Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 2, 691–698

Nonlinear Schrödinger Equations for Identical Particles and the Separation Property

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We investigate the separation property for hierarchies of Schrödinger operators for identical particles. We show that such hierarchies of translation invariant second order differential operators are necessarily linear. A weakened form of the separation property, related to a strong form of cluster decomposition, allows for homogeneous hierarchies of nonlinear differential operators. Some connection with field theoretic formalisms in Fock space are pointed out.

1 Introduction

In [1] we studied hierarchies of N-particle Schrödinger equations that satisfy the separation property. By this we mean that product functions evolve as product functions. The separation property was considered as a nonlinear version of the notion of non-interacting systems, as then uncorrelated states remain uncorrelated under time evolution. The motivation for studying such hierarchies came from concrete examples of nonlinear Schrödinger equations arising in problems of representations of the diffeomorphism group.

The hierarchies of Schrödinger operators that one encounters in such evolution equation satisfies a property that we called *tensor derivation* as the characteristic property is formally a derivation with respect to the tensor product of wave functions.

$$F_n(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_p) = F_{n_1}(\psi_1) \otimes \psi_2 \otimes \cdots \otimes \psi_p \\ + \psi_1 \otimes F_{n_2}(\psi_2) \otimes \cdots \otimes \psi_p + \cdots + \psi_1 \otimes \psi_2 \otimes \cdots \otimes F_{n_p}(\psi_p), \qquad (1)$$

where the F_m are m-particle operators, the ψ_k are n_k-particle wave function and $n = n_1 + \cdots + n_p$. Tensor derivations were fully classified in [1]. Canonical decompositions and constructions were also presented.

The analysis in [1] is incomplete in several aspects. One most apparent is that there one only considered N-particle systems in which the particles were all of different species. Thus there was no need to consider symmetric or antisymmetric wave functions. Since the world is made of bosons and fermions, one should reconsider the whole question for systems of identical particles. The tensor derivation property (1) must then be reformulated not with respect to the simple tensor product

$$\phi \otimes \psi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = \phi(x_1, \ldots, x_n)\psi(x_{n+1}, \ldots, x_{n+m})$$

of two wave functions, but with respect to the symmetric or anti-symmetric tensor product

$$\phi \hat{\otimes} \psi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \\ = \frac{1}{n!m!} \sum_{\pi} (-1)^{fs(\pi)} \phi(x_{\pi(1)}, \ldots, x_{\pi(n)}) \psi(x_{\pi(n+1)}, \ldots, x_{\pi(n+m)}),$$

where π is a permutation of $\{1, 2, \ldots, n+m\}$, $s(\pi)$ its parity, and f is the *Fermi number* equal to zero for bosons and one for fermions. The coefficient in front of the sum is conventional.

In [2] we explored the possibility of formulating a nonlinear relativistic quantum mechanics based on a nonlinear version of the consistent histories approach to quantum mechanics. A toy model led to a set of equations among which there were instances of the separation property for a symmetric tensor product. This showed once more that such a separation property is fundamental for understanding any nonlinear extension of ordinary quantum mechanics.

Given these motivations, this paper is dedicated to the beginning of a systematic exploration of the symmetric separation property.

2 Symmetric tensor derivations

A symmetric tensor derivation would be a hierarchy of operators that satisfies (1) with $\hat{\otimes}$ instead of $\otimes$. That is,

$$F_n(\psi_1\hat{\otimes}\psi_2\hat{\otimes}\cdots\hat{\otimes}\psi_p) = F_{n_1}(\psi_1)\hat{\otimes}\psi_2\hat{\otimes}\cdots\hat{\otimes}\psi_p + \psi_1\hat{\otimes}F_{n_2}(\psi_2)\hat{\otimes}\cdots\hat{\otimes}\psi_p + \cdots + \psi_1\hat{\otimes}\psi_2\hat{\otimes}\cdots\hat{\otimes}F_{n_p}(\psi_p). \tag{2}$$

One does not have a classification of these as one has for ordinary tensor derivations as given in [1]. It seems that the conditions to be a tensor derivation in the symmetric case is rather stringent, and as we shall now see, in the case of differential operators, implies linearity under some general conditions. We only treat the case of second order operators as these are the most common kind in physical applications.

Let us consider a possibly nonlinear differential operators of second order not depending explicitly on the position coordinates (dependence on time can be construed as simply dependence on a parameter), in the case $N = 2$. Such an operator has the form

$$H\left(\phi, \frac{\partial\phi}{\partial x_i}, \frac{\partial\phi}{\partial y_j}, \frac{\partial^2\phi}{\partial x_i\partial x_j}, \frac{\partial^2\phi}{\partial x_i\partial y_j}, \frac{\partial^2\phi}{\partial y_i\partial y_j}\right).$$

Introducing variable names for the arguments of H, we write $H(a, b_i, c_j, d_{ij}, e_{ij}, f_{ij})$. When ϕ is constrained to be a symmetrized product (here f is the Fermi number)

$$\phi(x, y) = \alpha(x)\beta(y) + f\beta(x)\alpha(y)$$

then the arguments of H are constrained to take on values of the form .

$$a = \alpha_0\beta_0 + f\tilde{\beta}_0\tilde{\alpha}_0, \qquad b_i = \alpha_i\beta_0 + f\tilde{\beta}_i\tilde{\alpha}_0, \qquad c_i = \alpha_0\beta_i + f\tilde{\beta}_0\tilde{\alpha}_i, \tag{3}$$

$$d_{ij} = \alpha_{ij}\beta_0 + f\tilde{\beta}_{ij}\tilde{\alpha}_0, \qquad e_{ij} = \alpha_i\beta_j + f\tilde{\beta}_i\tilde{\alpha}_j, \qquad f_{ij} = \alpha_0\beta_{ij} + f\tilde{\beta}_0\tilde{\alpha}_{ij} \tag{4}$$

where all the quantities on the right-hand sides: α_0, β_0, α_i, β_i, α_{ij}, β_{ij}, $\tilde{\alpha}_0$, $\tilde{\beta}_0$, $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\tilde{\alpha}_{ij}$, $\tilde{\beta}_{ij}$, which we shall call the $\alpha\beta$-quantities, can be given, by Borel's lemma, arbitrary complex values by an appropriate choice of the points x and y and functions α and β. Denote the right-hand sides of the above equations by $\hat{a}$, $\hat{b}_i$, $\hat{c}_i$, $\hat{d}_{ij}$, $\hat{e}_{ij}$, and $\hat{f}_{ij}$, respectfully.

The separability condition for the symmetrized tensor product now reads:

$$F_2(\hat{a}, \hat{b}_i, \hat{c}_i, \hat{d}_{ij}, \hat{e}_{ij}, \hat{f}_{ij}) = F_1(\alpha_0, \alpha_i, \alpha_{ij})\beta_0 + fF_1(\tilde{\alpha}_0, \tilde{\alpha}_i, \tilde{\alpha}_{ij})\tilde{\beta}_0 + F_1(\beta_0, \beta_i, \beta_{ij})\alpha_0 + fF_1(\tilde{\beta}_0, \tilde{\beta}_i, \tilde{\beta}_{ij})\tilde{\alpha}_0. \tag{5}$$

Based on the examples of separating hierarchies for the non symmetrized tensor product, we must admit that the differential operators F_1 and F_2 may be singular, so that in analyzing (5) we should avoid points in which the first argument vanishes. Aside from this we put no further restrictions the values of the $\alpha\beta$-quantities. The freedom of choice in these quantities is now such that we can give arbitrary values to a, with $a \neq 0$, b_i, c_i, d_{ij}, and f_{ij}. This is achieved by setting

$$\alpha_0 = \frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}, \qquad \alpha_i = \frac{b_i - f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}, \qquad \tilde{\alpha}_i = f\frac{\beta_0 c_i - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_i}{\beta_0\tilde{\beta}_0},$$
$$\alpha_{ij} = \frac{d_{ij} - f\tilde{\beta}_{ij}\tilde{\alpha}_0}{\beta_0}, \qquad \tilde{\alpha}_{ij} = f\frac{\beta_0 f_{ij} - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_{ij}}{\beta_0\tilde{\beta}_0}$$

with these substitutions one finds

$$\hat{e}_{ij} = \frac{\tilde{\beta}_0 b_i\beta_j + \beta_0\tilde{\beta}_i c_j - a\tilde{\beta}_i\beta_j}{\beta_0\tilde{\beta}_0}.$$

Equation (8) now becomes

$$\begin{aligned} F_2(a, b_i, c_i, d_{ij}, \hat{e}_{ij}, f_{ij}) &= F_1\left(\frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}, \frac{b_i - f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}, \frac{d_{ij} - f\tilde{\beta}_{ij}\tilde{\alpha}_0}{\beta_0}\right)\beta_0 \\ &+ fF_1\left(\tilde{\alpha}_0,\, f\frac{\beta_0 c_i - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_i}{\beta_0\tilde{\beta}_0},\, f\frac{\beta_0 f_{ij} - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_{ij}}{\beta_0\tilde{\beta}_0}\right)\tilde{\beta}_0 \\ &+ \frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}F_1\left(\beta_0,\, \beta_i,\, \beta_{ij}) + f\tilde{\alpha}_0 F_1(\tilde{\beta}_0,\, \tilde{\beta}_i,\, \tilde{\beta}_{ij}\right). \end{aligned} \tag{6}$$

The left-hand side of (6) is independent of $\tilde{\beta}_{ij}$ and the right-hand side has two terms that depend on it. Differentiating both sides with respect to β_{ij} one arrives at the following identity:

$$D_3^{ij}F_1\left(\frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}, \frac{b_i - f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}, \frac{d_{ij} - f\tilde{\beta}_{ij}\tilde{\alpha}_0}{\beta_0}\right) = D_3^{ij}F_1(\tilde{\beta}_0,\, \tilde{\beta}_i,\, \tilde{\beta}_{ij}) \tag{7}$$

which must hold for all values of the variables that appear. Here D_3^{ij} stands for the partial derivative with respect to the ij component of the third argument of F_1. Choosing $\tilde{\alpha}_0 = 1$, $\beta_0 = f\tilde{\beta}_0$, $a = 2f\tilde{\beta}_0$, $b_i = f\tilde{\beta}_i$, and $d_{ij} = f\tilde{\beta}_{ij}$ one gets

$$D_3^{ij}F_1(1, 0, 0) = D_3^{ij}F_1(\tilde{\beta}_0,\, \tilde{\beta}_i,\, \tilde{\beta}_{ij})$$

which means that

$$F_1(u, v_i, w_{ij}) = G(u, v_i) + \sum_{ij} k^{ij}w_{ij},$$

where k^{ij} are constants. After substituting this into (6) and simplifying, that equation now becomes

$$\begin{aligned} &F_2(a, b_i, c_i, d_{ij}, \hat{e}_{ij}, f_{ij}) \\ &\quad = G\left(\frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}, \frac{b_i - f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}\right)\beta_0 + fG\left(\tilde{\alpha}_0,\, f\frac{\beta_0 c_i - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_i}{\beta_0\tilde{\beta}_0}\right)\tilde{\beta}_0 \\ &\quad + \frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}G(\beta_0,\, \beta_i) + f\tilde{\alpha}_0 G(\tilde{\beta}_0,\, \tilde{\beta}_i) + \sum_{ij} k^{ij}(d_{ij} + f_{ij}). \end{aligned} \tag{8}$$

The linear differential operator represented by the term $\sum_{ij} k^{ij}(d_{ij}+f_{ij})$ is of the form $I\otimes L+L\otimes I$ and which is part of a $\hat{\otimes}$-separating hierarchy (in which the one-particle operator is L), so subtracting it from F_2 results in a new separating hierarchy with $k^{ij}=0$. We now note that the left-hand side of (8) is independent of $\tilde\alpha_0$ so differentiation both sides with respect to $\tilde\alpha_0$ results in the following identity:

$$\begin{aligned}
&-D_1G\left(\frac{a-f\tilde\alpha_0\tilde\beta_0}{\beta_0},\,\frac{b_i-f\tilde\beta_i\tilde\alpha_0}{\beta_0}\right)\tilde\beta_0\\
&\quad-\sum_i D_2^iG\left(\frac{a-f\tilde\alpha_0\tilde\beta_0}{\beta_0},\,\frac{b_i-f\tilde\beta_i\tilde\alpha_0}{\beta_0}\right)\tilde\beta_i+D_1G\left(\tilde\alpha_0,\,f\frac{\beta_0c_i-(a-f\tilde\alpha_0\tilde\beta_0)\beta_i}{\beta_0\tilde\beta_0}\right)\tilde\beta_0\\
&\quad+\frac{\tilde\beta_0}{\beta_0}\sum_i D_2^iG\left(\tilde\alpha_0,\,f\frac{\beta_0c_i-(a-f\tilde\alpha_0\tilde\beta_0)\beta_i}{\beta_0\tilde\beta_0}\right)\beta_i-\frac{\tilde\beta_0}{\beta_0}G(\beta_0,\beta_i)+G(\tilde\beta_0,\tilde\beta_i)=0. \qquad (9)
\end{aligned}$$

Choosing now as before $\tilde\alpha_0=1$, $\beta_0=f\tilde\beta_0$, $\beta_i=0$, $a=2f\tilde\beta_0$, $b_i=f\tilde\beta_i$, and $c_i=0$ one finds

$$G(\tilde\beta_0,\tilde\beta_i)+fG(f\tilde\beta_0,0)+\sum_i D_2^iG(1,0)\tilde\beta_i=0. \qquad (10)$$

This means that

$$G(u,v_i)=A(u)+\sum_i k^iv_i, \qquad (11)$$

where k^i are constants. Substituting this into (8) with $k^{ij}=0$ one gets

$$\begin{aligned}
F_2(a,b_i,c_i,d_{ij},\hat e_{ij},f_{ij})&=A\left(\frac{a-f\tilde\alpha_0\tilde\beta_0}{\beta_0}\right)\beta_0+fA(\tilde\alpha_0)\tilde\beta_0\\
&\quad+\frac{a-f\tilde\alpha_0\tilde\beta_0}{\beta_0}A(\beta_0)+f\tilde\alpha_0A(\tilde\beta_0)+\sum_i k^i(b_i+c_i). \qquad (12)
\end{aligned}$$

As before, the differential operator represented by the last term is part of a $\hat\otimes$-separating hierarchy, so subtracting it from F_2 results in a new separating hierarchy with $k^i=0$.

Also as before the right-hand side of (12) has to be independent of $\tilde\alpha_0$. Differentiating again both sides with respect to $\tilde\alpha_0$ one arrives at

$$-A'\left(\frac{a-f\tilde\alpha_0\tilde\beta_0}{\beta_0}\right)\tilde\beta_0+A'(\tilde\alpha_0)\tilde\beta_0-f\frac{\tilde\beta_0}{\beta_0}A(\beta_0)+fA(\tilde\beta_0)=0. \qquad (13)$$

As the first term is the only one that depends on a, this equation can only hold if $A'(u)$ is a constant, that is $A(u)=ku+\ell$ for constants k and ℓ. Substituting this into (12) now results in

$$F_2(a,b_i,c_i,d_{ij},\hat e_{ij},f_{ij})=2ka+\ell\left(\beta_0+f\tilde\beta_0+f\tilde\alpha_0+\frac{a-f\tilde\alpha_0\tilde\beta_0}{\beta_0}\right)$$

which seeing that the right-hand side must be independent of $\tilde\alpha_0$ means that $\ell=0$, and we conclude.

Lemma 1. *A $\hat\otimes$-derivation of translation invariant* second order differential operators *necessarily has F_1 a* linear *operator.*

Following the procedure in [1], we define $e_{00}=a$, $e_{0j}=c_j$, $e_{i0}=b_i$, and let the upper case indices I, J, K, L range over $0,1,\dots,d$, then the parameterization of our variety is given by $e_{IJ}=\alpha_I\beta_J+\tilde\alpha_I\tilde\beta_J$. This is equivalent to saying that e_{IJ} is at most a rank two matrix. By standard results about determinantal ideals, the ideal of polynomials over the complex numbers vanishing on the variety of such matrices is generated by the order-three minors

$$M_{IJKABC} = \begin{vmatrix} e_{IA} & e_{IB} & e_{IC} \\ e_{JA} & e_{JB} & e_{JC} \\ e_{KA} & e_{KB} & e_{KC} \end{vmatrix}.$$

A simple rotation-invariant example would be given by jp and kq contraction of

$$\begin{vmatrix} a & c_j & c_k \\ b_p & e_{pj} & e_{pk} \\ b_q & e_{qj} & e_{qk} \end{vmatrix}$$

that is,

$$\sum_{pq} (a(e_{pp}e_{qq} - e_{pq}e_{qp}) - 2c_p b_p e_{qq} + 2c_p b_q e_{pq}).$$

For the curious, written out explicitly as a differential operator for $\phi(x, y)$, using the summation convention, this is:

$$\phi \left(\frac{\partial^2 \phi}{\partial x^p \partial y^p} \right)^2 - \phi \frac{\partial^2 \phi}{\partial x^p \partial y^q} \frac{\partial^2 \phi}{\partial x^q \partial y^p} - 2 \frac{\partial \phi}{\partial x^p} \frac{\partial \phi}{\partial y^p} \frac{\partial^2 \phi}{\partial x^q \partial y^q} + 2 \frac{\partial \phi}{\partial x^p} \frac{\partial^2 \phi}{\partial x^p \partial y^q} \frac{\partial \phi}{\partial y^q}.$$

A somewhat more concise expression results if we use the mixed Hessian

$$H_{pq} = \frac{\partial^2 \phi}{\partial x^q \partial y^p}$$

then our operator becomes

$$\phi \mathrm{Tr}(H)^2 - \phi \mathrm{Tr}\left(H^2\right) - 2\nabla_x \phi \cdot \nabla_y \phi \, \mathrm{Tr}(H) + 2\nabla_x \phi \cdot H \cdot \nabla_y \phi.$$

This is not a homogeneous operator, but dividing it by ϕ^2 turns it into one.

If we were simply interested in only the one- and two-particle equations then a separating hierarchy would consist of a linear one-particle operator, and the two particle operator would be given by the sum of the canonically lifted one-particle operator [1] and an operator that vanishes identically on symmetrized tensor products of one-particle functions. If we want a full multiparticle hierarchy with N-particle operators for all N, the story is different. An N-particle wave-function for particles in $\mathbb{R}^d$ can be viewed as a one-particle wave-function for particles (let us call these *conglomerate* particles) in $\mathbb{R}^{Nd}$. We can now consider the consequences of the separating property for the hierarchy consisting of a $2N$ particle operator on a symmetrized tensor product of two N-particle wave functions reinterpreted as one consisting of an operator for two conglomerate particles and an operator for one conglomerate particle. A wave-function of two conglomerate particles does not have the same permutation symmetry as the wave-function of $2N$ particles, but the difference is such as to impose even stronger conditions due to the separation property. Let $\phi(x_1, \ldots, x_N)$ and $\psi(y_1, \ldots, y_N)$ be two properly symmetric N-particle wave-functions. One has

$$\phi \hat{\otimes} \psi(x_1, \ldots, x_{2N}) = C \sum_I (-1)^{fp(I)} \phi(x_{i_1}, \ldots, x_{i_N}) \psi(x_{j_1}, \ldots, x_{j_N}), \tag{15}$$

where C is a combinatorial factor, $I = (i_1, \ldots, i_N)$ are N numbers from $\{1, \ldots, 2N\}$, in ascending order, $(j_1, \ldots, j_N)$ the complementary numbers, also in ascending order, and $p(I)$ is the parity (0 or 1) of the permutation $(1, \ldots, 2N) \mapsto (i_1, \ldots, i_N, j_1, \ldots, j_N)$. For (15) the possible values that one can attribute to the wave-function and its derivatives at a point is now more complicated than that given by expressions (3), (4), but by an appropriate choice of coordinates and an appeal to Borel's lemma, we can again use, as a particular case, expressions (3), (4) for two conglomerate particles. Repeating the argument presented above for the two-particle case we see that the operator for one conglomerate particle must be linear and so the N-particle operator must be linear. With this the whole hierarchy must be linear. We thus have:

Theorem 1. *A $\hat{\otimes}$-derivation of translation invariant second order differential operators is linear.*

This result of course does not rule out the physical possibility of nonlinear quantum mechanics for identical particles, but points out a further subtlety in its manifestation. The separation property cannot be used as a generalization of the idea of non-interacting systems and the notion of lack of interaction becomes more subtle.

3 Strong cluster property

Given that separation cannot hold for identical particles in the nonlinear case, one can expect on intuitive grounds that it may hold for systems in which the subsystems are distant from each other. This is usually called the cluster decomposition property. This property however holds even in the interacting case, given short range interparticle potentials. A slightly strengthened version however eliminates interaction potentials in the linear case, and can be used as a generalization that can be extended to the symmetric nonlinear case. Consider an n-fold symmetric tensor product

$$(\phi_1\hat{\otimes}\phi_2\hat{\otimes}\cdots\hat{\otimes}\phi_n)(x) = C\sum_{\pi\in S}\pm\phi_1(x_{(1,\pi)})\phi_2(x_{(2,\pi)})\cdots\phi_n(x_{(n,\pi)}), \tag{16}$$

where C is a combinatorial coefficient x is an m-tuple of space points, S is a subset of the permutation group, and each $x_{(k,\pi)}$ is a subset of the m-tuple x ordered according to its original order in x. The sum is over all permutations that distribute x into the subsets $x_{(k,\pi)}$. We say such a product is *cluster-separated* if the supports of the summand in (16) are all disjoint. We say a hierarchy of operators has the strong cluster separation property if (2) holds for cluster-separated products. A simple verification with ordinary linear Schrödinger operators shows that these satisfy the strong cluster-separation property if and only if the interparticle potentials vanish, so this is indeed a proper generalization of lack of interaction. One sees immediately that the strong cluster separation property would hold if the ordinary separation property holds and if the operators were linear on sums of functions with disjoint supports. This linearity may at first sight seem contrary to the spirit of looking for nonlinear theories, but in fact, for *differential* operators it follows from the ordinary separation property in almost all cases. As was shown in [1] tensor derivations are for the most part homogeneous. Those that are not, differ from these by a fixed canonical term. Homogeneous differential operators have the remarkable property that they are linear on spaces generated by functions with disjoint supports:

Theorem 2. *If G be a differential operator which is homogeneous of degree $k \neq 0$ then it is additive on spaces generated by functions with disjoint support and for $k = 1$ it is linear on such spaces.*

Proof. By Euler's formula $DG(\phi)\phi = kG(\phi)$ where D denotes the Frechét derivative. Let ϕ_j, $j = 1, \ldots, r$ have disjoint supports. We have

$$G\left(\sum_j \phi_j\right) = k^{-1}DG\left(\sum_j \phi_j\right)\left(\sum_\ell \phi_\ell\right) = k^{-1}\sum_\ell DG\left(\sum_j \phi_j\right)\phi_\ell.$$

Now in a neighborhood of a point where $\phi_\ell \neq 0$ one has for $j \neq \ell$ that $\phi_j = 0$. Since the value at a point of a differential operator applied to a function depends only on the values of the function in any neighborhood of the point, we can write the last term as $k^{-1} \sum_\ell DG(\phi_\ell)\phi_\ell = \sum_j G(\phi_j)$ and we have additivity. If now $k = 1$, the operator will in fact be real-linear on the subspace generated by the ϕ_j. ■

From this we deduce

Theorem 3. *Homogeneous ordinary tensor derivations of differential operators satisfy the strong cluster separation property.*

This means that we can apply all the structural theorems of [1] to symmetric tensor derivation provided that we stay within the class of homogeneous differential operators.

4 Fock space considerations

In [2] we were led to consider the problem of finding a Lorentz invariant nonlinear operator K in a relativistic scalar free field Fock space for which

$$[[K, \phi(f)], \phi(g)] = 0 \tag{17}$$

provided the supports of f and g are space-like separated. In that reference we analyzed only the simplest consequence of this equation that arising from applying it to the vacuum state. One of the conditions was a symmetric separation property for space-like separated supports. We now address (17) more systematically. We here consider only the bosonic case as the fermionic one is entirely similar.

Let $\mathcal{H} = \bigoplus_{n=0}^\infty \mathcal{H}_n$ be the bosonic Fock space where $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_1$ is the 1-particle subspace, and $\mathcal{H}_n = \mathcal{H}_1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}_1$, the n-fold symmetric tensor product, is the n-particle subspace. We assume $\mathcal{H}_1$ has a antilinear involution $f \mapsto \bar{f}$ satisfying

$$(\bar{f}, g) = (\bar{g}, f). \tag{18}$$

For $f \in \mathcal{H}_1$ one defines the creation operator $a^+(f)$ and the annihilation operator $a(f)$ in $\mathcal{H}$ by

$$(a^+(f)\Psi)_n = \sqrt{n}\, f \hat{\otimes} \Psi_{n-1}, \tag{19}$$

$$(a(f)\Psi)_n = \sqrt{n+1}\, f \rfloor \Psi_{n+1}, \tag{20}$$

where the contraction operator $\rfloor$ is defined by

$$f \rfloor (g_1 \hat{\otimes} \cdots \hat{\otimes} g_n) = \frac{1}{n} \sum_{i=1}^n (f, g_i) g_1 \hat{\otimes} \cdots \hat{\otimes} \hat{g}_i \hat{\otimes} \cdots \hat{\otimes} g_n,$$

where by the hat over g_i we mean that that factor is missing. The *quantum field* is defined as

$$\phi(f) = \phi^{(+)}(f) + \phi^{(-)}(f) = a^+(f) + a(\bar{f}). \tag{21}$$

One has the famous *canonical commutation relations*

$$[\phi^{(+)}(f), \phi^{(+)}(g)] = 0,$$
$$[\phi^{(-)}(f), \phi^{(-)}(g)] = 0,$$
$$[\phi^{(-)}(f), \phi^{(+)}(g)] = (\bar{f}, g).$$

Assume that K respects particle number, that is, $(K\Psi)_n = K_n\Psi_n$ for a hierarchy of operators K_n. We analyze the equation

$$[[K,\phi(f)],\phi(g)] = 0 \tag{22}$$

by applying the left-hand side to a Fock space vector which has only an n-particle component Ψ_n. One arrives at the following three conditions

$$\begin{aligned}
&K_{n+2}\left(\sqrt{(n+2)(n+1)}\, f\hat{\otimes}g\hat{\otimes}\Psi_n\right) - \sqrt{n+2}\, f\hat{\otimes}K_{n+1}\left(\sqrt{n+1}\, g\hat{\otimes}\Psi_n\right)\\
&\qquad - \sqrt{n+2}\, g\hat{\otimes}K_{n+1}\left(\sqrt{n+1}\, f\hat{\otimes}\Psi_n\right) + \sqrt{(n+2)(n+1)}\, f\hat{\otimes}g\hat{\otimes}K_n(\Psi_n) = 0, \qquad (23)\\
&K_n\left((n+1)\bar{f}\rfloor g\hat{\otimes}\Psi_n + nf\hat{\otimes}\bar{g}\rfloor\Psi_n\right) - \sqrt{n+1}\,\bar{f}\rfloor K_{n+1}\left(\sqrt{n+1}\, g\hat{\otimes}\Psi_n\right)\\
&\qquad - \sqrt{n+1}\,\bar{g}\rfloor K_{n+1}\left(\sqrt{n+1}\, f\hat{\otimes}\Psi_n\right) - \sqrt{n}\, f\hat{\otimes}K_{n-1}\left(\sqrt{n}\,\bar{g}\rfloor\Psi_n\right)\\
&\qquad - \sqrt{n}\, g\hat{\otimes}K_{n-1}\left(\sqrt{n}\,\bar{f}\rfloor\Psi_n\right) + (n+1)\bar{g}\rfloor f\hat{\otimes}K_n(\Psi_n) + ng\hat{\otimes}\bar{f}\rfloor K_n(\Psi_n) = 0, \qquad (24)\\
&K_{n-2}\left(\sqrt{(n-1)n}\,\bar{f}\rfloor\bar{g}\rfloor\Psi_n\right) - \sqrt{n-1}\,\bar{f}\rfloor K_{n-1}\left(\sqrt{n}\,\bar{g}\rfloor\Psi_n\right)\\
&\qquad - \sqrt{n-1}\,\bar{g}\rfloor K_{n-1}\left(\sqrt{n}\,\bar{f}\rfloor\Psi_n\right) + \sqrt{(n-1)n}\,\bar{g}\rfloor\bar{f}\rfloor K_n(\Psi_n) = 0. \qquad (25)
\end{aligned}$$

In the relativistic case these conditions are to be satisfied whenever the smearing functions f and g have space-like separated supports.

We have

Theorem 4. *If K is a* linear *symmetric tensor derivation, then equations (23) and (25) are satisfied identically, while (24) is satisfied if*

$$(\bar{f}, K_1(g)) + (\bar{g}, K_1(f)) = 0. \tag{26}$$

This is a straightforward though tedious verification. It is enough to consider $\Psi_n = h_1\hat{\otimes}\cdots\hat{\otimes}h_n$ as linear operators are uniquely defined by their action on product functions.

Equation (26), imposed for all f and g says, using (18), that K_1 must be anti-symmetric, or that its exponential is unitary. This is an interesting consequence, as one of the requirements in [2] for a consistent history model is this unitarity which was states separately; here it is a consequence of the separation property and the commutation relation.

This result does not in itself provide us with an example of a nonlinear relativistic quantum mechanics, but it allows us to construct a theory, using the coherent histories approach, in which the quantum measurement process has properties similar to those we believe a nonlinear theory must have, that is, the future light-cone singular behavior pointed out in [2].

[1] Goldin G.A. and Svetlichny G., Nonlinear Schrödinger equations and the separation property, *J. Math. Phys.*, 1994, V.35, 3322–3332.

[2] Svetlichny G., On relativistic non-linear quantum mechanics, in Proceedings of Second International Conference "Symmetry in Nonlinear Mathematical Physics. Memorial Prof. W. Fushchych Conference" (7–13 July, 1997, Kyiv), Editors M. Shkil, A. Nikitin and V. Boyko, Institute of Mathematics, 1997, V.2, 262–269.