# On Pre-Novikov Algebras and Derived Zinbiel Variety

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**Abstract.** For a non-associative algebra $A$ with a derivation $d$, its derived algebra $A^{(d)}$ is the same space equipped with new operations $a \succ b = d(a)b$, $a \prec b = ad(b)$, $a, b \in A$. Given a variety Var of algebras, its derived variety is generated by all derived algebras $A^{(d)}$ for all $A$ in Var and for all derivations $d$ of $A$. The same terminology is applied to binary operads governing varieties of non-associative algebras. For example, the operad of Novikov algebras is the derived one for the operad of (associative) commutative algebras. We state a sufficient condition for every algebra from a derived variety to be embeddable into an appropriate differential algebra of the corresponding variety. We also find that for Var = Zinb, the variety of Zinbiel algebras, there exist algebras from the derived variety (which coincides with the class of pre-Novikov algebras) that cannot be embedded into a Zinbiel algebra with a derivation.

*Key words:* Novikov algebra; derivation; dendriform algebra; Zinbiel algebra

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## 1 Introduction

The class of nonassociative algebras with one binary operation satisfying the identities of left symmetry

$$(xy)z - x(yz) = (yx)z - y(xz) \tag{1.1}$$

and right commutativity

$$(xy)z = (xz)y \tag{1.2}$$

is known as the variety of *Novikov algebras*. Relations (1.1) and (1.2) emerged in [2, 10] as a tool for expressing certain conditions on the components of a tensor of rank 3 considered as a collection of structure constants of a finite-dimensional algebra with one bilinear operation. In [10], it was a sufficient condition for a differential operator to be Hamiltonian; in [2] it was a condition to guarantee the Jacobi identity for a generalized Poisson bracket in the framework of Hamiltonian formalism for partial differential equations of hydrodynamic type.

Novikov algebras may be obtained from (associative) commutative algebras with a derivation by means of the following operation-transforming functor (see [10]). Assume $A$ is a commutative algebra with a multiplication $*$, and let $d$ be a derivation of $A$, i.e., a linear operator $d\colon A \to A$ satisfying the Leibniz rule

$$d(a * b) = d(a) * b + a * d(b), \qquad a, b \in A.$$

Then the new operation

$$ab = a * d(b), \qquad a, b \in A, \tag{1.3}$$

satisfies the identities (1.1) and (1.2).

In [9], it was proved that the free Novikov algebra $\mathrm{Nov}\langle X\rangle$ generated by a set $X$ is a subalgebra of the free differential commutative algebra with respect to the operation (1.3). Moreover, it was shown in [4] that every Novikov algebra can be embedded into an appropriate commutative algebra with a derivation.

One may generalize the relation between commutative algebras with a derivation and Novikov algebras as follows. Let Var be a class of all algebras over a field $\Bbbk$ with one or more bilinear operations which is closed under direct products, subalgebras, and homomorphic images (HSP-class or *variety*). By the classical Birkhoff's theorem, a variety consists of all algebras that satisfy some family of identities. We will assume that Var is defined by multi-linear identities (this is not a restriction if $\operatorname{char}\Bbbk = 0$). For every $A \in \mathrm{Var}$, let $\mathrm{End}(A)$ stand for the set of all linear operators on the space $A$. Then a linear operator $d \in \mathrm{End}(A)$ is called a derivation if the analogue of the Leibniz rule holds for every binary operation in $A$. The set of all derivations of a given algebra $A$ forms a subspace (even a Lie subalgebra) $\mathrm{Der}(A)$ of $\mathrm{End}(A)$. The class of all pairs $(A, d)$, $A \in \mathrm{Var}$, $d \in \mathrm{Der}(A)$, is also a variety denoted $\mathrm{Var\,Der}$ defined by multi-linear identities. Given $(A, d) \in \mathrm{Var\,Der}$, denote by $A^{(d)}$ the same space $A$ equipped with two bilinear operations $\prec$, $\succ$ for each operation on $A$:

$$a \prec b = ad(b), \qquad a \succ b = d(a)b, \qquad a, b \in A.$$

The class of all systems $A^{(d)}$, $(A, d) \in \mathrm{Var\,Der}$, is closed under direct products, so all homomorphic images of all their subalgebras form a variety denoted $D\mathrm{Var}$, called the *derived variety* of Var (see [19]). Alternatively, $D\mathrm{Var}$ consists of all algebras with duplicated family of operations that satisfy all those identities that hold on all $A^{(d)}$ for all $(A, d) \in \mathrm{Var\,Der}$.

For example, if Com is the variety of commutative (and associative) algebras then $D\mathrm{Com} = \mathrm{Nov}$ since $x \prec y = y \succ x$. In general, the description of $D\mathrm{Var}$ may be obtained in the language of operads and their Manin products [19]. Namely, if we identify the notations for a variety and its governing operad [11] then the operad $D\mathrm{Var}$ coincides with the Manin white product of operads Var and Nov.

As a corollary, the free algebra $D\mathrm{Var}\langle X\rangle$ in the variety $D\mathrm{Var}$ generated by a set $X$ is isomorphic to the subalgebra in the free algebra in the variety Var Der generated by $X$. However, it is not clear that the following *embedding statement* holds in general: every $D\mathrm{Var}$-algebra can be embedded into an appropriate differential Var-algebra (or Var Der-algebra). In other words, the problem is to determine whether the class of all subalgebras of all algebras $A^{(d)}$, $(A, d) \in \mathrm{Var\,Der}$, is closed under homomorphic images. Positive answers were obtained for $\mathrm{Var} = \mathrm{Com}$ [4], $\mathrm{Var} = \mathrm{Lie}$ [19], $\mathrm{Var} = \mathrm{Perm}$ [18], $\mathrm{Var} = \mathrm{As}$ [21].

In this paper, we derive a sufficient condition for a positive answer to the embedding statement for a given Var. Namely, if the Manin white product $\mathrm{Nov} \circ \mathrm{Var}$ of the corresponding operads coincides with the Hadamard product $\mathrm{Nov} \otimes \mathrm{Var}$ then the embedding statement holds for Var.

We also find an example of a variety Var governed by a binary quadratic operad such that the embedding statement fails for Var. It turns out that the variety of Zinbiel algebras Zinb (also known as commutative dendriform algebras, pre-commutative algebras, dual Leibniz algebras, half-shuffle algebras) introduced in [20] provides such an example: there exists a $D\mathrm{Zinb}$-algebra which cannot be embedded into a Zinbiel algebra with a derivation. To our knowledge, this is the first example of such a variety Var that an algebra from $D$Var does not embed into a differential Var-algebra.

The operad Zinb governing the variety of Zinbiel algebras is a particular case of a general construction called *dendriform splitting* of an operad [1]. For every binary operad Var (not necessarily quadratic, see, e.g., [13]) there exists an operad preVar governing the class of systems with duplicated family of operations. The generic example of a preVar algebra may be obtained from a Var-algebra with a Rota–Baxter operator $R$ of weight zero (see, e.g., [14]). If $A \in \mathrm{Var}$ and $R\colon A \to A$ is such an operator then $(A, \vdash, \dashv)$, where

$$a \vdash b = R(a)b, \qquad a \dashv b = aR(b), \qquad a, b \in A,$$

is a preVar-algebra. In this context, preCom = Zinb (since $a \vdash b = b \dashv a$), preLie is the classical variety of left-symmetric algebras (relative to $\vdash$), preAs is exactly the variety of dendriform algebras [20].

The theory of pre-algebras and relations between them is similar in many aspects to the theory of "ordinary" algebras. For example, every preAs-algebra with respect to the "dendriform commutator" $a \vdash b - b \dashv a$ is a preLie-algebra. The properties of the corresponding left adjoint functor (universal envelope) are close to what we have for ordinary Lie algebras [6, 12]. The class of pre-Novikov algebras has been recently studied in [24]: it coincides with $D$Zinb. Therefore, our results show that the embedding statement cannot be transferred from ordinary algebras to pre-algebras.

## 2 Derived algebras and Manin white product of binary operads

The details about (symmetric) operads may be found, for example, in [5]. For an operad $\mathcal{P}$ denote $\mathcal{P}(n)$ the linear space (over a base field $\Bbbk$) of degree $n$ elements of $\mathcal{P}$, the action of a permutation $\sigma \in S_n$ on an element $f \in \mathcal{P}(n)$ is denoted $f^\sigma$, let $\mathrm{id} \in \mathcal{P}(1)$ stand for the identity element, and the composition rule

$$\mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n)$$

is denoted $\gamma^{m_1+\cdots+m_n}_{m_1,\ldots,m_n}$.

Recall that an operad $\mathcal{P}$ is said to be binary, if $\mathcal{P}(1) = \Bbbk\mathrm{id}$ and the entire $\mathcal{P}$ is generated (as an operad) by its degree 2 space $\mathcal{P}(2)$.

Let us fix a binary operad $\mathcal{P}$. For every linear space $A$, consider an operad $\mathrm{End}_A$ with $\mathrm{End}_A(n) = \mathrm{End}\,(A^{\otimes n}, A)$. A morphism of operads $\mathcal{P} \to \mathrm{End}_A$ defines an algebra structure on $A$ with a set of binary operations corresponding to the generators of $\mathcal{P}$ from $\mathcal{P}(2)$. The class of all such algebras is a variety of $\mathcal{P}$-algebras defined by multi-linear identities corresponding to the defining relations of the operad $\mathcal{P}$.

Conversely, every variety of algebras with binary operations defined by multi-linear identities gives rise to an operad $\mathcal{P}$ constructed in such a way that $\mathcal{P}(n)$ is the space of multi-linear elements of degree $n$ in the variables $x_1, \ldots, x_n$ of the free algebra in this variety generated by the countable set $\{x_1, x_2, \ldots\}$ (see, e.g., [17, Section 1.3.5] for details). In particular, $\mathrm{id} \in \mathcal{P}(1)$ is presented by the element $x_1$. Then the variety under consideration consists exactly of all $\mathcal{P}$-algebras, in other words, it is *governed* by the operad $\mathcal{P}$.

**Example 2.1.** Let $\mu, \nu \in \mathcal{P}(2)$. Suppose we have identified $\mu$ with $x_1x_2$ and $\nu$ with $x_1 * x_2$. Then

$$\gamma^3_{1,2}(\mu, \mathrm{id}, \nu) = \gamma^3_{1,2}(x_1x_2, x_1, x_1 * x_2) = x_1(x_2 * x_3),$$
$$\gamma^4_{2,2}(\nu^{(12)}, \mu, \nu) = \gamma^4_{2,2}(x_2 * x_1, x_1x_2, x_1 * x_2) = (x_3 * x_4) * (x_1x_2),$$

and so on.

We will not distinguish notations for an operad $\mathcal{P}$ and for the corresponding variety of $\mathcal{P}$-algebras.

**Example 2.2.** Suppose $\mathcal{F}_2$ is the free binary operad with $\mathcal{F}_2(2) \simeq \Bbbk S_2$ (as symmetric modules). Then the class of $\mathcal{F}_2$-algebras coincides with the variety of all nonassociative algebras with one non-symmetric binary operation, i.e., $\mathcal{F}_2(n)$ may be identified with the linear span of all bracketed monomials $\big(x_{\sigma(1)} \cdots x_{\sigma(n)}\big)$, $\sigma \in S_n$, so $\dim \mathcal{F}_2(n) = n!C_{n-1}$, where $C_k$ is the $k$th Catalan number.

If $\mathcal{P}$ is an operad governing a variety of algebras with one binary operation then $\mathcal{P}$ is a homomorphic image of $\mathcal{F}_2$. If the kernel of the morphism is generated (as an operadic ideal) by elements from $\mathcal{F}_2(3)$ then the operad $\mathcal{P}$ is said to be *quadratic*. The same definition works for operads with more than one generator.

**Example 2.3.** Let Zinb stand for the variety of algebras with one multiplication satisfying the identity

$$(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3 + x_3 \cdot x_2), \tag{2.1}$$

known as the Zinbiel identity [20]. Then the space $\mathrm{Zinb}(n)$, $n \geq 1$, is spanned by linearly independent monomials

$$\big(x_{\sigma^{-1}(1)} \cdot \big(x_{\sigma^{-1}(2)} \cdot \big(\cdots \big(x_{\sigma^{-1}(n-1)} \cdot x_{\sigma^{-1}(n)}\big) \cdots \big)\big)\big) = [x_1 x_2 \cdots x_{n-1} x_n]^\sigma, \qquad \sigma \in S_n,$$

so $\dim \mathrm{Zinb}(n) = n!$.

The product of two right-normed words in a Zinbiel algebra can be expressed via shuffle permutations $S_{n,m} \subset S_{n+m}$: if $u = x_1 \cdots x_n$, $v = x_{n+1} \cdots x_{n+m}$ then

$$[u] \cdot [v] = \sum_{\sigma \in S_{n,m}\colon \sigma(1)=1} [uv]^\sigma .$$

The new product $[u] \cdot [v] + [v] \cdot [u]$ on a Zinbiel algebra is associative and commutative (it is known as the *shuffle product* of words).

Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are two binary operads. Then the *Hadamard product* of $\mathcal{P}_1$ and $\mathcal{P}_2$ is the operad denoted $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$ such that $\mathcal{P}(n) = \mathcal{P}_1(n) \otimes \mathcal{P}_2(n)$, $n \geq 1$, the action of $S_n$ on $\mathcal{P}(n)$ and the composition rule are defined in the componentwize way.

The operad $\mathcal{P}_1 \otimes \mathcal{P}_2$ may not be a binary one (in this paper, we will deal with such an example below). The sub-operad of $\mathcal{P}_1 \otimes \mathcal{P}_2$ generated by $\mathcal{P}_1(2) \otimes \mathcal{P}_2(2)$ is called the *Manin white product* of $\mathcal{P}_1$ and $\mathcal{P}_2$ denoted by $\mathcal{P}_1 \circ \mathcal{P}_2$. (For some operads $\mathcal{P}_1$, it may happen that $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{P}_1 \otimes \mathcal{P}_2$ for all $\mathcal{P}_2$; in [22], all such operads were described.)

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are quadratic binary operads then so is $\mathcal{P}_1 \circ \mathcal{P}_2$. The defining relations of the last operad can be found as follows [11]. Suppose $R_i \subseteq \mathcal{P}_i(3)$, $i = 1, 2$, are the spaces of defining relations of $\mathcal{P}_i$ presented in the form of multi-linear identities in $x_1$, $x_2$, $x_3$. Consider the space $E(3)$ spanned by all possible compositions of degree 3 of operations from $\mathcal{P}_1(2) \otimes \mathcal{P}_2(2)$. Then the defining relations of $\mathcal{P}_1 \circ \mathcal{P}_2$ form the space $E(3) \cap (\mathcal{P}_1(3) \otimes R_2 + R_1 \otimes \mathcal{P}_2(3))$.

**Example 2.4.** Let $\mathcal{P}_1 = \mathrm{Nov}$, $\mathcal{P}_2 = \mathrm{Zinb}$. Then $\mathcal{P}_1(2) \otimes \mathcal{P}_2(2)$ is spanned by four elements

$$x_1 \prec x_2 = x_1x_2 \otimes x_1x_2, \qquad x_1 \succ x_2 = x_2x_1 \otimes x_1x_2,$$
$$x_2 \prec x_1 = x_2x_1 \otimes x_2x_1, \qquad x_2 \succ x_1 = x_1x_2 \otimes x_2x_1.$$

In order to find $E(3)$, calculate all monomials of degree 3 in $x_1$, $x_2$, $x_3$ with operations $\succ$, $\prec$. For example,

$$(x_1 \succ x_3) \prec x_2 = \gamma^3_{1,2}(x_2 \prec x_1, \mathrm{id}, x_1 \succ x_2)^{(12)}$$

$$= \gamma^3_{1,2}(x_2x_1, \mathrm{id}, x_2x_1)^{(12)} \otimes \gamma^3_{1,2}(x_2x_1, \mathrm{id}, x_1x_2)^{(12)}$$
$$= (x_3x_2)x_1^{(12)} \otimes (x_2x_3)x_1^{(12)} = (x_3x_1)x_2 \otimes (x_2x_3)x_1 \in \mathrm{Nov}(3) \otimes \mathrm{Zinb}(3).$$

In the same way, the expressions for all 48 monomials may be calculated in $\mathrm{Nov}(3) \otimes \mathrm{Zinb}(3)$. In order to get defining relations of $\mathrm{Nov} \circ \mathrm{Zinb}$ it is enough to find those linear combinations of these monomials that are zero in $\mathrm{Nov}(3) \otimes \mathrm{Zinb}(3)$. This is a routine problem of linear algebra. As a result, we obtain the following identities:

$$\begin{aligned}
&(x_1 \prec x_2) \prec x_3 = (x_1 \prec x_3) \prec x_2, \\
&x_1 \succ (x_2 \succ x_3) = (x_1 \succ x_3) \prec x_2 - x_1 \succ (x_3 \prec x_2), \\
&(x_1 \succ x_2) \succ x_3 = (x_1 \succ x_3) \succ x_2 + (x_1 \succ x_3) \prec x_2 - (x_1 \succ x_2) \prec x_3, \\
&(x_1 \prec x_2) \succ x_3 = x_1 \prec (x_2 \succ x_3) + x_1 \prec (x_3 \prec x_2) + (x_1 \succ x_3) \prec x_2 - (x_1 \prec x_3) \prec x_2.
\end{aligned} \tag{2.2}$$

**Remark 2.5.** Novikov algebras are closely related to conformal algebras in the sense of [16]. Namely, if $V$ is a Novikov algebra then the free $\Bbbk[\partial]$-module $C = \Bbbk[\partial] \otimes V$ equipped with a sesqui-linear $\lambda$-product

$$[u_{(\lambda)} v] = \partial(vu) + \lambda(vu + vu), \qquad u, v \in V,$$

is a Lie conformal algebra [23]. The reason of this relation is that the expression for the generalised Poisson bracket in [2] has the same form as the expression for the commutator of local chiral fields in [16].

**Remark 2.6.** The variety governed by the operad $\mathrm{Nov} \circ \mathrm{Zinb}$ is also related to conformal algebras. It is straightforward to check that if $V$ is an algebra over a field $\Bbbk$, $\mathrm{char}\,\Bbbk = 0$, with two operations $\prec$, $\succ$ satisfying the identities (2.2) then the free $\Bbbk[\partial]$-module $C = \Bbbk[\partial] \otimes V$ equipped with a sesqui-linear $\lambda$-product

$$\big(u_{(\lambda)} v\big) = \partial(v \prec u) + \lambda(v \succ u + v \prec u), \qquad u, v \in V,$$

is a left-symmetric conformal algebra [15]. We will explain this relation in the last section.

Let Var be a binary operad, we use the same notation for the corresponding variety of algebras. Given an algebra $A$ in Var, denote by $\mathrm{Der}(A)$ the set of all derivations of $A$. Recall that a derivation of $A$ is a linear map $d\colon A \to A$ such that

$$d(\mu(a, b)) = \mu(d(a), b) + \mu(a, d(b)), \qquad a, b \in A,$$

for all operations $\mu$ from $\mathrm{Var}(2)$. For a derivation $d$ of an algebra $A$, denote by $A^{(d)}$ the linear space $A$ equipped with *derived* operations

$$\mu^{\succ}(a, b) = \mu(d(a), b), \quad \mu^{\prec}(a, b) = \mu(a, d(b)), \qquad a, b \in A, \tag{2.3}$$

for all $\mu$ in $\mathrm{Var}(2)$. The variety generated by the class of systems $A^{(d)}$ for all $A \in \mathrm{Var}$ and $d \in \mathrm{Der}(A)$ is denoted by $D\mathrm{Var}$, the *derived variety* of Var. Obviously, the class of all such $A^{(d)}$ is closed under direct products, so, in order to get $D\mathrm{Var}$, one has to consider all homomorphic images of all subalgebras of these $A^{(d)}$. In many cases, it is enough to consider just subalgebras. The problem is to decide whether homomorphic images are actually needed.

**Theorem 2.7** ([19]). *For a binary operad* Var, *the variety* $D\mathrm{Var}$ *is governed by the operad* $\mathrm{Nov} \circ \mathrm{Var}$.

For example, all relations that hold on every Zinbiel algebra with a derivation relative to the operations $a \succ b = d(a)b$, $a \prec b = ad(b)$ follow from the identities (2.2).

If Var = Com is the variety of associative and commutative algebras then $D$Var = Nov, as follows from the construction of the free Novikov algebra [9]. Here we have to mention that commutativity implies $a \succ b = b \prec a$ in every algebra from $D$Com.

If Var = Lie then the algebras from $D$Var form exactly the class of all $\mathcal{F}_2$-algebras with one binary operation $a \succ b = -b \prec a$ [19]. Note that $\dim \mathrm{Nov}(n) = \binom{2n-2}{n-1}$ [7], and $\mathrm{Lie}(n) = (n-1)!$. Hence, $\dim(\mathrm{Nov} \otimes \mathrm{Lie})(n) = \frac{(2n-2)!}{(n-1)!}$ which is equal to $n!C_{n-1}$, where $C_n$ is the $n$th Catalan number. The number $n!C_{n-1}$ coincides with the $n$th dimension of the operad $\mathcal{F}_2$. Hence, $\mathrm{Nov} \circ \mathrm{Lie} = \mathrm{Nov} \otimes \mathrm{Lie}$.

Suppose Var is a binary operad, $D\mathrm{Var} = \mathrm{Nov} \circ \mathrm{Var}$, $D\mathrm{Var}\langle X\rangle$ is the free $D$Var-algebra generated by a countable set $X = \{x_1, x_2, \dots\}$. Denote by $F = \mathrm{Var}\,\mathrm{Der}\langle X, d\rangle$ the free differential Var-algebra generated by $X$ with one derivation $d$. Then there exists a homomorphism $\tau\colon D\mathrm{Var}\langle X\rangle \to F^{(d)}$ sending $X$ to $X$ identically. An element from $\ker \tau$ is an identity that holds on all Var-algebras with a derivation relative to the derived operations (2.3). Hence, $\tau$ is injective, i.e., the free $D$Var-algebra can be embedded into the free differential Var-algebra.

The next question is whether every $D$Var-algebra can be embedded into an appropriate differential Var-algebra. This is the same as to decide if the class of all subalgebras of $A^{(d)}$, $(A, d) \in \mathrm{Var}\,\mathrm{Der}$, is closed under homomorphisms. The answer is positive for Var = Com [4], Lie [19], Perm [18], and As [21]. In the following sections we derive a sufficient condition for Var to guarantee a positive answer. This condition is not necessary, but we find an example when the answer is negative.

## 3 The weight criterion and special derived algebras

Let Var be a binary operad. An algebra $V$ with two binary operations $\prec$, $\succ$ from the variety $D$Var is called *special* if it can be embedded into a Var-algebra $A$ with a derivation $d$ such that $u \prec v = ud(v)$ and $u \succ v = d(u)v$ in $A$ for all $u, v \in V$. The class of all Var-algebras with a derivation is a variety since it is defined by identities. The free differential Var-algebra $\mathrm{Var}\,\mathrm{Der}\langle X, d\rangle$ generated by a set $X$ is isomorphic as a Var-algebra to the free Var-algebra $\mathrm{Var}\big\langle X^{(\omega)}\big\rangle$ generated by the set

$$X^{(\omega)} = \big\{x^{(n)} \mid x \in X,\, n \in \mathbb{Z}_+\big\},$$

with the derivation $d$ defined by $d\big(x^{(n)}\big) = x^{(n+1)}$, $x \in X$, $n \in \mathbb{Z}_+$.

For a nonassociative monomial $u \in X^{(\omega)}$ define its weight $\mathrm{wt}(u) \in \mathbb{Z}$ as follows. For a single letter $u = x^{(n)}$, set $\mathrm{wt}(u) = n - 1$. If $u = u_1u_2$ then $\mathrm{wt}(u) = \mathrm{wt}(u_1) + \mathrm{wt}(u_2)$. Since the defining relations of $\mathrm{Var}\big\langle X^{(\omega)}\big\rangle$ are weight-homogeneous, we may define the weight function on $\mathrm{Var}\,\mathrm{Der}\langle X, d\rangle$. Note that if $f \in \mathrm{Var}\big\langle X^{(\omega)}\big\rangle$ is a weight-homogeneous polynomial then $\mathrm{wt}\, d(f) = \mathrm{wt}(f) + 1$.

**Lemma 3.1.** *Let* Var *be a binary operad such that* $\mathrm{Nov} \circ \mathrm{Var} = \mathrm{Nov} \otimes \mathrm{Var}$. *Then for every set* $X$ *an element* $f \in \mathrm{Var}\big\langle X^{(\omega)}\big\rangle$ *belongs to* $D\mathrm{Var}\langle X\rangle$ *if and only if* $\mathrm{wt}(f) = -1$.

**Proof.** The "only if" part of the statement does not depend on the particular operad Var. Indeed, every formal expression in the variables $X$ relative to binary operations $\prec$ and $\succ$ turns into a weight-homogeneous polynomial of weight $-1$ in $\mathrm{Var}\big\langle X^{(\omega)}\big\rangle$.

For the "if" part, assume $u$ is a monomial of weight $-1$ in the variables $X^{(\omega)}$. In the generic form,

$$u = \big(x_{i_1}^{(s_1)} \cdots x_{i_n}^{(s_n)}\big), \qquad x_{i_j} \in X, \quad s_j \geq 0,$$

with some bracketing. Here $s_1 + \dots + s_n = n - 1$. Consider the element

$$[u] = x_1^{(s_1)} \cdots x_n^{(s_n)} \otimes (x_1 \cdots x_n) \in \mathrm{Nov}(n) \otimes \mathrm{Var}(n).$$

Here the first tensor factor is a differential commutative monomial of degree $n$ and of weight $-1$ which belongs to $\mathrm{Nov}(n)$ by [9]. In the second factor, we put the nonassociative multi-linear word obtained from $u$ by removing all derivatives and consecutive re-numeration of variables (the bracketing remains the same as in $u$).

By the assumption, $[u]$ belongs to $(\mathrm{Nov} \circ \mathrm{Var})(n)$, i.e., can be obtained from $x_1 \prec x_2$ and $x_1 \succ x_2$ by compositions and symmetric groups actions. Hence, the monomial $\big(x_1^{(s_1)} \dots x_n^{(s_n)}\big)$ may be expressed in terms of operations $\succ$ and $\prec$ on the variables $x_1, \dots, x_n \in X$ in the differential algebra $\mathrm{Var}\big\langle X^{(\omega)}\big\rangle$. It remains to make the substitution $x_j \to x_{i_j}$ to get the desired expression for $u$ in $D\mathrm{Var}\langle X\rangle$. ∎

**Example 3.2.** For example, if $u = (x_1(x_1'' x_2)) \in \mathrm{Lie}\big\langle X^{(\omega)}\big\rangle$ then $[u] = x_1 x_2'' x_3 \otimes (x_1(x_2 x_3))$. It is straightforward to check that $[u] = x_1 \prec (x_2 \succ x_3) - x_2 \succ (x_1 \prec x_3) - (x_1 \prec x_2) \prec x_3$, where the monomials of degree 3 represent compositions of $x_1 \prec x_2$, $x_2 \prec x_1$, $x_1 \succ x_2$, $x_2 \succ x_1$, and $\mathrm{id} = x_1 \otimes x_1$ as in Example 2.4. Hence, $u = x_1 \prec (x_1 \succ x_2) - x_1 \succ (x_1 \prec x_2) - (x_1 \prec x_1) \prec x_2$.

**Remark 3.3.** The condition

$$D\mathrm{Var}\langle X\rangle = \big\{ f \in \mathrm{Var}\big\langle X^{(\omega)}\big\rangle \mid \mathrm{wt}(f) = -1 \big\}$$

for a binary operad Var implies $\mathrm{Nov} \circ \mathrm{Var} = \mathrm{Nov} \otimes \mathrm{Var}$. Indeed, the one-to-one correspondence

$$x_1^{(s_1)} \cdots x_n^{(s_n)} \otimes (x_{i_1} \cdots x_{i_n}) \leftrightarrow \big(x_{i_1}^{(s_{i_1})} \cdots x_{i_n}^{(s_{i_n})}\big), \qquad \sum_i s_i = n - 1,$$

between $(\mathrm{Nov} \otimes \mathrm{Var})(n)$ and $D\mathrm{Var}(n)$ preserves compositions and symmetric groups actions. Hence, if $\big(x_{i_1}^{(s_{i_1})} \cdots x_{i_n}^{(s_{i_n})}\big)$ may be expressed via $X$ in terms of operations $\mu^{\succ}$, $\mu^{\prec}$ ($\mu \in \mathrm{Var}(2)$) then $x_1^{(s_1)} \cdots x_n^{(s_n)} \otimes (x_{i_1} \cdots x_{i_n}) \in (\mathrm{Nov} \circ \mathrm{Var})(n)$.

**Proposition 3.4.** *The operad* $\mathrm{Var} = \mathrm{Nov}$ *satisfies the conditions of Lemma* 3.1*, i.e.,* $\mathrm{Nov} \circ \mathrm{Nov} = \mathrm{Nov} \otimes \mathrm{Nov}$.

**Proof.** In [7], a linear basis of the free Novikov algebra generated by an ordered set was described in terms of partitions and Young diagrams. To prove the assertion, we will use this basis, which consists of non-associative monomials constructed from Young diagrams with cells properly filled with generators, see [8, Section 4] for details.

Suppose $h$ is a non-associative monomial of degree $n$ in $\mathrm{Nov}\big\langle X^{(\omega)}\big\rangle$ of weight $-1$. The problem is to show that $h \in D\mathrm{Nov}\langle X\rangle$. Let us proceed by induction both on the degree of $h$ $\big($number of letters from $X^{(\omega)}\big)$ and on the number of "naked" letters $x = x^{(0)}$, $x \in X$, that appear in $h$. $\big($For brevity, letters of the form $x^{(n)}$ for $n > 0$ are called "derived".$\big)$

If $\deg h = 1$ then $h = x \in X \subset D\mathrm{Nov}\langle X\rangle$. If $\deg h > 1$ but $h$ contains only one "naked" letter $x \in X$ then all other letters are of the form $y_i' \in X'$ since $\mathrm{wt}(h) = -1$. Then the identities of left symmetry (1.1) and right commutativity (1.2) allow us to rewrite $h$ as a linear combination of nonassociative monomials with subwords of the form $xy'$ or $y'x$. The latter may be processed in a way described below: e.g., $xy'$ may be replaced with a new letter $(x \prec y)$ so that we get words of smaller degree in the extended alphabet.

*Case 1.* If the monomial $h$ has a subword $a^{(k)}b$ or $ab^{(k)}$ for some $a, b \in X$ and $k \geq 1$, then we may transform $h$ to an expression in the extended alphabet (adding a new letter $a \succ b$ to $X$) as

$$a^{(k)}b = (a \succ b)^{(k-1)} - \sum_{s \geq 1} \binom{k-1}{s} a^{(k-s)} b^{(s)},$$

or, similarly, for $ab^{(k)}$. The expression in the right-hand side contains monomials either of smaller degree or with a smaller number of "naked" letters. Hence, $h \in D\mathrm{Nov}\langle X\rangle$.

*Case 2.* For the general case, we need to recall the description of a linear basis of the free Novikov algebra (see [7, 8, 9]). Suppose $X^{(\omega)}$ is linearly ordered in such a way that every "naked" letter is smaller than every derived one.

Every element of $\mathrm{Nov}\big\langle X^{(\omega)}\big\rangle$ may be presented as a linear combination of non-associative words of the form

$$h = (\cdots(W_1W_2)\cdots W_{k-1})W_k, \tag{3.1}$$

where $W_1 = a_{1,r_1+1}(a_{1,r_1}(\cdots(a_{1,2}a_{1,1})\cdots))$, $W_l = a_{l,r_l}(a_{l,r_l-1}(\cdots(a_{l,2}a_{l,1})\cdots))$, $l = 2,\dots,k$, $r_1 \ge r_2 \ge \cdots \ge r_k$, $a_{i,j} \in X^{(\omega)}$. The letters are ordered in such a way that the following conditions hold:

- If $r_i = r_{i+1}$ then $a_{i,1} \ge a_{i+1,1}$ for $i = 1,\dots,k-1$;
- $a_{1,r_1+1} \ge a_{1,r_1} \ge \cdots \ge a_{1,2} \ge a_{2,r_2} \ge \cdots \ge a_{2,2} \ge \cdots \ge a_{k-1,2} \ge a_{k,r_k} \ge \cdots \ge a_{k,2}$.

In particular, if at least one of the words $W = W_l$ contains both "naked" and derived letters then there are two options: (i) the last letter $a_{l,1}$ is a derived one; (ii) $a_{l,1}$ is "naked". In the first case, the final subword $(a_{l,2}a_{l,1})$ of $W$ is of the form considered in Case 1 since $a_{l,2}$ has to be "naked" due to the choice of order on $X^{(\omega)}$. In the second case, we may find a suffix of $W$ which is of the following form:

$$y^{(n)}(x_1(x_2\cdots(x_{s-1}x_s)\cdots)), \qquad x_i, y \in X, \quad n > 0.$$

An easy induction on $s \ge 1$ shows that the suffix may be transformed (by means of left symmetry) to a sum of monomials considered in Case 1.

Hence, it remains to consider the case when each $W_l$ contains either only "naked" letters or only derived ones. Due to the ordering of letters in $X^{(\omega)}$, the word $h$ of the form (3.1) has the following property: there exists $1 \le l < k$ such that all $W_i$ for $i \le l$ consist of only derived letters and for $i > l$ all $W_i$ are nonassociative words in "naked" letters. Then use right-commutativity to transform $h$ to the form $h = (\cdots((\cdots((W_1W_{l+1})W_2)\cdots W_l)W_{l+2})\cdots W_k)$. Here $W_1 = y^{(n)}u$, $n > 0$, $u$ consists of derived letters, $W_{l+1} = xv$, $x \in X$, $v$ consists of "naked" letters. The subword $W_1W_{l+1}$ may be transformed to $\big(y^{(n)}(xv)\big)u$, $n > 0$, by right commutativity, and its prefix $y^{(n)}(xv)$ transforms (by induction on $\deg v$) to a form described in Case 1 by left symmetry:

$$y^{(n)}(xv) = \big(y^{(n)}x\big)v - \big(xy^{(n)}\big)v + x\big(y^{(n)}v\big). \qquad \blacksquare$$

**Theorem 3.5.** *If* Var *is a binary operad such that* $\mathrm{Var}\circ\mathrm{Nov} = \mathrm{Var}\otimes\mathrm{Nov}$ *then every* $D$Var*-algebra is special.*

**Proof.** Suppose $V$ is a $D$Var-algebra. Then $V$ may be presented as a quotient of a free algebra $D\mathrm{Var}\langle X\rangle$ modulo an ideal $I$. Consider the embedding $D\mathrm{Var}\langle X\rangle \subset \mathrm{Var}\big\langle X^{(\omega)}\big\rangle$ and denote by $J$ the differential ideal of $\mathrm{Var}\big\langle X^{(\omega)}\big\rangle$ generated by $I$. Then $U = \mathrm{Var}\big\langle X^{(\omega)}\big\rangle/J$ is the universal enveloping differential Var-algebra of $V$. It remains to prove that $J \cap D\mathrm{Var}\langle X\rangle = I$, namely, the "$\subseteq$" part is not a trivial one.

Assume $f \in J$. Then there exists a family of (differential) polynomials $\Phi_i \in \mathrm{Var}\big\langle (X\cup\{t\})^{(\omega)}\big\rangle$ such that $f = \sum_i \Phi_i|_{t=g_i}$, for some $g_i \in I$. If, in addition, $f \in D\mathrm{Var}\langle X\rangle$ then $\mathrm{wt}(f) = -1$. Since $\mathrm{wt}\, g_i = -1$, we should have $\mathrm{wt}\,\Phi_i = -1$ for all $i$. By Lemma 3.1, every polynomial $\Phi_i$ may be represented as an element of $D\mathrm{Var}\langle X\cup\{t\}\rangle$, so $\Phi_i|_{t=g_i} \in I$ for all $i$, and thus $f \in I$. $\blacksquare$

**Corollary 3.6.** *Every* $D$Nov*-algebra* $V$ *with operations* $\succ$ *and* $\prec$ *can be embedded into a commutative algebra with two commuting derivations* $d$ *and* $\partial$ *so that* $x \succ y = \partial(x)d(y)$, $x \prec y = x\partial d(y)$ *for all* $x, y \in V$.

**Proof.** For a free $D$Nov-algebra generated by a set $X$, we have the following chain of inclusions given by Proposition 3.4:

$$D\mathrm{Nov}\langle X\rangle \subset \mathrm{Nov}\,\mathrm{Der}\langle X,\partial\rangle = \mathrm{Nov}\big\langle X^{(\omega)}\big\rangle \subset \mathrm{Com}\,\mathrm{Der}\big\langle X^{(\omega)},d\big\rangle = \mathrm{Com}\big\langle X^{(\omega,\omega)}\big\rangle.$$

Here $X^{(\omega,\omega)} = \big(X^{(\omega)}\big)^{(\omega)} = \big\{x^{(n,m)} \mid x\in X, n,m\in\mathbb{Z}_+\big\}$, a variable $x^{(n,m)}$ represents $d^n\partial^m(x)$. The elements of $D\mathrm{Nov}\langle X\rangle$ are exactly those polynomials in $\mathrm{Com}\big\langle X^{(\omega,\omega)}\big\rangle$ that can be presented as linear combinations of monomials

$$x_1^{(n_1,m_1)}\cdots x_k^{(n_k,m_k)}, \qquad \sum_i n_i = \sum_i m_i = k+1.$$

The same arguments as in the proof of Theorem 3.5 imply the claim. ■

Apart from the operads Com and Lie considered above, the operads Pois and As governing the varieties of Poisson and associative algebras, respectively, also satisfy the conditions of Theorem 3.5 [19, 21]. However, even if $\mathrm{Nov}\circ\mathrm{Var}\neq\mathrm{Nov}\otimes\mathrm{Var}$ then it is still possible that every $D$Var-algebra is special. For example, if Var = Jord is the variety of Jordan algebras then the corresponding operad is not quadratic and, in particular, the element $[u]=x_1x_2'x_3'\otimes x_1(x_2x_3)\in \mathrm{Nov}(3)\otimes\mathrm{Jord}(3)$ does not belong to $(\mathrm{Nov}\circ\mathrm{Jord})(3)$. The operad $\mathrm{Nov}\circ\mathrm{Jord}$ is generated by a single operation $x_1\succ x_2 = x_2\prec x_1$ due to commutativity of Jord. Hence, $\mathrm{Nov}\circ\mathrm{Jord}$ is a homomorphic image of the free operad $\mathcal{F}_2$. On the other hand, we have

**Proposition 3.7.** *For every non-associative algebra $V$ with a multiplication $\nu\colon V\otimes V\to V$ there exists an associative algebra $(A,\cdot)$ with a derivation $d$ such that $V\subseteq A$ and $\nu(u,v) = d(u)\cdot v + v\cdot d(u)$ for all $u,v\in V$.*

In other words, we are going to show that every non-associative algebra $V$ embeds into the derived anti-commutator algebra $\big(A^{(+)}\big)^{(d)}$ for an appropriate associative differential algebra $(A,d)$.

**Proof.** Let us choose a linear basis $B$ of $V$ equipped with an arbitrary total order $\leq$ such that $(B,\leq)$ is a well-ordered set. Then define $F$ to be the free associative algebra generated by $B^{(\omega)}$. Induce the order $\leq$ on $B^{(\omega)}$ by the following rule:

$$a^{(n)}\leq b^{(m)} \iff (n,a)\leq(m,b) \text{ lexicographically},$$

and expand it to the words in $B^{(\omega)}$ by the deg-lex rule (first by length, then lexicographically).

Consider the set of defining relations

$$S=\left\{a^{(n)}b+\sum_{s\geq 1}\binom{n-1}{s}\big(a^{(n-s)}b^{(s)}+b^{(s)}a^{(n-s)}\big)+ba^{(n)}-\nu(a,b)^{(n-1)} \mid a,b\in B, n\geq 1\right\}.$$

All relations in the set $S$ are obtained from $a'b+ba'-\nu(a,b)$ by formal derivation $d\colon x^{(s)}\to x^{(s+1)}$, $x^{(s)}\in B^{(\omega)}$. Hence, $A=F/(S)$ is a differential associative algebra, and the map $\varphi\colon V\to A^{(+)}$, $\varphi(v)=v+(S)$, $v\in V$, preserves the operation, i.e., $\varphi(\nu(u,v)) = d(\varphi(u))\cdot\varphi(v)+\varphi(v)\cdot d(\varphi(u))$ for all $u,v\in V$.

The principal parts of $f\in S$ relative to the order $\leq$ are $a^{(n)}b$, $a,b\in B$, $n\geq 1$. These words have no compositions of inclusion or intersection, hence, $S$ is a Gröbner–Shirshov basis in $F$ and the images of all variables from $B$ are linearly independent in $A$ since they are $S$-reduced (see, e.g., [3] for the definitions). Therefore, $\varphi\colon V\to A^{(+)}$ is the desired embedding. ■

Consider the variety SJord generated by all special Jordan algebras (i.e., embeddable into associative ones with respect to the anti-commutator). In particular, for every associative algebra $(A, \cdot)$ with a derivation $d$ the same space $A$ equipped with a new operation $\nu(u,v) = d(u)\cdot v + v\cdot d(u)$, $u, v \in A$, is an algebra from $D$SJord. Proposition 3.7 implies that there are no identities that hold for the binary operation $\nu$ like that. Hence, the varieties $D\mathrm{SJord} = D\mathrm{Jord}$ coincide with the variety of all nonassociative algebras with one operation. However, again from Proposition 3.7 every $D$Jord-algebra embeds into an appropriate Jordan algebra (even a special Jordan algebra) with a derivation.

In the next section, we find an example of a variety Var for which the embedding statement fails.

## 4 Dendriform splitting and a non-special pre-Novikov algebra

Another example of a variety Var not satisfying the conditions of Theorem 3.5 is the class Zinb of Zinbiel (dual Leibniz or pre-commutative) algebras. This is a particular case of the dendriform splitting of a binary operad described in [1, 13]. Namely, if Var is a variety of algebras with (one or more) binary operation $\mu(x,y) = xy$ satisfying a family of multi-linear identities $\Sigma$ then preVar is a variety of algebras with duplicated set of binary operations $\mu_\vdash(x,y) = x \vdash y$, $\mu_\dashv(x,y) = x \dashv y$ satisfying a set of identities pre$\Sigma$ defined as follows. Assume $f = f(x_1,\dots,x_n)$ is a multi-linear polynomial of degree $\le n$, and let $k \in \{1,\dots,n\}$. Suppose $u$ is a nonassociative monomial in the variables $x_1,\dots,x_n$ such that each $x_i$ appears in $u$ no more than once. Define a polynomial $u^{[k]}$ in $x_1,\dots,x_n$ relative to the operations $\mu_\vdash$, $\mu_\dashv$ by induction on the degree. If $u = x_i$ then $u^{[k]} = x_i$; if $u = vw$ and $x_k$ appears in $v$ (or in $w$) then $u^{[k]} = v^{[k]} \dashv w^{[k]}$ (or, respectively, $v^{[k]} \vdash w^{[k]}$); if $x_k$ does not appear in $u$ then set $u^{[k]} = v^{[k]} \dashv w^{[k]} + v^{[k]} \vdash w^{[k]}$. Transforming each monomial $u$ in the multi-linear polynomial $f$ in this way, we get $f^{[k]}(x_1,\dots,x_n)$. The collection of all such $f^{[k]}$ for $f \in \Sigma$, $k = 1,\dots,\deg f$, forms the set of defining relations of a new variety denoted preVar.

For example, for $f(x_1,x_2,x_3) = (x_1x_2)x_3 - x_1(x_2x_3)$ the polynomials $f^{[k]}$, $k = 1,2,3$, are given by

$$\begin{aligned}
f^{[1]} &= (x_1 \dashv x_2) \dashv x_3 - x_1 \dashv (x_2 \dashv x_3 + x_2 \vdash x_3),\\
f^{[2]} &= (x_1 \vdash x_2) \dashv x_3 - x_1 \vdash (x_2 \dashv x_3),\\
f^{[3]} &= (x_1 \vdash x_2 + x_1 \dashv x_2) \vdash x_3 - x_1 \vdash (x_2 \vdash x_3).
\end{aligned} \tag{4.1}$$

These identities define the variety of pre-associative or dendriform algebras [20].

If the initial operation was commutative or anti-commutative then the set of identities pre$\Sigma$ includes $x_1 \vdash x_2 = \pm x_2 \dashv x_1$, so the operations in preVar are actually expressed via $\mu_\vdash$ or $\mu_\dashv$. For example, Var = Lie produces the variety preLie of left- or right-symmetric algebras (depending on the choice of $\vdash$ or $\dashv$). If Var = Com then, in terms of the operation $x \cdot y = x \dashv y = y \vdash x$, all three identities (4.1) of pre-associative algebras are equivalent to (2.1).

In a similar way, one may derive the identities of a preNov-algebra by means of the dendriform splitting applied to (1.1) and (1.2). Routine simplification leads us to the following definition: a preNov-algebra is a linear space with two bilinear operations $\vdash$, $\dashv$ satisfying

$$\begin{aligned}
&(x_1 \dashv x_2) \dashv x_3 = (x_1 \dashv x_3) \dashv x_2,\\
&(x_1 \vdash x_2) \dashv x_3 = (x_1 \vdash x_3) \vdash x_2 + (x_1 \dashv x_3) \vdash x_2,\\
&(x_1 \dashv x_2) \dashv x_3 - x_1 \dashv (x_2 \dashv x_3) - x_1 \dashv (x_2 \vdash x_3) = (x_2 \vdash x_1) \dashv x_3 - x_2 \vdash (x_1 \dashv x_3),\\
&(x_1 \vdash x_3) \dashv x_2 - x_1 \vdash (x_2 \vdash x_3) = (x_2 \vdash x_3) \dashv x_1 - x_2 \vdash (x_1 \vdash x_3).
\end{aligned} \tag{4.2}$$

The formal change of operations $x \dashv y = x \prec y$, $x \vdash y = y \succ x$ turns (4.2) exactly into (2.2). Hence, the operad preNov = pre$D$Com defines the same class of algebras as $D$Zinb = $D$preCom.

**Remark 4.1.** This is not hard to compute that pre$D$Lie $= D$preLie and pre$D$As $= D$preAs. In general, for every binary operad Var there exists a morphism of operads pre$D$Var $\to D$preVar (i.e., every $D$preVar-algebra is a pre$D$Var-algebra). We do not know an example when this morphism is not an isomorphism, i.e., when the operations pre and $D$ applied to a binary operad do not commute.

An equivalent way to define the variety preVar was proposed in [13]. Let Perm stand for the variety of associative algebras that satisfy left commutativity

$$x_1x_2x_3 = x_2x_1x_3.$$

An algebra $V$ with two operations $\dashv, \vdash$ is a preVar-algebra if and only if for every $P \in$ Perm the space $P \otimes V$ equipped with the single operation

$$(p \otimes u)(q \otimes v) = pq \otimes (u \vdash v) + qp \otimes (u \dashv v), \qquad p, q \in P, \quad u, v \in V, \tag{4.3}$$

is a Var-algebra. The same statement holds in the case when the binary operad Var is generated by several operations.

**Remark 4.2.** For an arbitrary binary operad, there is a morphism of operads $\zeta\colon$ preVar $\to$ Zinb $\circ$ Var. Namely, for every $A \in$ Var and for every $Z \in$ Zinb the space $Z \otimes A$ equipped with two operations

$$(z \otimes a) \vdash (w \otimes b) = (w \cdot z) \otimes ab, \qquad (z \otimes a) \dashv (w \otimes b) = (z \cdot w) \otimes ab,$$

for $z, w \in Z$, $a, b \in A$, is a preVar-algebra.

However, preVar and Zinb $\circ$ Var are not necessarily isomorphic. For example, if Var is defined by the identity $(x_1 \cdot x_2) \cdot x_3 = 0$ then the kernel of $\zeta$ is nonzero.

As a corollary, we obtain

**Proposition 4.3.** *The operad* preNov *is isomorphic to the Manin white product* Zinb $\circ$ Nov.

**Remark 4.4.** Let $V$ be a $D$Zinb-algebra. In terms of pre-Novikov operations $\dashv$ and $\vdash$, the conformal algebra structure mentioned in Remark 2.6 is expressed as

$$(u_{(\lambda)}v) = \partial(v \dashv u) + \lambda(u \vdash v + v \dashv u), \qquad u, v \in V.$$

This is indeed a left-symmetric conformal algebra which is easy to check via the conformal analogue of (4.3). By slight abuse of notations, for every Perm-algebra $P$ the operation

$$\begin{aligned}
[(p \otimes u)_{(\lambda)}(q \otimes v)] &= pq \otimes (u_{(\lambda)}v) - qp \otimes (v_{(-\partial-\lambda)}u) \\
&= pq \otimes (\partial(v \dashv u) + \lambda(u \vdash v + v \dashv u)) - qp \otimes (\partial(u \dashv v) - (\lambda + \partial)(v \vdash u + u \dashv v)) \\
&= \partial(qp \otimes v \vdash u + pq \otimes v \dashv u) + \lambda(pq \otimes u \vdash v + qp \otimes u \dashv v + qp \otimes v \vdash u + pq \otimes v \dashv u)
\end{aligned}$$

is exactly the quadratic Lie conformal algebra structure [23] on $\Bbbk[\partial] \otimes P \otimes V$ corresponding to the Novikov algebra $P \otimes V$:

$$[x_{(\lambda)}y] = \partial(yx) + \lambda(xy + yx)$$

for $x = p \otimes u$, $y = q \otimes v$, and the product is given by (4.3).

Hence the construction of a preLie conformal algebra from a $D$Zinb-algebra is a quite clear consequence of the commutativity of tensor product.

The final statement of this section shows a substantial difference between the properties of Novikov algebras and preNov-algebras. Although the defining identities of preNov are exactly those that hold on differential Zinbiel algebras with operations (2.3) (i.e., the dendriform analogue of [9, Theorem 7.8] holds), the general embedding statement (i.e., the dendriform analogue of [4, Theorem 3]) turns to be wrong.

**Theorem 4.5.** *If the characteristic of the base field $\Bbbk$ is not 2 or 3 then there exists a $D$Zinb-algebra which cannot be embedded into a differential Zinbiel algebra.*

**Proof.** Consider the free Zinbiel algebra $F$ generated by

$$\{a,b\}^{(\omega)} = \big\{a, b, a', b', \dots, a^{(n)}, b^{(n)}, \dots\big\}.$$

This is the free differential Zinbiel algebra with two generators $a$, $b$, its derivation $d$ maps $x^{(n)}$ to $x^{(n+1)}$ for $x \in \{a,b\}$. The product of two elements $f, g \in F$ is denoted $f \cdot g$.

For every $f, g \in F$, define $f \prec g$, $f \succ g$ by the rule (2.3):

$$f \prec g = f \cdot d(g), \qquad f \succ g = d(f) \cdot g.$$

Then $(F, \prec, \succ)$ is a $D$Zinb-algebra, and its subalgebra generated by $a$, $b$ is isomorphic to the free algebra $D\mathrm{Zinb}\langle a,b\rangle$.

Denote $f = b \prec b = b \cdot b' \in D\mathrm{Zinb}\langle a,b\rangle \subset F$, and let $J$ stand for the ideal in $F$ generated by $f$ and all its derivatives:

$$J = \big(f, f', f'', \dots\big) \lhd F, \qquad d(J) \subseteq J.$$

In particular,

$$h = a \cdot \big(f' \cdot b'\big) - a \cdot \big(f \cdot b''\big) \in J.$$

Let us show that $h \in D\mathrm{Zinb}\langle a,b\rangle \subset J$. Indeed,

$$\begin{aligned} h &= a \cdot \big(\big(b \cdot b'\big)' \cdot b'\big) - a \cdot \big(\big(b \cdot b'\big) \cdot b''\big) \\ &= a \cdot \big(\big(b' \cdot b'\big) \cdot b'\big) + a \cdot \big(\big(b \cdot b''\big) \cdot b'\big) - a \cdot \big(\big(b \cdot b'\big) \cdot b''\big) = a \cdot \big(\big(b' \cdot b'\big) \cdot b'\big) \end{aligned}$$

due to the right commutativity of Zinbiel algebras. Next, $(b' \cdot b') \cdot b' = 2b' \cdot (b' \cdot b')$, so $(b' \cdot b') \cdot b' + b' \cdot (b' \cdot b') = \frac{3}{2}(b' \cdot b') \cdot b'$. Therefore,

$$\begin{aligned} h &= a \cdot \big(\big(b' \cdot b'\big) \cdot b'\big) = \frac{2}{3} a \cdot \big(\big(b' \cdot b'\big) \cdot b' + b' \cdot \big(b' \cdot b'\big)\big) = \frac{2}{3}\big(a \cdot \big(b' \cdot b'\big)\big) \cdot b' \\ &= \frac{1}{3}\big(\big(a \cdot b'\big) \cdot b'\big) \cdot b' = \frac{1}{3}((a \prec b) \prec b) \prec b = 2\big[ab'b'b'\big] \in D\mathrm{Zinb}\langle a,b\rangle. \end{aligned}$$

As in Example 2.3, we denote by $[x_1 x_2 \cdots x_{n-1} x_n]$ the following expression in a Zinbiel algebra:

$$[x_1 x_2 \cdots x_{n-1} x_n] = x_1 \cdot (x_2 \cdot (\cdots (x_{n-1} \cdot x_n) \cdots)).$$

Recall [20] that all such expressions with $x_i$ from a set $X$ form a linear basis of the free Zinbiel algebra generated by $X$ (i.e., this is a normal form in $\mathrm{preCom}\langle X\rangle$).

Let $I$ be the ideal in $D\mathrm{Zinb}\langle a,b\rangle$ generated by $f$, and let $V = D\mathrm{Zinb}\langle a, b \mid f\rangle = D\mathrm{Zinb}\langle a,b\rangle/I$. Then $F/J$ is the universal differential Zinbiel envelope of $V$. To prove the theorem, it remains to show that $h \notin I$: if so then $h + I$ would lie in the kernel of every homomorphism from $V$ to the derived algebra $Z^{(d)}$ constructed from a differential Zinbiel algebra $Z$ with a derivation $d$.

Assume $[ab'b'b'] \in I$. The specific of the Zinbiel identity (2.1) is that the first letter remains unchanged in all terms. Hence, $[ab'b'b']$ should be a linear combination of the elements $(a * f \star b)$,

$(a * b \star f)$, where $*, \star \in \{\prec, \succ\}$, with two possible bracketing each, so we have in total 16 terms under consideration. Let us write them all in the normal form in $F$:

$$
\begin{aligned}
(a \prec f) \prec b &= 3\big[ab'b'b'\big] + \big[ab'bb''\big] + \big[abb'b''\big] + \big[abb''b'\big],\\
(a \prec f) \succ b &= \big[a'bb'b'\big] + \big[a'b'bb'\big] + \big[a'b'b'b\big] + 2\big[a'bbb''\big] + \big[a'bb''b\big] + \big[abb''b'\big] + \big[ab''bb'\big]\\
&\quad + \big[ab''b'b\big] + 2\big[abb'b''\big] + 2\big[ab'bb''\big] + 2\big[ab'b''b\big] + 2\big[abbb'''\big] + \big[abb'''b\big],\\
(a \succ f) \prec b &= \big[a'b'bb'\big] + 2\big[a'bb'b'\big],\\
(a \succ f) \succ b &= 2\big[a''bbb'\big] + \big[a''bb'b\big] + \big[a'bb'b'\big] + \big[a'b'bb'\big] + \big[a'b'b'b\big] + 2\big[a'bbb''\big] + \big[a'bb''b\big],\\
a \prec (f \prec b) &= 2\big[ab'b'b'\big] + 2\big[abb''b'\big] + 2\big[abb'b''\big],\\
a \prec (f \succ b) &= a\big[b'b'b\big]' + a\big[b'bb'\big]' + a\big[bb''b\big]' + a\big[bbb''\big]' = \big[ab''b'b\big] + 2\big[ab'b''b\big] + 2\big[ab'b'b'\big]\\
&\quad + \big[ab''bb'\big] + 2\big[ab'bb''\big] + \big[abb'''b\big] + \big[abb''b'\big] + \big[abb'b''\big] + \big[abbb'''\big],\\
a \succ (f \prec b) &= 2\big[a'bb'b'\big],\\
a \succ (f \succ b) &= \big[a'b'b'b\big] + \big[a'b'bb'\big] + \big[a'bb''b\big] + \big[a'bbb''\big],\\
(a \prec b) \prec f &= 3\big[ab'b'b'\big] + \big[ab'bb''\big] + \big[abb'b''\big] + \big[abb''b'\big],\\
(a \prec b) \succ f &= \big[a'b'bb'\big] + 2\big[a'bb'b'\big] + \big[ab''bb'\big] + \big[abb''b'\big] + \big[abb'b''\big],\\
(a \succ b) \prec f &= \big[a'bb'b'\big] + \big[a'b'bb'\big] + \big[a'b'b'b\big] + 2\big[a'bbb''\big] + \big[a'bb''b\big],\\
(a \succ b) \succ f &= 2\big[a''bbb'\big] + \big[a''bb'b\big] + \big[a'b'bb'\big] + 2\big[a'bb'b'\big],\\
a \prec (b \prec f) &= \big[ab'b'b'\big] + \big[abb''b'\big] + 2\big[abb'b''\big] + \big[ab'bb''\big] + \big[abbb'''\big],\\
a \prec (b \succ f) &= \big[ab''bb'\big] + \big[ab'b'b'\big] + \big[ab'bb''\big],\\
a \succ (b \prec f) &= \big[a'bb'b'\big] + \big[a'bbb''\big],\\
a \succ (b \succ f) &= \big[a'b'bb'\big].
\end{aligned}
$$

Arrange the normal Zinbiel words in the following order: $[ab'b'b']$, $[ab'bb'']$, $[abb'b'']$, $[abb''b']$, $[ab''bb']$, $[ab''b'b]$, $[ab'b''b]$, $[abbb''']$, $[abb'''b]$, $[a''bb'b]$, $[a''bbb']$, $[a'bb'b']$, $[a'b'bb']$, $[a'b'b'b]$, $[a'bbb'']$, $[a'bb''b]$. Then the assumption $[ab'b'b'] \in I$ is equivalent to the condition that the row vector $e_1 = (1, 0, \ldots, 0) \in \Bbbk^{16}$ belongs to the row space of the matrix

$$
\begin{pmatrix}
3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 2 & 1\\
2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0\\
1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

This matrix may be transformed by elementary row and column transformations with integer coefficients to the triangular form with 1, 2, or 0 on the diagonal, so that its rank equals 10 for $\operatorname{char} \Bbbk \neq 2$. Adding one more row $e_1$ increases the rank, so $[ab'b'b'] \notin I$. ∎

As a corollary, we obtain the following observation which is in some sense converse to Corollary 3.6. Suppose $W$ is a Novikov algebra equipped with a Rota–Baxter operator, that is, $R\colon W \to W$ is a linear operator such that

$$R(u)R(v) = R(uR(v) + R(u)v), \qquad u, v \in W.$$

Then, in general, there is no commutative algebra $(A, *)$ with a derivation $d$ and a Rota–Baxter operator $\rho$ such that $W \subseteq A$, $uv = u * d(v)$, $R(u) = \rho(u)$, for $u, v \in W$, and $\rho d = d\rho$. In other words, a Novikov Rota–Baxter algebra cannot be in general embedded into a commutative Rota–Baxter algebra with a derivation.

Indeed, assume such a system $(A, *, d, \rho)$ exists for every Novikov algebra with a Rota–Baxter operator. Every pre-Novikov algebra $V$ with operations $\vdash$ and $\dashv$ can be embedded into a Novikov algebra $W = \widehat{V}$ with a Rota–Baxter operator so that $u \vdash v = R(u)v$, $u \dashv v = uR(v)$, $u, v \in V$ (see, e.g., [13]). We may further embed this $W$ into a differential commutative Rota–Baxter algebra $(A, *, d, \rho)$ in which $a \vdash b = \rho(a) * d(b)$, $a \dashv b = a * d(\rho(b)) = a * \rho(d(b))$, for $a, b \in W$. On the other hand, the new operation $\cdot$ on $A$ given by $a \cdot b = a * \rho(b)$ turns $A$ into a Zinbiel algebra, $d$ remains a derivation relative to this new operation, and $u \vdash v = d(v) \cdot u = v \succ u$, $u \dashv v = u \cdot d(v) = u \prec v$ for $u, v \in V$. Therefore, we would embed a $D$Zinb-algebra into a differential Zinbiel algebra which is not the case.

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