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Transformation of measures in infinite-dimensional spaces by the flow induced by a stochastic differential equation

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Abstract. Let μ be a Gaussian measure in the space X, and H the Cameron–Martin space of the measure μ. Consider the stochastic differential equation

$$d\xi(u,t) = a_t(\xi(u,t))\,dt + \sum_n \sigma_t^n(\xi(u,t))\,d\omega_n(t), \quad t \in [0,T],$$
$$\xi(u,0) = u,$$

where $u \in X$, a and σ_n are functions taking values in H, and the $\omega_n(t)$, $n \geqslant 1$, are independent one-dimensional Wiener processes. Consider the measure-valued random process $\mu_t := \mu\circ\xi(\,\cdot\,,t)^{-1}$. It is shown that under certain natural conditions on the coefficients of the initial equation the measures $\mu_t(\omega)$ are equivalent to μ for almost all ω. Explicit expressions for their Radon–Nikodym densities are obtained.

Bibliography: 10 titles.

§ 1. Introduction

We consider the flow $\xi(u,t)$, $u \in \mathbb{R}^d$, $t \in [0,T]$, induced by the following stochastic differential equation in the Euclidean space $\mathbb{R}^d$:

$$d\xi(u,t) = a_t(\xi(u,t))\,dt + \sum_n \sigma_t^n(\xi(u,t))\,d\omega_n(t), \qquad \xi(u,0) = u. \tag{1.1}$$

Let μ be a measure in $\mathbb{R}^d$ having a smooth positive density with respect to Lebesgue measure. Kunita [1] proved that under certain natural assumptions about the smoothness and boundedness of the coefficients in (1.1) the measures $\mu_t(\omega) := \mu\circ\xi(\,\cdot\,,t,\omega)^{-1}$ are absolutely continuous for almost all ω, and a stochastic analogue of Liouville's theorem holds for the density $\rho_t(\omega)(u) = \dfrac{d\mu_t(\omega)}{d\mu}(u)$, $u \in \mathbb{R}^d$.

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In the present paper we consider a similar problem on the transformation of measures by the flow induced by a stochastic differential equation in an infinite-dimensional space. We point out several important differences between the finite-dimensional and infinite-dimensional cases. First, even a linear stochastic equation does not necessarily induce a flow in an infinite-dimensional space [2]. Second, this is related to the lack of a canonical infinite-dimensional Lebesgue measure and, accordingly, of measures with smooth density. Differential measures are a natural generalization of measures with smooth density in the infinite-dimensional case. The most interesting and widely discussed class of such measures is the class of Gaussian measures, which we consider in the present paper.

Transformations of measures by means of flows induced by ordinary (non-stochastic) differential equations in the infinite-dimensional case have been considered by many authors; see, for instance, [3]–[6] and the references in these papers.

The main result of the present paper is Theorem 3.1, in which we obtain conditions on the coefficients a_t, σ_t^n of equation (1.1) ensuring the equivalence of the measures μ and $\mu_t(\omega)$ for almost all ω in the case when μ is a Gaussian Radon measure. We obtain explicit expressions for the Radon–Nikodym densities, which coincide with the corresponding formulae in the finite-dimensional case [1].

§ 2. Definitions and auxiliary results

Let X be a locally convex space, μ a centered Gaussian Radon measure in X, and let $H \subset X$ be the Cameron–Martin space of μ with Hilbert norm $\|\cdot\|_H$.

Let E be a separable Hilbert space. Then we denote by $\mathcal{F}C_b^\infty(E)$ the set of cylindrical smooth functions of the following form:

$$F = \sum_{k=1}^{n} f_k(\langle x_1^*, \cdot\rangle, \dots, \langle x_n^*, \cdot\rangle) e_k \colon X \to E,$$

where $n \in \mathbb{N}$, $x_k^* \in X^*$, $e_k \in E$, $f_k \in C_b^\infty(\mathbb{R}^n)$.

Let D be the Fréchet derivative of functions in $\mathcal{F}C_b^\infty(E)$ along the directions in H. We define recursively the derivatives of higher orders.

By the Sobolev spaces $W_p^k(E)$ we understand the closures of $\mathcal{F}C_b^\infty(E)$ in the norm

$$\|F\|_{p,k} = \sum_{j=1}^{k} \|D^j F\|_{L_p(X,\mu,H^{\otimes j}\otimes E)}.$$

The closure of the operator D^k with respect to the norm $\|\cdot\|_{p,k}$ is called the *stochastic derivative*; we use for it the same symbol.

We also set

$$W_\infty^k(E) := \bigl\{f \mid \text{for each } p \geqslant 1,\ \|D^j f\|_{L_\infty(X,\mu,H^{\otimes j}\otimes E)} < \infty \\ \text{for each } f \in W_p^k(E)\ \ 0 \leqslant j \leqslant k\bigr\},$$

$$\|f\|_{\infty,k} := \max_{0\leqslant j\leqslant k} \|D^k f\|_{L_\infty(X,\mu,H^{\otimes k}\otimes E)}.$$

The adjoint operator $\delta = D^*$, where $\mathcal{D}(D) = W_p^1(E)$,

$$\delta = \delta_q : \mathcal{D}(\delta_q) \subset L_q(X, \mu, H \otimes E) \to L_q(X, \mu, H), \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

is called the *divergence operator* (or the extended Skorokhod operator).

We present some moment inequalities for the divergence operator and the stochastic derivative in the next statement.

Lemma 2.1 ([7], p. 50). *If $f \in W_p^k(H)$, then $\delta f \in W_p^{k-1}(\mathbb{R})$ and there exists a constant $c_{p,k}$ such that for each $f \in W_p^k(H)$,*

$$\|\delta f\|_{W_p^{k-1}} \leqslant c_{p,k} \|\delta f\|_{W_p^k(H)}.$$

Definition 2.1. A function $f : X \to E$ is said to be *H-Lipschitz with constant L* if

$$\|f(x+h) - f(x)\|_E \leqslant L\|h\|_H$$

for all $x \in X$ and $h \in H$.

Definition 2.2. A function $f : X \to E$ *is of class $\mathcal{H}C^n(E)$* if for each $x \in X$ the map $H \ni h \to f(x+h) \in E$ has continuous Fréchet derivatives of order n.

The following fact on the existence of an H-Lipschitz modification of functions of class W_∞^1 is well known (see, for instance, [8]).

Lemma 2.2. *Let $f \in W_\infty^1(E)$. Then f has an H-Lipschitz modification with constant* $\operatorname{ess\,sup}_\omega \|Df(\omega)\|$.

The Cameron–Martin space H can be identified in a natural fashion with the space $H(\mu)$ of square integrable functionals. Let $\widehat{h}$ be the element of $H(\mu)$ corresponding to a vector $h \in H$. We shall require the following well-known result on the action of the stochastic derivative on the conditional expectation of a differentiable random variable.

Lemma 2.3. *Let H_0 be a linear subspace of H, σ_0 the σ-algebra generated by the functionals $\{\widehat{h},\ h \in H_0\}$, and π the orthogonal projection onto H_0 in H. Then for each function $f \in W_p^k$, $p \geqslant 1$, $k \geqslant 1$, the conditional expectation $\mathsf{E}(f/\sigma_0)$ also belongs to W_p^k and*

$$D\mathsf{E}(f/\sigma_0) = \pi \mathsf{E}(Df/\sigma_0).$$

§ 3. Main results

Let μ be a centered Gaussian Radon measure in a locally convex space X, let $H \subset X$ be the Cameron-Martin space, and let $\{\omega_n(t),\ t \in [0,T],\ n \geqslant 1\}$ be a sequence of independent one-dimensional Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

Consider the following stochastic differential equation in X depending on the initial condition $u \in X$:

$$\varphi_{s,t}(u) = u + \int_s^t a_z(\varphi_{s,z}(u))\, dz + \sum_{n=1}^{\infty} \int_s^t \sigma_z^n(\varphi_{s,z}(u))\, d\omega_n(z), \tag{3.1}$$

where $a\colon [0,T]\times X\to H$ and the $\sigma^n\colon [0,T]\times X\to H$, $n\geqslant 1$, are functions jointly measurable in their variables.

It is sometimes more convenient to study equation (3.1) written in the Stratonovich form:

$$\varphi_{s,t}(u)=u+\int_s^t \widetilde{a}_z(\varphi_{s,z}(u))\,dz+\sum_{n=1}^{\infty}\int_s^t \sigma_z^n(\varphi_{s,z}(u))\circ d\omega_n(z), \tag{3.1$'$}$$

where

$$\widetilde{a}_t=a_t-\frac{1}{2}\sum_n\langle D\sigma_z^n,\sigma_z^n\rangle_H.$$

We consider also the generalized Stratonovich equation:

$$\varphi_{t,s}(u)=u-\int_s^t \widetilde{a}_z(\varphi_{z,t}(u))\,dz+\sum_{n=1}^{\infty}\int_s^t \sigma_z^n(\varphi_{z,t}(u))\circ d\widetilde{\omega}_n(z) \tag{3.1$''$}$$

with the new Wiener processes $\widetilde{\omega}_n(t)=\omega_n(T)-\omega_n(t)$. In a finite-dimensional space this equation defines the inverse flow $\varphi_{t,s}$, $t\geqslant s$ [1].

The main result of the present paper is as follows.

Theorem 3.1. I. *Assume that the following relations hold for all $t\in[0,T]$, $n\geqslant 1$:*

$$a_t\in W_\infty^2(H),\qquad \sigma_t^n\in W_\infty^3(H)$$

and

$$A_j:=\sup_t\operatorname*{ess\,sup}_u\|D^j a_t(u)\|_{H^{\otimes(j+1)}}<\infty,\qquad j\in\{0,1,2\},$$
$$S_l:=\sup_t\operatorname*{ess\,sup}_u\sum_n\|D^l\sigma_t^n(u)\|^2_{H^{\otimes(l+1)}}<\infty,\qquad l\in\{0,\dots,3\}.$$

Then there exists a random element $\varphi=\varphi_{s,t}(u,\omega)\colon [0,T]\times[0,T]\times X\times\Omega\to X$ jointly measurable in the four variables (s,t,u,ω) and a subset Ω_0 of Ω, $\mathsf{P}(\Omega_0)=1$, such that the following results hold for all $\omega\in\Omega_0$:

(a) *relations (3.1$'$), (3.1$''$) hold for all $s\leqslant t$, where the stochastic integrals are regarded as integrals of $L_2(X,\mu,H)$-valued random elements;*

(b) *for all $t_1,t_2,t_3\in[0,T]$,*

$$\varphi_{t_2,t_3}(\varphi_{t_1,t_2}(u,\omega),\omega)=\varphi_{t_1,t_3}(u,\omega)\quad \text{for }\ \mu\text{-almost all } u\in X,$$
$$\varphi_{t_1,t_1}(\,\cdot\,,\omega)=\mathrm{id}_X;$$

(c) *for all $s,t\in[0,T]$ the measures $\mu_{s,t}(\omega)=\mu\circ(\varphi_{s,t}(\cdot,\omega))^{-1}$ and μ are equivalent;*

(d) *the Radon–Nikodym density $\rho_{s,t}(\omega)(u):=\dfrac{d\mu_{s,t}(\omega)}{d\mu}(u)$ has the following form:*

$$\rho_{t,s}(u)=\exp\biggl\{-\int_s^t(\delta\widetilde{a}_z)(\varphi_{s,z}(u))\,dz-\sum_n\int_s^t(\delta\sigma_z^n)(\varphi_{s,z}(u))\circ d\omega_n(z)\biggr\}, \tag{3.2}$$

where $s\leqslant t$ and

$$\rho_{s,t}(u)=\left(\rho_{t,s}(\varphi_{s,t}(u))\right)^{-1}. \tag{3.3}$$

II. *Let* $a_t \in \mathcal{H}C^2(H)$, $\sigma_t^n \in \mathcal{H}C^3(H)$ *for all* $t \in [0,T]$, $n \geqslant 1$, *and assume that the quantities* A_j *and* S_k *defined in the first part of the theorem are finite. Then there exists a process* $\varphi_{s,t}$ *such that for each* $u \in X$ *relations* (3.1′) *and* (3.1″) *hold* P-*a.s. and* $\varphi_{s,t}(u)$ *is continuous in* s, t *for* P-*almost all* ω. *Moreover, each modification of* $\varphi_{s,t}(u)$ *continuous in* s, t *has properties* (a)–(d) *from the first part of the theorem.*

Remarks. (1) Sets of μ-measure zero in (b) depend in general on t_1, t_2, t_3, and ω.

(2) We consider composite functions in (3.3). However, $\varphi_{t,s}$ is defined up to sets of μ-measure zero. The composite is well defined because the measures μ and $\mu_{t,s}$ are equivalent by (c). The composite $\varphi_{t_2,t_3} \circ \varphi_{t_1,t_2}$ in (b) is well defined for the same reason.

(3) For $\omega \in \Omega_0$, $s,t \in [0,T]$, the map $\varphi_{s,t}(\,\cdot\,,\omega)\colon X \to X$ is μ-almost surely invertible (see (b)) and $\varphi_{t,s}(\,\cdot\,,\omega)$ defines the inverse map, that is,

$$\varphi_{t,s}(\varphi_{s,t}(\,\cdot\,,\omega),\omega) = \varphi_{s,t}(\varphi_{t,s}(\,\cdot\,,\omega),\omega) = \mathrm{id}_X \qquad \mu\text{-a.s.}$$

(4) The Stratonovich stochastic integrals in (3.1′), (3.1″), and (3.2) are used for simplicity of notation and are the Itô integrals of random elements ranging in L_2 with suitable additions. For instance, the expression $\displaystyle\int_s^t (\delta\sigma_z^n)(\varphi_{s,z}(u)) \circ d\omega_n(z)$ means

$$\int_s^t (\delta\sigma_z^n)(\varphi_{s,z}(u))\,d\omega_n(z) + \frac{1}{2}\int_s^t \langle D(\delta(\sigma_z^n)), \sigma_z^n\rangle(\varphi_{s,z})\,dz.$$

§ 4. Finite-dimensional approximations

Let $\{e_m\}$ be an orthonormal basis in H and assume that the square-integrable functionals $\{\widehat{e}_m\}$ are elements of $X^* \subset H(\mu)$. The linear map

$$j\colon X \ni u \mapsto (\langle \widehat{e}_1, u\rangle, \langle \widehat{e}_2, u\rangle, \dots) \in \mathbb{R}^\infty$$

transforms the measure μ into γ, the product of the Gaussian $\mathcal{N}(0,1)$ measures. In the process H is transformed into l_2, and the classes $W_p^k(X,\mu,H)$ into $W_p^k(\mathbb{R}^\infty,\gamma,l_2)$. One can verify the existence of a γ-measurable map $k\colon \mathbb{R}^\infty \to X$ such that $(kj)(u)=u$ μ-a.s. It is equally easy to see that the functions $\mathbb{R}^\infty \ni y \mapsto ja(ky)$, $\mathbb{R}^\infty \ni y \mapsto j\sigma_n(ky)$ satisfy all the assumptions of Theorem 3.1. Hence it is now sufficient to prove the theorem for $X = \mathbb{R}^\infty$ and $\mu = \gamma$. Let π_m be the projection onto the plane of the first m variables, and π^m the projection onto the space of the remaining variable. We shall approximate equation (3.1) by the following sequence:

$$\varphi_{s,t}^{(m)}(u) = u + \int_s^t a_z^m(\varphi_{s,z}^{(m)}(u))\,dz + \sum_n \int_s^t \sigma_z^{m,n}(\varphi_{s,z}^{(m)}(u))\,d\omega_n(z), \tag{4.1}$$

where the functions a_z^m and σ_z^{mn} satisfy the conditions

$$a_z^m(u) = \pi_m a_z^m(\pi_m u) \text{ and } \quad \sigma_z^{m,n} = \pi_m \sigma_z^{m,n}(\pi_m u). \tag{4.2}$$

Note that in this case the flow $\varphi_{s,t}^{(m)}$ changes only the coordinates of $\pi_m u$, but does not affect $\pi^m u$. Hence if the coefficients of (4.1) satisfy (4.2), then it remains to consider the finite-dimensional problem, which was thoroughly studied in [1] not only for Gaussian measures, but also for measures with smooth density.

Hence our aim is the approximation of the coefficients a and σ^n by 'finite-dimensional' functions a^m, $\sigma^{m,n}$ and after that the substantiation of the limit transition for the $\varphi_{s,t}^{(m)}$ and the Radon–Nikodym densities

$$\rho_{s,t}^{(m)} := \frac{d\mu \circ (\varphi_{s,t}^{m})^{-1}}{d\mu}.$$

We require the following result.

Lemma 4.1 [9]. *Let X_1 and X_2 be separable metric spaces, let μ_i be a probability measure in X_i, and let $\{f_n \colon X_1 \to X_2, n \geqslant 0\}$ be a sequence of measurable maps. Assume that*

(a) *the measures $\mu_1 \circ (f_n)^{-1}$ are absolutely continuous with respect to μ_2 for each $n \geqslant 1$,*

(b) *the sequence of Radon–Nikodym densities*

$$\left\{ \frac{d\mu_1 \circ (f_n)^{-1}}{d\mu_2},\ n \geqslant 1 \right\}$$

is uniformly integrable,

(c) *f_n approaches f_0 in measure as $n \to \infty$ with respect to μ_1.*

Then $\mu_1 \circ (f_0)^{-1} \ll \mu_2$.

Moreover, if the sequence $\left\{ \frac{d\mu_1 \circ (f_n)^{-1}}{d\mu_2},\ n \geqslant 1 \right\}$ converges to a function p in measure with respect to μ_2, then

$$p = \frac{d\mu_1 \circ (f_0)^{-1}}{d\mu_2}.$$

As follows by Lemma 4.1, uniform integrability of densities is very important for the substantiation of the absolute continuity of the limit measure. In the next result we give several estimates for $\rho_{s,t}^{(m)}$ independent of the space dimension m.

Theorem 4.1. *Let μ be a Gaussian measure in $\mathbb{R}^m$ with mean zero and identity covariance matrix. Let*

$$a \colon [0,T] \times \mathbb{R}^m \to \mathbb{R}^m \quad \text{and} \quad \sigma^n : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m, \quad n \geqslant 1,$$

be measurable functions satisfying the following conditions:

(i) *for each $t \in [0,T]$, $a_t \in C_b^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\sigma_t^n \in C_b^3(\mathbb{R}^m, \mathbb{R}^m)$;*

(ii) *$A_j := \sup_{t,x} \|\nabla^j a_t(x)\| < \infty$, $j \in \{0,1,2\}$,*
$S_l := \sup_{t,x} \sum_n \|\nabla^l \sigma_t^n(x)\|^2 < \infty$, $l \in \{0,1,2,3\}$.

Then there exist a random function $\varphi = \varphi_{s,t}(u)$, where $s,t \in [0,T]$ and $u \in \mathbb{R}^m$, satisfying equalities $(3.1')$ and $(3.1'')$ for each u and a subset Ω_0 of Ω, $\mathsf{P}(\Omega_0) = 1$, such that for all $\omega \in \Omega_0$:

(a) *φ is continuous in (s,t,u);*

(b) $\varphi_{s,t}(u)$ *is u-differentiable for all* s,t *and*

$$\nabla\varphi_{s,t}(u)=\mathrm{id}_{\mathbb{R}^m}+\int_s^t \nabla a_z(\varphi_{s,z}(u))\nabla\varphi_{s,z}(u)\,dz + \sum_n \int_s^t \nabla\sigma_z^n(\varphi_{s,z}(u))\nabla\varphi_{s,z}(u)\,d\omega_n(z) \tag{4.3}$$

for $s\leqslant t, u\in\mathbb{R}^m$;

(c) *for all* $u\in\mathbb{R}^m$ *and* $t_1,t_2,t_3\in[0,T]$,

$$\varphi_{t_1,t_1}(u,\omega)=u \quad \text{and} \quad \varphi_{t_2,t_3}(\varphi_{t_1,t_2}(u,\omega),\omega)=\varphi_{t_1,t_3}(u,\omega);$$

(d) *the measures* $\mu_{s,t}(\omega):=\mu\circ(\varphi_{s,t}(\,\cdot\,,\omega))^{-1}$ *and* μ *are equivalent and the Radon–Nikodym densities* $\rho_{s,t}=\dfrac{d\mu_{s,t}}{d\mu}$ *satisfy formulae* (3.2) *and* (3.3), *where*

$$(\delta f)(u)=-\sum_{k=1}^m\biggl(\frac{\partial f_k(u)}{\partial u_k}-f_k(u)u_k\biggr) \tag{4.4}$$

for $f=(f_1,\dots,f_m)\in C^1(\mathbb{R}^m,\mathbb{R}^m)$ *and* $u=(u_1,\dots,u_m)\in\mathbb{R}^m$;

(e) *there exists a constant* $C=C(A_0,A_1,S_0,S_1,S_2)$ *depending only on the* A_j *and the* S_l, *but not on the dimension of* $\mathbb{R}^m$ *such that*

$$\sup_{0\leqslant s,t\leqslant T}\int_\Omega\int_{\mathbb{R}^m}|\rho_{s,t}\ln\rho_{s,t}|\,d\mu\,d\mathsf{P}\leqslant C. \tag{4.5}$$

Remark. If in Theorem 3.1 $H=X=\mathbb{R}^m$, $\mu=\mathcal{N}(0,\mathrm{id}_{\mathbb{R}^m})$, and the coefficients a and σ^n satisfy the assumptions of Theorem 4.1, then the result of Theorem 3.1 holds in its full extent. For in this case the stochastic derivative is the gradient in $\mathbb{R}^m$ and the adjoint operator is defined by formula (4.4).

Proof of Theorem 4.1. For the proof of (a)–(d) see, for example, [1], §§ 4.3–4.6. We shall verify assertion (e). We consider the case $s\leqslant t$ first. We can assume without loss of generality that $s=0$. Let $\varphi_t:=\varphi_{0,t}$.

Lemma 4.2. *Let* $(A,\mathfrak{A},\nu)$ *be a probability space and let* $f\colon A\to A$ *be a measurable map with the following properties*:

(1) *the measures* ν *and* $\nu\circ f^{-1}$ *are equivalent*;
(2) *there exists a measurable map* $g:A\to A$ *such that* $g(f(\alpha))=f(g(\alpha))=\alpha$ *for* ν-*almost all* $\alpha\in A$.

Then $\nu\circ g^{-1}\sim\nu$,

$$\frac{d\nu\circ f^{-1}}{d\nu}=\biggl(\frac{d\nu\circ g^{-1}}{d\nu}(g)\biggr)^{-1}, \tag{4.6}$$

and

$$\int_A\biggl|\frac{d\nu\circ f^{-1}}{d\nu}\ln\frac{d\nu\circ f^{-1}}{d\nu}\biggr|\,d\nu=\int_A\biggl|\ln\frac{d\nu\circ g^{-1}}{d\nu}\biggr|\,d\nu. \tag{4.7}$$

Let E_μ, E_P, and $\mathsf{E}_{\mu\times\mathsf{P}}$ be expectations with respect to the measures μ, P, and $\mu\times\mathsf{P}$, respectively. Proceeding in (3.2) from the Stratonovich integral to the Itô integral and using (4.7) we obtain

$$\begin{aligned}\mathsf{E}_{\mu\times\mathsf{P}}|\rho_{0,t}\ln\rho_{0,t}| &= \mathsf{E}_{\mu\times\mathsf{P}}\,|\ln\rho_{t,0}| \\ &\leqslant \mathsf{E}_{\mu\times\mathsf{P}}\int_0^t|(\delta a_s)(\varphi_s)|\,ds \\ &\quad+\frac{1}{2}\mathsf{E}_{\mu\times\mathsf{P}}\int_0^t\biggl|\sum_n\delta(\nabla_{\sigma_s^n}\sigma_s^n)(\varphi_s)\biggr|\,ds \\ &\quad+\mathsf{E}_{\mu\times\mathsf{P}}\biggl|\sum_n\int_0^t(\delta\sigma_s^n)(\varphi_s)\,d\omega_n(s)\biggr| \\ &\quad+\frac{1}{2}\mathsf{E}_{\mu\times\mathsf{P}}\biggl|\int_0^t\sum_n(\nabla_{\sigma_s^n}(\delta\sigma_s^n))(\varphi_s)\,ds\biggr| \\ &= I_1+\frac{1}{2}I_2+I_3+\frac{1}{2}I_4. \end{aligned} \tag{4.8}$$

We now find estimates of the terms in (4.8). The fact that each integral in (4.8) is well defined will result from the proof. We shall require the following result on the action of the divergence operator δ on the composite of a smooth function and φ_t.

Lemma 4.3. *Let $b\in C^1(\mathbb{R}^m,\mathbb{R}^m)$. Then*

$$\begin{aligned}(\delta b)(\varphi_s) &= \delta(b(\varphi_s))+\biggl\langle b(\varphi_s),\int_0^s a_z(\varphi_z)\,dz+\sum_n\int_0^s\sigma_z^n(\varphi_z)\,d\omega_n(z)\biggr\rangle \\ &\quad+\operatorname{tr}\biggl((\nabla b)(\varphi_s)\biggl\{\int_0^s(\nabla a_z)(\varphi_z)\nabla\varphi_z\,dz \\ &\qquad+\sum_n\int_0^s(\nabla\sigma_z^n)(\varphi_z)\nabla\varphi_z\,d\omega_n(z)\biggr\}\biggr). \end{aligned} \tag{4.9}$$

Proof. Note that

$$\begin{aligned}(\delta b)(\varphi_s) &= \langle\varphi_s,b(\varphi_s)\rangle-\operatorname{tr}(\nabla b)(\varphi_s) \\ &= \langle u,b(\varphi_s)\rangle+\biggl\langle\int_0^s a_z(\varphi_z)\,dz+\sum_n\int_0^s\sigma_z^n(\varphi_z)\,d\omega_n(z),b(\varphi_s)\biggr\rangle \\ &\quad-\operatorname{tr}\nabla(b(\varphi_s))+\operatorname{tr}\bigl((\nabla b)(\varphi_s)\nabla\varphi_s-(\nabla b)(\varphi_s)\bigr). \end{aligned} \tag{4.10}$$

Applying (4.3) to (4.10) we obtain the expression (4.9). The proof of Lemma 4.3 is complete.

We now use (4.9) for the estimate of the term I_1 in (4.8):

$$\begin{aligned}I_1 &\leqslant \mathsf{E}_{\mu\times\mathsf{P}}\int_0^t |\delta(a_s(\varphi_s))|\,ds + \mathsf{E}_{\mu\times\mathsf{P}}\int_0^t \biggl|\biggl\langle \int_0^s a_z(\varphi_z)\,dz, a_s(\varphi_s)\biggr\rangle\biggr|\,ds \\ &\quad + \mathsf{E}_{\mu\times\mathsf{P}}\biggl|\int_0^t \biggl\langle a_s(\varphi_s), \sum_n \int_0^s \sigma_z^n(\varphi_z)\,d\omega_n(z)\biggr\rangle\,ds\biggr| \\ &\quad + \mathsf{E}_{\mu\times\mathsf{P}}\biggl|\int_0^t \operatorname{tr}\biggl\{\nabla a_s(\varphi_s)\int_0^s \nabla a_z(\varphi_z)\nabla\varphi_z\,dz\biggr\}\,ds\biggr| \\ &\quad + \mathsf{E}_{\mu\times\mathsf{P}}\biggl|\int_0^t \operatorname{tr}\biggl\{\nabla a_s(\varphi_s)\sum_n\int_0^s \nabla\sigma_z^n(\varphi_z)\nabla\varphi_z\,d\omega_n(z)\biggr\}\,ds\biggr| \\ &= I_{1,1} + \cdots + I_{1,5}.\end{aligned}$$

Note that

$$\begin{aligned}I_{1,2} &\leqslant \int_0^t\int_0^s \|a_z\|_\infty\|a_s\|_\infty\,dz\,ds \leqslant \frac{1}{2}A_0^2, \\ (I_{1,3})^2 &\leqslant t\int_0^t \|a_s\|_\infty^2\mathsf{E}_\mu\mathsf{E}_\mathsf{P}\biggl\|\sum_n\int_0^s \sigma_z^n(\varphi_z)\,d\omega_n(z)\biggr\|_{\mathbb{R}^m}^2\,dz\,ds \\ &\leqslant t\int_0^t \|a_s\|_\infty^2 \sum_n\int_0^s \|\sigma_z^n\|_\infty^2\,dz\,ds \leqslant t^2A_0^2S_0.\end{aligned}$$

For the estimate of $I_{1,1}$ we use Lemma 2.1:

$$\begin{aligned}(I_{1,1})^2 &\leqslant t\int_0^t \mathsf{E}_\mathsf{P}\{\mathsf{E}_\mu(\delta(a_s(\varphi_s)))^2\}\,ds \\ &\leqslant tc_{2,1}\int_0^t \mathsf{E}_\mathsf{P}\{\mathsf{E}_\mu[\|a_s(\varphi_s)\|^2 + \|\nabla a_s(\varphi_s)\nabla\varphi_s\|^2]\}\,ds \\ &\leqslant tc_{2,1}\int_0^t (\|a_s\|_\infty^2 + \|\nabla a_s\|_\infty^2\mathsf{E}_\mathsf{P}\mathsf{E}_\mu\|\nabla\varphi_s\|_{\mathrm{op}}^2)\,ds, \qquad (4.11)\end{aligned}$$

where $\|\nabla a_s\|_\infty = \sup_{x\in\mathbb{R}^m}\|\nabla a_s(x)\|_{\mathbb{R}^m\otimes\mathbb{R}^m}$, $\|\cdot\|_{\mathrm{op}}$ is the operator norm.

The required inequality for $I_{1,1}$ is a consequence of the following result.

Lemma 4.4. *For each $p \geqslant 1$ there exist constants $K_1(p) = K_1(p,T,A_0,A_1,S_0,S_1)$ and $K_2(p) = K_2(p,T,A_j,S_j,\ j=0,1,2)$ such that*

$$\mathsf{E}_{\mu\times\mathsf{P}}\|\nabla^j\varphi_t\|_{\mathrm{op}}^p \leqslant K_j(p), \qquad j=1,2. \tag{4.12}$$

Moreover, $\nabla\varphi_t$ is almost surely invertible and there exists $K_3(p) = K_3(p,T,A_j,S_j,$ $j=0,1,2)$ such that

$$\mathsf{E}_{\mu\times\mathsf{P}}\|(\nabla\varphi_t)^{-1}\|_{\mathrm{op}}^p \leqslant K_3(p).$$

Proof. Note that $\|\nabla\varphi_t\|_{\mathrm{op}} \leqslant 1 + \|\nabla\varphi_t - \mathrm{id}\,\|_{HS}$, where $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm. We set $\psi_t := \nabla\varphi_t - \mathrm{id}$.

Then

$$\psi_t = \int_0^t (\nabla a_s(\varphi_s)\psi_s + \nabla a_s(\varphi_s))\,ds + \sum_n \int_0^t (\nabla \sigma_s^n(\varphi_s)\psi_s + \nabla \sigma_s^n(\varphi_s))\,d\omega_n(s).$$

Applying Itô's formula to $\|\psi_t\|_{HS}^p$ and using arguments similar to [1], p. 156 we arrive at the following inequality:

$$\mathsf{E}\|\psi_t\|_{HS}^p \leqslant C(p, A_1, S_1) \int_0^t (1 + \mathsf{E}\|\psi_s\|_{HS}^p)\,ds,$$

with C independent of the space dimension. Inequality (4.12) for $j = 1$ follows by Gronwall's lemma. The case $j = 2$ is considered in a similar way. The invertibility of $\nabla\varphi_t$ is a known result and $(\nabla\varphi_t)^{-1}$ satisfies a certain linear stochastic equation. Estimates for the moments of $(\nabla\varphi_t)^{-1}$ can be obtained in the same way as for $\nabla\varphi_t$. The proof of Lemma 4.4 is now complete.

For the investigation of $I_{1,4}$ and $I_{1,5}$ we shall require the following fact. If H is a Hilbert space, $A, B \in \mathcal{HS}(H)$ are Hilbert–Schmidt operators, and $C \in \mathcal{L}(H)$ is a bounded operator, then

$$|\operatorname{tr} AB| \leqslant \|A\|_{HS}\|B\|_{HS}, \qquad \|AC\|_{HS} \leqslant \|A\|_{HS}\|C\|_{\mathrm{op}}. \tag{4.13}$$

Hence

$$\begin{aligned} I_{1,4} &\leqslant \int_0^t \|\nabla a_s\|_\infty \int_0^s \|\nabla a_z\|_\infty \mathsf{E}_{\mu\times\mathsf{P}}\|\nabla\varphi_z\|_{\mathrm{op}}\,dz\,ds \leqslant \frac{1}{2}K_1 A_0^2 t^2, \\ I_{1,5} &\leqslant t\int_0^t \|\nabla a_s\|_\infty^2 \mathsf{E}_\mu \mathsf{E}_\mathsf{P}\left\|\sum_n \int_0^s \nabla\sigma_z^n(\varphi_z)\nabla\varphi_z\,d\omega_n(z)\right\|_{HS}^2 ds \\ &\leqslant t\int_0^t A_1^2 \mathsf{E}_\mu \mathsf{E}_\mathsf{P} \int_0^s \sum_n \|\nabla\sigma_z^n(\varphi_z)\nabla\varphi_z\|_{HS}^2\,dz\,ds \\ &\leqslant A_1^2 t \int_0^t \int_0^s \sum_n \|\nabla\sigma_z^n\|_\infty^2 \mathsf{E}_{\mu\times\mathsf{P}}\|\nabla\varphi_z\|_{\mathrm{op}}^2\,dz\,ds \leqslant \frac{1}{2}A_1^2 t^3 S_0 K_2. \end{aligned}$$

Thus, the above inequality completes the proof of the estimate

$$I_1 \leqslant K(T, A_0, A_1, S_0, S_1).$$

Similarly to our previous arguments we can show that $I_2 \leqslant K(T, A_0, A_1, S_0, S_1, S_2)$. It is sufficient to observe that for each $t \in [0, T]$ the function $\sum_n \nabla_{\sigma_t^n}\sigma_t^n$ has continuous derivatives and

$$\sup_t \left(\left\|\sum_n \nabla_{\sigma_t^n}\sigma_t^n\right\|_{\mathbb{R}^m} + \left\|\nabla\sum_n (\nabla_{\sigma_t^n}\sigma_t^n)\right\|_{\mathbb{R}^m\otimes\mathbb{R}^m} \right) < \infty.$$

In fact,

$$\Big\|\sum_n \nabla_{\sigma_t^n}\sigma_t^n\Big\| \leqslant \sum_n \|\nabla\sigma_t^n\|\,\|\sigma_t^n\| \leqslant \sum_n(\|\nabla\sigma_t^n\|^2+\|\sigma_t^n\|^2) \leqslant S_0+S_1,$$

$$\Big\|\nabla\Big(\sum_n \nabla_{\sigma_t^n}\sigma_t^n\Big)\Big\| \leqslant \sum_n(\|\nabla^2\sigma_t^n\|\|\sigma_t^n\|+\|\nabla\sigma_t^n\|^2) \leqslant S_0+S_1+S_2.$$

Consider now I_3:

$$\begin{aligned}
|I_3| \leqslant{}& \mathsf{E}_{\mu\times\mathsf{P}}\Big|\sum_n\int_0^t \delta(\sigma_s^n(\varphi_s))\,d\omega_n(s)\Big| \\
&+\mathsf{E}_{\mu\times\mathsf{P}}\Big|\sum_n\int_0^t\Big\langle \sigma_s^n(\varphi_s),\int_0^s a_z(\varphi_z)\,dz\Big\rangle\,d\omega_n(s)\Big| \\
&+\mathsf{E}_{\mu\times\mathsf{P}}\Big|\sum_n\int_0^t\Big\langle \sigma_s^n(\varphi_s),\sum_m\int_0^s \sigma_z^m(\varphi_z)\,d\omega_m(z)\Big\rangle\,d\omega_n(s)\Big| \\
&+\mathsf{E}_{\mu\times\mathsf{P}}\Big|\mathrm{tr}\Big\{\sum_n\int_0^t\nabla\sigma_s^n(\varphi_s)\int_0^s\nabla a_z(\varphi_z)\nabla\varphi_z\,dz\,d\omega_n(s)\Big\}\Big| \\
&+\mathsf{E}_{\mu\times\mathsf{P}}\Big|\mathrm{tr}\Big\{\sum_n\int_0^t\nabla\sigma_s^n(\varphi_s)\int_0^s\sum_m\nabla\sigma_z^m(\varphi_z)\nabla\varphi_z\,d\omega_m(z)\,d\omega_n(s)\Big\}\Big| \\
={}& I_{3,1}+\cdots+I_{3,5};
\end{aligned}$$

$$\begin{aligned}
(I_{3,1})^2 &\leqslant \mathsf{E}_\mu\Big\{\mathsf{E}_\mathsf{P}\Big(\sum_n\int_0^t\delta(\sigma_s^n(\varphi_s))\,d\omega_n(s)\Big)^2\Big\} \\
&= \mathsf{E}_{\mu\times\mathsf{P}}\sum_n\int_0^t(\delta(\sigma_s^n(\varphi_s)))^2\,ds.
\end{aligned}$$

Our further arguments for $I_{3,1}$ proceed in the same fashion as for $I_{1,1}$:

$$\begin{aligned}
(I_{3,2})^2 &\leqslant \mathsf{E}_{\mu\times\mathsf{P}}\sum_n\int_0^t\Big(\Big\langle\sigma_s^n(\varphi_s),\int_0^s a_z(\varphi_z)\,dz\Big\rangle\Big)^2\,ds \\
&\leqslant \sum_n\int_0^t\|\sigma_s^n\|_\infty^2\,ds\Big(\int_0^t\|a_z\|_\infty\,dz\Big)^2 \leqslant S_0A_0^2t^3; \\
(I_{3,3})^2 &\leqslant \mathsf{E}_\mu\mathsf{E}_\mathsf{P}\sum_n\int_0^t\Big(\Big\langle\sigma_s^n(\varphi_s),\sum_m\int_0^s\sigma_z^m(\varphi_z)\,d\omega_m(z)\Big\rangle\Big)\,ds \\
&\leqslant \int_0^t\sum_n\|\sigma_s^n\|_\infty^2\,\mathsf{E}_\mu\Big(\mathsf{E}_\mathsf{P}\Big\|\sum_m\int_0^s\sigma_z^m(\varphi_z)\,d\omega_m(z)\Big\|^2\Big)\,ds \leqslant \frac{1}{2}S_0^2t^2.
\end{aligned}$$

Applying (4.13) and Lemma 4.4 to $I_{3,4}$ we obtain

$$
\begin{aligned}
(I_{3,4})^2 &\leqslant \mathsf{E}_\mu \mathsf{E}_\mathsf{P} \sum_n \int_0^t \biggl(\operatorname{tr}\biggl\{\nabla\sigma_s^n(\varphi_s)\int_0^s \nabla a_z(\varphi_z)\nabla\varphi_z\,dz\biggr\}\biggr)^2\,ds \\
&\leqslant \sum_n \int_0^t \|\nabla\sigma_s^n\|_\infty^2 s\int_0^s \|\nabla a_z\|_\infty^2 \mathsf{E}_{\mu\times\mathsf{P}}\|\nabla\varphi_z\|_{\mathrm{op}}^2\,dz\,ds \leqslant \frac{1}{3}K_1(2)S_1A_1^2t^3,
\end{aligned}
$$

where the constant $K_1(2)$ is defined in Lemma 4.4.

In a similar way,

$$
(I_{3,5})^2 \leqslant \sum_n \int_0^t \|\nabla\sigma_s^n\|_\infty^2 \int_0^s \sum_m \|\nabla\sigma_z^m\|_\infty^2 \mathsf{E}_{\mu\times\mathsf{P}}\|\nabla\varphi_z\|_{\mathrm{op}}^2\,dz\,ds \leqslant \frac{1}{2}S_1^2t^2\cdot K_1(2).
$$

We find an estimate of the last term in (4.8):

$$
\begin{aligned}
I_4 &= \mathsf{E}_{\mu\times\mathsf{P}}\int_0^t \biggl|\sum_n \langle\nabla((\delta\sigma_s^n)(\varphi_s)), (\nabla\varphi_s)^{-1}\sigma_s^n(\varphi_s)\rangle\biggr|\,ds \\
&\leqslant \int_0^t \sum_n \mathsf{E}_{\mu\times\mathsf{P}}\|\nabla((\delta\sigma_s^n)(\varphi_s))\|^2\,ds + \int_0^t \sum_n \|\sigma_s^n\|_\infty^2 \mathsf{E}_{\mu\times\mathsf{P}}\|(\nabla\varphi_s)^{-1}\|_{\mathrm{op}}^2\,ds.
\end{aligned}
$$

The second term is bounded by Lemma 4.4. We consider now the integrand in the first term. From Lemma 4.3 we obtain

$$
\begin{aligned}
&\sum_n \mathsf{E}_{\mu\times\mathsf{P}}\|\nabla((\delta\sigma_n^s)(\varphi_s))\|^2 \\
&\qquad\leqslant 5\sum_n \mathsf{E}_{\mu\times\mathsf{P}}\biggl(\|(\nabla\delta)(\sigma_s^n(\varphi_s))\|^2 + \biggl\|\nabla\biggl(\biggl\langle\sigma_s^n(\varphi_s), \int_0^s a_z(\varphi_z)\,dz\biggr\rangle\biggr)\biggr\|^2 \\
&\qquad\quad + \biggl\|\nabla\biggl(\biggl\langle\sigma_s^n(\varphi_s), \int_0^s \sum_m \sigma_z^m(\varphi_z)\,d\omega_m(z)\biggr\rangle\biggr)\biggr\|^2 \\
&\qquad\quad + \biggl\|\nabla\biggl(\operatorname{tr}\biggl\{\biggl\langle\sigma_s^n(\varphi_s), \int_0^s a_z(\varphi_z)\,dz\biggr\rangle\biggr\}\biggr)\biggr\|^2 \\
&\qquad\quad + \biggl\|\nabla\biggl(\operatorname{tr}\biggl\{\biggl\langle\sigma_s^n(\varphi_s), \sum_m\int_0^s \sigma_z^m(\varphi_z)\,d\omega_m(z)\biggr\rangle\biggr\}\biggr)\biggr\|^2\biggr) \\
&\qquad = 5(I_{4,1}+\dots+I_{4,5}).
\end{aligned}
$$

Applying Lemmas 4.4 and 2.1 to $I_{4,1}$ we obtain the inequality $I_{4,1} \leqslant C(T, A_j, S_j, j=0,1,2)$. The boundedness of $I_{4,2}$ follows by the same arguments as in the case of $I_{1,3}$. One can show ([1], § 4.6) that

$$
\nabla\biggl(\sum_m \int_0^s \sigma_z^m(\varphi_z)\,d\omega_m(z)\biggr) = \sum_m \int_0^s \nabla\sigma_z^m(\varphi_z)\nabla\varphi_z\,d\omega_m(z).
$$

Hence,

$$
\begin{aligned}
I_{4,3} &\leqslant \sum_n \mathsf{E}_{\mu\times\mathsf{P}}\biggl\|\biggl\langle \nabla\sigma_s^n(\varphi_s)\nabla\varphi_s, \sum_m \int_0^s \sigma_z^m(\varphi_z)\,d\omega_m(z)\biggr\rangle\biggr\|^2 \\
&\quad + \sum_n \mathsf{E}_{\mu\times\mathsf{P}}\biggl\|\biggl\langle \sigma_s^n(\varphi_s), \sum_m \int_0^s \nabla\sigma_z^m(\varphi_z)\nabla\varphi_z\,d\omega_m(z)\biggr\rangle\biggr\|^2 \\
&\leqslant S_1\mathsf{E}_{\mu\times\mathsf{P}}\|\nabla\varphi_s\|_{\mathrm{op}}^2\biggl\|\sum_m \int_0^s \sigma_z^m(\varphi_z)\,d\omega_m(z)\biggr\|^2 \\
&\quad + S_0\mathsf{E}_{\mu\times\mathsf{P}}\int_0^s \sum_m \|\nabla\sigma_z^m(\varphi_z)\|_{\mathbb{R}^m\otimes\mathbb{R}^m}^2\|\nabla\varphi_z\|_{\mathrm{op}}^2\,dz \\
&\leqslant S_1(\mathsf{E}_{\mu\times\mathsf{P}}\|\nabla\varphi_s\|_{\mathrm{op}}^4)^{1/2}\biggl(\mathsf{E}_{\mu\times\mathsf{P}}\biggl\|\sum_m \int_0^s \sigma_z^m(\varphi_z)\,d\omega_m(z)\biggr\|^4\biggr)^{1/2} \\
&\quad + S_0S_1sK_1(2).
\end{aligned}
$$

We shall require the following well-known martingale inequality:

$$
\mathsf{E}\|M_t\|^4 \leqslant K\mathsf{E}(\langle M_t, M_t\rangle)^2,
$$

where M_t is a continuous square-integrable martingale of mean zero ranging in the Hilbert space. Here K is a universal constant.

Consequently,

$$
\mathsf{E}_{\mu\times\mathsf{P}}\biggl\|\sum_m \int_0^s \sigma_z^m(\varphi_z)\,d\omega_m(z)\biggr\|^4 \leqslant K\biggl(\sum_m \int_0^s \|\sigma_z^m\|_\infty^2\,dz\biggr)^2 \leqslant Ks^2S_0^2.
$$

Taking all the above inequalities into account we obtain the boundedness of $I_{4,3}$. The expressions $I_{4,4}$ and $I_{4,5}$ are treated similarly to $I_{4,3}$, with the use of inequality (4.13).

The proof of Theorem 4.1 is now complete.

§ 5. Existence of a measurable solution

Assume that for each $t \in [0,T]$ and $n \geqslant 1$,

(a) the functions a_t, $\sigma_t^n : X \to H$ are H-Lipschitz and

(b)
$$
\sup_{t\in[0,T]}\sup_{x\in X}\sup_{h\in H,\,h\neq 0}\biggl(\|a_t(x)\| + \sum_n \|\sigma_t^n(x)\|^2 + \frac{\|a_t(x+h)-a_t(x)\|}{\|h\|} + \sum_n \frac{\|\sigma_t^n(x+h)-\sigma_t^n(x)\|^2}{\|h\|^2}\biggr) < \infty.
$$

In this section we establish the existence of a random element

$$
\varphi = \varphi_{s,t}(u,\omega), \qquad 0 \leqslant s \leqslant t \leqslant T, \quad u \in X, \quad \omega \in \Omega,
$$

such that

(1) φ is jointly measurable in the four variables (s,t,u,ω),

(2) relation (3.1) holds for each $u \in X, 0 \leqslant s \leqslant t \leqslant T$,

(3) for all $u \in X$ and $\omega \in \Omega$ the function $\varphi_{s,t}(u)$ is continuous in s, t.

Note that if the functions a_t, σ_t^n satisfy the assumptions of Theorem 3.1.II, then they satisfy conditions (a) and (b).

Note that the solution to (3.1) exists for each fixed $u \in X$. For in that case $\varphi_{s,t}(u) = u + \psi_{s,t}(u)$, where $\psi_{s,t}$ is the solution of the following stochastic differential equation in H with H-Lipschitz coefficients:

$$\psi_{s,t}(u) = \int_s^t a_z(\psi_{s,z}(u) + u)\, dz + \sum_n \int_s^t \sigma_z^n(\psi_{s,z}(u) + u)\, d\omega_n(z).$$

Applying Kolmogorov's theorem it is easy to verify the existence of a modification continuous in s, t. However, the resulting process is not necessarily measurable in (s, t, u, ω). For the construction of a process measurable in the four variables we require the following result.

Theorem 5.1 [10]. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, Y a separable metric space, and $(A, \mathfrak{A})$ a measurable space. Consider a sequence of $\mathcal{F} \otimes \mathfrak{A}$-measurable processes $X_n = X_n(a)$, $a \in A$, $n \geqslant 1$, taking values in Y. Assume that there exists a limit of $\{X_n(a),\ n \geqslant 1\}$ in probability for each $a \in A$.*

Then there exists an $\mathcal{F} \otimes \mathfrak{A}$-measurable random element $X(a)$, $a \in A$, taking values in Y such that

$$X_n(a) \xrightarrow{\mathsf{P}} X(a) \quad \text{as } n \to \infty \text{ for each } a \in A.$$

Let $s \in [0, T]$ be some fixed time. We shall start with the construction of a random process $\varphi_{(s),t}(u, \omega)$, $t \in [s, T]$, measurable in (t, u, ω), satisfying (3.1), and continuous in t for all u and ω.

Without loss of generality we assume that $s = 0$. Let φ_t be the solution of (3.1) with $s = 0$. We shall construct a measurable version of φ_t by means of a limiting process, with the use of Theorem 5.1.

We set

$$\varphi_t^{(0)}(u, \omega) := u, \quad \varphi_t^{(k+1)}(u, \omega) := u + \int_0^t a_z(\varphi_z^{(k)}(u))\, dz + \sum_n \int_0^t \sigma_z^n(\varphi_z^{(k)}(u))\, d\omega_n(z).$$

We take for A the space $C([0, T], H)$ and verify the existence of an $\mathcal{F} \otimes \mathcal{B}(X)$-measurable version of the integral $\int_0^t \sigma_z^n(\varphi_z^{(k)}(u))\, d\omega_n(z)$ regarded as a random element taking values in $C([0, T], H)$.

Consider the following sequence of $C([0, T], H)$-valued $\mathcal{F} \otimes \mathcal{B}(X)$-measurable random elements:

$$\begin{aligned} \xi_t^{k,m,n}(u) = \sum_{j=0}^{l-1} \sigma_{jT/m}^{(n)}\big(\varphi_{jT/m}^{(k)}(u)\big) \left(\omega_n\left(\frac{(j+1)T}{m}\right) - \omega_n\left(\frac{jT}{m}\right)\right) \\ + \sigma_{lT/m}^{(n)}\big(\varphi_{lT/m}^{(n)}(u)\big) \left(\omega_n(t) - \omega_n\left(\frac{lT}{m}\right)\right), \end{aligned}$$

where l is an integer such that $t \in [lT/m, (l+1)T/m]$.

It is easy to verify that the $\xi^{k,m,n}_{\cdot}(u)$ converge in measure in $C([0,T],H)$ for each $u\in X$ with respect to P. Hence by Theorem 5.1 the integral $\int_0^{\cdot}\sigma_z^{(n)}(\varphi_z^{(k)}(u))\,d\omega_n(z)$ has a version measurable in (u,ω), regarded as a random element of $C([0,T],H)$. In a similar fashion one shows the existence of a measurable version of the sum

$$\sum_n\int_0^t\sigma_z^n(\varphi_z^{(k)}(u))\,d\omega_n(z).$$

For each $u\in X$ the sequence of iterates $(\varphi_t^{(k)}(u)-u)$ converges in measure to $(\varphi_t(u)-u)$ in the space $C([0,T],H)$ with respect to P. Applying Theorem 5.1 again we establish the existence of a measurable modification of $\varphi_{(s),t}(u)$ for each $s\in[0,T]$.

We consider now the $\mathcal{B}(X)\otimes\mathcal{F}$-measurable $C(\{(s,t),\ 0\leqslant s\leqslant t\leqslant T\},H)$-valued random element

$$\varphi_{s,t}^{(n)}=\begin{cases}\varphi_{(0),t} & \text{if } s\in[0,T/n],\\ \varphi_{((k-1)T/n),t} & \text{if } s=kT/n,\\ \varphi_{((k-1)T/n),t}\theta_n+\varphi_{(kT/n),t}(1-\theta_n) & \text{if } s\in(kT/n,(k+1)T/n),\end{cases}$$

where $\theta_n=(s-kT/n)n$.

For each $u\in X$ there exists a version of $\varphi_{s,t}(u)$ continuous in (s,t), $0\leqslant s\leqslant t\leqslant T$. Hence for each $u\in X$ we obtain

$$\sup_{0\leqslant s\leqslant t\leqslant T}\|\varphi_{s,t}^{(n)}(u,\omega)-\varphi_{s,t}(u,\omega)\|_H\to 0\quad\text{as } n\to\infty \text{ for almost all } \omega.$$

By Theorem 5.1 there exists a modification of $\varphi_{s,t}$ continuous in s, t and measurable in u, ω. A process with continuous trajectories is measurable, therefore the map $(s,t,u,\omega)\mapsto\varphi_{s,t}(u,\omega)$, $s\leqslant t$, $u\in X$, $\omega\in\Omega$, is measurable.

Remark. Using arguments similar to the above one can demonstrate the existence of a process $\varphi_{s,t}$, $0\leqslant s,t\leqslant T$, measurable in (s,t,u,ω) and satisfying (3.1$'$), (3.1$''$) for $s\leqslant t$ and all $u\in X$.

§ 6. Proof of Theorem 3.1

Let $\widehat{a}$ and $\widehat{\sigma}^n$ be H-Lipschitz modifications of a and σ^n, respectively (Lemma 2.2). We consider the stochastic equation

$$\varphi_{s,t}(u)=u+\int_s^t\widehat{a}_z(\varphi_{s,z}(u))\,dz+\sum_n\int_s^t\widehat{\sigma}_z^n(\varphi_{s,z}(u))\,d\omega_n(z).\qquad(6.1)$$

Assume that $\varphi_{s,t}$ satisfies the measurability conditions presented in § 5. Note that if $\mu\circ\varphi_{s,t}^{-1}\sim\mu$, then $\varphi_{s,t}$ is also a solution of (3.1); moreover, $\varphi_{s,t}(u)$ satisfies (3.1) for μ-almost all $u\in X$. Hence we shall assume throughout that $a=\widehat{a}$ and $\sigma^n=\widehat{\sigma}^n$. We claim that $\varphi_{s,t}(u)$ is the required process.

As in § 4, let $X=\mathbb{R}^\infty$, let μ be a product of standard Gaussian measures, and $H=l_2$. Let π_m be the projection onto the space of the first m variables, and π^m

the projection onto the space of the other variables (we shall regard π_m and π^m as operators into $\mathbb{R}^\infty$ or into l_2). We identify $\pi_m(\mathbb{R}^\infty)$ with $\mathbb{R}^m$ and each element $u \in \mathbb{R}^\infty$ with the pair $(u_m, u^m) = (\pi_m(u), \pi^m(u)) \in \mathbb{R}^m \times \pi^m(\mathbb{R}^\infty)$. We denote the σ-algebra generated by the first m coordinate functions by $\mathcal{F}_m$.

Consider now the functions

$$a^m = \mathsf{E}_\mu(\pi_m a/\mathcal{F}_m) \quad \text{and} \quad \sigma^{m,n} = \mathsf{E}_\mu(\pi_m \sigma^n/\mathcal{F}_m),$$

where $\mathsf{E}_\mu(\,\cdot\,/\mathcal{F}_m)$ is the conditional expectation in $(X, \mathcal{B}(X), \mu)$ with respect to $\mathcal{F}_m$.

It follows by Lemma 2.3 that the a^m and the $\sigma^{m,n}$, $n \geqslant 1$, satisfy the assumptions of Theorem 3.1 with the same constants. Moreover, for each $t \in [0,T]$ we have the convergence

$$\begin{aligned} D^i a_t^m &\to D^i a_t \quad \text{as } m \to \infty, \quad i = 0,1, \\ D^j \sigma_t^{m,n} &\to D^j \sigma_t^n \quad \text{as } m \to \infty, \quad j = 0,1,2, \end{aligned}$$

for μ-almost all $u \in X$.

Note that the functions a_t^m, $\sigma_t^{m,n}$ take values in $\mathbb{R}^m$ and depend only on the $\mathbb{R}^m$-variable, that is, for all $v_1, v_2 \in \pi^m(\mathbb{R}^\infty)$, $u_m \in \mathbb{R}^m$, and $t \in [0,T]$ we have

$$a_t^m(u_m, v_1) = a_t^m(u_m, v_2), \qquad \sigma_t^{m,n}(u_m, v_1) = \sigma_t^{m,n}(u_m, v_2).$$

Hence it will be sometimes convenient to regard a^m and $\sigma^{m,n}$ as functions from $[0,T] \times \mathbb{R}^m$ into $\mathbb{R}^m$.

We cannot yet apply Theorem 4.1 to the equation

$$d\varphi_{s,t}^m = a_t^m(\varphi_{s,t}^m)\,dt + \sum_n \sigma_t^{m,n}(\varphi_{s,t}^m)\,d\omega_n(t), \tag{6.2}$$

$$\varphi_{s,s}^m(u_m) = u_m, \quad u_m \in \mathbb{R}^m, \tag{6.3}$$

since for each $t \in [0,T]$ the functions a_t^m and $\sigma_t^{m,n}$, regarded as functions on $\mathbb{R}^m$, are not in the space C_b^2, but only in W_∞^2.

As is known, an element $f \in W_\infty^k(\mathbb{R}^m, \mu_m, \mathbb{R}^m)$, $\mu_m \sim \mathcal{N}(0, \mathrm{id}_{\mathbb{R}^m})$, has a modification with continuous derivatives of order $k-1$ such that $\nabla^{k-1} f$ is globally Lipschitz with constant not exceeding $\|f\|_{\infty,k}$.

We approximate a_t^m and $\sigma_t^{m,n}$ by functions with two continuous derivatives. We shall require the following well-known result on the approximation of a function by convolutions.

Lemma 6.1. *Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be a non-negative infinitely differentiable function and let* $\displaystyle\int_{\mathbb{R}^m} \varphi(x)\,dx = 1$. *Let $\varphi_n(x) = n^m \varphi(x/n)$, $x \in \mathbb{R}^m$, $n \geqslant 1$. Then for each bounded measurable function $f\colon \mathbb{R}^m \to \mathbb{R}^m$,*

$$f * \varphi_n \to f \text{ in measure as } n \to \infty \quad (\text{with respect to } \mu_m).$$

If f is uniformly continuous, then the $f * \varphi_n$ converge uniformly to f.

If $f \in W^k_\infty(\mathbb{R}^m, \mu_m, \mathbb{R}^m)$, *then*

$$\nabla^j (f * \varphi_n) \to \nabla^j f \quad \textit{as } n \to \infty, \quad \textit{for } j = 0, \dots, k-1$$

uniformly on $\mathbb{R}^m$ *and*

$$\nabla^k (f * \varphi_n) \xrightarrow{\mu_m} \nabla^k f \quad \textit{as } n \to \infty.$$

Remark. If $f \in W^k_\infty$, then for each $j \in \{0, \dots, k\}$ and $n \geqslant 1$,

$$\|\nabla^j (f * \varphi_n)\|_\infty \leqslant \|\nabla^j f\|_\infty.$$

Applying Cantor's diagonal procedure, Lebesgue's bounded convergence theorem, and Lemma 6.1 we can select a subsequence $\{k_m\}$ such that the functions $a^{m,k_m} = a^m * \varphi_{k_m}$ and $\sigma^{n,m,k_m} = \sigma^{n,m} * \varphi_{k_m}$ satisfy the following conditions:

$$\int_0^T \|a_t^{m,k_m} - a_t\|_H \, dt \to 0 \quad \text{as } m \to \infty \quad \mu\text{-a.s.}, \tag{6.4}$$
$$\int_0^T \|\sigma_t^{n,m,k_m} - \sigma_t^n\|_H^2 \, dt \to 0 \quad \text{as } m \to \infty \quad \mu\text{-a.s.}$$

Remark. Here we regard a^{m,k_m} and σ^{n,m,k_m} as functions on X.

Below we consider only the functions a^{m,k_m} and σ^{n,m,k_m} and do not consider a^m and $\sigma^{n,m}$. For that reason we shall drop the superscript k_m for convenience.

We shall verify that for all s, t, $0 \leqslant s \leqslant t \leqslant T$,

$$\mathsf{E}_{\mu \times \mathsf{P}} \|\varphi^m_{s,t} - \varphi_{s,t}\|_H^2 \to 0 \quad \text{as } m \to \infty, \tag{6.5}$$

where $\varphi^m_{s,t}$ is the solution of (6.2) with

$$\varphi^m_{s,s}(u) = u, \quad a^m_t(u) = (a^m_t(u_m), 0), \quad \sigma^{m,n}(u) = (\sigma^{m,n}(u_m), 0).$$

In fact,

$$\mathsf{E}_{\mu \times \mathsf{P}} \|\varphi^m_{s,t} - \varphi_{s,t}\|_H^2 \leqslant 2 \int_s^t \mathsf{E}_{\mu \times \mathsf{P}} \|a^m(\varphi^m_{s,z}) - a(\varphi_{s,z})\|_H^2 \, dz$$
$$+ 2 \int_s^t \sum_n \mathsf{E}_{\mu \times \mathsf{P}} \|\sigma^{m,n}(\varphi^m_{s,z}) - \sigma^n(\varphi_{s,z})\|_H^2 \, dz. \tag{6.6}$$

The first term in (6.6) has the estimate

$$2 \int_s^t \mathsf{E}_{\mu \times \mathsf{P}} \|a^m_z(\varphi^m_{s,z}) - a_z(\varphi^m_{s,z})\|_H^2 \, dz + 2 \int_s^t \mathsf{E}_{\mu \times \mathsf{P}} \|a_z(\varphi^m_{s,z}) - a_z(\varphi_{s,z})\|_H^2 \, dz$$
$$\leqslant 2 \int_s^t \mathsf{E}_{\mu \times \mathsf{P}} \big(\|a^m_z - a_z\|_H^2 \rho^m_{s,z}\big) \, dz + 2 \int_s^t \|D a_z\|_\infty^2 \mathsf{E}_{\mu \times \mathsf{P}} \|\varphi^m_{s,z} - \varphi_{s,z}\|_H^2 \, dz. \tag{6.7}$$

The first term on the right-hand side of (6.7) converges to zero uniformly in s, t in view of (6.4), the uniform boundedness of $\{a_t^m, a_t;\ t \in [0,T],\ m \geqslant 1\}$, and the uniform integrability of $\{\rho_{s,z}^m;\ m \geqslant 1,\ s,z \in [0,T]\}$. The second term on the right-hand side of (6.6) is treated in a similar way.

Hence we obtain the inequality

$$\mathsf{E}_{\mu\times\mathsf{P}}\|\varphi_{s,t}^m - \varphi_{s,t}\|_H^2 \leqslant \varepsilon_m + 2(A_1^2 + S_1)\int_s^t \mathsf{E}_{\mu\times\mathsf{P}}\|\varphi_{s,z}^m - \varphi_{s,z}\|_H^2\,dz,$$

where $\varepsilon_m \to 0$ as $m \to \infty$. The convergence (6.5) is a consequence of Gronwall's lemma.

By Lemma 4.1 and (6.5) the measure $(\mu\times\mathsf{P})\circ(\varphi_{s,t})^{-1}$ is absolutely continuous with respect to $\mu\times\mathsf{P}$. Hence for all $s \leqslant t$ and almost all $\omega\in\Omega$,

$$\mu\circ(\varphi_{s,t}(\omega,\cdot))^{-1} \ll \mu.$$

Remark. The corresponding set of P-measure zero depends in general on s and t.

To prove formulae (3.2), (3.3), and assertion (b) of Theorem 3.1 we shall require the following easy result on the limit of composite functions (see, for instance, [6]).

Lemma 6.2. *Let X and Y be complete separable metric spaces, μ a probability measure in X, and let $\varphi_n\colon X\to X$, $f_n\colon X\to Y$, $n\geqslant 0$, be measurable functions.*

Assume that

(i) $\varphi_n \xrightarrow{\mu} \varphi_0$, $n\to\infty$; $f_n \xrightarrow{\mu} f_0$ *as* $n\to\infty$;

(ii) *for each $n\geqslant 1$ the image of the measure $\mu\circ\varphi_n^{-1}$ is absolutely continuous with respect to μ;*

(iii) *the sequence of densities* $\left\{\dfrac{d\mu\circ\varphi_n^{-1}}{d\mu},\ n\geqslant 1\right\}$ *is uniformly integrable.*

Then $f_n\circ\varphi_n \xrightarrow{\mu} f_0\circ\varphi_0$ as $n\to\infty$.

We consider now the backward stochastic equation with reversed Wiener process $\widetilde{\omega}_n(t) = \omega_n(T) - \omega_n(t)$. Then similarly to the above arguments one can demonstrate the existence of a measurable 'inverse flow' $\varphi_{t,s}$, $s\leqslant t$, and the absolute continuity of $\mu\circ(\varphi_{t,s})^{-1}\ll\mu$ for almost all ω.

Using Lemma 6.2 and the properties of flows for stochastic differential equations in finite-dimensional spaces we obtain that

$$\begin{gathered}\text{for all } t_1,t_2,t_3\in[0,T] \quad \text{there exists } \Omega_{t_1,t_2,t_3} \quad \text{such that} \quad \mathsf{P}(\Omega_{t_1,t_2,t_3}) = 1 \\ \text{and} \quad \varphi_{t_2,t_3}\circ\varphi_{t_1,t_2} = \varphi_{t_1,t_3} \quad \mu\text{-a.s. for all } \omega\in\Omega_{t_1,t_2,t_3}.\end{gathered} \tag{6.8}$$

For the proof of (3.2), (3.3) we require an infinite-dimensional version of Lemmas 4.3 and 4.4.

Lemma 6.3. *The following results hold for all $s,t\in[0,T]$, $s\leqslant t$, and almost all ω:*

(a) *for each $p\geqslant 1$ the map $u\mapsto(\varphi_{s,t}(u,\omega)-u)$ belongs to the space $W_p^2(H)$,*

(b) *the derivatives*

$$D\varphi_{s,t}(u) := \mathrm{id}_H + D(\varphi_{s,t}(u) - u)$$

satisfy all the estimates in Lemma 4.4 with the same constants,

(c) *the derivative* $D\varphi_{s,t}$ *satisfies the relation*

$$D\varphi_{s,t} = \mathrm{id}_H + \int_s^t Da_z(\varphi_{s,z})D\varphi_{s,z}\,dz + \sum_n \int_s^t D\sigma_z^n(\varphi_{s,z})D\varphi_{s,z}\,d\omega_n(z) \tag{6.9}$$

for almost all $(\omega, u) \in \Omega \times \mathbb{R}^\infty$.

Proof. Note that by Lemma 4.4,

$$(\varphi_{s,t}^m(u) - u) \in W_p^2(H) \quad \text{for almost all } \omega,$$

and $\pi_m D(\varphi_{s,t}^m(u) - u) = D(\varphi_{s,t}^m(u) - u)$ because the function $u \mapsto \varphi_{s,t}^m(u,\omega) - u$ (as a function of $u \in X$ for fixed ω) depends only on u_m. For simplicity let $s = 0$ and let $\varphi_t := \varphi_{0,t}$. We shall prove the Cauchy property for the sequence $\{D(\varphi_t^m(u) - u), m \geqslant 1\}$. We apply Itô's formula to $\|D\varphi_t^m - D\varphi_t^k\|_{HS}^p$:

$$\begin{aligned}
&\mathsf{E}_{\mu\times\mathsf{P}}\|D\varphi_t^m - D\varphi_t^k\|_{HS}^p \\
&\quad\leqslant K_{.}\mathsf{E}_{\mu\times\mathsf{P}}\biggl\{\int_0^t \|D\varphi_s^m - D\varphi_s^k\|^{p-1}\|(Da_s^m)(\varphi_s^m)D\varphi_s^m - (Da_s^k)(\varphi_s^k)D\varphi_s^k\| \\
&\qquad + \sum_n \|D\varphi_s^m - D\varphi_s^k\|^{p-2}\|D\sigma_s^{m,n}(\varphi_s^m)D\varphi_s^m - D\sigma_s^{k,n}(\varphi_s^k)D\varphi_s^k\|^2\biggr\}\,ds \\
&\quad\leqslant K_{.}\mathsf{E}_{\mu\times\mathsf{P}}\int_0^t \|D\varphi_s^m - D\varphi_s^k\|^p\,ds \\
&\qquad + K_{.}\mathsf{E}_{\mu\times\mathsf{P}}\int_0^t \biggl(\|Da_s^m(\varphi_s^m) - Da_s^k(\varphi_s^k)\| \\
&\qquad + \sum_n \|D\sigma_s^{m,n}(\varphi_s^m) - D\sigma^{k,n}(\varphi_s^k)\|\biggr)\xi_{k,m}(s)\,ds \\
&\quad= I_1(m,k) + I_2(m,k),
\end{aligned} \tag{6.10}$$

where $K_{.} = K_{.}(p,T)$ is a constant and $\xi_{k,m}(s)$, $s \in [0,T]$, is a positive process (see Lemma 4.4) such that $\sup_{s,k,m} \mathsf{E}_{\mu\times\mathsf{P}}(\xi_{k,m}(s))^p < \infty$ for each $p > 0$. By Lemma 6.2 and Lebesgue's dominated convergence theorem $I_2(m,k) \to 0$ as $m,k \to \infty$.

Applying Gronwall's lemma to (6.10) we obtain

$$\mathsf{E}_{\mu\times\mathsf{P}}\|D\varphi_t^m - D\varphi_t^n\|_{HS}^p \to 0 \quad \text{as } m,n \to \infty. \tag{6.11}$$

We denote the limit of the $D\varphi_t^m$ by ψ_t. We have also already proved that

$$\mathsf{E}_{\mu\times\mathsf{P}}\|\varphi_t^m - \varphi_t\|^p \to 0 \quad \text{as } m \to \infty.$$

We now select a subsequence $\{m_k\}$ such that

$$\mathsf{E}_\mu\|\varphi_t^{m_k} - \varphi_t\|^p \to 0 \quad \text{as } m_k \to \infty$$

and

$$\mathsf{E}_\mu \|D\varphi_t^{m_k} - \psi_t\|^p \to 0, \qquad m_k \to \infty,$$

for all $\omega \in \Omega_t$, where Ω_t is a subset of full P-measure. Then $\psi_t = D\varphi_t$, the map $u \mapsto (\varphi_t(u) - u)$ belongs to $W_p^1(H)$, and the proof of assertion (b) is complete. To prove (c) we observe that $D\varphi_{s,t}^m$ satisfies equation (6.9) with the functions a^m and $\sigma^{m,n}$ in place of a and σ^n, respectively. The limiting process as $m \to \infty$ and the application of Lemmas 6.2 and 6.3 (a), (b) completes the proof of (c). The second-order derivatives are treated in a similar manner.

Lemma 6.4. *Assume that $b\colon X \to H$ belongs to the space $W_\infty^1(H)$. Then (4.9) holds for $\mu \times \mathsf{P}$-almost all (u, ω) if one replaces the operator ∇ by D.*

Remark. We assume here that $D\varphi_s(u) := \mathrm{id}_H + D(\varphi_s(u) - u)$. Note that $D\varphi_s$ is not a Hilbert–Schmidt operator, but $(D\sigma_z^n)(\varphi_z)D\varphi_z$ is, and therefore the stochastic operators and the trace are well defined since

$$\sum_n \int_0^T \mathsf{E}_{\mu\times\mathsf{P}} \|\nabla\sigma_z^n(\varphi_z)\nabla\varphi_z\|_{HS}^2 \, dz < \infty.$$

Proof. Let $\{b_m\colon X \to H,\ m \geqslant 1\}$ be a sequence of functions such that for all u and $b_m(u) = \pi_m b_m(\pi_m u)$,

$$\sup_{m,u} \|b_m(u)\| \leqslant \operatorname*{ess\,sup}_u \|b(u)\|, \qquad \sup_{m,u} \|Db_m(u)\| \leqslant \operatorname*{ess\,sup}_u \|Db(u)\|$$

and for μ-almost all $u \in X$ we have $b_m(u) \to b(u)$ and $Db_m(u) \to Db(u)$ as $m \to \infty$. This sequence can be selected similarly to $\{a^{m,n,k_m}\}$ at the beginning of the proof of Theorem 4.1.

Lemmas 6.2 and 6.3 yield the convergence

$$b_m(\varphi_s^m) \to b(\varphi_s) \quad \text{as } m \to \infty,$$
$$D(b_m(\varphi_s^m)) = (Db_m)(\varphi_s^m)D\varphi_s^m \to (Db)(\varphi_s)D\varphi_s \quad \text{as } m \to \infty$$

in each space L_p, $p \geqslant 1$. In particular,

$$b_m \to b \quad \text{as } m \to \infty \quad \text{in } W_2^1(H)$$

and

$$b_m(\varphi_s^m) \to b(\varphi_s) \quad \text{as } m \to \infty \quad \text{in } L_2(\Omega, \mathsf{P}, W_2^1(X, \mu)).$$

Hence $(\delta b^m)(\varphi_s^m) \to (\delta b)(\varphi_s)$ in measure with respect to $\mu \times \mathsf{P}$ and

$$\delta(b^m(\varphi_s^m)) \to \delta(b(\varphi_s)) \quad \text{in } L_2(X \times \Omega, \mu \times \mathsf{P}).$$

Applying Lemmas 6.2 and 6.3 to the other terms in (4.9) we complete the proof of Lemma 6.4.

We now prove (3.2) for all $\omega \in \Omega_{s,t}$, where $\Omega_{s,t}$ is a set of full P-measure. It is sufficient to verify (see Lemma 4.1) that

$$\mathsf{E}_{\mu\times\mathsf{P}}(\ln\rho_{s,t}^m - \ln\rho_{s,t})^2 \to 0 \quad \text{as } m \to \infty, \tag{6.12}$$

where $\rho_{s,t}$ is defined by formula (3.2).

We assume for simplicity that $t=0$. Let $\varphi_s:=\varphi_{0,s}$. Then

$$\begin{aligned}
&\mathsf{E}_{\mu\times\mathsf{P}}(\ln\rho^m_{s,0}-\ln\rho_{s,0})^2\\
&\quad\leqslant 4\mathsf{E}_{\mu\times\mathsf{P}}\int_0^t\biggl\{((\delta a_s)(\varphi_s)-(\delta a^m_s)(\varphi^m_s))^2\\
&\qquad+\sum_n\bigl(\delta(D_{\sigma^n_s}\sigma^n_s)(\varphi_s)-\delta(D_{\sigma^{m,n}_s}\sigma^{m,n}_s)(\varphi^m_s)\bigr)^2\\
&\qquad+\sum_n\bigl((\delta\sigma^n_s)(\varphi_s)-(\delta\sigma^{m,n}_s)(\varphi^m_s)\bigr)^2\\
&\qquad+\sum_n\bigl((D_{\sigma^n_s}(\delta\sigma^n_s))(\varphi_s)-(D_{\sigma^{m,n}_s}(\delta\sigma^{m,n}_s))(\varphi^m_s)\bigr)^2\biggr\}\,ds. \qquad (6.13)
\end{aligned}$$

The convergence to zero of the right-hand side of (6.13) as $m\to\infty$ can be verified with the help of (6.5), (6.11), and Lemmas 2.1, 6.2–6.4. Formula (3.3) for $s<t$ follows by (4.6), (3.2), and (6.8).

We claim that assertions (b)–(d) of Theorem 3.1 hold on some common subset Ω_0, $\mathsf{P}(\Omega_0)=1$, independent of t_1,t_2,t_3,s,t. It is sufficient to prove that there exist a modification of the function $\rho_{s,t}$ and a set $\Omega_1,\mathsf{P}(\Omega_1)=1$, such that for each $\omega\in\Omega$ the system of functions $\{\rho_{s,t}(\,\cdot\,,\omega),\ s,t\in[0,T]\}$, defined in (3.2), (3.3) is uniformly integrable with respect to μ, and $\rho_{s,t}(\,\cdot\,,\omega)$ is continuous in s, t in the topology of convergence in measure with respect to μ. In fact, assume that this holds. Recall that $\varphi_{s,t}(u,\omega)$ is continuous in s and t for all $\omega\in\Omega$ and $u\in X$. Let Ω_2 be a set such that $\rho_{s,t}(\,\cdot\,,\omega)$ defined in (3.2) coincides with the Radon–Nikodym density $\dfrac{d\mu\circ\varphi^{-1}_{s,t}}{d\mu}$ for all rational s and t. Note that $\mathsf{P}(\Omega_2)=1$. Then by Lemma 4.1, $\rho_{s,t}$ is the required Radon–Nikodym density for all $s,t\in[0,T]$ and $\omega\in\Omega_0=\Omega_1\cap\Omega_2$. Assertions (b) and (c) of Theorem 3.1 hold for almost all ω for rational t_1,t_2,t_3,s,t. Hence they hold for all t_1,t_2,t_3,s,t by Lemma 6.2.

To prove the existence of a modification of $\{\rho_{s,t}(\,\cdot\,,\omega),\ 0\leqslant s\leqslant t\}$ that is uniformly integrable with respect to μ for almost all ω it is sufficient to verify the existence of a version of

$$\mathsf{E}_\mu|\rho_{s,t}\ln\rho_{s,t}|=\mathsf{E}_\mu|\ln\rho_{t,s}|=\mathsf{E}_\mu\biggl|\int_s^t(\delta\widetilde{a}_z)(\varphi_{s,z})\,dz+\sum_n\int_s^t\delta\sigma^n_z(\varphi_{s,z})\circ d\omega_n(z)\biggr|$$

that is continuous in (s,t), $s\leqslant t$. We use Kolmogorov's theorem. For convenience assume that $0\leqslant s\leqslant s_1\leqslant t_1\leqslant t\leqslant T$. Then

$$\begin{aligned}
&\mathsf{E}_p(\mathsf{E}_\mu|\rho_{s,t}\ln\rho_{s,t}|-\mathsf{E}_\mu|\rho_{s_1,t_1}\ln\rho_{s_1,t_1}|)^6\\
&\quad\leqslant 10\sup_{z\geqslant s}\mathsf{E}_{\mu\times\mathsf{P}}\biggl\{\bigl((\delta\widetilde{a}_z)(\varphi_{s,z})\bigr)^6+\biggl(\sum_n D_{\sigma^n_z}(\delta\sigma^n_z)(\varphi_{s,z})\biggr)^6\biggr\}(|s-s_1|^6+|t-t_1|^6)\\
&\qquad+\sup_{s_1\leqslant z\leqslant t_1}\mathsf{E}_{\mu\times\mathsf{P}}\bigl((\delta a_z)(\varphi_{s,z})-(\delta a_z)(\varphi_{s_1,z})\bigr)^6\\
&\qquad+\sup_{s_1\leqslant z\leqslant t_1}\mathsf{E}_{\mu\times\mathsf{P}}\biggl(\sum_n\bigl((\delta\sigma^n_z)(\varphi_{s,z})-(\delta\sigma^n_z)(\varphi_{s_1,z})\bigr)^2\biggr)^3\\
&\quad=I_1+I_2+I_3.
\end{aligned}$$

The boundedness of the supremum of the expectation in I_1 is demonstrated similarly to the proof of Theorem 4.1. Our reasoning for I_2 and I_3 is almost identical. We content ourselves with the discussion of I_2. We use Lemma 6.4:

$$\begin{aligned}
&\mathsf{E}_{\mu\times\mathsf{P}}((\delta a_z)(\varphi_{s,z}) - (\delta a_z)(\varphi_{s_1,z}))^6 \\
&\quad\leqslant 3\biggl\{\mathsf{E}_{\mu\times\mathsf{P}}[\delta(a_z(\varphi_{s,z}) - a_z(\varphi_{s_1,z}))]^6 \\
&\quad + \mathsf{E}_{\mu\times\mathsf{P}}\biggl[\biggl\langle a_z(\varphi_{s,z}), \int_s^z a_\tau(\varphi_{s,\tau})\,d\tau + \sum_n \int_s^z \sigma_\tau^n(\varphi_{s,\tau})\,d\omega_n(\tau)\biggr\rangle \\
&\quad - \biggl\langle a_z(\varphi_{s_1,z}), \int_{s_1}^z a_\tau(\varphi_{s_1,\tau})\,d\tau + \sum_n \int_{s_1}^z \sigma_\tau^n(\varphi_{s_1,\tau})\,d\omega_n(\tau)\biggr\rangle\biggr]^6 \\
&\quad + \mathsf{E}_{\mu\times\mathsf{P}}\biggl[\operatorname{tr}\biggl((Da_z)(\varphi_{s,z})\biggl\{\int_s^z (Da_\tau)(\varphi_{s,\tau})D\varphi_{s,\tau}\,d\tau \\
&\quad + \sum_n \int_s^z (D\sigma_\tau^n)(\varphi_{s,\tau})D\varphi_{s,\tau}\,d\omega_n(\tau)\biggr\} \\
&\quad - (Da_z)(\varphi_{s_1,z})\biggl\{\int_{s_1}^z (Da_\tau)(\varphi_{s_1,\tau})D\varphi_{s_1,\tau}\,d\tau \\
&\quad + \sum_n \int_s^z (D\sigma_\tau^n)(\varphi_{s_1,\tau})D\varphi_{s_1,\tau}\,d\omega_n(\tau)\biggr\}\biggr)\biggr]^6\biggr\}.
\end{aligned} \tag{6.14}$$

Lemma 6.5. *For each $p \geqslant 2$ there exists a constant K_p such that for $j \in \{0,1,2\}$ and all s, s_1, z, $s < s_1 < z$,*

$$\mathsf{E}_{\mu\times\mathsf{P}}\|D^j\varphi_{s,z} - D^j\varphi_{s_1,z}\|^p \leqslant K_p|s - s_1|^{p/2}.$$

The proof proceeds as in the finite-dimensional case (see, for instance, [1]).

Applying Lemma 6.5 to the right-hand side of (6.14) and using Lemma 2.1 for the estimate of the moments of the divergence we obtain the following inequality:

$$\mathsf{E}_{\mathsf{P}}(\mathsf{E}_\mu|\rho_{s,t}\ln\rho_{s,t}| - \mathsf{E}_\mu|\rho_{s_1,t_1}\ln\rho_{s_1,t_1}|)^6 \leqslant C(|s-s_1|^3 + |t-t_1|^3), \quad s \leqslant t, \ \ s_1 \leqslant t_1.$$

Hence by Kolmogorov's theorem $\mathsf{E}_\mu|\rho_{s,t}\ln\rho_{s,t}|$ is continuous in (s,t) and therefore $\sup_{s\leqslant t}\mathsf{E}_\mu|\rho_{s,t}\ln\rho_{s,t}| < \infty$ for almost all ω. Consequently, for almost all ω and all $s \leqslant t$ we have the absolute continuity $\mu\circ\varphi_{s,t}^{-1} \ll \mu$.

In a similar fashion, by considering the backward stochastic equation we obtain $\mu\circ\varphi_{t,s}^{-1} \ll \mu$ for $s \leqslant t$ and also the uniform integrability of $\biggl\{\dfrac{d\mu\circ\varphi_{t,s}(\,\cdot\,,\omega)^{-1}}{d\mu},\ s \leqslant t\biggr\}$ for almost all ω. Assertion (b) of Theorem 3.1 is thus established. To prove (3.2) and (3.3) for almost all ω and all s and t it is sufficient to observe that for almost all ω the function $\{\rho_{t,s}(\,\cdot\,,\omega),\ s \leqslant t\}$ is continuous in s, t in the topology of convergence in measure with respect to μ.

The proof of Theorem 3.1 is now complete.

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