphi_{t_1 t_2} = \varphi_{t_1 t_3}. \tag{2}$$

More complex stochastic systems can also generate the flow (cf.[3]), however the mapping $\varphi_{st}(\cdot,\omega)$ could be non-smooth and non-injective.

The study of coupling for Markov processes is actively studying, nowadays. Note that the qualitative behaviour even of Brownian flows with reflection or with membrane is not trivial [4-7].

In this article we consider a flow φ_{st} generated by stochastic equation with normal reflection in the halfspace $\mathbb{R}^d_+ := \mathbb{R}^{d-1} \times [0,\infty)$.

The Section 1 is technical. There we prove the existence of continuous modification in (s,t,x) of a flow $\varphi_{st}(x)$ is proved. Also some moments estimations are given.

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The main result of §2 and the article consists in the following characterization of the flow. The particle, which starts from a point $x \in \mathbb{R}^{d-1} \times (0, \infty)$, does not coalesce with any other particle until a moment $\tau(x)$ of hitting the hyperplane $\mathbb{R}^{d-1} \times \{0\}$. In an instant $\tau(x)$ the particle occupies the common place with some particle, that started from $\mathbb{R}^{d-1} \times \{0\}$.

§1. Existence and properties of a flow generated by SDE in $\mathbb{R}^d_+$ with normal reflection at the boundary

Consider the following stochastic equation in $\mathbb{R}^d_+$ with normal reflection in the boundary [8]

$$\begin{cases} d\varphi_{st}(x) = & a_0(\varphi_{st}(x))dt + \sum_{k=1}^m a_k(\varphi_{st}(x))dw_k(t) + \\ & +\overline{n}\xi_{s,dt}(x), \ \ s \le t, \\ \varphi_{ss}(x) = x, & \xi_{ss}(x) = 0, \ x \in \mathbb{R}^d_+, \end{cases} \tag{3}$$

where $a_k : \mathbb{R}^d_+ \to \mathbb{R}^d, k = 0, \ldots, m$, $\{w_k(t), k = 1, \ldots, m\}$ are independent Wiener processes, $\overline{n} = (0, \ldots, 0, 1)$ is a normal vector to a hyperplane $\mathbb{R}^{d-1} \times \{0\}$, $\varphi_{st}(x)$ is a reflected process, $\xi_{st}(x)$ is not decreasing in t process, which increases only on moments when $\varphi_{st}(x) \in \mathbb{R}^{d-1} \times \{0\}$, i.e.

$$\xi_{st}(x) = \int_s^t 1\!\!\mathrm{I}_{\{\varphi_{sz}(x) \in \mathbb{R}^{d-1} \times \{0\}\}} \xi_{s,dz}(x).$$

It is well-known (cf. [9]), that if functions $a_k, k = 0, \ldots, m$ satisfy Lipschitz condition

$$\exists L > 0 \ \forall k = 0, \ldots, m \ \forall x, y \in \mathbb{R}^d_+ : \|a_k(x) - a_k(y)\| \le L\|x - y\|, \tag{4}$$

then for each initial point $x \in \mathbb{R}^d_+$ and initial moment of time $s \in \mathbb{R}$ there exists and unique strong solution of (3).

Let us establish the existence of continuous in (s, t, x) modificaion of $\varphi_{st}(x)$, which satisfy the property of the flow (2).

Theorem 1. *Assume that functions $a_k, k = 0, \ldots, m$ satisfy Lipschitz condition (4). Then there exists a modification of processes $\varphi_{st}(x), \xi_{st}(x)$ (it will be denoted by the same symbols), such that*

1) for all $s \in \mathbb{R}, x \in \mathbb{R}^d_+$ the pair $(\varphi_{st}(x), \xi_{st}(x))$, $t \ge s$ is a solution of (3);

2) for all $\omega \in \Omega$ processes $\varphi_{st}(x), \xi_{st}(x)$ are continuous in a triple of arguments $(s, t, x), s \le t, x \in \mathbb{R}^d_+$;

3) φ is a flow, i.e. for all $\omega \in \Omega, x \in \mathbb{R}^d_+$, $t_1 \le t_2 \le t_3$:

$$\varphi_{t_1 t_1}(x) = x,$$
$$\varphi_{t_1 t_3}(x) = \varphi_{t_2 t_3}(\varphi_{t_1 t_2}(x)).$$

The proof of this theorem is very similar to the proof that SDE without reflection with Lipschitz coefficients generates a flow of diffeomorphisms (see Theorem 4.5.1 [1]). So, let us only sketch the main steps of the proof.

Lemma 1.
1) $\forall p \in \mathbb{R}\ \exists c\ \forall s \le t\ \forall x \in \mathbb{R}^d_+$:

$$E(1 + \|\varphi_{st}(x)\|^2)^p \le ce^{c(t-s)}(1 + |x|^2)^p. \tag{5}$$

2) $\forall p > 1\ \exists c\ \forall s \le t_1 \le t_2\ \forall x \in \mathbb{R}^d_+$:

$$E\|\varphi_{st_1}(x) - \varphi_{st_2}(x)\|^{2p} \le ce^{c(t_2-s)}(1 + \|x\|^2)^p(t_2 - t_1)^p; \tag{6}$$

3) $\forall p > 1\ \exists c\ \forall x, y \in \mathbb{R}^d_+\ \ \forall s \le t$:

$$E\|\varphi_{st}(x) - \varphi_{st}(y)\|^{2p} \le ce^{c(t_2-s)}\|x - y\|^{2p}. \tag{7}$$

Corollary 1. $\forall p > 1\ \exists c\ \forall x, y \in \mathbb{R}^d_+\ \ \forall s_1, s_2, t_1, t_2,\ s_1 \le t_1, s_2 \le t_2$:

$$E\|\varphi_{s_1t_1}(x) - \varphi_{s_2t_2}(y)\|^{2p} \le ce^{c(t_1 \vee t_2 - s_1 \wedge s_2)}\{\|x - y\|^{2p}+$$

$$+(1 + \|x\|^2 + \|y\|^2)^p(|t_1 - t_2|^p + |s_1 - s_2|^p)\}. \tag{8}$$

The inequalities (5) – (7) are obtained by using Ito formula and Gronwall lemma, similarly to [1], Lemmas 4.5.3 – 4.5.6.

Verify (5), for example.

$$(1 + \|\varphi_{st}(x)\|^2)^p \le (1 + \|x\|^2)^p + c\int_s^t (1 + \|\varphi_{sz}(x)\|^2)^p dz+$$

$$+2p\int_s^t (1 + \|\varphi_{sz}(x)\|^2)^{p-1}(\varphi_{sz}(x), \overline{n})\xi_{s,dz} + M_t, \tag{9}$$

where M_t is some martingale with $EM_t = 0$.

Remark. It is well-known that under Lemma's assumptions the random variable $\|\varphi_{st}(x)\|$ has all moments.

Since $(\varphi_{sz}(x), \overline{n}) = 0$, for each z, the point of growth of the process $\xi_{s,\cdot},\ z \ge s$, then the third item in (9) is equal to zero. Taking the expectation in both sides of (9) and applying Gronwall lemma, we get (5).

Analogously the inequalities (6), (7) can be obtained.

Kolmogorov's continuity criterion and (8) imply the existence of continuous modification of$\varphi_{st}(x)$ in a triple of its arguments. Applying (8) and Burkhoelder inequality to a process

$$\xi_{st}(x) = \left(\varphi_{st}(x) - \int_s^t a_0(\varphi_{sz}(x))dz - \sum_{k=1}^m \int_s^t a_k(\varphi_{sz}(x))d\omega_k(z), \overline{n}\right),$$

we get the following inequality

$$\forall p > 1\ \exists c\ \forall x, y \in \mathbb{R}^d_+\ \ \forall s_1, s_2, t_1, t_2, s_1 \le t_1, s_2 \le t_2 :$$

$$E\|\xi_{s_1t_1}(x)-\xi_{s_2t_2}(y)\|^{2p}\le$$
$$\le ce^{c(t_1\vee t_2-s_1\wedge s_2)}\{\|x-y\|^{2p}+(1+\|x\|^2+\|y\|^2)^p(|t_1-t_2|^p+|s_1-s_2|^p)\}.$$

So, ξ also has continuous in (s,t,x) modification. Reasoning as in Theorem 4.5.1 [1] completes the proof of the theorem.

§**2. Properties of the flow** $\varphi_t(x)$

We characterize some properties of joint motion of particles. Further we consider only the continuous in (s,t,x) modification of $\varphi_{st}(x)$, which also satisfies (2) for each ω.

Put $\varphi_t(x):=\varphi_{0,t}(x)$. Observe that (2) implies the following. If for some t,x,y,ω we have the equality

$$\varphi_t(x,\omega)=\varphi_t(y,\omega),$$

then

$$\forall z\ge t:\ \varphi_z(x,\omega)=\varphi_z(y,\omega). \tag{10}$$

I.e. if the position of particles started from x,y, coincides in some instant of time t, then they coalesce and move together.

In this section we study the moment and the position of coalescence.

Let $x\in\mathbb{R}^d_+$. Denote by

$$\tau(x)=\inf\{t\ge 0:\varphi_t(x)\in\mathbb{R}^{d-1}\times\{0\}\}$$

the moment of hitting a hyperplane $\mathbb{R}^{d-1}\times\{0\}$ by a particle that started from x.

Consider some of examples of flows.

Example 1. Let $d=1,m=1,a_0=0,a_1=1$. Then $\varphi_t(x)$ is a reflected Brownian motion in $\mathbb{R}^1_+$:

$$\varphi_t(x)=x+w(t)+\xi_t(x),x\ge 0.$$

It is easy to verify that $\varphi_t(x),\xi_t(x)$ has a form

$$\varphi_t(x)=\begin{cases} w(t)+\min_{0\le s\le t}w(s),\ x=0,\\ w(t)+x,\ x>0 \text{ and } \tau(x)\ge t,\\ \varphi_t(0),\ x>0 \text{ and } \tau(x)<t,\end{cases}$$

$$\xi_t(x)=\begin{cases} -\min_{\tau(x)\le s\le t}w(s),\tau(x)<t,\\ 0,\ \tau(x)\ge t.\end{cases}$$

I.e. $\varphi_t(x)$ is moving as $w(t)+x$ until it get to zero. Then the motion of $\varphi_t(x)$ coinsides with reflected Brownian motion $\varphi_t(0)$ started from zero.

Example 2. The situation of the previos example can be generalized to a solution of SDE with reflection in $\mathbb{R}^1_+$. If

$$\varphi_t(x)=x+\int_0^t a_0(\varphi_s(x))ds+\sum_{k=1}^m\int_0^t a_k(\varphi_s(x))dw_k(s)+\xi_t(x),$$

then for each $t \geq \tau(x)$ we have the equality $\varphi_t(x) = \varphi_t(0)$. Really, for each $t \geq 0$ by comparison theorem (similarly to [10]) we have the inequality

$$\varphi_t(x) \geq \varphi_t(0)$$

for all $t \geq 0, x \geq 0$. Thus for $t = \tau(x)$ we have

$$\varphi_{\tau(x)}(x) = 0 \geq \varphi_{\tau(x)}(0) \geq 0.$$

Therefore $\varphi_{\tau(x)}(x) = \varphi_{\tau(x)}(0) = 0$ and so $\varphi_t(x) = \varphi_t(0)$ when $t \geq \tau(x)$ by (10).

It can be proved that situation similar to Examples 1, 2 is true in multidimensional space also. But for the proof of the corresponding fact we need the more delicate topological reasoning then comparison theorem. It happens because multidimensional space does not possess the property of linear order.

The main result is the following.

Theorem 2. *Assume that functions $a_k, k = \overline{0,m}$ are Lipschitzian. Then there exists a set Ω_0 of probability 1 such that for all $\omega \in \Omega_0$ the following statement holds true:*

1) for all $x, y \in \mathbb{R}^d_+, x \neq y$ and $t < \max\{\tau(x), \tau(y)\}$ the following inequality is satisfied

$$\varphi_t(x) \neq \varphi_t(y);$$

2) for each $x \in \mathbb{R}^d_+$ there exists $y = y(x,\omega) \in \mathbb{R}^{d-1} \times \{0\}$ such that, if $\tau(x) < \infty$, then

$$\varphi_{\tau(x)}(x) = \varphi_{\tau(x)}(y),$$

moreover $\varphi_t(x) = \varphi_t(y)$ as $t \geq \tau(x)$.

Remark. Informal this theorem can be formulated in such a way. A particle that started from a point $x \in \mathbb{R}^{d-1} \times (0, \infty)$, does not hit any other particle, until the moment of hitting the hyperplane $\mathbb{R}^{d-1} \times \{0\}$. At the instant $\tau(x)$ it coalesce with some particle, that was started from $\mathbb{R}^{d-1} \times \{0\}$. After this they move together.

Remark. Theorem 2 can be easily generalized for SDE in "nice" closed set $U \subset \mathbb{R}^d$ (smooth boundary, finite number of holes, etc.) It can be shown that if there exists $x \in \partial U$ such that $\sum_{k=1}^m \langle a_k(x), n(x)\rangle^2 \neq 0$, where $n(x)$ is the normal vector, then

$$\forall t > 0 \ : \ \ P\left(\exists y \in U \backslash \partial U : \ \tau(y) < t\right) = 1.$$

Thus, it results from the Theorem 2 that

$$\forall t > 0 \ : \ \ P\left(\exists y, z \in U, \ y \neq z : \varphi_t(y) = \varphi_t(z)\right) = 1.$$

It is interesting to compare this with results of [4,5] for planar Brownian motion with normal reflection in a circle or in convex set U with some assumptions on the curvature of the boundary ∂U. It is proved that two solutions starting from different points do not meet in finite time with probability 1, i.e.

$$\forall t \geq 0 \forall y, z \in U, \ y \neq z : \ \ P\left(\varphi_t(y) = \varphi_t(z)\right) = 0.$$

Proof. Let $\widetilde{\varphi}_{st}(x)$ is a solution of the SDE

$$\begin{cases} d\widetilde{\varphi}_{st}(x) = a_0(\widetilde{\varphi}_{st}(x))dt + \sum_{k=1}^m a_k(\widetilde{\varphi}_{st}(x))dw_k(t), \\ \widetilde{\varphi}_{ss}(x) = x, x \in \mathbb{R}^{d-1} \times (0,\infty), \end{cases} \tag{11}$$

defined for $t \in [s, \tau_s(x))$, where

$$\tau_s(x) = \inf\{z \geq s : \widetilde{\varphi}_{sz}(x) \in \mathbb{R}^{d-1} \times \{0\}\}.$$

The uniqueness of solutions of (11) and (3), and the continuity in (s,t,x) implies that

$$\widetilde{\varphi}_{st}(x) = \varphi_{st}(x) \tag{12}$$

for all $s, x, t \in [s, \tau_s(x))$ and all ω from some set of complete probability. It is known [1] that for a.a. ω the mapping

$$\widetilde{\varphi}_{st} : \{x \in \mathbb{R}^{d-1} \times (0,\infty), \tau_s(x) > t\} \to \mathbb{R}^d$$

generates a flow of local homeomorphisms. I.e. for all ω from some set of complete probability $\widetilde{\Omega}$:

1) $\widetilde{\varphi}_{t_1t_3}(x) = \widetilde{\varphi}_{t_2t_3}(\widetilde{\varphi}_{t_1t_2}(x))$, when $\tau_{t_1}(x) > t_3, t_1 < t_2 < t_3$;

2) $\widetilde{\varphi}_{ss}$ is identity map;

3) $\widetilde{\varphi}_{st} : \{x \in \mathbb{R}^{d-1} \times (0,\infty) : \tau_s(x) > t\} \to \{\widetilde{\varphi}_{st}(x) : x \in \mathbb{R}^{d-1} \times (0,\infty), \tau_s(x) > t\}$ is a homeomorphism for all $s < t$.

It can be assumed, without loss of generality, that this properties and (12) are satisfied for all ω.

Assume that there exist $x, y, x \neq y$ and ω such that

$$\varphi_{t_0}(x,\omega) = \varphi_{t_0}(y,\omega)$$

for some $t_0 < \tau(x,\omega)$.

It follows from (12) and the injectivity of $\widetilde{\varphi}_{st}$ that $t_0 > \tau(y)$.

Put

$$\sigma(y) = \sup\{s \in [0,t_0] : \varphi_s(y) \in \mathbb{R}^{d-1} \times \{0\}\}.$$

Let us choose some $z \in (\sigma(y), t_0)$ such that $\varphi_z(x) \neq \varphi_z(y)$.

Remark 1. The variable $\sigma(y)$ is not a stopping time.

Remark 2. The choise of z is not unique. However, we even don't need the measurable dependence z on ω.

Observe that for $t \in [z, t_0]$ we have $\varphi_t(y) \notin \mathbb{R}^{d-1} \times \{0\}$. So,

$$\varphi_t(y) = \varphi_{zt}(\varphi_{0z}(y)) = \widetilde{\varphi}_{zt}(\varphi_{0z}(y)) \neq$$

$$\neq \widetilde{\varphi}_{zt}(\varphi_{0z}(x)) = \varphi_{zt}(\varphi_{0z}(x)) = \varphi_t(x).$$

The first statement of the Theorem is proved.

To prove the second part we need two auxiliary results.

Lemma 2. *For each* $T \geq 0:$

$$P\left(\lim_{\|x\|\to\infty, x\in\mathbb{R}^d_+} \inf_{t\in[0,T]} \|\varphi_t(x)\| = \infty \right) = 1.$$

Proof. Put $\widehat{x} = \|x\|^{-2}x$ for $x \neq 0$ and define

$$\eta_{st}(\widehat{x}) = \begin{cases} (1 + \|\varphi_{st}(x)\|)^{-1} & \text{when } \widehat{x} \neq 0 \\ 0 & \text{when } \widehat{x} \neq 0. \end{cases}$$

Then for each $p > 1$ there exists $c > 0$ such that

$$E[\|\eta_{s,t}(\widehat{x}) - \eta_{s't'}(\widehat{x}')\|^{2p}] \leq c(\|\widehat{x} - \widehat{x}'\|^{2p} + |t - t'|^p + |s - s'|^p) \tag{13}$$

for all $\widehat{x}, \widehat{x}', \|\widehat{x}\| \leq 1, \|\widehat{x}'\| \leq 1, t, t', s, s' \in [0, T]$.

The proof of (13) can be done by using Lemma 1 and is similar to the reasoning of Lemma 4.5.8 [1].

Thus, the random process $\eta_{st}(\widehat{x})$ has continuous in $(s, t, \widehat{x}), s \leq t, \|\widehat{x}\| \leq 1$ modification by Kolmogorov's theorem. QED.

Let $\omega \in \Omega$ and $x \in \mathbb{R}^{d-1} \times (0, \infty)$ are such that $\tau = \tau(x, \omega) < \infty$ and $\lim\limits_{\|y\|\to\infty} \inf\limits_{t\in[0,\tau]} \|\varphi_t(y, \omega)\| = \infty$.

Put $U(R) := \{y \in \mathbb{R}^d_+ : \|y\| \leq R\}$. Let us choose such $R = R(\omega) > 0$, that for all $t \in [0, \tau]$ and $\|y\| = R, y \in \mathbb{R}^d_+$ the inequality holds

$$\varphi_t(y) \neq \varphi_t(x).$$

Assume the contrary to the statement 2) of the Theorem. Then for each $y \in \partial U(R)$ and each $t \in [0, \tau]$ we have the inequality $\varphi_t(y) \neq \varphi_t(x)$.

Since $\varphi_t(x)$ is continuous on t, x, then there exists a collection of points $\{t_k\}_{k=\overline{0,n}}$ such that

$$0 = t_0 < t_1 < \ldots < t_n = \tau$$

and $\varphi_s(y) \neq \varphi_{t_{k+1}}(x)$, $\varphi_s(x) \neq \varphi_{t_k}(y)$ for all $s \in [t_k, t_{k+1}], y \in \partial U(R)$.

Let us briefly recall some properties of the topological degree [11] §1.4. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function, $U \subset \mathbb{R}^d$ be a closed set, $x \in \mathbb{R}^d$.Then

1) if there is no $y \in U$ such that $f(y) = x$ then $\deg(x, f, U) = 0$;

2) if f is identity map and $x \in U \backslash \partial U$ then $\deg(x, f, U) = 1$;

3) if $\varphi = \varphi_s(y), s \in [0, t], y \in \mathbb{R}^d$ is continuous in (s, y) and a family of sets $U_s \subset \mathbb{R}^d$ is continuous in s, and $x \notin \varphi_s(\partial U)$ for any s then $\deg(x, \varphi_0, U) = \deg(x, \varphi_t, U)$.

Thus, we have the sequence of equalities of topological degrees :

$$\deg(x, \varphi_0, U(R)) = \deg(\varphi_{t_1}(x), \varphi_0, U(R)) =$$

$$= \deg(\varphi_{t_1}(x), \varphi_{t_1}, U(R)) = \deg(\varphi_{t_2}(x), \varphi_{t_1}, U(R)) =$$

$$= \deg(\varphi_{t_2}(x), \varphi_{t_2}, U(R)) = \ldots = \deg(\varphi_\tau(x), \varphi_\tau, U(R)).$$

Observe that for each $s > 0$:

$$\deg(\varphi_\tau(x), \varphi_\tau, U(R)) = \deg(\varphi_\tau(x) - s\overline{n}, \varphi_\tau, U(R)), \tag{14}$$

because $\varphi_{\tau(x)} \notin \varphi_\tau(\partial U(R))$ for assumption and for each $s > 0$ a point $\varphi_\tau(x) - s\overline{n}$ lies in a lower halfplane (so, it does not belong to $\varphi_\tau(\partial U(R))$).

The the equation (as the equation with respect to y)

$$\varphi_\tau(y) = \varphi_s(x) - s\overline{n}$$

does not have any solution in $U(R)$ when $s > 0$. Therefore, the right hand side of (14) is equal to zero.

In the other hand, φ_0 is the identity map and $x \in U(R)$. So $\deg(x, \varphi_0, U(R)) = 1$.

The obtained contradiction proves the correctness of the statement 2) of the Theorem 2.

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