STATIONARY MEASURE-VALUED PROCESSES GENERATED BY A FLOW OF INTERACTED PARTICLES *

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Abstract. Let $x(u,t), t \in \mathbb{R}, u \in \mathbb{R}^d$ be a stationary stochastic flow in $\mathbb{R}^d$ that generated by a system of SDEs with interaction. Here $x(u,t)$ is a position of a particle indexed by $u \in \mathbb{R}$ in a moment of time $t \in \mathbb{R}$. Let ν be an absolute continuous measure in $\mathbb{R}^d$. Consider a stationary measure-valued process $\nu_t(\omega) = \nu \circ x(\cdot, t)^{-1}$. The conditions which ensure that $\nu_t(\omega)$ is absolute continuous w.r.t. the Lebesgue measure are studied.

Let $x(u,t)$ be a stationary solution of the following system of SDE's with interaction:

$$\begin{aligned} dx(u,t) = &- \alpha(x(u,t) - u)dt + \int_{\mathbb{R}^d} a(x(u,t), x(v,t))\mu(dv)dt + \\ &+ \sum_{k=1}^{m} b_k(x(u,t))dw_k(t), \end{aligned} \tag{1.1}$$

where $u \in \mathbb{R}^{2d}, \alpha > 0, a : \mathbb{R}^d \to \mathbb{R}^d, b_k : \mathbb{R}^d \to \mathbb{R}^d, \mu$ is a probability measure on $\mathbb{R}^d, \omega_1(t), \dots, \omega_m(t), t \in \mathbb{R}$ are independent one-dimensional Wiener processes given on a probability space $(\Omega, \mathcal{F}, P)$.

The value of $x(u,t)$ is identified with a position of the particle indexed by $u \in \mathbb{R}^d$ in a moment of time $t \in \mathbb{R}$. The dynamics of random flow $x(u,t)$ is the following. The flow attracts to identity flow $\bar{x}(u,t) \equiv u$ (see term $-\alpha(x(u,t) - u)$), there is some interaction between particles described by function a and measure μ, each particle is randomly perturbed.

Let ν be an absolute continuous probability measure on $\mathbb{R}^d$. Consider the evolution of ν driven by the flow $x(u,t)$. Put $\nu_t(\omega) := \nu \circ (x(\cdot, t, \omega))^{-1}, \nu_t$ is a random measure-valued process.

We study conditions that ensures the absolute continuity of $\nu_t(\omega)$ for a.a.ω.

The corresponding result in a case that "close" to one-dimensional one was researched in [1]. A case of additive noise (i.e. b_k = const) was considered in [2]. Moreover, conditions of [2] imply that if initial measure ν has a smooth density then $\nu_t(\omega)$ also possesses smooth density. In §3 we show that, generally, this is not true if $b_k \neq$ const.

1991 *Mathematics Subject Classification.* 60K35, 60H99.

Key words and phrases. Stationary measure-valued processes, Stochastic flows, Absolute continuity.

* This article was supported (in part) by Ministry of Education and Science of Ukraine, project N 01.07/103.

§2. Main results

Definition 2.1. A process $x(u,t)$ is said to be a solution of the equation (1.1) if

1) $x = x(u,t,\omega)$ is measurable in a triple (t,u,ω);
2) for all $u \in \mathbb{R}^d, s,t \in \mathbb{R}, s \le t:$

$$x(u,t) = x(u,s) + \int_s^t \left(-\alpha(x(u,z) - u) + \int_{\mathbb{R}^d} a(x(u,z), x(v,z))\mu(dv) \right) dz$$
$$+ \sum_{k=1}^n \int_s^t b_k(x(u,z))dw_k(z);$$

3) $E\int_{\mathbb{R}^d} \|x(u,t)\|^2\mu(du) < \infty, t \in \mathbb{R}.$

Remark. Condition 3) is technical. The function a will satisfy linear growth condition. So we demand 3) to ensure the existence of integrals.

Definition 2.2. A process $x(u,t), u \in \mathbb{R}^d, t \in \mathbb{R}$ is called stationary if for all $n \in \mathbb{N}, \delta \in \mathbb{R}, u_1, \dots, u_n \in \mathbb{R}^d, t_1, \dots, t_n \in \mathbb{R}$ the distribution of random vectors $(x(u_1,t_1), \dots, x(u_n,t_n))$ and $(x(u_1,t_1+\delta), \dots, x(u_n,t_n+\delta))$ coincides.

The main result of the article is the following theorem.

Theorem 2.1. *Assume that*

1) *probability measure μ has all positive moments;*
2) *function a is twice continuously differentiable in the first argument and bounded;*
3) *functions $b_k, k = 1, \dots, m$ are twice continuously differentiable and first derivative is globally bounded;*
4) *a function a satisfy Lipschitz condition:*

$$\exists L\ \forall u_1, u_2, v_1, v_2 \in \mathbb{R}^d:\ \|a(u_1,v_1) - a(u_2,v_2)\| \le L(\|u_1 - u_2\| + \|v_1 - v_2\|)$$

5) *for each $x \in \mathbb{R}^d$ the rank of the matrix $\|b_k(x), \nabla b_k(x)\|_{k=\overline{1,m}}$ is equal to $d + d^2$;*
6) $\alpha > 2\sup_{x,y} \|\partial_x a(x,y)\| + 2\sum_k \sup_x \|\nabla b_k(x)\|^2$.

Then there exists the unique stationary solution of (1.1) and for each finite measure $\nu \ll \lambda^d$, for each $t \in \mathbb{R}$ the absolute continuity

$$\nu_t(\omega) = \nu \circ (x(\cdot,t,\omega))^{-1} \ll \lambda^d$$

holds true for a.a.ω.

Proof. The existence and uniqueness is proved in [1]. Moreover, there exists a modification of $x(u,t)$ which is continuously differentiable in a parameter $u \in \mathbb{R}^d$ and this derivative $y(u,t) := \frac{\partial x(u,t)}{\partial u}$ is the unique stationary solution of the equation:

$$dy(u,t) = -\alpha(y(u,t) - \mathbb{I})dt + \int_{\mathbb{R}^d} a_1'(x(u,t), x(v,t))\mu(dv)y(u,t)dt + \tag{2.1}$$
$$+ \sum_{k=1}^m \nabla b_k(x(u,t))y(u,t)dw_k(t),$$

where $\mathbb{I}$ is identity $d \times d$ matrix.

Remark. Note that if x_1 is some modification of x, i.e.

$$\forall u \in \mathbb{R}^d\, \forall t \in \mathbb{R}: x_1(u,t) = x(u,t) \text{ a.e.}$$

then $\nu \circ (x_1(\cdot,t))^{-1} = \nu \circ (x(\cdot,t))^{-1}$ a.e. So, properties of a process ν_t does not depend on a modification of $x(u,t)$. Further, only continuous modification of $x(u,t)$ is considered.

To prove the theorem we need the following well-known statement.

Proposition 2.1. *Assume that $\nu \ll \lambda^d$ is a finite measure in $\mathbb{R}^d, f : \mathbb{R}^d \to \mathbb{R}^d$ is differentiable mapping. If $\det \|\nabla f(x)\| \neq 0$ for ν-a.a. $x \in \mathbb{R}^d$ then the image-measure $\nu \circ f^{-1}$ is absolute continuous w.r.t. λ^d.*

So, to prove the theorem it suffices to show that for all $u \in \mathbb{R}^d, t \in \mathbb{R}$ the matrix $y(u,t)$ is not degenerative a.s. To do this we will verify that the distribution of $y(u,t)$ in $\mathbb{R}^{d^2}$ is absolute continuous with respect to λ^{d^2}.

Proposition 2.2. *Suppose that assumptions 1) - 5) of the Theorem 2.1 are satisfied. Let $x(u,t), y(u,t)$ be solutions of (1.1), (2.1) , $t \in [0,T]$ with initial conditions:*

$$x(u,0) = \varphi(u), \;\; y(u,0) = \psi(u), \tag{2.2}$$

where φ, ψ are continuous differentiable and

$$\int_{\mathbb{R}^d} (\|\varphi(u)\|^2 + \|\psi(u)\|^2)\mu(du) < \infty. \tag{2.3}$$

Assume that

$$\det \|\psi(u_0)\| \neq 0. \tag{2.4}$$

Then for each $t \in [0,T]$ the distribution of the pair $(x(u_0,t), y(u_0,t))$ in $\mathbb{R}^d \times \mathbb{R}^{d^2} = \mathbb{R}^{d+d^2}$ is absolute continuous w.r.t. λ^{d+d^2}.

Corollary. *If conditions of the Proposition 2.2 are fulfilled. Then for each $t > 0$ the distribution of $y(u_0,t)$ is absolute continuous w.r.t. λ^{d^2}. In particular, $\det y(u_0,t) \neq 0$ a.s.*

Proof. Let $R > 0$ be fixed, $\tau = \tau(R)$ be a moment of exit of a process $x(u_0,t)$ from the ball with a radius R and a center in zero. Due to condition 2,3 and 5 of the Theorem 2.1 and continuity in t of trajectories of $(x(u_0,t), y(u_0,t))$ there exists a system of $\mathcal{F}_t = \sigma(\omega_k(s), k = 1,\dots,m, s \in [0,t])$ adapted processes $\alpha_k(t), 1 \le k \le m, t \in [0,T]$ such that

$$\sum_{k=1}^m \nabla^j b_k(x(u_0,t))\alpha_k(t) = -\int_{\mathbb{R}^d} [(\partial_1)^j a](x(u_0,t), x(v,t))\mu(dv), j \in \{0,1\}, t \in [0, \tau(R) \wedge T]$$

and

$$\sup_{\omega} \max_{k} |\alpha_k(t)| < \infty, t \in [0, \tau(R) \wedge T].$$

Remark. Generally, the choice of processes $\alpha_k(t), t = 1,\dots,m$ is not unique.

By Girsanov theorem the processes

$$\bar{\omega}_k(t) = \omega_k(t) - \int_0^{t \wedge \tau(R)} \alpha_k(z)dz, t \in [0,T]$$

are independent Wiener processes w.r.t. the measure

$$d\tilde{P} = \exp\left\{ \sum_{k=1}^m \left(\int_0^{T\wedge\tau(R)} \alpha_k(s)dw_k(s) - \frac{1}{2}\int_0^{T\wedge\tau(R)} \alpha_k^2(s)ds \right) \right\} dP.$$

Observe that for all $t \in [0, \tau(R)]$:

$$x(u_0, t) = \bar{x}(t),\ y(u_0, t) = \bar{y}(t),$$

where $(\bar{x}(t), \bar{y}(t))$ is a solution of a system

$$\begin{cases} d\bar{x}(t) = -\alpha(\bar{x}(t) - u_0)dt + \sum_{k=1}^m b_k(\bar{x}(t))d\bar{w}_k(t) \\ d\bar{y}(t) = -\alpha(\bar{y}(t) - \mathbb{I})dt + \sum_{k=1}^m \nabla b_k(\bar{x}(t))\bar{y}(t)d\bar{w}_k(t) \\ \bar{x}(0) = \varphi(u_0) \\ \bar{y}(0) = \psi(u_0) \end{cases}$$

It is well-known [3] that if coefficient of SDE are smooth, have linear growth and diffusion matrix is not degenerative at the initial moment of time $t_0 = 0$ then the distribution of a solution in each moment $t > 0$ has a density.

Due to condition 5) of the Theorem 2.1 and (2.4):

$$d^2 + d = \mathrm{rank}\|b_k(u_0), \nabla b_k(u_0)\|_{k=\overline{1,m}} = \mathrm{rank}\|b_k(u_0), \nabla b_k(u_0)\psi(u_0)\|_{k=\overline{1,m}}.$$

So the distribution of $(\bar{x}(t), \bar{y}(t))$ is absolute continuous. Consider $(x(u_0, t), y(u_0, t))$ as a function on a space Ω with a measure $\mathbb{I}_{\{t \le \tau(R)\}}dP$. Its distribution is absolute continuous also. Observe that $\tau(R) \to \infty$ as $R \to \infty$ a.s. Therefore the distribution of $(x(u_0, t), y(u_0, t))$, considered in a space Ω with a measure P, has a density and the Proposition 2.2 is proved.

Let us continue the proof of the Theorem 2.1 Let $x(u, t), y(u, t)$ be stationary solution of (1.1), (2.1). Observe that
$\forall t_1, t_2, t_1 > t_1 \forall u \in \mathbb{R}^d$:

$$P(\det y(u, t_2) \neq 0 / \det y(u, t_1) \neq 0) = 1$$

by the Corollary to the Proposition 2.2.

Remark. Markov property for stochastic flows, which generated by SDE with interaction, was shown in [7]

The stationarity of $y(u, t)$ implies that the probability $p(u) = P(\det y(u, t) = 0)$ does not depend on t. By complete probability formula

$$\forall t_1 < t_2\, \forall u \in \mathbb{R}^d :$$

$$1 - p(u) = P(\det y(u, t_2) \neq 0) =$$

$$= P(\det y(u, t_2) \neq 0 / \det y(u, t_1) \neq 0)P(\det y(u, t_1) \neq 0) +$$

$$+ P(\det y(u, t_2) \neq 0 / \det y(u, t_1) = 0)P(\det y(u, t_1) = 0) =$$

$$= 1 - p(u) + P(\det y(u, t_2) \neq 0 / \det y(u, t_1) = 0)p(u) \tag{2.5}$$

Observe that

$$P(\det y(u, t_2) \neq 0 / \det y(u, t_1) = 0) =$$

$$= \int_{\{(\phi,\psi)|\det\psi(u)=0)\}} P(\det y(u,t) \neq 0/y(\cdot,0) = \psi(\cdot), x(\cdot,0) = \phi) P_0(d\phi, d\psi),$$

where P_0 is a distribution of $x(\cdot,0), y(\cdot,0), t = t_2 - t_1$.

Remark. Of course only continuous differentiable functions ϕ, ψ are considered, that satisfy (2.3). Note that for a.a. ω functions $x(\cdot,t,\omega)$ and $y(\cdot,t,\omega)$ satisfy this condition.

Analogously to [1] it can be proved that for each pair (ϕ,ψ) the integrand converges to $P(\det y(u,t) \neq 0)$ as $t \to \infty$.

Due to (2.5), to prove $p(u) = 0$ and hence a Theorem 2.1, it is sufficient to show that

$$\forall u \in \mathbb{R}^d \ \forall t \ : P(\det y(u,t) \neq 0) > 0.$$

Proposition 2.3. *Let $x(u,t), y(u,t)$ be solutions of (1.1), (2.1) with initial conditions (2.2), which satisfy (2.3). Then for each $u \in \mathbb{R}^d$ there exists $t_0 > 0$ and $\varepsilon > 0$ such that*

$$\forall t \geq t_0: \ E\|y(u,t) - \mathbb{I}\|^2 < 1 - \varepsilon. \tag{2.7}$$

Remark. If (2.7) holds then for stationary solution $y(u,t)$ of (2.1) the inequality $E\|y(u,t) - \mathbb{I}\|^2 \leq 1 - \varepsilon < 1$ is satisfied. Therefore $\|y(u,t) - \mathbb{I}\| < 1$ with positive probability and so $y(u,t)$ is invertible with positive probability.

Proof. Write the differential for $\|y(u,t) - \mathbb{I}\|^2$, where $\|\cdot\|$ is Hilbert-Schmidt norm.

$$\begin{aligned} d\|y(u,t) - \mathbb{I}\|^2 &= -2\alpha\|y(u,t) - \mathbb{I}\|^2 dt + \\ &+ 2\left\langle \int_{\mathbb{R}^d} a_1'(x(u,t), x(v,t))\mu(dv) y(u,t), y(u,t) - \mathbb{I} \right\rangle dt + \\ &+ \sum_{k=1}^m \|\nabla b_k(x(u,t)), y(u,t)\|^2 dt + dM_t, \end{aligned}$$

where M_t is a martingale.

So

$$\begin{aligned} E\|y(u,t) - \mathbb{I}\|^2 &\leq E\|y(u,0) - \mathbb{I}\|^2 + \\ &+ \int_0^t \left(-2\alpha + 2\sup_{x,y}(\|a_1'(x,y)\| + \sum_{k=1}^m \|b_k(x)\|^2) \right) \|y(u,s) - \mathbb{I}\|^2 ds + \\ &+ \int_0^t 2\sup_x \sum_k \|\nabla b_k(x)\|^2 + 2\sup_{x,y}\|a_1'(x,y)\| E\|y(u,s) - \mathbb{I}\| ds. \end{aligned}$$

Set

$$\gamma(t) := \sqrt{E\|y(u,t) - \mathbb{I}\|^2}, a := 2\sup_{x,y}\|a_1'(x,y)\|, b := 2\sup_x \sum_{k=1}^m \|\nabla b_k(x)\|^2.$$

Then

$$\gamma^2(t) \leq \gamma^2(0) + 2\int_0^t ((-\alpha + a + b)\gamma^2(s) + a\gamma(s) + b) ds.$$

The value of a function $f(x) = (-\alpha + a + b)x^2 + ax + b$ in the point $x = 1$ is negative. Moreover $f(x) < f(1)$ if $x > 1$. So the inequality $\gamma^2(t) \geq 1$ implies $f(\gamma(t)) \leq f(1) < 0$.

Thus there exists $t_0 \in \left(0; \frac{\gamma^2(0)}{-2f(1)}\right)$ and $\varepsilon > 0$ such that $\gamma(t) < 1 - \varepsilon$ for all $t \geq t_0$, i.e. $E\|y(u,t) - \mathbb{I}\|^2 < 1 - \varepsilon$.

Proposition 2.3 and Theorem 2.1 are proved.

§3. Example of erratic behavior

It was shown in §2 that if $x(u,t)$ is solution of (1.1) with initial condition $x(u,0) = \varphi(u)$, where $\det \nabla\varphi(u) \neq 0$ for λ^d-a.a $u \in \mathbb{R}^d$ and $\nu \ll \lambda^d$, then under some additional assumptions

$$\forall t > 0: \ \nu_t \ll \lambda^d \ P\text{-a.s.}$$

It is natural to expect that if coefficients of the equation (1.1) are C^∞, $\det \nabla\varphi(u) \neq 0, u \in \mathbb{R}^d$ and ν has C^∞-density then ν_t has also C^∞ density. This supposition is true in one dimensional case [1] ($d = 1$) or in a case of additive noise [2], i.e. $b_k = \text{const}$. The purpose of this section is to present one counterexample in the case $b_k \neq \text{const}$. Consider the analogue of (1.1) without interaction

$$\begin{aligned} dx(u,t) = &- \alpha(x(u,t) - u)dt + a(x(u,t))dt + \\ &+ \sum_{k=1}^m b_k(x(u,t)) \circ dw_k(t), \ t > 0 \end{aligned} \tag{3.1}$$

with identity initial condition

$$x(u,0) = u, \ u \in \mathbb{R}^d. \tag{3.2}$$

All functions $a, b_k : \mathbb{R}^d \to \mathbb{R}^d$ are supposed to be infinitely differentiable and bounded with all their derivatives. There exists a modification [4] of $x(u,t)$ which is infinitely differentiable in u and $\partial_u^{(k)} x(u,t)$ is continuous in (u,t) for each $k \geq 0$. Assume that the conditions of the Proposition 2.2 are satisfied. Then

$$\det \frac{\partial x(u,t)}{\partial u} \neq 0 \ \ \lambda^d\text{-a.a} \ \ u \in \mathbb{R}^d, \text{a.a.}\omega.$$

So if $\frac{d\nu}{d\lambda^d} = p \in C(\mathbb{R}^d)$ then

$$\frac{d\nu_t(u)}{d\lambda^d} = \sum_{v:x(v,t)=u} p(v) \left| \det \frac{\partial x(v,t)}{\partial v} \right|^{-1}.$$

Suppose that p is a positive function. If there exists $u \in \mathbb{R}^d$ such that $\det \frac{\partial x(u,t)}{\partial u} = 0$ then $\frac{d\nu_t}{d\lambda^d}$ cannot be even continuous. So we construct our counterexample to conjecture $p \in C^\infty \implies \frac{d\nu_t}{d\lambda^d} \in C^\infty$ if we find some conditions to ensure

$$\forall t_0 > 0: \ P\left(\exists u \in \mathbb{R}^d : \det \frac{\partial x(u,t_0)}{\partial u} = 0\right) > 0. \tag{3.3}$$

Remark. A set $\{\exists u \in \mathbb{R}^d : \det \frac{\partial x(u,t_0)}{\partial u} = 0\}$ is measurable because $\frac{\partial x(u,t_0)}{\partial u}$ is continuous in u.

Remark. If we have homogeneous in space equation

$$\begin{aligned} &dx(u,t) = a(x(u,t))dt + \sum_{k=1}^m b_k(x(u,t)) \circ dw_k(t) \\ &x(u,0) = u \end{aligned}$$

then [4,5] $x(u,t)$ generates a flow of diffeomorphisms. Obviously, in this situation $p \in C^\infty$ implies $\frac{d\nu_t}{d\lambda^d} \in C^\infty$ a.s.

Proposition 3.1. *Assume that $a, b_k \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and*

$$\forall x, y \in \mathbb{R}^d\, x \neq y,\ \operatorname{rank}\|b_k(x), \nabla b_k(x), b_k(y), \nabla b_k(y)\|_{k=\overline{1,m}} = 2d + 2d^2.$$

Then (3.3) holds true.

Proof. Let us verify that

$$\forall t > 0 :\ P(\exists u_1 \neq u_2 : x(u_1, t) = x(u_2, t)) > 0. \tag{3.4}$$

Observe that (3.4) implies (3.3). Really, assume that

$$x(u_1, t) = x(u_2, t),\ \det \frac{\partial x(u_1, t)}{\partial u} \det \frac{\partial x(u_2, t)}{\partial u} > 0.$$

Let $\hat{\mathbb{R}}^d \equiv \mathbb{R}^d \cup \{\infty\}$ be a one point compactification of $\mathbb{R}^d$. Analogously to [4] (see Lemmas 4.5.6, 4.5.8 and a reasoning on a p.161) it can be proved that for a.a. ω a mapping $\hat{x}(\cdot, \cdot, \omega) : \hat{\mathbb{R}}^d \times [0, \infty) \to \hat{\mathbb{R}}^d$ defined as following

$$\hat{x}(u, t) = x(u, t), u \in \mathbb{R}^d$$

$$\hat{x}(\infty, t) = \infty, u = \infty$$

is continuous in (u, t). So a function $u \to x(u, t)$ is homotopic to $u \to x(u, 0) = u$, i.e. the identity map. Thus the topological degree of a point $x(u_1, t) = x(u_2, t)$ is equal to one. So there exists $u_3 \in \mathbb{R}^d$ such that $x(u_3, t) = x(u_1, t)$ and

$$\det \frac{\partial x(u_3, t)}{\partial u} \det \frac{\partial x(u_1, t)}{\partial u} < 0.$$

A function $\det \frac{\partial x(u,t)}{\partial u}$ is continuous in u. Hence

$$\exists u :\ \det \frac{\partial x(u, t)}{\partial u} = 0$$

and (3.3) is a consequence of (3.4).

Let us prove (3.4).

Let $\varphi = (\varphi_1, \dots, \varphi_m) \in C([0, \infty), \mathbb{R}^m)$.

Consider the equations:

$$\frac{\partial x^\varphi(u, t)}{\partial t} = -\alpha(x^\varphi(u, t) - u) + a(x^\varphi(u, t)) + \sum_{k=1}^m b_k(x^\varphi(u, t))\varphi_k(t), \tag{3.5}$$

$$\begin{aligned} \frac{\partial y^\varphi(u, t)}{\partial t} &= -\alpha(y^\varphi(u, t) - \mathbb{1}) + \nabla a(x^\varphi(u, t))y^\varphi(u, t) + \\ &+ \sum_{k=1}^n \nabla b_k(x^\varphi(u, t))y^\varphi(u, t)\varphi_k(t), \end{aligned} \tag{3.6}$$

$$x^\varphi(u, 0) = u,\ y^\varphi(u, 0) = \mathbb{1}.$$

Let $t_0 > 0$ be fixed, $u_1, u_2 \in \mathbb{R}^d, u_1 \neq u_2$. Find a piece wise smooth function φ such that

$$\det y^{\varphi}(u_1, t_0) = \det y^{\varphi}(u_2, t_0) = \mathbb{I},$$

$$x^{\varphi}(u_1, t_0) = x^{\varphi}(u_2, t_0).$$

The choice of such function can be done by the following way. At first choose a point $x_0 \in \mathbb{R}^d$ such that

$$\alpha(x_0 - u_1) + a(u_1) \neq \alpha(x_0 - u_2) + a(u_2).$$

Then consider backward equation for (3.5), (3.6) with $\varphi \equiv 0$ and initial condition:

$$x^{\varphi}(u_1, t_0) = x^{\varphi}(u_2, t_0) = x_0,$$
$$y^{\varphi}(u_1, t_0) = y^{\varphi}(u_2, t_0) = \mathbb{I}.$$

Assume that $(t_0 - t_1)$ is small enough to guarantee the existence of a smooth curve

$$\psi(t) = (\psi^{x_1}(t), \psi^{y_1}(t), \psi^{x_2}(t), \psi^{y_2}(t)), t \in [0, t_1]$$

such that
(i) it connects points $(u_1, \mathbb{I}, u_2, \mathbb{I})$ and $(x^{\varphi}(u_1, t_1), y^{\varphi}(u_1, t_1), x^{\varphi}(u_2, t_1), y^{\varphi}(u_2, t_1))$,
(ii) for each $t \in [0, t_1]$ matrixes $\psi^{y_1}(t)$ and $\psi^{y_2}(t)$ are not degenerative,
(iii) $\forall t \in [0, t_1] \ : \ x^{\varphi}(u_1, t) \neq x^{\varphi}(u_2, t), \ y^{\varphi}(u_1, t) \neq y^{\varphi}(u_2, t).$

It is not difficult to show the existence of piece-wise smooth function $\varphi(t), t \in [0, t_1]$ such that

$$\psi^{x_j}(t) = x^{\varphi}(u_j, t), \psi^{y_j}(t) = y^{\varphi}(u_j, t), j \in \{1, 2\}$$

where x^{φ}, y^{φ} are defined by (3.4), (3.5).

By theorem on support of stochastic flows [4,6]:
$\forall R > 0 \, \forall \delta > 0$

$$P\left(\sup_{\|u\| \le R} \big(\|x(u, t_0) - x^{\varphi}(u, t_0)\| + \|y(u, t_0) - y^{\varphi}(u, t_0)\| \big) < \delta \right) > 0. \tag{3.7}$$

Remark. Only homogeneous in space equations were considered in [4,6], but all reasoning can be extended to our situation.

Let $R > \max(\|u_1\|, \|u_2\|)$. The equalities

$$x^{\varphi}(u_1, t_0) = x^{\varphi}(u_2, t_0), \ \frac{\partial x^{\varphi}(u_j, t_0)}{\partial u} = y^{\varphi}(u_j, t_0) = \mathbb{I}$$

and (3.7) imply:

$$\forall \varepsilon > 0 \ P\big(\exists \tilde{u}_1, \tilde{u}_2, |\tilde{u}_j - u_j| < \varepsilon, j = 1, 2 : x(\tilde{u}_1, t_0) = x(\tilde{u}_2, t_0) = x^{\varphi}(u_1, t_0)\big) > 0.$$

The inequality (3.4) and Proposition 3.1 are proved.

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