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Approximation theorem for stochastic differential equations with interaction[1 2]

A. Yu. PILIPENKO
Institute of Mathematics, National Academy of Science of Ukraine
Tereshchenkivska 3, 252601, Kiev, Ukraine

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Abstract— The approximation theorem for an infinite system of interacting stochastic differential equations is proved. This result is applied then for describing the support of the distribution of the solution of this system

The approximation of the solution of a stochastic differential equation by a sequence of solutions of ordinary differential equations is a subject of interest in the theory of stochastic processes for a long time, see for ex. [1,2,3,4].

Let us consider the following system of stochastic differential equations with interaction

$$\begin{cases} dx(u,t) & = \int_{\mathbb{R}^d} a(x(u,t),x(v,t),t)\mu_0(dv)dt+ \\ & + \sum_{k=1}^N \int_{\mathbb{R}^d} b_k(x(u,t),x(v,t),t)\mu_k(dv)) \circ dB^k(t),\ t\in[0,T] \\ x(u,0) & = \varphi(u),\ u\in\mathbb{R}^d, \end{cases} \tag{1}$$

where $a, b_k : \mathbb{R}^{2d}\times[0,T]\to\mathbb{R}^d, k=\overline{1,N},\ \ \varphi:\mathbb{R}^d\to\mathbb{R}^d,\ \ \mu_k$ are probability measures on $\mathbb{R}^d$, $B^1(t),\dots,B^N(t)$ are independent one-dimensional Wiener processes.

Under some natural conditions on the smoothness and growth of the coefficients a and b_k of the equation and the initial condition φ, there exists [5] a unique solution of equation (1). In this case a certain modification of $x(u,t)$ has derivatives with respect to u, which are continuous in (u,t) almost surely. Thus a flow $\{x(u,t), u\in\mathbb{R}^d, t\in[0,T]\}$ can be regarded as a random element in the space $C([0,T],C^n(\mathbb{R}^d;\mathbb{R}^d))$, where the space $C^n(\mathbb{R}^d;\mathbb{R}^d)$ of n times continuously differentiable functions is equipped with the topology of uniform convergence of derivatives on compact sets.

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Let $(\xi_\varepsilon^1(t),\ldots,\xi_\varepsilon^m(t)),\varepsilon>0$ be such processes that a process

$$\left(\int\limits_0^t \xi_\varepsilon^k(s)\,ds\right)_{k=\overline{1,N}},\quad t\in[0,T]$$

converges in some sense to a Brownian motion. Let $x_\varepsilon(u,t)$ be a solution to the following equation

$$\begin{cases} \frac{\partial x_\varepsilon(u,t)}{\partial t} & =\int a(x_\varepsilon(u,t),x_\varepsilon(v,t),t)\mu_0(dv)+ \\ & +\sum_{k=1}^N\int b_k(x_\varepsilon(u,t),x_\varepsilon(v,t),t)\mu_k(dv)\xi_\varepsilon^k(t), t\in[0,T], \\ x_\varepsilon(u,0) & =\varphi(u), \end{cases} \tag{2}$$

$$x_\varepsilon(u,0)=u,\ u\in\mathbb{R}^d,$$

The aim of the article is to prove convergence $x_\varepsilon(u,t)$ to $x(u,t)$ as $\varepsilon\to0$ in the space $C([0,T],C^n(\mathbb{R}^d;\mathbb{R}^d))$.

Set

$$B_\varepsilon(t)=(B_\varepsilon^1(t),\ldots,B_\varepsilon^N(t))=\left(\int\limits_0^t\xi_\varepsilon^k(s)ds\right)_{k=\overline{1,N}}$$

and define σ-algebras

$$\mathcal{G}_t^\varepsilon:=\sigma(\xi_\varepsilon^k(z):0\le z\le t,\ k=1,\ldots,N).$$

Theorem 1. *Assume that a function $\varphi\in C^n(\mathbb{R}^d,\mathbb{R}^d)$ has bounded derivatives of $1,...,n$ orders and coefficients of the equation (2) satisfy the following conditions*

A1.

$$\sup_{k,u,v,t}(\|\partial_u^\alpha\partial_v^\beta a(u,v,t)\|\vee\|\partial_u^\alpha\partial_v^\beta b_k(u,v,t)\|)<\infty$$

for each multi-indexes $\alpha=(\alpha_1,\ldots,\alpha_d),\ \beta=(\beta_1,\ldots,\beta_d)$,

$$|\alpha_1|+\ldots+|\alpha_d|\le n,\quad |\beta_1|+\ldots+|\beta_d|\le n.$$

A2. The collection of processes

$$K_\varepsilon(s)=\sum_{l,j=1}^N\left(\int_s^T\left|\mathbf{E}\left(\xi_\varepsilon^l(r)/\mathcal{G}_s^\varepsilon\right)\right|dr\right)\left(1+|\xi_\varepsilon^j(s)|\right),\quad\varepsilon>0$$

is uniformly L_p-bounded for each $p\ge1$:

$$\forall p\ge1\quad\sup_\varepsilon\sup_s\mathbf{E}\,K_\varepsilon^p(s)=K_p<\infty$$

and is uniformly exponential bounded in mean:

$$\forall \lambda > 0 \quad \sup_{\varepsilon} \mathbf{E} \exp\{\lambda \int_0^T K_\varepsilon(s)\, ds\} < \infty.$$

A3. For each $s \in [0,T]$ *the following convergence holds in* L_2 :

$$\int_s^T \left|\mathbf{E}\left(\xi_\varepsilon^l(z)/\mathcal{G}_s^\varepsilon\right)\right| dz \to 0,\ \varepsilon \to 0.$$

A4. There exist deterministic bounded functions $\sigma^{lm}(t), 1 \le l, m \le N$ *such that for all* $s < t$ *we have the following convergence in* L_1 :

$$\mathbf{E}\left(\int_s^t \xi_\varepsilon^l(\tau) d\tau \int_s^\tau \xi_\varepsilon^m(v) dv/\mathcal{G}_s^\varepsilon\right) \underset{\varepsilon \to 0}{\longrightarrow} \int_s^t \sigma^{lm}(z) dz.$$

Then the distribution P_ε *of the family of pairs* $\{(x_\varepsilon(u,t), B_\varepsilon(t)) : u \in \mathbb{R}^d, t \in [0,T]\}$ *converges weakly as* $\varepsilon \to 0$ *in the space* $\mathcal{C}_{n-1} \times C([0,T], \mathbb{R}^N)$ *to some limit measure* P_0*, that has the properties*
1) the distribution of the second coordinate $B(t)$ *in* $C([0,T],\mathbb{R}^N)$ *coincides with the distribution of* N*-dimensional Brownian motion with mean value 0 and covariation matrix being equal to*

$$\int_0^t (\sigma^{lm}(z) + \sigma^{ml}(z)) dz.$$

2) $x(u,t)$ *and* $B(t)$ *satisfy the following Ito equation*

$$\begin{aligned} dx(u,t) &= \int a(x(u,t), x(v,t), t)\mu_0(dv) dt \\ &+ \sum_{k,j} \Bigg(\int \int \Big\{ (b_k)'_1(x(u,t), x(v,t), t) b_j(x(u,t), x(w,t), t) \\ &+ (b_k)'_2(x(u,t), x(v,t), t) b_j(x(v,t), x(w,t), t) \Big\} \mu_k(dv)\mu_j(dw)\sigma^{kj}(t)\, dt \Bigg) \\ &+ \sum_k \int b_k(x(u,t), x(v,t), t)\mu_k(dv) dB^k(t), \quad x(u,0) = \varphi(u). \end{aligned} \tag{3}$$

Remark. The existence and uniqueness for the solution of the equation (3) is proved in [5] in more general situation, this solution belongs to the space $\mathcal{C}_{n-1}$, also.

Analogously to the classical results on approximation of solutions of stochastic differential equations (without interaction) it can be verified that a.s. convergence $B_\varepsilon(t) \to B(t)$ together with pathwise uniqueness of the solution imply a.s. convergence $x_\varepsilon(u,t)$ to $x(u,t)$.

Theorem 2. *Assume that the conditions of the Theorem 1 are satisfied and for each* $t \in [0,T]$ *we have the convergence* $B^{\varepsilon}(t) \xrightarrow{\mathbf{P}} B(t),\ \varepsilon \to 0$ *a.s.*

Then

$$\forall p \geq 1\ \forall R > 0\ \forall k,\ 1 \leq k \leq n-1 \quad :$$
$$\mathbf{E} \sup_{\|u\| \leq R} \sup_{t \in [0,T]} \|\nabla^k \big(x^{\varepsilon}(u,t) - x(u,t)\big)\|^p \to 0,\ \varepsilon \to 0,$$

Example 1. Let $B(t) = (B^1(t), \dots, B^N(t))$ be N-dimensional Wiener process, $\langle B^i(t), B^j(t)\rangle = \delta_{ij} t$. Consider the polygonal approximation $B_{\varepsilon}(t)$ for $B(t)$. Put

$$\xi_{\varepsilon}(t) = \frac{1}{\varepsilon}[B((k+1)\varepsilon) - B(k\varepsilon)],$$

if $t \in [k\varepsilon, (k+1)\varepsilon)$.

Then ([4], Ex.5.7.4.) the processes $\xi_{\varepsilon}(t)$ satisfy the conditions A1–A4 of the Theorem 1, where $\sigma^{kl} = \frac{1}{2}\delta_{kl}$ and the equation (3) has a form (1).

It follows from [6] that under some conditions on the coefficients of the homogeneous analogue of (1):

$$\begin{aligned} dx(u,t) &= \int_{\mathbb{R}^d} a(x(u,t), x(v,t))\mu_0(dv)dt \\ &\quad + \sum_{k=1}^{N} \int_{\mathbb{R}^d} b_k(x(u,t), x(v,t))\mu_k(dv) \circ dB^k(t),\ t \in [0,T] \\ x(u,0) &= \varphi(u),\ u \in \mathbb{R}^d, \end{aligned} \tag{1'}$$

the support of the flow $x(u,t)$ contains a set of functions $x^{\psi} : [0,T] \to \mathbb{R}^N$, where ψ is piecewise smooth function, x^{ψ} is determined by the following system of differential equations:

$$\begin{cases} \frac{\partial x^{\psi}(u,t)}{\partial t} & = \int_{\mathbb{R}^d} a(x^{\psi}(u,t), x^{\psi}(v,t))\mu_0(dv) + \\ & \quad + \sum_{k=1}^{N} \int_{\mathbb{R}^d} b_k(x^{\psi}(u,t), x^{\psi}(v,t))\mu_k(dv)(\psi^k)'(t),\ t \in [0,T] \\ x^{\psi}(u,0) & = \varphi(u). \end{cases}$$

Combining the corresponding result with the Theorem 1 and Example 1 we obtain the following statement.

Theorem 3. *Assume that a function* φ *satisfies the conditions of the Theorem 1, functions* $a, b_k : \mathbb{R}^{2d} \to \mathbb{R}^d, k = 1, \dots, N$ *are (n+2), (n+4) times continuously differentiable, respectively, and bounded together with all their derivatives.*

Then a support of $(1')$ *is equal to the closure of the set* $\{x^{\psi} | \psi \text{ is piecewise smooth function}\}$ *in the space* $\mathcal{C}_n$.

Proof of the Theorem 1. The proof will be done according to a scheme, which is similar to the case of the flows that generated by stochastic differential equations without interaction [4] (i.e. $b_k(u,v,t) = b_k(u,t)$). Let us prove, at first, that the family of measures $P_\varepsilon, \varepsilon > 0$ is weakly relatively compact. Then choose a subsequence $\{P_{\varepsilon_k}\}$ which converges weakly to some limit measure P_0. Finally we will show that $B(t) = (B^1(t), \dots, B^N(t))$ is a Wiener process with required covariances and for each $u \in \mathbb{R}^d$ a process $M(u,t) = (M^{(1)}(u,t), \cdots, M^{(d)}(u,t))$,

$$\begin{aligned} M(u,t) &= x(u,t) - \varphi(u) - \int_0^t \int a(x(u,s), x(v,s), s)\mu_0(dv)ds \\ &- \sum_{k,j} \int_0^t \Big(\int \int \Big\{ (b_k)'_1(x(u,s), x(v,s), s) b_j(x(u,s), x(w,s), s) \\ &- (b_k)'_2(x(u,s), x(v,s), s) b_j(x(v,s), x(w,s), s) \Big\} \mu_k(dv)\mu_j(dw)\sigma^{kj}(s)ds \Big) \end{aligned} \tag{4}$$

is a continuous L_2-martingale with respect to the $\sigma\big(B(s), s \in [0,t]\big), t \in [0,T]$, with square characteristics

$$\begin{aligned} \langle M^{(i)}(u,t), M^{(j)}(v,t) \rangle = \sum_{k,l} \int_0^t \int b_k^{(i)}(x(u,s), x(v_1,s), s)\mu_k(dv_1) \\ \times \int b_l^{(j)}(x(v,s), x(v_2,s), s)\mu_l(dv_2)(\sigma^{kl}(s) + \sigma^{lk}(s))ds \end{aligned} \tag{5}$$

$$\langle M^{(i)}(u,t), B^j(t) \rangle = \sum_k \int_0^t \int b_k^{(i)}(x(u,s), x(v,s))\mu_k(dv)(\sigma^{kj}(s) + \sigma^{jk}(s))ds. \tag{6}$$

From this it can be concluded easily that $x(u,t)$ coincides with the strong solution of (3).

In order to prove that $\{P_\varepsilon, \varepsilon > 0\}$ is relatively compact in $\mathcal{C}_{n-1} \times C([0,T], \mathbb{R}^d)$ it suffices to verify that $\{P_\varepsilon\}$ is tight [7], i.e. to show that for each $\delta > 0$ there exists a compact C_δ such that

$$\forall \varepsilon > 0 : P((x_\varepsilon, B_\varepsilon) \in C_\delta) \geq 1 - \delta. \tag{7}$$

The proof of the fact that the distributions of B_ε converges weakly to the distribution of the corresponding Brownian motion as $\varepsilon \to 0$ in $C([0,T], \mathbb{R}^N)$ is given in [4] (Theorem 5.7.1, p. 277–279), so we will restrict ourselves to the verification of the tightness of $\{x_\varepsilon\}$ distributions in $\mathcal{C}_{n-1}$.

It is well known that Sobolev space $W^k_{p,loc}$ is embedded continuously to $C^{k-1}(\mathbb{R}^d, \mathbb{R}^d)$ if $p > d$, thus by Arzela-Ascoli criterion and Theorem 1.4.7 [4] to prove (7) it suffices to show that

$$\forall m \in \mathbb{N} \ \ \forall R > 0 \ \ \exists C = C_{R,m} > 0 : \sup_\varepsilon \mathbf{E} \, \|x_\varepsilon(\cdot, t)\|^{2m}_{W^k_{2m}(B_R)} \leq C, \tag{8}$$

$$\forall t', t'' : \sup_\varepsilon \mathbf{E} \, \|x_\varepsilon(\cdot, t') - x_\varepsilon(\cdot, t'')\|^{2m}_{W^k_{2m}(B_R)} \leq C|t' - t''|^{2 - \frac{1}{m}}, \tag{9}$$

where B_R is a ball in $\mathbb{R}^d$ with a center in zero and a radius R.

Let us verify (8), (9) only in the case $n = 2, d = 1, m = 1$. Higher order derivatives or a case $m > 1$ can be considered analogously.

A lot of cumbersome expression will appear in further reasoning. To make our calculations easier we will use additional functions $a_\varepsilon^{(k)}(s), b_\varepsilon^{(k)}(s), \beta_k^{(\varepsilon)}(s), \beta_k^{(\varepsilon)}(s,t)$, $s \le t$. Next, an index ε will be omitted as usual. It will be always supposed that $a^{(k)}(s), b^{(k)}(s), \beta_k(s), \beta_k(s,t)$, $s \le t$ are $\sigma\{x_\varepsilon(v,z) : v \in \mathbb{R}^d, 0 \le z \le s\}$-measurable and

$$A = \max_k \sup_s \sup_\varepsilon \operatorname{esssup}_\omega |a^{(k)}(s)| \vee |b^{(k)}(s)| < \infty,$$

$$B = \max_k \sup_{s \le t} \sup_\varepsilon \operatorname{esssup}_\omega |\beta_k(s,t)| < \infty.$$

Let us differentiate $x^{2m}(u,t) = (x_\varepsilon(u,t))^{2m}, m \in \mathbb{N}$ with respect to t:

$$\begin{aligned} x^{2m}(u,t) - \varphi^{2m}(u) = 2m \int_0^t \Big[\int a(x(u,s), x(v,s), s)\mu_0(dv) \\ + \int b(x(u,s), x(v,s), s)\mu_1(dv)\xi(s) \Big] x(u,s)^{2m-1} ds. \end{aligned} \tag{10}$$

Let us write a differential of a function $b(x(u,s), x(v,s), s)x(u,s)^{2m-1}$, considering that the third coordinate of the function b is fixed. Then the expression in (10) equals

$$\begin{aligned} &2m \int_0^t \Big[\int a(x(u,s), x(v,s), s)\mu_0(dv) x(u,s)^{2m-1} \\ &+ \int b(\varphi(u), \varphi(v), s)\mu_1(dv)\xi(s)(\varphi(u))^{2m-1} \\ &+ \int_0^s \int \Big\{ b_1'(x(u,r), x(v,r), s)[a^{(1)}(r) + b^{(1)}(r,v)\xi(r)] x^{2m-1}(u,r) \\ &+ b_2'(x(u,r), x(v,r), s)[a^{(2)}(r) + b^{(2)}(r,v)\xi(r)] x^{2m-1}(u,r) \Big\} \mu_1(dv) dr \xi(s) \Big] ds \\ &+ \int_0^t \int_0^s 2m(2m-1) \Big[\int a(x(u,r), x(v,r), r)\mu_0(dv) \\ &+ \int b(x(u,r), x(v,r), r)\mu(dv)\xi(r) \Big] b(x(u,r), x(v,r), s) x^{2m-2}(u,r)\xi(s) dr ds. \end{aligned} \tag{11}$$

Change the order of integration in terms of the form $\int_0^t \int_0^s$ in (11). Thus

$$\begin{aligned} x^{2m}(u,t) - \varphi^{2m}(u) = \Big(\int_0^t a^{(0)}(s) x^{2m-1}(u,s) ds + |\varphi^{2m-1}(u)| \int_0^t b^{(0)}(s)\xi(s) ds \Big) \\ + \sum_{j=1}^2 \int_0^t \int_r^t \beta_j(r,s)\xi(s) ds x^{2m-j}(u,r) dr \end{aligned}$$

$$+\sum_{j=1}^{2}\int_0^t\int_r^t \beta_{2+j}(r,s)\xi(s)ds\xi(r)x^{2m-j}(u,r)dr$$
$$= I_1+\sum_{j=1}^{2} I_{2,j}+\sum_{j=1}^{2} I_{3,j}, \tag{12}$$

where $b^{(0)}(s)$ is $\mathcal{G}_0-$measurable.

Consider $I_{2,1}$:

$$\begin{aligned}\mathbf{E}\, I_{2,1} &= \int_0^t \mathbf{E}\left(\int_r^t \beta_1(r,s)\mathbf{E}\left(\xi(s)/\mathcal{G}_r^{\varepsilon}\right)ds x^{2m-1}(u,r)\right)dr\\ &\le B\int_0^t\left(\mathbf{E}\left(\int_r^t |\mathbf{E}\left(\xi(s)/\mathcal{G}_r^{\varepsilon}\right)|ds\right)^{2m}\right)^{\frac{1}{2m}}\left(\mathbf{E}\, x^{2m}(u,r)\right)^{1-\frac{1}{2m}}dr\\ &\le B\int_0^t K_{2m}^{\frac{1}{2m}}(Ex^{2m}(u,r))^{1-\frac{1}{2m}}dr,\end{aligned}$$

where K_{2m} is defined in $A2$.

We have the following estimation for the expectation of $I_{3,1}$:

$$\begin{aligned}\mathbf{E}\, I_{3,1} &= \mathbf{E}\int_0^t\int_r^t \beta_3(r,s)\mathbf{E}\left(\xi(s)/\mathcal{G}_r^{\varepsilon}\right)ds\xi(r)x^{2m-1}(u,r)dr\\ &\le B\int_0^t\left(\mathbf{E}\left(\int_r^t |\mathbf{E}\left(\xi(s)/\mathcal{G}_r^{\varepsilon}\right)|ds|\xi(r)|\right)^{2m}\right)^{\frac{1}{2m}}\left(\mathbf{E}\, x^{2m}(u,r)\right)^{1-\frac{1}{2m}}dr\\ &\le BK_{2m}^{\frac{1}{2m}}\int_0^t(\mathbf{E}\, x^{2m}(u,r))^{1-\frac{1}{2m}}dr.\end{aligned}$$

Furthermore

$$\mathbf{E}\int_0^t b^{(0)}(s)\xi(s)ds \le \mathbf{E}\int_0^t |b^{(0)}(s)||E(\xi(s)/\mathcal{G}_0)|ds \le AK_1.$$

Terms $I_{2,2}, I_{3,2}$ are considered as $I_{2,1}$ and $I_{3,1}$. Let us apply the inequality

$$\forall\alpha\in(0,1),\quad \xi\ge 0:\xi^{1-\alpha}\le 1+\xi,$$

for estimations of $I_{2,k}, I_{3,k}$. We have

$$\mathbf{E}\, x^{2m}(u,t)\le C\left(1+\int_0^t(1+\mathbf{E}\, x^{2m}(u,r))dr\right),$$

where $C=\text{const}$ does not depends on $u\in B_R$ and $t\in[0,T]$. The Gronwall lemma implies

$$\sup_{u\in B_R,t\in[0,T]}\mathbf{E}\, x^{2m}(u,t)<\infty.$$

Reasoning similarly as in the proof of (12), we get the following equality for the derivative $\frac{\partial x(u,t)}{\partial u}$:

$$\begin{aligned}\left(\frac{\partial x(u,t)}{\partial u}\right)^{2m} &= \left(\frac{\partial \varphi(u)}{\partial u}\right)^{2m} + \int_0^t a^{(1)}(s)\left(\frac{\partial x(u,s)}{\partial u}\right)^{2m} ds \\ &+ \int_0^t b^{(1)}(s)\xi(s)ds\left(\frac{\partial \varphi(u)}{\partial u}\right)^{2m} \\ &+ \int_0^t \int_r^t \beta_5(r,s)\xi(s)ds\left(\frac{\partial x(u,r)}{\partial u}\right)^{2m} dr \\ &+ \int_0^t \int_r^t \beta_6(r,s)\xi(s)ds\left(\frac{\partial x(u,r)}{\partial u}\right)^{2m} \xi(r)dr \\ &\le \left(\frac{\partial \varphi(u)}{\partial u}\right)^{2m} + \int_0^t \Big| a^{(1)}(r) + \int_r^t \beta_5(r,s)\mathbf{E}\left(\xi(s)/\mathcal{G}_r\right)ds \\ &+ \int_r^t \beta_6(r,s)\mathbf{E}\left(\xi(s)/\mathcal{G}_r\right)ds\xi(r)\Big| \left(\frac{\partial x(u,r)}{\partial u}\right)^{2m} dr \\ &+ \int_0^t b^{(1)}(r)\xi(r)dr\left(\frac{\partial \varphi(u)}{\partial u}\right)^{2m} + \int_0^t Y_r\, dr,\end{aligned} \tag{13}$$

where $b^{(1)}(r)$ is $\mathcal{G}_0$-measurable, a random process Y_r is such that

$$\forall r \in [0,T] \;\; \mathbf{E}\left(Y_r/\mathcal{G}_r\right) = 0 \text{ and } \sup_r \mathbf{E}\,|Y_r|^p < \infty.$$

Lemma 4. *Assume that non-negative measurable function y satisfies the inequality*

$$y(t) \le A + \int_0^t \left[B(s)y(s) + C(s)\right] ds, t \ge 0,$$

where $B(t) \ge 0$ when $t \ge 0$, B, C are continuous functions.

Then for each $t \ge 0$ the value of the function $y(t)$ does not exceed $\bar{y}(t)$, where $\bar{y}$ is a solution of the equation

$$\bar{y}(t) = A + \int_0^t \left[B(s)\bar{y}(s) + C(s)\right] ds,$$

i.e.

$$y(t) \le e^{\int_0^t B(s)\,ds}(A + \int_0^t C(s)\exp\{-\int_0^s B(z)\,dz\}\,ds).$$

Put

$$\begin{aligned}\tilde{K}(r) := |a^{(1)}(r) &+ \int_r^t \beta_5(r,s)\mathbf{E}\left(\xi(s)/\mathcal{G}_r\right) ds \\ &+ \int_r^t \beta_6(r,s)\mathbf{E}\left(\xi(s)/\mathcal{G}_r\right) ds\xi(r)|.\end{aligned}$$

Note that

$$\tilde{K}(r) \leq A + 2BK(r),$$

and therefore $\tilde{K} = \tilde{K}_\varepsilon$ satisfies the uniformly L_p-boundedness condition for all $p \geq 0$ and also it is uniformly exponential bounded in mean. It follows from Lemma 4 and (13) that

$$\begin{aligned} e^{-\int_0^t \tilde{K}(r)dr} \left(\frac{\partial x(u,t)}{\partial u}\right)^{2m} &\leq \left(\frac{\partial \varphi(u)}{\partial u}\right)^{2m} \left(1 + \int_0^t b^{(1)}(r)\xi(r)e^{-\int_0^r \tilde{K}(s)ds}dr\right) \\ &\quad + \int_0^t e^{-\int_0^r \tilde{K}(s)ds} Y_r dr. \end{aligned} \tag{14}$$

Applying the integration by parts, observe that

$$\begin{aligned} \left|\int_0^t b^{(1)}(r)\xi(r)e^{-\int_0^r \tilde{K}(s)ds}\, dr\right| &\leq \left|\int_0^t b^{(1)}(r)\xi(r)dr\right| e^{-\int_0^t \tilde{K}(s)ds} \\ &\quad + \int_0^t \tilde{K}(r)e^{-\int_0^r \tilde{K}(s)ds} \left|\int_0^r b^{(1)}(s)\xi(s)\, ds\right| dr. \end{aligned}$$

Moreover $\mathbf{E}\int_0^t \psi_r Y_r dr = 0$. Thus, evaluating the expectation in (14), one has

$$\begin{aligned} &\mathbf{E}\left[\left(\frac{\partial x(u,r)}{\partial u}\right)^{2m} e^{-\int_0^t \tilde{K}(r)dr}\right] \\ &\leq \left(\frac{\partial \varphi(u)}{\partial u}\right)^{2m} \left(1 + \left(1 + T\sup_r(\mathbf{E}\,\tilde{K}^2(r))^{\frac{1}{2}}\right) \sup_r \left(\mathbf{E}\left(\int_0^r b^{(1)}(s)\xi(s)ds\right)^2\right)^{1/2}\right). \end{aligned}$$

In order to prove that

$$\forall m \geq 1 : \sup_{u \in B_{R,\varepsilon,t}} \mathbf{E}\left|\frac{\partial x_\varepsilon(u,r)}{\partial u}\right|^m < \infty, \tag{15}$$

it remains to note that

$$\begin{aligned} \sup_{t,\varepsilon} \mathbf{E}\left(\int_0^t b^{(1)}(s)\xi_\varepsilon(s)ds\right)^2 &= 2\sup_{t,\varepsilon} \mathbf{E}\int_0^t \int_r^t b^{(1)}(s)b^{(1)}(r)\xi_\varepsilon(s)\xi_\varepsilon(r)dsdr \\ &\leq 2\sup_{t,\varepsilon} B^2 \mathbf{E}\int_0^t \int_r^t |\mathbf{E}\left(\xi_\varepsilon(s)/\mathcal{G}_r^\varepsilon\right)| ds |\xi_\varepsilon(r)| dr \\ &\leq 2B^2 K_1 t. \end{aligned}$$

Thus

$$\begin{aligned} &\sup_{t,\varepsilon} \mathbf{E}\left|\frac{\partial x_\varepsilon(u,t)}{\partial u}\right|^m \\ &\leq \sup_{t,\varepsilon}\left(\mathbf{E}\left\{\left(\frac{\partial x_\varepsilon(u,t)}{\partial u}\right)^{2m} e^{-\int_0^t \tilde{K}_\varepsilon(r)dr}\right\} \mathbf{E}\, e^{\int_0^t \tilde{K}_\varepsilon(r)dr}\right)^{1/2} < \infty \end{aligned}$$

Similarly to (12), (13) we obtain the following equality for the second derivative $\frac{\partial^2 x(u,t)}{\partial u^2}$

$$\left(\frac{\partial^2 x(u,t)}{\partial u^2}\right)^{2m} = (\varphi''(u))^{2m}\left(1+\int_0^t b^{(2)}(r)\xi(r)dr\right)$$
$$+\sum_{j=0}^{2}\int_0^t\int_r^t\left(\frac{\partial^2 x(u,t)}{\partial u^2}\right)^{2m-j}\Big(\beta_{7+j}(r,s)+\beta_{10+j}(r,s)\mathbf{E}\left(\xi(s)/\mathcal{G}_r\right) \tag{16}$$
$$+\beta_{13+j}(r,s)\mathbf{E}\left(\xi(s)/\mathcal{G}_r\right)\xi(r)\Big)\left(\frac{\partial x(u,t)}{\partial u}\right)^{2j} dsdr+\int_0^t \tilde{Y}_r dr,$$

where $b^{(2)}(r)$ is $\mathcal{G}_0$-measurable and

$$\forall r\in[0,T]\ \ \mathbf{E}\,(\tilde{Y}_r/\mathcal{G}_r)=0,\ \sup_r \mathbf{E}\,|\tilde{Y}_r|<\infty.$$

Using the inequality

$$|ab|\le\frac{|a|^p}{p}+\frac{|b|^q}{q},$$

where

$$\frac{1}{p}+\frac{1}{q}=1$$

for items with $j=1,2$ and $p_j=\frac{2m}{2m-j}$ we can derive from (16), that

$$\left(\frac{\partial^2 x(u,t)}{\partial u^2}\right)^{2m} \le (\varphi''(u))^{2m}\left(1+\int_0^t b^{(2)}(r)\xi(r)dr\right)$$
$$+C_1\int_0^t\left(\frac{\partial^2 x(u,t)}{\partial u^2}\right)^{2m}(1+K(r))dr$$
$$+C_2\int_0^t(K^m(r)+K^{2m}(r))\left(\frac{\partial x(u,t)}{\partial u}\right)^{2m}dr+\int_0^t\tilde{Y}_r\,dr,$$

where $C_1, C_2>0$ are some constants. Further reasoning can be done with the use of Lemma 4 and (15) similarly to the case of the first derivative $\frac{\partial x(u,t)}{\partial u}$.

Thus, the inequality (8) is proved(derivatives with $k\ge 3$ are considered analogously).

Let us verify (9). Let $t'<t$. Set

$$y(t)=\frac{\partial x(u,t)}{\partial u}.$$

Then, as in the proof of (13), we obtain

$$(y(t)-y(t'))^{2m} = \int_{t'}^{t} a^{(3)}(s)(y(s)-y(t'))^{2m-1}y(s)\,ds$$
$$+\int_{t'}^{t} b^{(3)}(s)(y(s)-y(t'))^{2m-1}y(s)\xi(s)\,ds$$
$$=\int_{t'}^{t} a^{(3)}(s)(y(s)-y(t'))^{2m-1}y(s)ds$$
$$+\sum_{j=1}^{2}\int_{t'}^{t}\int_{r}^{t}(\beta_{15+j}(r,s)+\beta_{17+j}(r,s)\xi(r))(y(r)-y(t'))^{2m-j}y^j(r)\xi(s)\,ds\,dr.$$

So

$$\mathbf{E}\,(y(t)-y(t'))^{2m} \le A\int_{t'}^{t}\mathbf{E}\,(|y(s)-y(t')|^{2m-1}|y(s)|)ds$$
$$+B\sum_{j=1}^{r}\int_{t'}^{t}\mathbf{E}\,|y(r)-y(t')|^{2m-j}|y^j(r)|K(r)\,dr.$$

Further argumentation is similar to [4] (see the deriving of (35) from (34), p.254). Higher order derivatives are considered analogously. The inequality (9) and hence a relatively compactness of $\{P_\varepsilon, \varepsilon>0\}$ is proved.

Let $\varepsilon_n \to 0, n\to\infty$ is a sequence such that the distribution of P_{ε_n} converges weakly to some limit measure P_0 in the space $\mathcal{C}_{n-1}\times C([0,T],\mathbb{R}^N)$. Let $x(u,t), B(t)$ be the first and second coordinates correspondingly in the space $\mathcal{C}_{n-1}\times C([0,T],\mathbb{R}^N)$ with the measure P_0. As has been mentioned above [4] p.277, $B(t)$ is continuous L_2-martingale with square characteristics

$$\int_0^t(\sigma^{lm}(z)+\sigma^{ml}(z))dz.$$

Let

$$\mathcal{F}_t=\sigma(B_s, 0\le s\le t).$$

Consider a process $M(u,t)$ defined in (4). To prove Theorem 1 it suffices to verify (analogously to [4], the proof of Theorem 5.2.1, pp.231–232) that the processes $M(u,t), u\in\mathbb{R}^d$ are continuous L_2-martingales according to the filtration $\mathcal{F}_t$ with square characteristics, that satisfy (5) and (6).

Let $m\in\mathbb{N}, u_1,\dots,u_m\in\mathbb{R}^d$. Put

$$\vec{u}:=(u_1,\dots,u_m)\in\mathbb{R}^{nd},\quad \vec{x}(\vec{u},t):=(x(u_1,t),\dots,x(u_m,t)).$$

Lemma 5. *Assume that $\beta : \mathbb{R}^{md} \times \mathbb{R}^{md} \times [0,T]^2 \to \mathbb{R}$ is continuous function and the order of growth of β in the first argument is not greater than polynomial, uniformly on the other arguments:*

$$\exists l \geq 0 \exists C \ \forall \vec{u}, \vec{v}, s, t : |\beta(\vec{u}, \vec{v}, s, t)| \leq C(1 + \|\vec{u}\|^l).$$

Let $f : \mathbb{R}^{2md} \to \mathbb{R}$ be an arbitrary continuous bounded function, $s, t \in [0,T]$, $s_1, \dots, s_m \in [0, s]$, $\vec{u} \in \mathbb{R}^{md}$.

Set

$$\Phi_\varepsilon = f(x_\varepsilon(u_1, s_1), \dots, x_\varepsilon(u_m, s_m), B_\varepsilon(s_1), \dots, B_\varepsilon(s_m)),$$

$$\Phi = f(x(u_1, s_1), \dots, x(u_m, s_m), B(s_1), \dots, B(s_m)).$$

Denote by $\mathbf{E}_0$ the expectation with respect to the measure P_0.

Then for each probability measure ν in $\mathbb{R}^{md}$ the following convergence holds true

$$\begin{aligned} &\mathbf{E} \int_s^t \int \beta(\vec{x}_{\varepsilon_n}(\vec{u}, r), \vec{x}_{\varepsilon_n}(\vec{v}, r), r, r) \nu(d\vec{v}) dr \Phi_{\varepsilon_n} \\ &\to \mathbf{E}_0 \int_s^t \int \beta(\vec{x}(\vec{u}, r), \vec{x}(\vec{v}, r), r, r) \nu(d\vec{v}) dr \Phi, n \to \infty; \end{aligned} \tag{17}$$

$$\mathbf{E} \int_s^t \int \beta(\vec{x}_{\varepsilon_n}(\vec{u}, s), \vec{x}_{\varepsilon_n}(\vec{v}, s), r, r) \nu(d\vec{v}) \xi^i_{\varepsilon_n}(r) dr \Phi_{\varepsilon_n} \to 0, \quad n \to \infty. \tag{18}$$

If β is bounded then

$$\begin{aligned} &\mathbf{E} \int_s^t \int_s^z \int \beta(\vec{x}_{\varepsilon_n}(\vec{u}, r), \vec{x}_{\varepsilon_n}(\vec{v}, r), r, z) \nu(d\vec{v}) \xi^i_{\varepsilon_n}(z) \xi^j_{\varepsilon_n}(r) \, dr \, dz \, \Phi_{\varepsilon_n} \\ &\to \mathbf{E}_0 \int_s^t \int \beta(\vec{x}(\vec{u}, r), x(\vec{v}, r), r, r) \nu(d\vec{v}) \sigma^{ij}(r) \, dr \, \Phi, n \to \infty. \end{aligned} \tag{19}$$

Proof. The convergence of x_{ε_n} and (8) implies (17). To prove (18) it suffices to observe that the absolute value of the left hand side of (18) does not exceed

$$\|f\|_\infty C \mathbf{E} \left((1 + \|x_\varepsilon(u, s)\|^l) \int_s^t \|\mathbf{E}\, (\xi^\varepsilon(r) / \mathcal{G}^\varepsilon_s)\| dr \right)$$

and tends to zero as $n \to \infty$ due to A3 and (8).

Let us prove (19) only in the case $m = 1, d = 1$. It follows from relatively compactness of the distributions of $\{x_\varepsilon, \varepsilon > 0\}$ in $\mathcal{C}_{n-1} = C([0,T], C^{n-1}(\mathbb{R}^d, \mathbb{R}^d))$ and Arzela-Ascoli criterion that
$\forall \delta > 0, \ \forall R > 0 \ \exists M \ \forall \varepsilon > 0 :$

$$\mathbf{P} \left(\sup_{|v| \leq R} \sup_{t \in [0,T]} |x_\varepsilon(v, t)| \leq M \right) \geq 1 - \delta, \tag{20}$$

$\forall\delta > 0 \forall\eta > 0\ \exists\theta > 0:$

$$\mathbf{P}\left(\sup_{|v|\le\eta}\ \sup_{|s'-s''|\le\theta}|x_\varepsilon(v,s')-x_\varepsilon(v,s'')|\le\theta\right)\ge 1-\delta. \tag{21}$$

A function β is continuous, so it is uniformly continuous on compact sets, i.e. $\forall\delta>0\ \forall M>0\ \exists\eta>0\ \forall u',u'',v',v'',|u'|,|u''|,|v'|,|v''|\le M$, $|u'-u''|<\eta,\ |v'-v''|<\eta,\ \forall s',s'',t',t''\ |s'-s''|<\eta,\ |t'-t''|<\eta:$

$$|\beta(u',v',s',t')-\beta(u'',v'',s'',t'')|<\delta. \tag{22}$$

Let $\delta>0$ be given. Choose R in (20) such that $\nu(v:|v|\ge R)<\delta$ and $R>|u|$.

Take M in (22) from (20), η in (21) from (22). Fix a partition $\Delta=\{s=t_0<t_1<\ldots<t_l=t\}$ of the interval $[s,t]$ with a diameter $|\delta|<\min\{\eta,\theta\}$ (where θ is given in (21)). Put

$$B^\varepsilon(r,z):=\int\beta(x_\varepsilon(u,r),x_\varepsilon(v,r),r,z)\nu(dv),$$

$$B^\varepsilon_R(r,z):=\int_{|v|\le R}\beta(x_\varepsilon(u,r),x_\varepsilon(v,r),r,z)\nu(dv).$$

Then changing the limits of integration in (19), we obtain the following estimate for the absolute value of the difference between left and right sides of (19)

$$\begin{aligned}
&\left|\mathbf{E}\sum_k\int_{t_k}^{t_{k+1}}\int_r^t(B^\varepsilon_R(r,z)-B^\varepsilon_R(t_k,z))\mathbf{E}\,(\xi_\varepsilon(z)/\mathcal{G}^\varepsilon_r)\xi_\varepsilon(r)dzdr\Phi_\varepsilon\right|\\
&+\left|\mathbf{E}\sum_k\int_{t_k}^{t_{k+1}}\left(\int_r^t B^\varepsilon_R(t_k,z)\xi_\varepsilon(z)\xi_\varepsilon(r)dz-B^\varepsilon_R(t_k,r)\sigma(r)\right)dr\Phi_\varepsilon\right|\\
&+\left|\mathbf{E}\sum_k\int_{t_k}^{t_{k+1}}B^\varepsilon_R(t_k,r)\sigma(r)dr\Phi_\varepsilon-\mathbf{E}_0\sum_k\int_{t_k}^{t_{k+1}}B_R(t_k,r)\sigma(r)dr\Phi\right|\\
&+\left|\mathbf{E}_0\left(\sum_k\int_{t_k}^{t_{k+1}}B_R(t_k,r)\sigma(r)dr-\int_s^t B_R(r,r)\sigma(r)dr\right)\Phi\right|\\
&+\|\beta\|_\infty\delta\|f\|_\infty\left(\mathbf{E}\int_s^t\int_r^t|\mathbf{E}\,(\xi_\varepsilon(z)/\mathcal{G}_r)||\xi_\varepsilon(r)|dzdr+\int_s^t\sigma(r)dr\right)\\
&=I_1+\ldots+I_5.
\end{aligned}$$

Let us estimate each of the items $I_1,\ldots,I_5$.

Divide the expectations of I_1 and I_4 into integrals over the set

$$\left\{\sup_{|v|\le R}\ \sup_{|s'-s''|\le\theta}|x(v,s')-x(v,s'')|\le\theta,\ \sup_{|v|\le R}\ \sup_{t\in[0,T]}|x(v,t)|\le M\right\}$$

and its complement. Then

$$I_1 + I_4 \leq 2\delta\|f\|_\infty(1+\|\beta\|_\infty)\left[\mathbf{E}\int_s^t\int_r^t |\mathbf{E}\left(\xi_\varepsilon(z)/\mathcal{G}_r\right)\xi_\varepsilon(r)|\,dz\,dr + \int_s^t \sigma(r)dr\right]$$
$$\leq 2\delta\|f\|_\infty(1+\|\beta\|_\infty)(K_1 + \int_s^t \sigma(r)dr).$$

Due to weak convergence of x_{ε_n} distributions to the distribution of x one has a convergence to 0 of I_3 as $n\to\infty$.

The value of I_5 does not exceed than

$$\|\beta\|_\infty\delta\|f\|_\infty(K_1 + \int\limits_s^t \sigma(r)dr).$$

Represent an integral $\int\limits_{t_k}^{t_{k+1}}\int\limits_r^t$ in I_2 as a sum

$$\int\limits_{t_k}^{t_{k+1}}\int\limits_r^{t_{k+1}} + \int\limits_{t_k}^{t_{k+1}}\int\limits_{t_{k+1}}^{t}.$$

So

$$I_2 \leq \|f\|_\infty\mathbf{E}\sum_k\int_{t_k}^{t_{k+1}}\sup_{z\in[t_k,t_{k+1}]}|B_R^\varepsilon(t_k,z) - B_R^\varepsilon(t_k,t_k)|(K(r)+\sigma(r))dr$$
$$+\left|\mathbf{E}\sum_k B_R^\varepsilon(t_k,t_k)\left(\int_{t_k}^{t_{k+1}}\int_{t_k}^{z}\xi_\varepsilon(r)\xi_\varepsilon(z)drdz - \int_{t_k}^{t_{k+1}}\sigma(r)dr\right)\Phi_\varepsilon\right|$$
$$+\left|\sum_k\mathbf{E}\int_{t_k}^{t_{k+1}}\int_{t_{k+1}}^{t} B_R^\varepsilon(t_k,z)\xi_\varepsilon(r)\xi_\varepsilon(z)dzdr\Phi_\varepsilon\right| = I_{2,1}+I_{2,2}+I_{2,3}.$$

Similarly to I_1 and I_3 :

$$I_{2,1} \leq \|f\|_\infty(2\|\beta\|_\infty+1)2\delta\left(K_1 + \int_s^t\sigma(r)dr\right).$$

By A4 we obtain that $I_{2,2}\to 0, n\to\infty$.

Take the conditional expectation of k-th member of $I_{2,3}$ with respect to $\mathcal{G}_{t_k}^\varepsilon$ and apply Cauchy inequality

$$I_{2,3} \leq \|f\|_\infty\sum_k\left[\mathbf{E}\left(\int_{t_k}^{t_{k+1}}\xi_\varepsilon(r)dr\right)^2\mathbf{E}\left(\int_{t_{k+1}}^t|\mathbf{E}\left(\xi_\varepsilon(z)/\mathcal{G}_{t_k}^\varepsilon\right)|\|\beta\|_\infty dz\right)^2\right]^{1/2}.$$

The right hand side converges to zero as $\varepsilon \to 0$ due to A3 and the fact that

$$\mathbf{E}\left(\int_{t_k}^{t_{k+1}} \xi_\varepsilon(r)dr\right)^2 = 2\mathbf{E}\int_{t_k}^{t_{k+1}}\int_r^{t_{k+1}} \mathbf{E}\left(\xi_\varepsilon(z)/\mathcal{G}_r\right)\xi_\varepsilon(r)\,dz\,dr \le 2K_1.$$

Convergence (19) follows from the above reasoning. Lemma 5 is proved.

Let us verify that $M(u,t)$ is a $\mathcal{G}_t$-martingale. Consider the expectation

$$\begin{aligned}
&\mathbf{E}\Big(x_\varepsilon(u,t) - x_\varepsilon(u,s) - \int_s^t \int a(x_\varepsilon(u,z), x_\varepsilon(v,z), z)\mu_0(dv)dz\Big)\Phi_\varepsilon \\
&= \mathbf{E}\int_s^t \sum_k \int b_k(x_\varepsilon(u,z), x_\varepsilon(v,z), z)\mu_k(dv)\xi_\varepsilon^k(z)dz\Phi_\varepsilon \\
&= \mathbf{E}\int_s^t \sum_k \int b_k(x_\varepsilon(u,s), x_\varepsilon(v,s), z)\mu_k(dv)\xi_\varepsilon^k(z)dz\Phi_\varepsilon \\
&+ \mathbf{E}\int_s^t \int_s^z \sum_{k,j}\Big(\int\int \big((b_k)_1'(x_\varepsilon(u,r), x_\varepsilon(v,r), z)b_j(x_\varepsilon(u,r), x_\varepsilon(w,r), r) \\
&+ (b_k)_2'(x_\varepsilon(u,r), x_\varepsilon(v,r), z)b_j(x_\varepsilon(v,r), x_\varepsilon(w,r), r)\big)\mu_k(dv)\mu_j(dw)\Big) \\
&\quad \times \xi_\varepsilon^k(z)\xi_\varepsilon^j(r)\,dr\,dz.
\end{aligned} \tag{23}$$

Due to Lemma 5 the right hand side of (23) converges to

$$\begin{aligned}
&\mathbf{E}\int_s^t \sum_{k,j}\Big(\int\int \big((b_k)_1'(x(u,r), x(v,r), r)b_j(x(u,r), x(w,r), r) \\
&+ (b_k)_2'(x(u,r), x(v,r), r)b_j(x(v,r), x(w,r), r)\big)\mu_k(dv)\mu_j(dw)\Big)\sigma^{kj}(r)dr,
\end{aligned}$$

and left hand side to

$$\mathbf{E}\left(x(u,t) - x(u,s) - \int_s^t \int a(x(u,z), x(v,z), z)\mu_0(dv)dz\right)\Phi.$$

Therefore, $M(u,t)$ is a martingale. The proof of the facts, that square characteristics of $M(u,t), B(t)$ are connected by relations (5), (6), that $x(u,t), B(t)$ satisfy (3) and also the uniqueness of the limit measure is similar to [4, Theorem 5.4.1 and Lemma 5.3.3]. Theorem 1 is proved.

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