Seiberg–Witten Geometry of Four-Dimensional $\mathcal{N} = 2$ Quiver Gauge Theories

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Abstract. Seiberg–Witten geometry of mass deformed $\mathcal{N} = 2$ superconformal ADE quiver gauge theories in four dimensions is determined. We solve the limit shape equations derived from the gauge theory and identify the space $\mathfrak{M}$ of vacua of the theory with the moduli space of the genus zero holomorphic (quasi)maps to the moduli space $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ of holomorphic $G^{\mathbb{C}}$-bundles on a (possibly degenerate) elliptic curve $\mathcal{E}$ defined in terms of the microscopic gauge couplings, for the corresponding simple ADE Lie group G. The integrable systems $\mathfrak{P}$ underlying the special geometry of $\mathfrak{M}$ are identified. The moduli spaces of framed G-instantons on $\mathbb{R}^2 \times \mathbb{T}^2$, of G-monopoles with singularities on $\mathbb{R}^2 \times \mathbb{S}^1$, the Hitchin systems on curves with punctures, as well as various spin chains play an important rôle in our story. We also comment on the higher-dimensional theories.

Key words: low-energy theory; instantons; monopoles; integrability

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A note added ten years later. This is a minimally edited version of the manuscript [150], which was never previously submitted to a peer-reviewed journal for no reason in particular. We can observe that [150] laid foundation to several important developments in supersymmetric gauge theories and two-dimensional conformal/integrable theories, in BPS/CFT correspondence, including connections to quantum groups and integrability [23, 151], generalizations [102, 103, 104, 160] of W-algebras, generalizations [143, 144] of q-characters [54], gauge origami [145], studies of surface defects [88, 91, 146, 147, 158], connections to (and generalizations of) geometric Langlands program [99] and representation theory, refinements of Donaldson theory [123], and many many more. The connection of geometry and physics explored here was always appreciated by our former Director Jean-Pierre Bourguignon. We are happy to dedicate our work to his anniversary.

This paper is a contribution to the Special Issue on Differential Geometry Inspired by Mathematical Physics in honor of Jean-Pierre Bourguignon for his 75th birthday. The full collection is available at https://www.emis.de/journals/SIGMA/Bourguignon.html

Contents

1 Introduction

In this work a class of quiver $\mathcal{N} = 2$ supersymmetric theories in four dimensions is analyzed. The first problem of this sort was solved in [166, 167] for the SU(2) gauge theory in four dimensions with eight supercharges.

We study mass perturbed $\mathcal{N} = 2$ superconformal theories, and compute the exact metric

$$\mathrm{d}s^2_{\mathfrak{M}} = G_{\mathcal{I}\bar{\mathcal{J}}}\mathrm{d}u^{\mathcal{I}}\mathrm{d}\bar{u}^{\bar{\mathcal{J}}}$$

on the moduli space $\mathfrak{M}$ of vacua of the low-energy effective theory. We also compute the vacuum expectation values

$$\langle \mathcal{O}_{i,n} \rangle_u$$

of all gauge invariant $\mathcal{N} = 2$ chiral operators.

Our theories have the gauge group G_{g} which is a product of a finite number of special unitary groups. The technique we use is the saddle point approach to the calculation of the supersymmetric partition function of the theory in Ω-background [140]. The partition function is given by the sum over special instanton configurations. In the limit, where the Ω-deformation is removed so that the theory approaches the original flat space theory, the sum over the special instantons is dominated by the contribution of one particular special instanton configuration, of a very large instanton charge (with the expected small effective density of instanton charge). This configuration, the so-called limit shape, is found in this work using a novel approach, built on the analytic techniques of [149]. Namely, we interpret the limit shape equations as the conditions defining the analytic continuations of the generating functions

$$\mathcal{Y}_i(x) = \exp\langle \mathrm{tr}\log(x - \Phi_i)\rangle_u,$$

where i labels the simple factors in the gauge group G_{g}, and Φ_i is the corresponding complex adjoint Higgs scalar field. We get the system of (algebraic) equations determining these functions by fixing the set of basic invariants of the monodromy of the analytic continuation.

Recall that a complex Lie group $\mathbf{G}_{\mathrm{q}}$ is naturally associated with the quiver gauge theory. This group is different from the original gauge group G_{g} of the theory. Roughly speaking the Dynkin diagram of $\mathbf{G}_{\mathrm{q}}$ is the universal cover of the quiver of the gauge theory. The group $\mathbf{G}_{\mathrm{q}}$ may be infinite-dimensional. In fact, for the $\mathcal{N} = 2$ superconformal theories the corresponding Lie algebra $\mathfrak{g}_{\mathrm{q}}$ is the finite-dimensional simple Lie algebra of the ADE type $\mathfrak{g}$, or its affine version $\widehat{\mathfrak{g}}$, or the algebra $\widehat{\mathfrak{gl}}_\infty$.

Our main construct is the x-dependent element $g(x)$ of the maximal torus $\mathbf{T}^{\mathrm{ad}}_{\mathrm{q}}$ of $\mathbf{G}^{\mathrm{ad}}_{\mathrm{q}}$, which can also be viewed as the multi-valued $\mathbf{T}_{\mathrm{q}}$-valued function. The group element is locally analytic in x,

$$g(x) = \prod_{i\in\widetilde{\mathrm{Vert}_\gamma}} \mathcal{P}_i(x+\mu_i)^{-\lambda_i^\vee}\mathcal{Y}_i(x+\mu_i)^{\alpha_i^\vee}, \tag{1.1}$$

where i runs over the set $\widetilde{\mathrm{Vert}_\gamma}$ of vertices of the universal cover of the quiver graph γ, the polynomials $\mathcal{P}_i(x)$ and the complex parameters μ_i are determined by the gauge couplings and the masses of matter hypermultiplets, and $\alpha_i^\vee$ and $\lambda_i^\vee$ are the simple coroots and the fundamental coweights of $\mathfrak{g}_{\mathrm{q}}$. It is also convenient to introduce another group element

$$g_\infty(x) = \prod_{i\in\widetilde{\mathrm{Vert}_\gamma}} \mathcal{P}_i(x+\mu_i)^{-\lambda_i^\vee}. \tag{1.2}$$

The notation $z_i^{\alpha_i^\vee}$, $w_i^{\lambda_i^\vee}$ *used in* (1.1), (1.2) *and below should be understood with the help of the exponential map* $\mathrm{Lie}(\mathbf{T}_\mathrm{q}) \to \mathbf{T}_\mathrm{q}$, *see Appendix* C. *It is well-defined for the coroots* $\alpha^\vee$, *while for the coweights* $\lambda^\vee$ *it is well-defined as valued in the conformal extension discussed below.*

Our main claim is that the conjugacy class $[g(x)] \in \mathbf{T}_\mathrm{q}^\mathrm{ad}/W(\mathfrak{g}_\mathrm{q})$ is holomorphic in x, so that the basic adjoint invariants of $\mathbf{G}_\mathrm{q}$, evaluated on $[g(x)]$ (up to some twist discussed further) are polynomials of x, leading to a system of equations relating $\mathcal{Y}$ and x:

$$\mathcal{X}_i(\mathcal{Y}(x)) = T_i(x) = T_{i,0}x^{\mathbf{v}_i} + T_{i,1}x^{\mathbf{v}_i-1} + \cdots + T_{i,\mathbf{v}_i}, \tag{1.3}$$

which define what we call the *cameral curve*

$$\mathcal{C}_u \subset \mathbf{C}_{\langle x\rangle} \times \left(\mathbb{C}^\times\right)^{\mathrm{Vert}_\gamma}.$$

The invariants $\mathcal{X}_i$ are normalized characters of $g(x)$ in the fundamental representations $\mathbf{R}_i$ of $\mathbf{G}_\mathrm{q}$, of the highest weight λ_i:

$$\mathcal{X}_i(\mathcal{Y}(x)) = g_\infty(x)^{-\lambda_i}\,\mathrm{Tr}_{R_i}\, g(x).$$

Moreover, from the work of Steinberg [174] (see also [58]) we know that for the finite-dimensional $\mathbf{G}_\mathrm{q}$ one can conjugate $g(x)$ in $\mathbf{G}_\mathrm{q}$ to obtain a smooth $\mathbf{G}_\mathrm{q}$-valued function $\mathbf{g}(x)$ of x. Further inspection shows that $\mathbf{g}(x)$ is a quasi-classical limit of an element of the Yangian algebra $Y(\mathfrak{g}_\mathrm{q})$, built on the Lie algebra $\mathfrak{g}_\mathrm{q}$ of $\mathbf{G}_\mathrm{q}$. Hopefully the analogous statements hold for all $\mathbf{G}_\mathrm{q}$'s we encounter.

In this way one recovers all known results about the Seiberg–Witten geometries of the $\mathcal{N} = 2$ theories in four dimensions (we do not review all of them in this work) as well as finds new results. *We do not claim to reproduce all conjectured Seiberg–Witten geometries, as, e.g., theories with non-classical gauge groups are outside the realm of our methods.* In particular, we find the families of curves describing the geometry of the moduli space of vacua for the theories which were previously believed not to have such description. We also find that the special geometry of the quiver theories with unitary groups is captured in general by a polylogarithmic system of differentials on these curves.

Higher dimensions

The gauge theories we discuss can be also lifted to five-dimensional theories compactified on a circle $\mathbb{S}^1_{\langle\beta\rangle}$ of circumference β, or even to the six-dimensional theories, compactified on a two-torus, of the area β^2. In the limit $\beta \to 0$ one recovers the original four-dimensional theory. In the five-dimensional case the polynomials $T_i(x)$ in equation (1.3) are replaced by Laurent polynomials in $\mathrm{e}^{\mathrm{i}\beta x}$, while in the six-dimensional case the functions $T_i(x)$ become elliptic.

Defreezing

One of the initial questions which led us to the subject of this work was the following. Consider the SU(2) theory with $N_f = 4$ hypermultiplets in the fundamental representation, with the coupling $\mathfrak{q}$. By now there is an overwhelming evidence [1] of connection this theory has to Liouville conformal blocks on a sphere with four punctures. The momenta of Liouville vertex operators at the punctures are related to the masses of the hypermultiplets, the locations of the vertex operators are, e.g., 0, 1, $\mathfrak{q}$, ∞, and the momentum at the intermediate channel is the Coulomb parameter a.

Let us view this theory as a $U(2)$ theory, and let us single out the maximal torus $U(1)^4$ of the Spin(8) flavor symmetry group. Let us gauge these $U(1)$ groups. This gauging is possible in the noncommutative geometry setup. One acquires four additional coupling constants. What will happen to the Liouville theory?

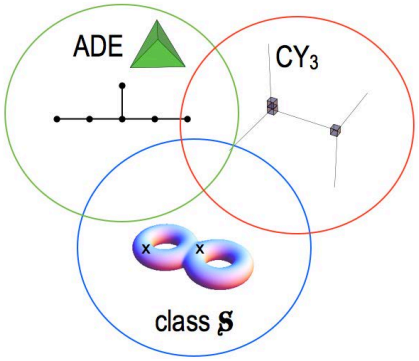


Figure 1.1. The three major ways to construct $\mathcal{N} = 2$ theories.

Upon some reflection one concludes that the resulting theory is a particular case of the $\widehat{D}_4$ theory, with $\mathbf{v}_0 = \mathbf{v}_1 = \mathbf{v}_3 = \mathbf{v}_4 = 1$, and $\mathbf{v}_2 = 2$. We then decided to solve the general quiver superconformal theory which led us to discover many other interesting things.

Classification

Another motivation comes from the question whether Hitchin's system exhaust the list of all reasonable Seiberg–Witten integrable systems. From the early discovery [38] that the $\mathcal{N} = 2^*$ theory with $G_{\rm g} = {\rm SU}(N)$ is governed by the ${\rm SU}(N)$ Hitchin system on a one-punctured torus (which is nothing but the elliptic Calogero–Moser system, as shown previously in [75]), proposals in [124], and subsequent developments culminating in the introduction of the "S-class" theories [1, 61, 63, 175] there was a lot of activity with experimental evidence suggesting that $\mathcal{N} = 2$ theories can be described by some version of Hitchin's system. The underlying construction in these approaches is the compactification of the six-dimensional superconformal $(0, 2)$-theory on some Riemann surface embedded as a supersymmetric cycle in some ambient geometry, and it is believed that the global features of the embedding should play virtually no rôle in the effective gauge theory dynamics.

Another way of engineering $\mathcal{N} = 2$ theories, using string theory, is the so-called geometric engineering [100, 113], which is the study of the gravity-decoupled limit of the IIA compactification on a Calabi–Yau threefold, with the Calabi–Yau becoming effectively non-compact. A large class of models comes from toric Calabi–Yau's. One then employs the local mirror symmetry to generate curves with differentials, whose periods capture the special geometry of the $\mathcal{N} = 2$ theory.

In our work, we presented another characterization of the integrable systems underlying the special geometry of the $\mathcal{N} = 2$ theories with the superconformal ultraviolet limit. Namely, we identify these systems with the moduli spaces of some gauge/Higgs configurations, such as monopoles or instantons, with the gauge group $\mathbf{G}_{\rm q}$ corresponding to the quiver diagram encoding, among other things, the matter sector of the theory. Unlike all previous approaches, see Figure 1.1, which involved some reference to the non-perturbative dualities, or even embedding of the gauge theory to string theory and M-theory, we derive these statements within the quantum field theory, by analyzing the instanton contributions to the low-energy effective action.

In some cases (e.g., in a simple fashion for the A_r type theories, in a more subtle way for the D_r type theories) our phase spaces can be identified with the phase spaces of Hitchin systems on the low genus curves with punctures, using some version of Nahm–Fourier–Mukai transform, but in general we don't have such a duality. Provided a complete description of the Seiberg–Witten curves and algebraic integrable systems for the $\mathcal{N} = 2$ ADE quiver theories it would be interesting to further investigate this ADE quiver class, along the lines of [62, 63] or [2] for the "S-class".

				classical $\epsilon_1 = 0$ $\epsilon_2 = 0$	quantum $\epsilon_1 \neq 0$ $\epsilon_2 = 0$	double quantum $\epsilon_1 \neq 0$ $\epsilon_2 \neq 0$
4d	$\mathbf{C}_{\langle x\rangle} = \mathbb{C}$	XXX	rational	$\mathbf{G}_{\mathrm{q}}(\mathbb{C})$	$Y(\mathfrak{g}_{\mathrm{q}})$	...
5d	$\mathbf{C}_{\langle x\rangle} = \mathbb{C}^\times$	XXZ	trigonometric	$\mathbf{G}_{\mathrm{q}}(\mathbb{C}^\times)$	$U_q(\widehat{\mathfrak{g}_{\mathrm{q}}})$	...
6d	$\mathbf{C}_{\langle x\rangle} = E_T$	XYZ	elliptic	$\mathbf{G}_{\mathrm{q}}(E)$	$E_{T,\eta}(\mathfrak{g}_{\mathrm{q}})$	...

Table 1.1. The (rational/trigonometric/elliptic) by (classical/quantum/double quantum).

In the other cases, e.g., the class II E_r type theories we can use the relation between the moduli of del Pezzo surfaces and the moduli of E-bundles on elliptic curve to assign to our version of the Seiberg–Witten curve a one-parametric family of del Pezzo surfaces, which can be viewed as an example of the mirror noncompact threefold of [101].

Outlook

In the companion paper [151] the connection between the class of ADE quiver gauge theories and *quantum* integrable ADE spin chains was studied in some detail. In particular we explain there that the five-dimensional version of the ADE quiver gauge theory on the twisted bundle $\mathbb{R}^4 \tilde{\times} \mathbb{S}^1_{\langle \epsilon_1, \epsilon_2; \beta\rangle}$ [140] with the equivariant parameters set to $\epsilon_1 = \epsilon$, $\epsilon_2 = 0$ as in [156] is associated with the XXZ spin chain $\mathfrak{g}_{\mathrm{q}}$. The theory is solved by the quantum version of the master equation (1.3): the group $\mathbf{G}_{\mathrm{q}}$ is replaced by the *quantum affine algebra* $U_q(\widehat{\mathfrak{g}_{\mathrm{q}}})$ with the quantum parameter $q = \mathrm{e}^{\mathrm{i}\beta\epsilon}$, while the characters $\mathfrak{X}_i$ are promoted to the q-characters of Frenkel–Reshetikhin [54]. (If $\mathbf{G}_{\mathrm{q}}$ is itself affine Kac–Moody group $\mathbf{G}_{\mathrm{q}} = \widehat{\mathbf{G}}$ then $U_q(\widehat{\mathfrak{g}_{\mathrm{q}}})$ is naturally *quantum toroidal algebra*). In the four-dimensional limit the XXZ $\mathfrak{g}_{\mathrm{q}}$ spin chain turns into the XXX $\mathfrak{g}_{\mathrm{q}}$ spin chain, the quantum affine algebra $U_q(\widehat{\mathfrak{g}_{\mathrm{q}}})$ degenerates into Yangian $Y(\mathfrak{g}_{\mathrm{q}})$, and the gauge theory on the twisted bundle becomes the four-dimensional theory subject to a two-dimensional Ω-background. Finally, the six-dimensional theory compactified on a torus E_T corresponds to the XYZ $\mathfrak{g}_{\mathrm{q}}$ spin chain, with the quantum affine group $U_q(\widehat{\mathfrak{g}_{\mathrm{q}}})$ elevated to the quantum elliptic group $E_{T,\eta}(\mathfrak{g}_{\mathrm{q}})$ [44, 49, 51], with $\eta = \beta\epsilon/2\pi$.

It is clear that there is an even larger picture in which the algebraic integrable systems we encountered in this work are quantized, or ϵ_2-deformed, cf. Table 1.1, with the rational/trigonometric/elliptic trichotomy in the vertical direction established in [11, 47, 173] and connected with the gauge theories in [138]. It would be exciting to explore the connection with H. Nakajima's work [130] on quiver varieties and quantum affine algebras as well as the connection with elliptic cohomology [70, 77, 122] of moduli spaces. Notice that the quantum or double quantum exploration of the ADE quiver world is in a sense orthogonal to the approach of [1] dealing with the "S-class" world in Figure 1.1. Classically, on the overlap, the relation between the corresponding algebraic integrable systems comes from the Nahm–Fourier–Mukai/Corrigan–Goddard/ADHM reciprocity relating the moduli space of $\mathbf{G}$-bundles and Hitchin systems. The (doubly) quantum version of this Nahm transform, if it exists, seems to cover the "quantum" geometric Langlands duality, separation of variables for quantum systems [46, 48, 52, 171, 172]. The new ingredient [157] in this relatively classic field of research are the supersymmetric gauge theories in four dimensions. Table 1.1 has been basically filled in recent years.

1.1 Organization of the material

Section 2 introduces the quiver supersymmetric gauge theories which we shall study.

Section 3 presents the classification of the gauge theories which are $\mathcal{N} = 2$ superconformal in the ultraviolet. We distinguish three classes of such theories, I, II, and II*. The I and II classes have an ADE classification so that for class I $\mathbf{G}_{\mathrm{q}} = \mathbf{G}$ and for class II $\mathbf{G}_{\mathrm{q}} = \widehat{\mathbf{G}}$ where $\mathbf{G}$ is ADE group, the II* theories correspond to $\widehat{\mathrm{GL}}_\infty$ group.

Section 4 reviews the special Kähler geometry of the vectormultiplet moduli spaces $\mathfrak{M}$ of vacua of $\mathcal{N} = 2$ theories. We also recall the relation of $\mathfrak{M}$ to the algebraic integrable systems and the hyperKähler manifolds. We give some examples to be used later.

Section 5 introduces our main tool: the limit shape equations, which summarize the microscopic gauge theory calculation leading to the effective low-energy action, i.e., the prepotential $\mathcal{F}$.

Section 9 presents the solution of the limit shape equation. We reformulate the equations as the Riemann–Hilbert problem for the set of functions $\mathcal{Y}(x)$ and solve it by equating the invariants $\mathcal{X}(\mathcal{Y}(x))$ of the monodromy group, the *iWeyl group* which we attach to every $\mathcal{N} = 2$ gauge theory, to some polynomials $T(x)$. In this manner we find an (algebraic) curve $\mathcal{C}$ and a system of differentials, whose periods give the special coordinates $\mathfrak{a}$ and the derivatives $\partial\mathcal{F}/\partial\mathfrak{a}$ of the prepotential $\mathcal{F}$.

Section 10 analyzes the solution in some detail. We interpret the data for the solution of the class I theories as describing a holomorphic map with prescribed singularities of $\mathbb{CP}^1$ to the space of conjugacy classes $\mathbf{T}/W(\mathbf{g})$ in a complex Lie group $\mathbf{G}$, which can be also viewed as the moduli space of holomorphic $\mathbf{G}$-bundles on a degenerate elliptic curve. For the class II theories the analogous data parametrizes (quasi)maps to the moduli space $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ of holomorphic $\mathbf{G}$-bundles on elliptic curve. In some cases we relate the curve $\mathcal{C}$ to the more familiar Seiberg–Witten curves. For the theories corresponding to the A series we manage to relate our curves $\mathcal{C}$ to the spectral curves of rational and elliptic Gaudin models (the Hitchin system on the genus zero and one curves with punctures), and also reproduce the results of [169, 175]. For the class II D type theories we reproduce the results of [95]. For the class II E type theories we find yet another interpretation of our solution, in terms of families of del Pezzo surfaces. In this way we get a field theory understanding of some of the local mirror symmetry predictions [101] and brane construction [96, 98].

Section 11 discusses the moduli spaces $\mathfrak{P}$ of vacua of the gauge theory compactified on a circle $\mathbb{S}^1$. We don't present the full analysis of the hyperKähler metric on $\mathfrak{P}$ in this work. Instead, we focus on the geometry of $\mathfrak{P}$ in the complex structure inherited from four dimensions (this complex structure is sometimes called the complex structure $\mathbf{I}$), in which it presents itself as an (algebraic) integrable system. Our solution of the four-dimensional theory comes in a form which leads to a natural guess for the phase space $\mathfrak{P}$ of the integrable systems corresponding to our theories. For the class I theories it is the moduli space of G-monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$ with singularities, for the class II theories it is the moduli space of G-instantons on $\mathbb{R}^2 \times \mathbb{T}^2$, and for the class II* $\widehat{A}_r$ theories it is the moduli space of noncommutative $U(r+1)$ instantons on $\mathbb{R}^2 \times \mathbb{T}^2$. Of course these spaces have a natural hyperkähler structure which depends in the expected fashion on all the parameters of the theory and its compactification. Although our motivation comes from the field theory analysis in the previous chapters, our results confirm the conjectures of [16, 18, 19, 20, 81, 95, 96, 98] which are motivated by the string theory analysis, and in particular by the brane constructions.

Section 12 discusses the modifications of our solutions in the five and six-dimensional cases.

In Appendix A, we review the affine ADE graphs, the McKay correspondence and the M-theory/D-brane picture for the present work; in Appendix B, we put our conventions on the partitions and representations by free fermions; in Appendix C, we review some standard material on Lie groups and Lie algebras which we use in solving our theories. We recall the notions of the (co)root and the (co)weight lattices, Weyl groups, and the integrable highest weight representations; in Appendix M, we collect our conventions for elliptic functions; in Appendix P, we give some technical details on spectral curves of affine E-series.

1.2 Notations

Quivers, Section 2

Vert_γ	set of vertices
Edge_γ	set of edges
$s(e) \in \mathrm{Vert}_\gamma$	the source of the edge $e \in \mathrm{Edge}_\gamma$
$t(e) \in \mathrm{Vert}_\gamma$	the target of the edge $e \in \mathrm{Edge}_\gamma$
$\mathfrak{q}_i = \mathrm{e}^{2\pi \mathrm{i} \tau_i}$	gauge coupling constants
$\tilde{\mathfrak{q}}$	$(\mathfrak{q}_i)_{i \in \mathrm{Vert}_\gamma}$
$\mathbf{v}_i$	number of colors for i-th node gauge group $\mathrm{SU}(\mathbf{v}_i)$
$\mathbf{w}_i$	number of flavors for i-th node fundamental matter
$\mathfrak{a}_{i,\mathbf{a}}$	eigenvalues of the complex scalars
$\mathfrak{a}_{i,\mathbf{a}} \leftrightarrow \mathfrak{a}^{\mathcal{I}}$	the special coordinates on Coulomb moduli space
C_{ij}	Cartan matrix associated to the quiver by its Dynkin graph
$r = \mathrm{rk}(C)$	$\lvert\mathrm{Vert}_\gamma\rvert$ if γ is finite ADE or $\lvert\mathrm{Vert}_\gamma\rvert - 1$ if γ is affine ADE

Lie groups

i	$\sqrt{-1}$
G_{g}	Gauge group of the four-dimensional theory
$\mathbf{G}_{\mathrm{q}}$	Kac–Moody group associated with quiver Dynkin diagram
G_M	the flavor group
$\mathbf{G} = G^{\mathbb{C}}$	finite-dimensional complex Lie group
$\widehat{\mathbf{G}}$	affine Kac–Moody group for $\mathbf{G}$
G	maximal compact subgroup of $\mathbf{G}$
$\mathbf{T}$	maximal torus of $\mathbf{G}$
T	maximal torus of G
Z	the center of both G and $\mathbf{G}$
$\mathbf{G}^{\mathrm{ad}} = \mathbf{G}/Z$	adjoint form of the complex Lie group $\mathbf{G}$
$\mathbf{T}^{\mathrm{ad}} = \mathbf{T}/Z$	the maximal torus of $\mathbf{G}^{\mathrm{ad}}$
$\mathbf{G}_{\mathrm{q}}^{\mathrm{ad}} = \mathbf{G}_{\mathrm{q}}/Z$	adjoint form of the complex Lie group $\mathbf{G}_{\mathrm{q}}$
$\mathbf{T}_{\mathrm{q}}^{\mathrm{ad}} = \mathbf{T}_{\mathrm{q}}/Z$	its maximal torus

Lie algebras

$\mathfrak{g}_{\mathrm{q}}$	Kac–Moody Lie algebra associated with quiver Dynkin diagram
$\mathfrak{g}$	$\mathrm{Lie}(\mathbf{G})$
$\mathfrak{h}$	$\mathrm{Lie}(\mathbf{T})$

Representation theory, Appendix C

a_i	Kac–Dynkin marks
$\mathrm{Q},\ \Lambda^\vee$	root lattice, coweight lattice

$\Lambda,\ \mathrm{Q}^\vee$	weight lattice, coroot lattice
R_i	i'th fundamental representation of $\mathbf{G}$
$\widehat{R}_i$	i'th fundamental representation of $\widehat{\mathbf{G}}$
$\mathcal{R}_i$	i'th fundamental representation of $\widehat{\mathrm{GL}}_\infty$

Spaces

$B(\mathfrak{g})$	$\mathbf{T}/W(\mathfrak{g})$ the space of conjugacy classes in $\mathbf{G}$
$B(\mathfrak{g}_\mathrm{q})$	$\mathbf{T}_\mathrm{q}/W(\mathfrak{g}_\mathrm{q})$ the space of conjugacy classes in $\mathbf{G}_\mathrm{q}$
$B^\mathrm{ad}(\mathfrak{g})$	$\mathbf{T}/(Z\times W(\mathfrak{g}))$ the space of conjugacy classes in $\mathbf{G}^\mathrm{ad}$
$B^\mathrm{ad}(\mathfrak{g}_\mathrm{q})$	$\mathbf{T}_\mathrm{q}/(Z\times W(\mathfrak{g}_\mathrm{q}))$ the space of conjugacy classes in $\mathbf{G}^\mathrm{ad}_\mathrm{q}$
$\mathbf{C}_{\langle x\rangle}$	complex plane $\mathbb{C}$ in 4d, cylinder $\mathbb{C}^\times$ in 5d, torus E in 6d
$\mathbb{CP}^1_{\langle x\rangle}$	$=\mathbf{C}_{\langle x\rangle}\cup\{\infty\}$
$\mathcal{E}(\mathfrak{q})$	elliptic curve $\mathbb{C}^\times/\mathfrak{q}^\mathbb{Z}$
$\mathfrak{q}$	$\prod_{i\in\mathrm{Vert}_\gamma}\mathfrak{q}_i^{a_i}$ for class II theories
$\mathrm{Bun}_\mathbf{G}(\mathcal{E})$	coarse moduli space of semistable holomorphic $\mathbf{G}$-bundles on $\mathcal{E}$
$\mathfrak{M}$	the Coulomb moduli space of the 4d $\times\mathrm{SU}(\mathbf{v}_i)$ gauge theory
$\mathfrak{M}^\mathrm{ext}$	the Coulomb moduli space of the 4d $\times U(\mathbf{v}_i)$ gauge theory
$\mathfrak{P}\to\mathfrak{M}$	the algebraic integrable system $\dim_\mathbb{C}\mathfrak{P}=2\dim_\mathbb{C}\mathfrak{M}$
$\mathfrak{P}^\mathrm{ext}\to\mathfrak{M}^\mathrm{ext}$	the complex integrable system $\dim_\mathbb{C}\mathfrak{P}^\mathrm{ext}=2\dim_\mathbb{C}\mathfrak{M}^\mathrm{ext}$

Seiberg–Witten curves

$\mathcal{C}_u$	cameral curve: Section 9.6
C_u	spectral curve: Section 9.7
$\mathcal{C}$	obscure curve: Section 9.8
x	flat coordinate on $\mathbf{C}_{\langle x\rangle}$
$\mathcal{Y}_i(x)$	*amplitudes* (the solution of the theory): Section 5.1
$T_i(x)$	*gauge polynomials* of degree $\mathbf{v}_i$
$\mathcal{P}_i(x)$	*matter polynomials* of degree $\mathbf{w}_i$
$g(x)$	$\mathbf{T}^\mathrm{ad}_\mathrm{q}$ valued analytic function on $\mathbf{C}_{\langle x\rangle}$
$\mathbf{g}(x)$	$\mathbf{G}_\mathrm{q}$ valued analytic function on $\mathbf{C}_{\langle x\rangle}$
$\mathcal{X}_j$	$\mathbf{G}_\mathrm{q}$ character (or Weyl invariant) for i-th fundamental weight of $\mathbf{G}_\mathrm{q}$

Partitions

λ	partition $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{\ell(\lambda)}>0$, $\lambda_i\in\mathbb{Z}_{\geq 0}$
$\ell(\lambda)$	the length of the partition λ
$\lvert\lambda\rvert$	the size of the partition $\lvert\lambda\rvert=\sum_{i=1}^{\ell(\lambda)}\lambda_i$

Let $x\in R^\mathbb{Z}$ be a sequence $(x_i)_{i\in\mathbb{Z}}$, with x_i in some ring R.

Consecutive products

$$x_j^{[i]} \equiv \frac{\prod_{k=-\infty}^{k=j+i-1} x_k}{\prod_{k=-\infty}^{j-1} x_k},$$
$$x^{[i]} = x_1^{[i]},$$
$$x^{[i+1]} = x^{[i]} x_{i+1},$$

for example

$$x^{[0]} = 1,$$
$$x_4^{[2]} = x_4 x_5,$$
$$x^{[-1]} = x_0^{-1},$$
$$x_5^{[-4]} = x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1}.$$

Consecutive sums

$$x_{(i)} = \sum_{j=-\infty}^{i} x_j - \sum_{j=-\infty}^{0} x_j,$$
$$x_{(i+1)} = x_{(i)} + x_{i+1},$$

for example

$$x_{(0)} = 0,$$
$$x_{(3)} = x_1 + x_2 + x_3,$$
$$x_{(-2)} = -x_0 - x_{-1},$$
$$\lambda_{(\ell(\lambda))} = |\lambda|.$$

2 Supersymmetric quiver theories

Consider any $\mathcal{N} = 2$ supersymmetric field theory in four dimensions whose gauge group is a product of special unitary groups, while the matter hypermultiplets are in the fundamental, bifundamental, and adjoint representations. The field content, the parameters of the Lagrangian, and the choice of the vacuum are conveniently encoded in the quiver data, which is

1. An oriented graph γ with the set Vert_γ of vertices and the set $\mathrm{Edge}_\gamma \subset \mathrm{Vert}_\gamma \times \mathrm{Vert}_\gamma$ of oriented edges. Let $s, t\colon \mathrm{Vert}_\gamma \times \mathrm{Vert}_\gamma \to \mathrm{Vert}_\gamma$ by the projections onto the first and the second factors. They define the two maps $s, t\colon \mathrm{Edge}_\gamma \to \mathrm{Vert}_\gamma$ which assign to an oriented edge its source and the target, respectively. In what follows we shall use the notation

   $$|\mathrm{Vert}_\gamma| = \#\mathrm{Vert}_\gamma$$

   for the number of vertices in the quiver.

2. An assignment of polynomials to the vertices:

   $$\mathcal{T}, \mathcal{P}\colon\ \mathrm{Vert}_\gamma \to \mathbb{C}[x], \qquad i \mapsto \mathcal{T}_i(x), \mathcal{P}_i(x)$$

   and $\mathbf{v}, \mathbf{w}\colon \mathrm{Vert}_\gamma \to \mathbb{Z}_+$, where

   $$\mathbf{v}_i = \deg \mathcal{T}_i, \qquad \mathbf{w}_i = \deg \mathcal{P}_i, \qquad i \in \mathrm{Vert}_\gamma.$$

   The polynomials $\mathcal{T}_i(x)$ are monic, the highest-order term coefficients $\mathfrak{q}_i$ of the polynomials $\mathcal{P}_i(x)$ are required to obey: $|\mathfrak{q}_i| < 1$.

3. A 1-cocycle $\mathbf{m} \in \mathcal{C}^1(\gamma, \mathbb{C})$, in other words an assignment

$$e \in \mathrm{Edge}_\gamma \mapsto m_e \in \mathbb{C}.$$

We now proceed with the explanation of the rôles of the polynomials $\mathcal{T}$, $\mathcal{P}$, as well as that of the cocycle $\mathbf{m}$.

2.1 Gauge group, matter fields, couplings, parameters

2.2 The gauge group

We denote the gauge group by G_{g}. It is the product

$$G_{\mathrm{g}} = \prod_{i \in \mathrm{Vert}_\gamma} \mathrm{SU}(\mathbf{v}_i). \tag{2.1}$$

The vector multiplet therefore splits into a collection of vector multiplets for the $\mathrm{SU}(\mathbf{v}_i)$ gauge factors

$$\Phi_{\mathrm{g}} = (\Phi_i)_{i \in \mathrm{Vert}_\gamma}.$$

We have a gauge coupling e_i and the theta angle ϑ_i for each $i \in \mathrm{Vert}_\gamma$. As usual, we combine them into the complexified gauge couplings,

$$\tau_i = \frac{1}{2\pi \mathrm{i}} \log(\mathfrak{q}_i) = \frac{\vartheta_i}{2\pi} + \frac{4\pi \mathrm{i}}{e_i^2}.$$

The bosonic part of the action for gauge fields is given by

$$\mathcal{L}_{\mathrm{YM}} = \sum_{i \in \mathrm{Vert}_\gamma} \left(\frac{1}{e_i^2} \int \mathrm{tr}_{\mathbf{v}_i} F_{A_i} \wedge \star F_{A_i} + \frac{\mathrm{i}\vartheta_i}{8\pi^2} \int \mathrm{tr}_{\mathbf{v}_i} F_{A_i} \wedge F_{A_i} \right),$$

where tr_v denotes the trace of a $v \times v$ matrix. The exponentiated coupling

$$\mathfrak{q}_i = \mathrm{e}^{2\pi \mathrm{i} \tau_i}$$

enters the path integral measure. The perturbative effects do not depend on ϑ_i, while the non-perturbative effects, which are the contributions of the gauge fields with non-trivial instanton charge, depend on $\mathfrak{q}_i$, $\bar{\mathfrak{q}}_i$. In other words, the partition function is expected to be invariant under the shifts

$$\tau_i \longrightarrow \tau_i + 1.$$

For $i \in \mathrm{Vert}_\gamma$ let Φ_i denote the corresponding complex scalar in the adjoint representation. The bosonic potential of the vector multiplet field Φ_i contains a universal term

$$\mathrm{tr}_{\mathbf{v}_i} \left[\Phi_i, \Phi_i^\dagger \right]^2$$

plus some possible non-negative terms coming from interactions with matter fields. If the matter fields are massive then this term alone forces Φ_i to commute with its conjugate at low energies. Therefore, at low energy the field Φ_i can be diagonalized:

$$\Phi_i \longrightarrow \mathrm{diag}(\mathfrak{a}_{i,\alpha})_{\alpha=1}^{\mathbf{v}_i}. \tag{2.2}$$

The gauge invariant order parameters are the vacuum expectation values of the coefficients of the characteristic polynomial $\mathcal{T}_i$ of Φ_i:

$$\mathcal{T}_i(x) = \langle \det_{\mathbf{v}_i}(x - \Phi_i) \rangle_u \,, \tag{2.3}$$

where we assume the normalization $\langle 1 \rangle_u = 1$, and $\det_v$ denotes the determinant of a $v \times v$-matrix. Therefore the polynomials $\mathcal{T}_i(x)$ in (2.3) are monic.

Thus, a collection of polynomials $\mathcal{T}_i(x)$, $i \in \mathrm{Vert}_\gamma$ fixes the choice of the vacuum $u = (u_{i,\mathbf{a}}) \in \mathfrak{M}$:

$$\mathcal{T}_i(x) = x^{\mathbf{v}_i} + \sum_{\mathbf{a}=2}^{\mathbf{v}_i} u_{i,\mathbf{a}} x^{\mathbf{v}_i - \mathbf{a}}. \tag{2.4}$$

Because of the non-perturbative (instanton) effects the relation between $u_{i,\mathbf{a}}$ and $\mathfrak{a}_{i,\mathbf{a}}$ is not polynomial, and for the same reason $\mathcal{T}_i(x) \neq \mathcal{Y}_i(x)$.

2.3 The hypermultiplets in the bi-fundamental, or adjoint representations

The bifundamental or adjoint hypermultiplet H_e, $e \in \mathrm{Edge}_\gamma$ transforms in the following G_{g} representation:

$$\begin{array}{ll} \left(\overline{\mathbf{v}}_{s(e)}, \mathbf{v}_{t(e)}\right) \oplus \left(\overline{\mathbf{v}}_{t(e)}, \mathbf{v}_{s(e)}\right) & \text{for} \quad s(e) \neq t(e), \\ \mathrm{Adj}(su(\mathbf{v}_i)) & \text{for} \quad s(e) = t(e) = i. \end{array}$$

The masses of the bi-fundamental hypermultiplets are conveniently represented by the 1-cocycle: $\mathbf{m} \in \mathcal{C}^1(\gamma, \mathbb{C})$, $e \mapsto \mathbf{m}_e$. Let $[\mathbf{m}] \in H^1(\gamma, \mathbb{C})$ be the corresponding cohomology class. If we denote by m^* a particular representative of $[\mathbf{m}]$ in $\mathcal{C}^1(\gamma, \mathbb{C})$, then

$$\mathbf{m} = m^* + \delta\mu, \qquad \mu \in \mathcal{C}^0(\gamma, \mathbb{C}) \tag{2.5}$$

or, in components,

$$\mathbf{m}_e = m^*_e + \mu_{t(e)} - \mu_{s(e)}.$$

2.4 The hypermultiplets in the fundamental representations

These are assigned to the vertices $i \in \mathrm{Vert}_\gamma$. We have $\mathbf{w}_i$ such multiplets. Write

$$\mathcal{P}_i(x) = \mathfrak{q}_i \prod_{\mathfrak{f}=1}^{\mathbf{w}_i} (x - m_{i,\mathfrak{f}}).$$

Then $m_{i,\mathfrak{f}}$ are the masses of the fundamental hypermultiplets, charged under $\mathrm{SU}(\mathbf{v}_i)$. A $\mathbf{w}_i$-tuplet of fundamental hypermultiplets can be thought as a bifundamental $(\mathbf{v}_i, \bar{\mathbf{w}}_i)$ for $\mathrm{SU}(\mathbf{v}_i)$ and an auxiliary frozen $U(\mathbf{w}_i)$, so that $m_{i,\mathfrak{f}}$ can be interpreted as the values of the frozen scalar field in the vector multiplet of $U(\mathbf{w}_i)$.

3 The ADE classifications of superconformal $\mathcal{N} = 2$ theories

In quantum gauge theory the coupling constants e_i are subject to the renormalization which leads to their dependence on the energy scale at which one measures the interaction between the charged particles. The consistent theories have the gauge couplings which tend to zero as the scale approaches ultraviolet, or approach some fixed values. These theories are called asymptotically free and asymptotically conformal, respectively. Moreover, starting with the

asymptotically conformal theory, one can perturb it by the mass terms. Then, by tuning the masses and the bare couplings, one arrives at the asymptotically free theory. All asymptotically free quiver theories arise in this way. Therefore it suffices to solve the asymptotically conformal theories.

From the perspective of geometrical engineering the ADE quiver theories were studied in [101], and three-dimensional ADE quiver theories were studied in [64].

3.1 Beta functions and Cartan matrix

The running of the gauge coupling τ_i is described by the Gell-Mann–Low equations which are one-loop exact for the $\mathcal{N} = 2$ supersymmetric theories, the result of [161]. The actual contributions of the matter and gauge multiplets to the gauge couplings are

$$\beta_i = 2\pi \mathrm{i} \frac{\mathrm{d}\tau_i}{\mathrm{d}\log\Lambda} = \mathbf{w}_i - 2\mathbf{v}_i + \sum_{e\colon t(e)=i} \mathbf{v}_{s(e)} + \sum_{e\colon s(e)=i} \mathbf{v}_{t(e)}, \tag{3.1}$$

where Λ is the energy scale. The theory is asymptotically conformal if $\beta_i = 0$ vanishes for all $i \in \mathrm{Vert}_\gamma$. The theory is asymptotically free if $\beta_i < 0$.

Let us define the incidence of the pair of vertices I_{ij} to be the number of edges e connecting the vertices i and j:

$$I_{ij} = \#\{e \mid s(e) = i,\, t(e) = j, \text{ or } s(e) = j,\, t(e) = i\}$$

with the understanding that if the vertex $i \in \mathrm{Vert}_\gamma$ is connected to itself by a loop, then the corresponding edge contributes 2 to the incidence matrix element I_{ii}. Define, for all quivers, the Cartan matrix of size $|\mathrm{Vert}_\gamma| \times |\mathrm{Vert}_\gamma|$

$$C_{ij} = 2\delta_{ij} - I_{ij}. \tag{3.2}$$

Then

$$\beta_i \propto (\mathbf{w} - C\mathbf{v})_i,$$

where

$$(C\mathbf{v})_i = \sum_{j \in \mathrm{Vert}_\gamma} C_{ij}\mathbf{v}_j.$$

Let us solve the $\beta_i = 0$ conditions (cf. [85, 101, 111]. It is convenient to separate the solutions into three cases, which we shall call the theories of class I, the theories of class II and the theories of class II*. By r we shall denote the rank of the Cartan matrix C

$$r = \mathrm{rk}(C).$$

The main difference between the class I and class II, II* theories is that the Cartan matrix of class I theories has the maximal rank

$$r(C_{\gamma \text{ of class I}}) = |\mathrm{Vert}_\gamma|$$

while for the theories of class II and class II* the Cartan matrix has one-dimensional kernel,

$$r(C_{\gamma \text{ of class II}}) = |\mathrm{Vert}_\gamma| - 1.$$

3.2 Class I theories

The solutions to the equations $\beta_i = 0$ with $\mathbf{w} \neq 0$ are the theories of class I. It is well-known that the graph γ is in this case a Dynkin diagram of a finite-dimensional simple simply-laced Lie algebra $\mathfrak{g}_{\rm q}$, of the ADE type, with Vert_γ labeling the simple roots of $\mathfrak{g}_{\rm q}$:

$$i \in \mathrm{Vert}_\gamma \mapsto \alpha_i.$$

To solve the equation $\beta_i = 0$ is equivalent to finding two vectors

$$\mathbf{v} = \sum_{i \in \mathrm{Vert}_\gamma} \mathbf{v}_i \alpha_i^\vee, \qquad \mathbf{w} = \sum_{i \in \mathrm{Vert}_\gamma} \mathbf{w}_i \alpha_i^\vee$$

with non-negative components $\mathbf{v}_i, \mathbf{w}_i \in \mathbb{Z}_{\geq 0}$, such that (cf. (3.2)

$$\mathbf{w} = C^{\mathfrak{g}_{\rm q}} \mathbf{v}, \tag{3.3}$$

where $C^{\mathfrak{g}_{\rm q}}$ is the Cartan matrix of the corresponding finite-dimensional Lie algebra $\mathfrak{g}_{\rm q}$ of the ADE type. Equivalently

$$\mathbf{v} = \sum_{i \in \mathrm{Vert}_\gamma} \mathbf{w}_i \lambda_i^\vee,$$

where $\lambda_i^\vee$ are the fundamental coweights of $\mathfrak{g}_{\rm q}$.

For class I theories we set $\mathbf{G}_{\rm q} = \mathbf{G}$ where $\mathbf{G}$ is finite dimensional complex ADE group.

Remark 3.1. In the case of $\mathfrak{g}_{\rm q}$ of the A_r type the dimensions $\mathbf{v}_i$ must be a convex function of i. In particular, they grow with i, for $i = 1, \ldots, i_*$, and then decrease:

$$\mathbf{v}_1 \leq \mathbf{v}_2 \leq \cdots \leq \mathbf{v}_{i_*-1} \leq \mathbf{v}_{i_*} \geq \mathbf{v}_{i_*+1} \geq \cdots \geq \mathbf{v}_r. \tag{3.4}$$

Remark 3.2. The graphs of the D_r and E_r Dynkin type have a single tri-valent vertex, let us call it i_*. One can easily show using the $\beta_i = 0$ equations that $\mathbf{v}_{i_*}$ is the maximal value of $\mathbf{v}_i$ on $i \in \mathrm{Vert}_\gamma$, and that $\mathbf{v}_i$ decrease along each leg emanating from the tri-valent vertex i_*.

3.3 Class II theories

The class II theories have $\mathbf{w} \equiv 0$, and $[\mathbf{m}] = 0$. It is well-known that the graphs γ, such that the corresponding Cartan matrix C has a zero eigenvector with positive integer entries are in one-to-one correspondence with the simply laced affine Dynkin diagrams (see Appendix A for our conventions on ADE graphs and McKay correspondence):

1) $\widehat{A}_r$, $r \geq 2$,
2) $\widehat{D}_r$, $r \geq 5$,
3) $\widehat{E}_r$, $r = 7, 8, 9$.

These Dynkin diagrams correspond to the affine Lie algebras $\widehat{\mathfrak{g}}$ associated to finite-dimensional Lie algebras $\mathfrak{g}$ of rank r. We set $\mathbf{G}_{\rm q} = \widehat{\mathbf{G}}$ and $\mathfrak{g}_{\rm q} = \widehat{\mathfrak{g}}$. We discuss the relevant aspects of the theory of affine Kac–Moody algebras in the following subsections.

Note that the $\widehat{A}_0$ case (its quiver has one vertex and one edge connecting it to itself), given our constraint $[\mathbf{m}] = 0$ for the class II theory, corresponds to the $\mathcal{N} = 4$ superconformal theory. It is well known that the classical moduli space of vacua gets no quantum corrections in this theory.

The dimensions $\mathbf{v}$ are uniquely specified, up to a single multiple:

$$\mathbf{v}_i = N a_i,$$

where a_i are the so-called Dynkin labels. We shall recall several interpretations of these numbers below.

3.4 Class II* theories

The class II* theories have $\mathbf{w} = 0$ and $[\mathbf{m}] \neq 0$. The first condition reduces our choice of γ to the affine Dynkin diagrams (including the $\widehat{A}_0$ case of the quiver with one vertex and one loop connecting this vertex with itself). The second condition implies that γ is the Dynkin diagram of the $\widehat{A}_r$ type for some $r \geq 1$. Indeed, only in this the affine Dynkin diagram has $H^1(\gamma, \mathbb{C}) = \mathbb{C}$, the diagram being a regular $(r+1)$-gon. The dimensions $\mathbf{v}_i$ are all equal to N, a non-negative integer.

In particular, the class II* $r = 0$, $\widehat{A}_0$-theory with $[\mathbf{m}] \neq 0$ is the celebrated $\mathcal{N} = 2^*$ theory, the SU(N) theory with massive adjoint hypermultiplet.

We shall see that the Kac–Moody Lie algebra which corresponds to the theories of class II* is the $\widehat{\mathfrak{gl}(\infty)}$ algebra, which contains $\widehat{\mathfrak{u}(r)}$ as a subalgebra of r-periodic matrices.

4 Low-energy effective theory

We now can proceed with the main subject of our study. Our goal is to determine the two-derivative/four-fermions terms in the low-energy effective action of our theory.

The low-energy effective theory of the $\mathcal{N} = 2$ supersymmetric quiver theory with generic masses $(m_e), (m_{i,\mathfrak{f}})$ is the abelian $\mathcal{N} = 2$ theory of $\mathbf{r}$ vector multiplets,

$$\mathbf{r} = \sum_{i \in \mathrm{Vert}_\gamma} (\mathbf{v}_i - 1).$$

For generic masses the theory has the manifold $\mathfrak{M}$ of vacua, which is a complex variety of complex dimension $\mathbf{r}$:

$$\dim_{\mathbb{C}} \mathfrak{M} = \mathbf{r}.$$

The effective theory is a sigma model on $\mathfrak{M}$, interacting with $\mathbf{r}$ abelian gauge fields $A_\mu^{\mathcal{I}}$, $\mathcal{I} = 1, \dots, \mathbf{r}$, and some fermionic fields. Our goal is to determine the metric on $\mathfrak{M}$, the effective gauge couplings $\operatorname{Im} \tau_{\mathcal{I}\mathcal{J}}$ and the effective theta-angles $\operatorname{Re} \tau_{\mathcal{I}\mathcal{J}}$ of these gauge fields.

4.1 Special Kähler geometry

One can interpret the eigenvalues (2.2) obeying

$$\sum_{\mathbf{a}=1}^{\mathbf{v}_i} \mathfrak{a}_{i,\mathbf{a}} = 0$$

as the *special coordinates* on the moduli space $\mathfrak{M}$ of vacua. As is well known, $\mathfrak{M}$ is a Kähler manifold, with a peculiar metric, and a rigid system of local coordinate systems. The corresponding geometry is called the *rigid special geometry*, and it is a limit [166] of the *special geometry* of $\mathcal{N} = 2$ supergravity, studied in [24].

Let us label the effective abelian vector multiplets $\mathcal{A}^{\mathcal{I}}$ by $\mathcal{I} = (i, \mathbf{a})$, $i \in \mathrm{Vert}_\gamma$, $\mathbf{a} = 2, \dots, \mathbf{v}_i$. In components

$$\mathcal{A}^{\mathcal{I}} = \mathfrak{a}^{\mathcal{I}} + \vartheta\psi^{\mathcal{I}} + \vartheta\vartheta\big(F_A^{\mathcal{I}}\big)^- + \cdots,$$

where

$$\big(F_A^{\mathcal{I}}\big)^\pm = \frac{1}{2}\big(F_A^{\mathcal{I}} \pm \star F_A^{\mathcal{I}}\big).$$

The scalar components $\mathfrak{a}^{\mathcal{I}} \leftrightarrow \mathfrak{a}_{i,\mathbf{a}}$, $\mathbf{a} = 2, \dots, \mathbf{v}_i$, more precisely their vacuum expectation values are the local *special coordinates*. Globally they are subject to monodromy transformations, unlike the global coordinates $(u_{i,\mathbf{a}})$ in equation (2.4), which are defined via the expectation values of the gauge invariant local operators of the microscopic theory.

The monodromy transformations act by symplectic transformations mixing the special coordinates $\mathfrak{a}^{\mathcal{I}}$ and their duals $\mathfrak{a}^D_{\mathcal{I}}$ together with the masses (m_e), $e \in \mathrm{Edge}_\gamma$ and $(m_{i,\mathfrak{f}})$, $i \in \mathrm{Vert}_\gamma$, $\mathfrak{f} = 1, \dots, \mathbf{w}_i$. The dual coordinates are the derivatives of the prepotential $\mathcal{F}$,

$$\mathfrak{a}^D_{\mathcal{I}} = \frac{\partial \mathcal{F}}{\partial \mathfrak{a}^{\mathcal{I}}}. \tag{4.1}$$

The prepotential is a multi-valued analytic function of $\mathfrak{a}^{\mathcal{I}}$, it is the superspace action which determines the low-energy effective action in the approximation we are working:

$$\mathcal{L}^{\mathrm{eff}} = \frac{\mathrm{i}}{4\pi} \int \tau_{\mathcal{I}\mathcal{J}} \left(F^{\mathcal{I}}_A\right)^- \wedge \left(F^{\mathcal{J}}_A\right)^- + \bar{\tau}_{\mathcal{I}\mathcal{J}} \left(F^{\mathcal{I}}_A\right)^+ \wedge \left(F^{\mathcal{J}}_A\right)^+ - \mathrm{i}\,\mathrm{Im}\,\tau_{\mathcal{I}\mathcal{J}}\, \mathrm{d}\mathfrak{a}^{\mathcal{I}} \wedge \star \mathrm{d}\bar{\mathfrak{a}}^{\mathcal{J}}, \tag{4.2}$$

where

$$\tau_{\mathcal{I}\mathcal{J}} = \frac{\partial^2 \mathcal{F}}{\partial \mathfrak{a}^{\mathcal{I}} \partial \mathfrak{a}^{\mathcal{J}}}. \tag{4.3}$$

The invariant formulation of equation (4.1) is that the two-form

$$\sum_{\mathcal{I}} \mathrm{d}\mathfrak{a}^{\mathcal{I}} \wedge \mathrm{d}\mathfrak{a}^D_{\mathcal{I}}$$

identically vanishes on $\mathfrak{M}$. The proper formulation of this condition uses the additional structure which we review below.

4.2 Extended moduli space

In our solution of the theories of class I and class II it would be sometimes convenient to trade the bifundamental masses formally with the $U(1)$ factors as explained in (2.5) if one considers $\times_{i \in \mathrm{Vert}_\gamma} U(\mathbf{v}_i)$ gauge group instead of $\times_{i \in \mathrm{Vert}_\gamma} \mathrm{SU}(\mathbf{v}_i)$. The $|\mathrm{Vert}_\gamma| - 1$ bifundamental masses[1] and one overall $U(1)$ factor add $|\mathrm{Vert}_\gamma|$ parameters to $\mathfrak{M}$. We set

$$\mathfrak{M}^{\mathrm{ext}} = \mathfrak{M} \times \mathbb{C}^{\mathrm{Vert}_\gamma} \tag{4.4}$$

with

$$\dim_{\mathbb{C}} \mathfrak{M}^{\mathrm{ext}} = \dim_{\mathbb{C}} \mathfrak{M} + |\mathrm{Vert}_\gamma| = \sum_{i \in \mathrm{Vert}_\gamma} \mathbf{v}_i.$$

For the class II theories

$$\dim_{\mathbb{C}} \mathfrak{M}^{\mathrm{ext}} = \sum_{i \in \mathrm{Vert}_\gamma} \mathbf{v}_i = Nh,$$

where $h = \sum_{i \in \mathrm{Vert}_\gamma} a_i$ is the Coxeter number of $\mathbf{G}$. Recall that we only encounter simply-laced Lie algebras for which $h = h^\vee$.

We should emphasize that only the true moduli space $\mathfrak{M}$ of vacua has the special geometry with (4.3) defining a positive (outside the loci of singularities which signal the appearance

[1] Recall that in the class II $\widehat{A}_r$ theory there are r mass parameters that can be traded for the $U(1)$ scalars and 1 additional "twist" mass parameter m^* promoting the class II to class II*.

of massless BPS particles) metric in the appropriate duality frame. On the extended moduli space $\mathfrak{M}^{\text{ext}}$ the prepotential $\mathcal{F}$ still defines some kind of metric, but it cannot be positive everywhere throughout the variety of masses. This is because the dependence on masses is purely perturbative. Once we gauge the flavor symmetry (an example of such gauging, promoting the A_1 class I theory to the $\widehat{D}_4$ class II theory, will be discussed in Section 10.20), we correct the metric by the instanton contributions.

4.3 Finite size effects

Subjecting the gauge theory to some boundary conditions reveals more structure.

For example, we can compactify the four-dimensional $\mathcal{N} = 2$ gauge theory on a circle $\mathbb{S}^1$ of radius R. The resulting theory looks like a three-dimensional sigma model with the target space $\mathfrak{P}$ which is a hyperkähler manifold of real dimension $4\mathbf{r}$. The hyperkähler metric on $\mathfrak{P}$ contains a lot of interesting information about the particle content of the original four-dimensional theory.

The hyperkähler structure on $\mathfrak{P}$ is a triplet of integrable complex structures, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, such that every linear combination $(a\mathbf{I} + b\mathbf{J} + c\mathbf{K})$ for $a^2 + b^2 + c^2 = 1$ is also an integrable complex structure, and a triplet of the corresponding symplectic forms $\omega_{\mathbf{I}}$, $\omega_{\mathbf{J}}$, $\omega_{\mathbf{K}}$ which are the Kähler forms for the metric g on $\mathfrak{P}$ in the corresponding complex structures.

Among the two-sphere of complex structures, one complex structure, which is usually called $\mathbf{I}$, plays a special rôle. This complex structure and the corresponding $(2,0)$ symplectic form $\Omega_{\mathbf{I}} = \omega_{\mathbf{J}} + \mathrm{i}\omega_{\mathbf{K}}$ are visible in the limit $R \to \infty$, where $\mathfrak{P}$ as a metric space collapses to $\mathfrak{M}$. For very large but finite R the manifold $\mathfrak{P}$ looks like a fibration over $\mathfrak{M}$ whose fibers $\mathcal{A}_u$, $u \in \mathfrak{M}$ are the abelian varieties (complex tori, which we describe in more detail momentarily) of diameter which scales like R^{-1}.

These fibers $\mathcal{A}_u$ parametrize the $\mathbb{S}^1$ holonomy of the abelian gauge fields $A^{\mathcal{I}}$ and their duals $A^D_{\mathcal{I}}$. The reduction of the action (4.2) on $\mathbb{S}^1$ gives

$$\begin{aligned}\mathcal{L}^{\text{eff3d}} = &\int \mathrm{i}\,\mathrm{Re}\,\tau_{\mathcal{I}\mathcal{J}}\,\mathrm{d}\alpha^{\mathcal{I}} \wedge B^{\mathcal{J}} \\ &+ \frac{1}{2}\,\mathrm{Im}\,\tau_{\mathcal{I}\mathcal{J}}\big(R\mathrm{d}\mathfrak{a}^{\mathcal{I}} \wedge *\mathrm{d}\bar{\mathfrak{a}}^{\mathcal{J}} + R^{-1}\mathrm{d}\alpha^{\mathcal{I}} \wedge *\mathrm{d}\alpha^{\mathcal{J}} + RB^{\mathcal{I}} \wedge *B^{\mathcal{J}}\big),\end{aligned}$$

where we denote by $*$ the three-dimensional Hodge star, and by $B^{\mathcal{I}}$ the curvature of the three-dimensional gauge field $B^{\mathcal{I}} = dA^{\mathcal{I}}_{3\mathrm{d}}$ which is obtained by decomposing $A^{\mathcal{I}}_{4\mathrm{d}} = \alpha^{\mathcal{I}} d\theta + A^{\mathcal{I}}_{3\mathrm{d}}$. The scalar $\alpha^{\mathcal{I}}$ is actually circle-valued, since the gauge transformations $\mathrm{e}^{2\pi\mathrm{i}n_{\mathcal{I}}\theta}$ shift it by $2\pi\mathrm{i}n^{\mathcal{I}}$, $n^{\mathcal{I}} \in \mathbb{Z}$. Next we dualize the three-dimensional abelian gauge field, by promoting $B^{\mathcal{I}}$ to the independent 2-form, and coupling it to the dual scalar $\beta_{\mathcal{I}}$, which is also circle-valued, in order to ensure the flux quantization of the original gauge curvature $B_{\mathcal{I}}$:

$$\begin{aligned}\mathcal{L}^{\text{eff3dd}} = &\int \mathrm{i}\big(\mathrm{d}\beta_{\mathcal{J}} + \mathrm{Re}\,\tau_{\mathcal{I}\mathcal{J}}\,\mathrm{d}\alpha^{\mathcal{I}}\big) \wedge B^{\mathcal{J}} \\ &+ \frac{1}{2}\,\mathrm{Im}\,\tau_{\mathcal{I}\mathcal{J}}\big(R\,\mathrm{d}\mathfrak{a}^{\mathcal{I}} \wedge *\mathrm{d}\bar{\mathfrak{a}}^{\mathcal{J}} + R^{-1}\mathrm{d}\alpha^{\mathcal{I}} \wedge *\mathrm{d}\alpha^{\mathcal{J}} + RB^{\mathcal{I}} \wedge *B^{\mathcal{J}}\big) \\ \longrightarrow\ &\frac{R}{2}\int \mathrm{Im}\,\tau_{\mathcal{I}\mathcal{J}}\,\mathrm{d}\mathfrak{a}^{\mathcal{I}} \wedge *\mathrm{d}\bar{\mathfrak{a}}^{\mathcal{J}} + \frac{1}{2R}\int \big(\mathrm{Im}\,\tau^{-1}\big)^{\mathcal{I}\mathcal{J}}\mathrm{d}z_{\mathcal{I}} \wedge *\mathrm{d}\bar{z}_{\mathcal{J}}.\end{aligned} \tag{4.5}$$

In the last line we have integrated out the unconstrained Gaussian field $B_{\mathcal{I}}$. We also introduced the holomorphic coordinates

$$z_{\mathcal{I}} = \beta_{\mathcal{I}} + \tau_{\mathcal{I}\mathcal{J}}\alpha^{\mathcal{J}}, \qquad I = 1, \ldots, \mathbf{r}$$

on the fibers $\mathcal{A}_u$ of the fibration $\mathfrak{P} \to \mathfrak{M}$. Both $\mathfrak{a}_\mathcal{I}$ and $z_\mathcal{I}$ are the $\mathbf{I}$-holomorphic coordinates on $\mathfrak{P}$. By construction, the coordinates $z_\mathcal{I}$ are subject to the periodic identifications:

$$z_\mathcal{I} \to z_\mathcal{I} + 2\pi\mathrm{i}\big(n_\mathcal{I} + \tau_{\mathcal{I}\mathcal{J}} m^\mathcal{J}\big), \qquad n_\mathcal{I}, m^\mathcal{I} \in \mathbb{Z}, \tag{4.6}$$

which confirm our assertion that the fibers $\mathcal{A}_u$ of the map $\mathfrak{P} \to \mathfrak{M}$ are abelian varieties (recall that the metric $\operatorname{Im}\tau d\mathfrak{a}\otimes d\bar{\mathfrak{a}}$ is positive definite, the unitarity requirement). The coordinates $\mathfrak{a}_\mathcal{I}, z_\mathcal{I}$ are the Darboux coordinates for the $(2,0)$ form $\Omega_\mathbf{I}$:

$$\Omega_\mathbf{I} = \sum_{\mathcal{I}=1}^{\mathbf{r}} \mathrm{d}\mathfrak{a}^\mathcal{I} \wedge \mathrm{d}z_\mathcal{I} = \sum_{\mathcal{I}=1}^{\mathbf{r}} \mathrm{d}\mathfrak{a}^D_\mathcal{I} \wedge \mathrm{d}z^\mathcal{I}_D \tag{4.7}$$

as well as the electric-magnetic duals $\mathfrak{a}^D_\mathcal{I}$ and $z^\mathcal{I}_D = \big(\tau^{-1}\big)^{\mathcal{I}\mathcal{J}}\beta_\mathcal{J} + \alpha^\mathcal{I}$. The fibers $\mathcal{A}_u$ are Lagrangian with respect to $\Omega_\mathbf{I}$.

The metric on $\mathfrak{P}$, which enters the kinetic term in the equation (4.5) is actually not the correct hyperkähler metric on $\mathfrak{P}$ for finite R. It receives corrections which are exponentially small with R,

$$\sim \mathrm{e}^{-M(\mathfrak{a})R}, \tag{4.8}$$

where $M(\mathfrak{a})$ is the mass of a BPS particle in the Hilbert space of the theory in four dimensions built over the vacuum $u \in \mathfrak{M}$. As is well-known, the masses of some BPS particles vanish along some loci in $\mathfrak{M}$, where the corrections (4.8) become significant. One can show, however, that $\Omega_\mathbf{I}$ does not get corrected by the finite size effects of these BPS particles.

One can also compactify the theory on a two-dimensional Riemann surface Σ (with a partial twist along Σ, to preserve some supersymmetry). For Σ other then two-torus this leads to the two-dimensional theory with $\mathcal{N} = 2$ supersymmetry. One has various sectors labeled by the electric and magnetic fluxes $\mathbf{e} = (e_\mathcal{I})$, $\mathbf{m} = \big(m^\mathcal{I}\big)$ through Σ. In the sector where $(\mathbf{e},\mathbf{m}) \neq (0,0)$ one gets an effective superpotential [120]:

$$W_{(\mathbf{e},\mathbf{m})} = \sum_{\mathcal{I}=1}^{\mathbf{r}} e_\mathcal{I}\mathfrak{a}^\mathcal{I} + m^\mathcal{I}\mathfrak{a}^D_\mathcal{I},$$

which in four-dimensional theory is the central charge of the $\mathcal{N} = 2$ superalgebra. It is also equal to one of the *action variables* of the Seiberg–Witten integrable system [32, 38, 72].

If one compactifies on a two-torus, then the resulting two-dimensional theory is the $\mathcal{N} = 4$ supersymmetric sigma model whose target space $\mathfrak{P}$ is the hyperkähler manifold.

It turns out to be quite useful to interpret the $\mathcal{N} = 2$ theory on a four-dimensional manifold X which can be viewed as a two-torus fibration over some base B, as an effective sigma model with B as a world sheet. In case where the fibration has singularities of real codimension one (for example, if X is a product of a disk and a cylinder), then B has a boundary, and the smoothness of the four-dimensional field configurations translates to particular boundary conditions in the two-dimensional sigma model [159]. An interesting class of such boundary conditions come from the so-called canonical coisotropic branes [97, 99, 159]. The algebra of the open string vertex operators corresponding to such a brane turns out to be the deformation quantization [107] of the algebra of holomorphic (in the appropriate complex structure) functions on $\mathfrak{P}$. Remarkably, when $\mathfrak{P}$ is an algebraic integrable system in one of the complex structures, one can apply the fiberwise T-duality along the Liouville fibers, leading to the mirror perspective on the quantization procedure. First of all, in the case of the Hitchin system the mirror manifold turns out to be the Hitchin system for the Langlands dual group. In the general case the mirror $\mathfrak{P}^\vee$ of the original hyperkähler manifold $\mathfrak{P}$ is also expected to be an integrable system.

The mirror of the canonical coisotropic brane is believed to be a holomorphic (in appropriate complex structure) Lagrangian brane. In the case of Hitchin system this brane is argued [99] to be the so-called brane $\mathcal{B}_{\mathcal{O}}$ of opers, with evidence supported by exact computations in [87, 88, 89, 90, 112, 148, 158].

4.4 The appearance of an integrable system

The complex symplectic manifold $(\mathfrak{P}, \Omega_{\mathbf{I}})$, its projection $\pi\colon \mathfrak{P} \to \mathfrak{M}$ with Lagrangian fibers $\mathcal{A}_u = \pi^{-1}(u)$, $u \in \mathfrak{M}$, which are principally polarized abelian varieties (the principal polarization comes from the restriction of $\omega_{\mathbf{I}}$ onto the fibers) define what is known as the *algebraic integrable system* [32, 35, 38]. It is one of the possible complexifications of the familiar notion of the completely integrable system in the classical mechanics.

The other possibility, namely a complex symplectic manifold with the Lagrangian fibration whose fibers are the complex tori $(\mathbb{C}^\times)^{\mathbf{r}}$, is also realized in the context of gauge theories. However, the base of such a system typically parametrizes the space of mass parameters of the gauge theory.

The fibers $\mathcal{A}_u$ are the Liouville tori, while $(\mathfrak{a}_{\mathcal{I}}, z_{\mathcal{I}})$ are the action-angle variables. The novelty of the complex case is the doubling of the possible choices of the action-angle variables with fixed Liouville fibration. Indeed, the fibers $\mathcal{A}_u$ are the $2\mathbf{r}$-real-dimensional tori, therefore in producing the action variables as in the Arnol'd–Liouville theorem one has a choice of $\mathbf{r}$ out of $2\mathbf{r}$ cycles in $H_1(\mathcal{A}_u, \mathbb{Z})$. The lattice $H_1(\mathcal{A}_u, \mathbb{Z})$ has a symplectic form ϖ, which comes from the polarization, i.e., a properly normalized class of the restriction $\omega_{\mathbf{I}}|_{\mathcal{A}_u}$. It turns out that any Lagrangian sublattice L in $H_1(\mathcal{A}_u, \mathbb{Z})$ defines a system of local coordinates $(\mathfrak{a}_I)$ on the base $\mathfrak{M}$ near the point $u \in \mathfrak{M}$, as well as the conjugate angle-like coordinates $(z_{\mathcal{I}})$ on the fiber $\mathcal{A}_u$ itself. Let $\mathcal{A}_{\mathcal{I}}$ be the integral basis of this sublattice $L \subset H_1(\mathcal{A}_u, \mathbb{Z})$. Then

$$\mathrm{d}\mathfrak{a}^{\mathcal{I}} = \oint_{\mathcal{A}_{\mathcal{I}}} \Omega_{\mathbf{I}}. \tag{4.9}$$

One can also define

$$\mathrm{d}\mathfrak{a}^D_{\mathcal{I}} = \oint_{\mathcal{B}^{\mathcal{I}}} \Omega_{\mathbf{I}}, \tag{4.10}$$

where $\mathcal{B}^{\mathcal{I}}$ is the basis in the dual sublattice $L^\vee \subset H_1(\mathcal{A}_u, \mathbb{Z})$, such that

$$\varpi(\mathcal{A}_{\mathcal{I}}, \mathcal{A}_{\mathcal{J}}) = \varpi\big(\mathcal{B}^{\mathcal{I}}, \mathcal{B}^{\mathcal{J}}\big) = 0, \qquad \varpi\big(\mathcal{A}_{\mathcal{I}}, \mathcal{B}^{\mathcal{J}}\big) = \delta^{\mathcal{J}}_{\mathcal{I}}. \tag{4.11}$$

One then shows that

$$\sum_{\mathcal{I}=1}^{\mathbf{r}} \mathrm{d}\mathfrak{a}^{\mathcal{I}} \wedge \mathrm{d}\mathfrak{a}^D_{\mathcal{I}} \equiv 0 \tag{4.12}$$

on $\mathfrak{M}$, which, in turn, implies (4.1). The coordinates $z_{\mathcal{I}}$ along $\mathcal{A}_u$ are defined using (4.7) with the normalization (4.6) that half of the periods of $z_{\mathcal{I}}$ are in $2\pi\mathrm{i}\mathbb{Z}$.

The integrable systems which one encounters in the classical mechanics are rarely given in the form of the action-angle variables. Usually one has the phase space $\mathfrak{P}$, the symplectic form $\Omega_{\mathbf{I}}$, perhaps some Darboux coordinates

$$\Omega_{\mathbf{I}} = \sum_{\mathcal{I}=1}^{\mathbf{r}} \mathrm{d}p_{\mathcal{I}} \wedge \mathrm{d}q^{\mathcal{I}}$$

and the collection of Poisson-commuting functionally independent Hamiltonians $U_1(p,q), \dots, U_{\mathbf{r}}(p,q)$. One then looks for the action-angle coordinates, i.e., the Darboux coordinates $(\mathfrak{a}, z)$,

such that the Hamiltonians $U_{\mathcal{I}}(p,q) = u_{\mathcal{I}}(\mathfrak{a})$ depend only on $\mathfrak{a}$, the action variables. The Hamiltonian evolution then linearizes on the fibers $\mathcal{A}_u$, which are the level sets of the Hamiltonians. The motion is a constant velocity motion in the z coordinates:

$$z_{\mathcal{I}}(t) = z_{\mathcal{I}}(0) + \sum_{\mathcal{J}=1}^{\mathbf{r}} t_{\mathcal{J}} \frac{\partial u_{\mathcal{J}}}{\partial \mathfrak{a}_{\mathcal{I}}}.$$

It is interesting to study the level sets $\mathcal{A}_u$ of the Hamiltonians, the Liouville tori. The algebraic integrable systems are such, that the fibers can be compactified to become the polarized abelian varieties. Where do the polarized abelian varieties come from?

4.5 Integrable systems from classical gauge theories

One source of the polarized abelian varieties are the Jacobians of the algebraic curves. The Liouville tori of algebraic integrable systems can be often found inside the Jacobians of the algebraic curves, constructed while solving some classical gauge field equations.

4.5.1 Hitchin system

There is an interesting class of algebraic integrable systems for which the Liouville tori are precisely these Jacobians. Take the $U(N)$ Hitchin system on a genus g Riemann surface. The phase space $\mathfrak{P}$ is the cotangent bundle (up to a birational transformation) to the moduli space $M_{N,c}$ of holomorphic rank N vector bundles E over Σ with fixed first Chern class $c = c_1(E)$. It is convenient to take $(c,N) = 1$ to avoid complications coming from the reducible connections.

In the complex structure $\mathbf{I}$ the holomorphic coordinates on $\mathfrak{P}$ are $\big(\bar{A}, \Phi\big)$, where $\bar{\partial} + \bar{A}$ is the $(0,1)$-connection on the smooth vector bundle E which endows it with the complex structure, and $\Phi \in \mathrm{End}(E) \otimes \Omega^{1,0}(\Sigma)$ is the holomorphic Higgs field

$$\bar{\partial}\Phi + \big[\bar{A}, \Phi\big] = 0. \tag{4.13}$$

The symplectic form on $\mathfrak{P}$ comes from the $(2,0)$ symplectic form on the space of all smooth pairs $(\bar{A}, \Phi)$

$$\Omega_{\mathbf{I}} = \int_{\Sigma} \mathrm{tr}\, \delta\Phi \wedge \delta\bar{A}$$

by the symplectic reduction with respect to the action of the gauge group:

$$g\colon\ \big(\bar{A}, \Phi\big) \longrightarrow \big(g^{-1}\bar{A}g + g^{-1}\bar{\partial}g, g^{-1}\Phi g\big).$$

The set of Poisson-commuting Hamiltonians is given by

$$U_{i,\mathbf{a}} = \int_{\Sigma} \nu_{i,\mathbf{a}}\, \mathrm{tr}\, \Phi^i, \qquad i = 1, \dots, N, \tag{4.14}$$

where $\nu_{i,\mathbf{a}} \in H^{0,1}\big(\Sigma, K_{\Sigma}^{\otimes(1-i)}\big)$, $\mathbf{a} = 1, \dots, (2i-1)(g-1) + \delta_{i,1}$ form a basis in the space of holomorphic $(1-i,1)$-differentials. Fixing the values $u_{i,\mathbf{a}}$ of all the Hamiltonians $U_{i,\mathbf{a}}$ gives us a point $u \in \mathfrak{M}$ in the vector space

$$\mathfrak{M} = \bigoplus_i H^{0,1}\big(\Sigma, K_{\Sigma}^{\otimes(1-i)}\big).$$

One defines the *spectral curve* $C_u \subset T^*\Sigma$ as the zero locus of the characteristic polynomial of Φ:

$$\mathrm{Det}(\Phi - \lambda) = 0. \tag{4.15}$$

It is a holomorphic curve thanks to (4.13), which is invariant under the Hamiltonian flows generated by the Hamiltonians (4.14). The curve C is an N-sheeted cover of Σ

$$\pi\colon\ C_u \to \Sigma.$$

Its genus can be computed using the Riemann–Hurwitz formula

$$2-2g_{C_u} = N(2-2g_\Sigma) - \delta,$$

where $\delta = 2N(N-1)(g_\Sigma - 1)$ is the number of branch points. The latter is the number of zeroes of the discriminant of the polynomial (4.15), which is a holomorphic $N(N-1)$-differential on Σ. Thus

$$g_{C_u} = N^2(g_\Sigma - 1) + 1.$$

The Jacobian of C is thus an abelian variety of dimension

$$g_{C_u} = g_\Sigma + \sum_{j=2}^{N} (2j-1)(g_\Sigma - 1), \tag{4.16}$$

which is equal to the dimension of the base $\mathfrak{M}$ of the Hitchin fibration. The fibers $\mathcal{A}_u$ of the Hitchin fibration are thus the Jacobians of the corresponding spectral curves.

One generalization is to study the $\mathrm{SL}(N)$ Hitchin system. In this case the corresponding rank N vector bundles have the trivial determinant, and the corresponding Higgs field is traceless. The base of the Hitchin fibration now has the dimension $\big(N^2-1\big)(g_\Sigma-1)$, the equation (4.15) has vanishing $\propto \lambda^{N-1}$ term, and the fibers $\mathcal{A}_u$ are not the full Jacobians of the spectral curve C_u, which still has the genus g_{C_u} (4.16) but the kernel J_0 of the map $\pi_*\colon\ \mathrm{Jac}(C_u) \to \mathrm{Jac}(\Sigma)$, which sends the degree zero line bundle L on C_u to the line bundle $\mathcal{L} = \mathrm{Det}\,\pi_* L$ on Σ, whose fiber $\mathcal{L}_z$ over the point $z \in \Sigma$ is the tensor product of the fibers L_y of L over all preimages of z:

$$\mathcal{L}_z = \bigotimes_{y \in \pi^{-1}(z)} L_y.$$

The Hitchin system can be defined [83] for any algebraic Lie group G, with the maximal torus T. Let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(T)$. The Hitchin space is the moduli space of stable pairs $(\mathcal{P}, \Phi)$, where $\mathcal{P}$ is a holomorphic G-bundle over Σ, and Φ is a holomorphic $(1,0)$-form on Σ, valued in the bundle of Lie algebras $\mathfrak{g}$, associated with $\mathcal{P}$ via the adjoint representation:

$$\Phi \in H^0(\Sigma, K_\Sigma \otimes \mathrm{ad}(\mathcal{P})).$$

The Hitchin fibration is defined by fixing the gauge-invariant polynomials $P_j(\Phi) \in H^0\big(\Sigma, K_\Sigma^{\otimes d_j}\big)$ of the $\mathfrak{g}$-valued Higgs field Φ:

$$u = (P_j(\Phi))_{j=1}^r \in \bigoplus_{j=1}^{r} \mathbb{C}^{(2d_j-1)(g_\Sigma-1)} = \mathfrak{M},$$

where d_j's are the degrees of basic Ad-invariant polynomials on $\mathfrak{g}$.

The fibers of the Hitchin fibration are now trickier to define. First of all, there is no preferred notion of the spectral curve. For some gauge groups one can use the minuscule representation, but this is not always available.

One option is to consider the so-called *cameral curve* $\mathcal{C}_u$, which is a $W(\mathfrak{g})$-cover of the base curve Σ. The points of the cameral curve $\mathcal{C}_u$ are, over generic z, the pairs (φ, z), where $z \in \Sigma$ and $\varphi \in \mathfrak{h}$ is the element of the fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is conjugate to the

Higgs field $\Phi(z)$. This definition makes sense for the points $z \in \Sigma$ for which $\Phi(z)$ is semi-simple, i.e., belongs to the $\mathrm{ad}(G)$-orbit of an element in $\mathfrak{h}$. If this is not the case (e.g., $\Phi(z)$ is conjugate to a Jordan block in the $\mathrm{GL}(N)$ case), one can find an appropriate representative in $\mathfrak{h}$ by modifying the equivalence relation (e.g., two matrices are equivalent if their characteristic polynomials coincide). To stress the fact that $\mathcal{C}_u$ depends on u which is the set of holomorphic d_j-differentials $P_j(\Phi)$.

Over $\mathcal{C}_u$ so defined one has r line bundles, $\mathcal{L}_i$, $i = 1, \dots, r$, which correspond to the fundamental weights $\lambda_i \in \mathfrak{h}^*$. The line bundle $\mathcal{L}_i$ is a subbundle in the holomorphic vector bundle $\mathbf{R}_i = R_i \times_G \mathcal{P}$, associated with $\mathcal{P}$ via the i-th fundamental representation R_i of G. The fiber of $\mathcal{L}_i \subset \mathbf{R}_i$ over (φ, z) is the eigenspace corresponding to the eigenvalue $\lambda_i(\varphi)$.

To any weight vector $\lambda \in \Lambda$ a line bundle $\mathcal{L}_\lambda$ over $\mathcal{C}_u$ can be associated:

$$\lambda = \sum_{i=1}^{r} n_i \lambda_i \mapsto \mathcal{L}_\lambda = \bigotimes_{i=1}^{r} \mathcal{L}_i^{\otimes n_i}.$$

In a more physical language, the Hitchin moduli space is the quotient of the space of pairs $\big(\bar{A}, \Phi\big)$, where $\bar{A}$ is a $(0,1)$-connection on smooth principal G-bundle $\mathcal{P}$ over Σ, and Φ is a $(1,0)$ $\mathfrak{g}$-valued form, which are compatible, i.e., solve the equation (4.13), and are considered up to the G-gauge transformations:

$$g\colon\ \big(\bar{A}, \Phi\big) \mapsto \big(g^{-1}\bar{\partial} g + \mathrm{Ad}_g \bar{A}, \mathrm{Ad}_g \Phi\big).$$

By fixing the partial gauge $\Phi = \varphi \in \mathfrak{h}$ for fixed $\mathfrak{h} \subset \mathfrak{g}$, one reduces the gauge invariance from G to $N(T)$. The equation (4.13) imply that in this gauge $\bar{A}$ is a T-connection $\bar{A} = \bar{a}$, with the T subgroup of $N(T)$ acting by the T-gauge transformations $\bar{a} \mapsto \bar{a} + \bar{\partial}\chi$, $\mathrm{e}^{\chi} \in T$. On Σ the T-valued gauge field and the $\mathfrak{h}$-valued Higgs field φ are not well-defined, since there are the $W(\mathfrak{g}) = N(T)/T$ remaining gauge transformations. On $\mathcal{C}$, however, both φ and $\bar{a}$ are well-defined. In fact, $\bar{a}$ defines on $\mathcal{C}_u$ a holomorphic principal T-bundle $\mathcal{T}$, so that $L_i = \mathcal{T}^{\lambda_i}$. The T-bundle $\mathcal{T}$ is $W(\mathfrak{g})$-equivariant. This is the translation of the fact that the Weyl group $W(\mathfrak{g})$ acts simultaneously on φ and $\bar{a}$. The isomorphism (properly understood at the ramification points)

Holomorphic		$W(\mathfrak{g})$-covers $\mathcal{C}$ of Σ,
principal G-bundles $\mathcal{P}$ on Σ,		$\mathcal{C} \subset T^*\Sigma \otimes \mathfrak{h}$,
holomorphic Higgs fields	$\Leftrightarrow$	holomorphic $W(\mathfrak{g})$-equivariant
$\Phi \in H^0(\Sigma, K_\Sigma \otimes \mathrm{ad}(\mathcal{P}))$		principal T-bundles on $\mathcal{C}$

allows to represent the Hitchin moduli space as a fibration over the vector space $\mathfrak{M}$, whose points are the $W(\mathfrak{g})$-invariant curves $\mathcal{C}_u$ sitting in the tensor product $T^*\Sigma \otimes \mathfrak{h}$ (this is almost a tautology: a $W(\mathfrak{g})$-invariant curve in $T^*\Sigma \otimes \mathfrak{h}$ is a curve in $T^*\Sigma \otimes \mathfrak{h}/W(\mathfrak{g})$, i.e., a holomorphic section of the vector bundle $T^*\Sigma \otimes \mathbb{C}[\mathfrak{h}]^{W(\mathfrak{g})}$).

The fiber $\mathcal{A}_u$ of the Liouville fibration (which is called Hitchin's fibration in this case) is a generalized Prym variety, which is, roughly speaking,

$$\mathcal{A}_u \approx \mathrm{Hom}_{W(\mathfrak{g})}\left(\Lambda, \mathrm{Pic}(\mathcal{C}_u)\right) = \mathrm{Bun}_T(\mathcal{C}_u)^{W(\mathfrak{g})}. \tag{4.17}$$

The papers [30, 33, 34] correct the equation (4.17) in a couple of subtle points as well as provide the additional theory.

4.5.2 Instanton moduli spaces as integrable systems

Hitchin's equations (4.13), for flat Σ, are the dimensional reduction of the instanton (or anti-self-duality) equations from four dimensions. It turns out that one can get an integrable system

directly from the moduli spaces of four-dimensional instantons, or three-dimensional monopoles (examples of integrable systems on moduli spaces of instantons were found in [136]).

We only briefly sketch the constructions here.

Let S be an elliptic K3 manifold, i.e., an algebraic surface, with the holomorphic $\omega_S^{2,0}$ form, and with the projection $\pi\colon S \to \mathbb{CP}^1$ whose fibers $\pi^{-1}(z)$, $z \in \mathbb{CP}^1$ are the elliptic curves $\mathcal{E}_z$ (generically nonsingular). One can endow S with the hyperkähler metric. Consider the moduli space $\mathfrak{P} = \mathcal{M}_N(G)$ of charge N G-instantons on S, i.e., the solutions to the system of partial differential equations

$$F_A \wedge \omega_{\mathbf{I}} = F_A \wedge \omega_{\mathbf{J}} = F_A \wedge \omega_{\mathbf{K}} = 0,$$
$$F_A^{0,2} = 0$$

(the last equation is a linear combination of the **J** and **K** equations from the first line) of fixed instanton charge $N \geq 0$:

$$-\frac{1}{8\pi^2}\int_S \operatorname{tr} F_A \wedge F_A = N.$$

Here G is some compact simply-connected simple Lie group, which has a simply-laced Lie algebra $\mathfrak{g}$. The moduli space $\mathfrak{P} = \mathcal{M}_N(G)$ is also hyperkähler, in particular it is holomorphic symplectic, with the $(2,0)$-form given by

$$\Omega_{\mathbf{I}}^{N,G} = \int_S \omega_S^{2,0} \wedge \operatorname{tr} \delta\bar{A} \wedge \delta\bar{A}.$$

The integrable system structure is obtained by studying the restriction of the instanton gauge field on the elliptic fibers, where generically they define a point in the coarse moduli space $\operatorname{Bun}_{\mathbf{G}}(\mathcal{E}_z)$ of semi-stable principal holomorphic **G**-bundles on the fiber, see Appendix F. Thanks to E. Loojienga's theorem, this moduli space is a weighted projective space, which can be identified for different non-singular fibers. One gets thus a section of the locally trivial bundle of

$$\Pi\colon\ \mathcal{P} = \bigcup_{z\in\mathbb{CP}^1} \operatorname{Bun}_{\mathbf{G}}(\mathcal{E}_z) \longrightarrow \mathbb{CP}^1.$$

One has to be careful at the singular fibers. The base $\mathfrak{M}$ of the integrable systems is the properly compactified moduli space of the holomorphic sections $\sigma\colon \mathbb{CP}^1 \longrightarrow \mathcal{P}$ of appropriate degree with some ramification conditions at the discriminant locus of the original elliptic fibration π.

In this work we shall not encounter these difficulties.

In fact, as we shall explain in more detail in Section 11, the moduli spaces of vacua of the quiver gauge theories we study lead to the integrable systems which arise from the the moduli spaces of G-monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$ for class I theories with $\mathbf{G}_{\mathrm{q}} = \mathbf{G}$, or from the moduli spaces of G-instantons on $\mathbb{R}^2 \times \mathbb{T}^2$ for class II theories with $\mathbf{G}_{\mathrm{q}} = \widehat{\mathbf{G}}$. Here G is a compact Lie group, whose complexification is the complex simple Lie group **G**.

The moduli space $\mathfrak{P}$ of G-instantons, viewed in the complex structure where $\mathbb{R}^2 \times \mathbb{T}^2 = \mathbb{C}^1 \times \mathcal{E}$, is birational to the moduli space of semi-stable holomorphic **G**-bundles on $\mathbb{CP}^1_{\langle x\rangle} \times \mathcal{E}$, with fixed trivialization at $\infty \times \mathcal{E}$. The moduli space $\mathfrak{P}$ projects down to the moduli space $\mathfrak{M}$ of quasimaps from $\mathbb{CP}^1_{\langle x\rangle}$ to the moduli space of semi-stable holomorphic bundles $\operatorname{Bun}_{\mathbf{G}}(\mathcal{E})$ on a fixed elliptic curve $\mathcal{E}$. The moduli space of monopoles maps to the moduli space of quasimaps with prescribed singularities on $\mathbb{CP}^1_{\langle x\rangle}$ to $B(\mathfrak{g}) = \mathbf{G}/\operatorname{Ad}(\mathbf{G}) = \mathbf{T}/W(\mathfrak{g})$.

4.6 Extended moduli space as a complex integrable system

The extended moduli space $\mathfrak{M}^{\mathrm{ext}}$ is a base of a complex, but not algebraic, integrable system $\mathfrak{P}^{\mathrm{ext}} \to \mathfrak{M}^{\mathrm{ext}}$. The Liouville tori of this integrable system are acted on by an algebraic

torus $(\mathbb{C}^\times)^{\mathrm{Vert}_\gamma}$, so that the quotients are the compact abelian varieties, the Liouville tori fibered over $\mathfrak{M}$. The symplectic quotient of $\mathfrak{P}^{\mathrm{ext}}$ with respect to $(\mathbb{C}^\times)^{\mathrm{Vert}_\gamma}$ at some level of the moment map, which is linearly determined by the values of the bi-fundamental masses, gives $\mathfrak{P}$. Recall that Duistermaat–Heckmann theorem [45] then implies that the cohomology class $[\Omega_{\mathbf{I}}]$ of the $(2,0)$-symplectic form $\Omega_{\mathbf{I}}$ on $\mathfrak{M}$ is linear with masses.

The physics behind the reason $\mathfrak{P}^{\mathrm{ext}} \to \mathfrak{M}^{\mathrm{ext}}$ is not an algebraic integrable system is that the kinetic term for the instanton/monopole zero modes in the $\big(\mathbb{R}^2 \times T^2\big)/\big(\mathbb{R}^2 \times S^1\big)$ geometry diverges. These modes are non-dynamical in the effective three dynamical theory obtained by compactifying our $\mathcal{N} = 2$ theory from four to three dimensions. The electric-magnetic duality of dynamical vector multiplets in four dimensions leading to the algebraic integrable system on the moduli space of vacua of the corresponding three-dimensional theory is therefore broken.

5 The limit shape equations

In this section we return to the microscopic analysis of our gauge theory. Recall that the $\mathcal{N} = 2$ supersymmetry algebra is generated by four supercharges $\mathcal{Q}_{\alpha i}$, $\alpha = 1, 2$, $i = 1, 2$ of the left and by four supercharges $\bar{\mathcal{Q}}^i_{\dot\alpha}$, $\dot\alpha = 1, 2$ of the right chirality. The prepotential $\mathcal{F}(\mathcal{A})$ of the theory is a function of the superfield $\mathcal{A}$ which is annihilated by $\mathcal{Q}_{\alpha i}$'s. We shall now focus on the observables which are in the cohomology of one of the $\mathcal{Q}_{\alpha i}$ supercharges, which we shall call simply $\mathcal{Q}$.

5.1 The amplitude functions

The basic such observable is the scalar Φ_i in the vector multiplet. More precisely, any gauge invariant functional, in particular the local operator $P(\Phi_i(\mathbf{x}))$, where P is some invariant polynomial on the Lie algebra of $\mathrm{SU}(\mathbb{B}_i)$, and $\mathbf{x}$ is a point in space-time, is annihilated by $\mathcal{Q}$. Moreover, the observables $P(\Phi_i(\mathbf{x}))$ and $P(\Phi_i(\mathbf{x}'))$ for two different points $\mathbf{x}$ and $\mathbf{x}'$ are in the same $\mathcal{Q}$-cohomology class. Therefore, one may talk about the vacuum expectation value of $P(\Phi_i)$ without specifying the point $\mathbf{x}$.

Consider the observables $\mathcal{O}_{n,i} = \mathrm{tr}_{\mathbf{v}_i} \Phi_i^n$. Form the generating function

$$\mathcal{Y}_i(x) = x^{\mathbf{v}_i} \exp\left(-\sum_{n=1}^{\infty} \frac{\langle \mathcal{O}_{n,i} \rangle_u}{n} x^{-n} \right), \tag{5.1}$$

which turns out to be well-behaved for sufficiently large x. We shall denote the x-plane where $\mathcal{Y}_i(x)$ are defined, by $\mathbf{C}_{\langle x \rangle}$. Actually, the analytic continuation in the x variable gives us the set $\mathcal{Y}(x)$ of multi-valued analytic functions on $\mathbf{C}_{\langle x \rangle}$,

$$\mathcal{Y}(x) = (\mathcal{Y}_i(x))_{i \in \mathrm{Vert}_\gamma}.$$

This set of multi-valued functions captures the vacuum expectation values of all the local gauge invariant observables commuting with the supercharge $\mathcal{Q}$.

Remark 5.1. The general relation between the amplitude functions $\mathcal{Y}_i(x)$ and the polynomials $\mathcal{T}_i(x)$ generating the first non-trivial Casimirs of the gauge group G_{g} is

$$\mathcal{T}_i(x) = (\mathcal{Y}_i(x))_+,$$

where $(\dots)_+$ denotes the polynomial part.

In what follows we shall use another set of $(T_i(x))_{i \in \mathrm{Vert}_\gamma}$ polynomials, $\deg T_i(x) = \mathbf{v}_i$, which are not monic. The coefficients of $T_i(x)$ are related to the coefficients of $\mathcal{T}_i(x)$ by a "mirror map" change of variables, which will become clear in the course of our exposition.

Remark 5.2. In the way we defined these functions, the information about the vacuum expectation values of these observables is contained in the expansion of $\mathcal{Y}_i(x)$ near $x=\infty$ on the physical sheet of these functions. It would be interesting to see whether the expansion at $x=\infty$ of the branches of $\mathcal{Y}_i(x)$ contains information about the vevs of the chiral observables of the theories, related to the one we started with via some version of S-duality.

The functions $\mathcal{Y}_i(x)$ are the integral transforms of the densities

$$\rho_i(\mathrm{x}), \qquad \mathrm{x}\in\mathbb{R}, \quad i\in \mathrm{Vert}_\gamma,$$

which describe the combinatorics of the set of fixed points of the symmetry group action on the instanton moduli space used in the localization approach to the calculation of the supersymmetric partition function of the gauge theory. For the introduction to the subject see [119, 140, 141, 142] and for the novel applications and refinements [1, 163].

We now write down the equations obeyed by the amplitude functions, the so-called *limit shape equations*, generalizing the limit shape equations studied in [149, 154, 168, 169, 170]. We shall solve the limit shape equations using the analytic properties of the amplitude functions. One finds that the analytic continuation of these functions is governed by the monodromy group, which we shall call the iWeyl group (the *instanton Weyl group*).

The iWeyl group is the Weyl group $W(\mathfrak{g}_\mathrm{q})$. For the class I theories $W(\mathfrak{g}_\mathrm{q})$ is the finite Weyl group $W(\mathfrak{g})$ of the corresponding ADE simple Lie algebra $\mathfrak{g}$, for the class II theories the iWeyl group turns out to be the affine Weyl group $W(\widehat{\mathfrak{g}})$ of the corresponding affine Lie algebra $\widehat{\mathfrak{g}}=\mathfrak{g}_\mathrm{q}$. The Weyl group of $\widehat{\mathrm{GL}}_\infty$ shows up in the class II* theories.

We solve the limit shape equations by constructing the iWeyl invariants $\mathcal{X}_j(\mathcal{Y}(x))$ of $\mathcal{Y}_i(x+\mu_i)$, for the appropriate shifts μ_i, and showing that these invariants are polynomials in x,

$$\mathcal{X}_j(\mathcal{Y}(x))=T_j(x), \qquad j\in\mathrm{Vert}_\gamma.$$

For the class II theories the invariants $\mathcal{X}_j$ are convergent power series in $\mathfrak{q}_j$. Moreover, in each order in expansion in $\mathfrak{q}=\prod_i \mathfrak{q}_i^{a_i}$ they are finite Laurent polynomials in $\mathcal{Y}_j$'s. For the class II* theories the invariants $\mathcal{X}_j$ are convergent power series in $\mathfrak{q}_j$, and finite Laurent polynomials in $\mathcal{Y}_j(x+\mu_j+lm^*)$, for a finite collection of integers $l\in\mathbb{Z}$, again in every order in $\mathfrak{q}$ expansion. For the class I theories the functions $\mathcal{X}_j$ are polynomials in $\mathcal{P}_j(x)$ and Laurent polynomials in $\mathcal{Y}_j(x)$.

5.2 The densities and the amplitude functions

The amplitude functions $\mathcal{Y}_i(x)$ are the multi-valued analytic functions, which we defined, for large x, via equation (5.1):

$$\mathcal{Y}_i(x)=\exp\langle \mathrm{tr}_{\mathbf{v}_i}\log(x-\Phi_i)\rangle_u. \tag{5.2}$$

One shows, using the fixed point techniques that

$$\mathcal{Y}_i(x)=\exp\int_{\mathbb{R}}\mathrm{dx}\rho_i(\mathrm{x})\log(x-\mathrm{x}), \tag{5.3}$$

where the *density* function $\rho_i(x)$ has compact support which consists of $\mathbf{v}_i$ intervals

$$\mathrm{supp}\,\rho_i=\bigcup_{\mathbf{a}=1}^{\mathbf{v}_i} I_{i,\mathbf{a}}.$$

The intervals $I_{i,\mathbf{a}}$ should be thought of as the "fattened" versions of the eigenvalues $\mathfrak{a}_{i,\mathbf{a}}$. More precisely,

$$1=\int_{I_{i,\mathbf{a}}}\rho_i(\mathrm{x})\,\mathrm{dx}, \qquad \mathfrak{a}_{i,\mathbf{a}}=\int_{I_{i,\mathbf{a}}}\mathrm{x}\rho_i(\mathrm{x})\,\mathrm{dx}. \tag{5.4}$$

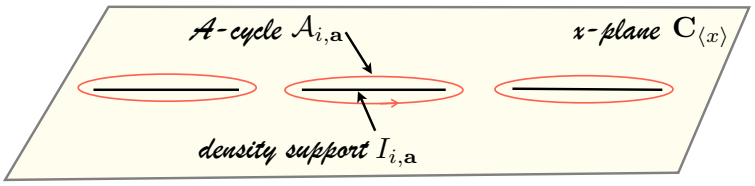


Figure 5.1. Support intervals $I_{i,\mathbf{a}}$ of the densities ρ_i and the cycles $A_{i\mathbf{a}}$.

The functions $\mathcal{Y}_i(x)$ have, therefore, the cuts at the intervals $I_{i,\mathbf{a}}$, with the limit values $\mathcal{Y}_i^{\pm}$ of the $\mathcal{Y}_i$ function at the top and the bottom banks of the interval $I_{i,\mathbf{a}}$ being related via

$$\mathcal{Y}_i^+(x)/\mathcal{Y}_i^-(x) = \exp\left(2\pi\mathrm{i}\int_{-\infty}^{x}\rho_i(\mathrm{x})\,\mathrm{dx}\right).$$

One then analytically continues $\mathcal{Y}_i(x)$ across the cuts, which leads to the set of the multi-valued analytic functions. We shall describe this analytic continuation in detail in the coming section.

5.3 The special coordinates

From the equations (5.2) and (5.4) one derives

$$\mathfrak{a}_{i,\mathbf{a}} = \frac{1}{2\pi\mathrm{i}}\oint_{A_{i\mathbf{a}}} x\,\mathrm{d}\log\mathcal{Y}_i(x), \tag{5.5}$$

where $A_{i\mathbf{a}}$ is a small loop surrounding the cut $I_{i,\mathbf{a}}$, see Figure 5.1.

6 The limit shape prepotential

The prepotential of the low-energy theory is expressed in terms of the densities as follows:

$$\begin{aligned}\mathcal{F}(\mathfrak{a};m;\tau) = &-\iint_{\mathbb{R}^2}\mathrm{dx'dx''}\sum_{i\in\mathrm{Vert}_\gamma}\rho_i(\mathrm{x'})\rho_i(\mathrm{x''})\mathcal{K}(\mathrm{x'}-\mathrm{x''})\\ &+\iint_{\mathbb{R}^2}\mathrm{dx'dx''}\sum_{e\in\mathrm{Edge}_\gamma}\rho_{t(e)}(\mathrm{x'})\rho_{s(e)}(\mathrm{x''})\mathcal{K}(\mathrm{x'}-\mathrm{x''}+m_e)\\ &+\sum_{i\in\mathrm{Vert}_\gamma}\int_{\mathbb{R}}\mathrm{dx}\,\rho_i(\mathrm{x})\left((\log\mathfrak{q}_i)\frac{\mathrm{x}^2}{2}+\sum_{\mathfrak{f}=1}^{\mathbf{w}_i}\mathcal{K}(\mathrm{x}-m_{i,\mathfrak{f}})\right)\\ &+\sum_{i,\mathbf{a}}\mathfrak{b}_{i,\mathbf{a}}\left(1-\int_{I_{i,\mathbf{a}}}\rho_i(\mathrm{x})\,\mathrm{dx}\right)+\sum_{i,\mathbf{a}}\mathfrak{a}^D_{i,\mathbf{a}}\left(\mathfrak{a}_{i,\mathbf{a}}-\int_{I_{i,\mathbf{a}}}\mathrm{x}\rho_i(\mathrm{x})\,\mathrm{dx}\right),\end{aligned}$$

where the constraints (5.4) are incorporated by the last two lines via Lagrangian multipliers $\mathfrak{b}_{i,\mathbf{a}}$, $\mathfrak{a}^D_{i,\mathbf{a}}$,

$$\mathcal{K}(x) = \frac{x^2}{2}\left(\log\left(\frac{x}{\Lambda_{\mathrm{UV}}}\right)-\frac{3}{2}\right)$$

and Λ_{UV} is the UV cutoff scale. In fact, the Λ_{UV}-dependence drops out for the theories solving the $\beta_i = 0$ equations. However, in the intermediate formulae we keep the explicit Λ_{UV}-dependence.

6.1 The limit shape equations

Originally the limit shape equations were derived as the variational equations on $\mathcal{F}$ [149, 154]. These are linear integral equations on the densities $\rho_i(x)$: for any $x \in I_{i,\mathbf{a}}$ the following should hold

$$-2\int_{\mathbb{R}} \rho_i(\mathrm{x})\mathcal{K}(x-\mathrm{x})\,\mathrm{dx} + \sum_{e\colon\, t(e)=i}\int_{\mathbb{R}} \rho_{s(e)}(\mathrm{x})\mathcal{K}(x-\mathrm{x}+m_e)\,\mathrm{dx} \tag{6.1}$$

$$+\sum_{e\colon\, s(e)=i}\int_{\mathbb{R}} \rho_{t(e)}(\mathrm{x})\mathcal{K}(x-\mathrm{x}-m_e)\,\mathrm{dx} + \sum_{\mathfrak{f}=1}^{\mathbf{w}_i}\mathcal{K}(x-m_{i,\mathfrak{f}}) + \frac{x^2}{2}\log(\mathfrak{q}_i) = x\mathfrak{a}^D_{i,\mathbf{a}} + \mathfrak{b}_{i,\mathbf{a}},$$

where $\mathfrak{a}^D_{i,\mathbf{a}}$, $\mathfrak{b}_{i,\mathbf{a}}$ are some constants, the Lagrange multipliers for the conditions (5.4) which are determined from the solution. Actually, $\mathfrak{a}^D_{i,\mathbf{a}}$ *is* the dual special coordinate, cf. (4.1),

$$\frac{\partial\mathcal{F}}{\partial\mathfrak{a}_{i,\mathbf{a}}} = \mathfrak{a}^D_{i,\mathbf{a}}. \tag{6.2}$$

We find it useful to rewrite the second derivative with respect to x of the linear integral equations (6.1) on $\rho_i(\mathrm{x})$ as the non-linear polynomial difference equations on the amplitudes $\mathcal{Y}_i(x)$:

$$\mathcal{Y}_i^+(x)\mathcal{Y}_i^-(x) = \mathcal{P}_i(x)\prod_{e\colon\, t(e)=i}\mathcal{Y}_{s(e)}(x+m_e)\prod_{e\colon\, s(e)=i}\mathcal{Y}_{t(e)}(x-m_e) \tag{6.3}$$

for $x \in I_{i,\mathbf{a}}$, $\mathbf{a}=1,\dots,\mathbf{v}_i$, where we used the notation:

$$\mathcal{Y}_i^{\pm}(x) = \mathcal{Y}_i(x\pm\mathrm{i}0), \qquad x\in I_{i,\mathbf{a}} \qquad \text{and} \qquad \mathcal{P}_i(x) = \prod_{\mathfrak{f}=1}^{\mathbf{w}_i}(x-m_{i,\mathfrak{f}}).$$

6.2 The mass cocycles

In what follows we shall redefine the amplitude functions and the $\mathcal{P}$-polynomials

$$\mathcal{Y}_i(x)\to\mathcal{Y}_i(x+\mu_i), \qquad \mathcal{P}_i(x)\to\mathcal{P}_i(x+\mu_i), \tag{6.4}$$

so as to simplify the shifts of the arguments by the masses m_e of the bi-fundamental hypermultiplets:

$$m_e \longrightarrow m_e+\mu_{t(e)}-\mu_{s(e)}.$$

The equations (6.3) are the main equations which determine the low-energy effective action as well as the expectation values of all gauge invariant chiral observables. One can view the equations (6.3) as a Riemann–Hilbert problem. They are also similar, but not identical, to the so-called Y-systems and discrete Hirota equations.

For the class I and class II theories the shift (6.4) maps the equations (6.3) to

$$\mathcal{Y}_i^+(x)\mathcal{Y}_i^-(x) = \mathcal{P}_i(x)\prod_{j\in\mathrm{Vert}_\gamma, j\neq i}\mathcal{Y}_j(x)^{I_{ij}}, \tag{6.5}$$

where for the class II theories $\mathcal{P}_i(x)=\mathfrak{q}_i$. For the class II* theory $\widehat{A}^*_r$, with the clockwise, say, orientation of the quiver, we can make all masses m_e to be equal, $m_e=\frac{1}{r+1}\mathfrak{m}$, by using the shift (6.4). More precisely, in writing (6.6) we chose the representative m^* such that if all the edges are oriented so that $t(e)=(s(e)+1)\bmod(r+1)$, then $m^*_e=\frac{1}{r+1}\mathfrak{m}$. Then,

$$\mathcal{Y}_i^+(x)\mathcal{Y}_i^-(x) = \mathfrak{q}_i\mathcal{Y}_{i-1}\left(x-\frac{\mathfrak{m}}{r+1}\right)\mathcal{Y}_{i+1}\left(x+\frac{\mathfrak{m}}{r+1}\right), \tag{6.6}$$

where $\mathcal{Y}_{i+r+1}(x)=\mathcal{Y}_i(x)$.

6.3 Analytic continuation

We use equation (5.2) to analytically continue the functions $\mathcal{Y}_i(x)$ through the cuts $I_{i,\mathbf{a}}$: for the class I and II theories, after the redefinition (6.4) which eliminates m_e,

$$r_i\colon\ \mathcal{Y}_i(x) \longrightarrow \mathcal{Y}_i(x)\mathcal{P}_i(x)\prod_{j\in \mathrm{Vert}_\gamma}\mathcal{Y}_j(x)^{-C_{ij}}. \tag{6.7}$$

For the class II* theories after the redefinition (6.4), we have the following analogue of (6.7):

$$r_i\colon\ \mathcal{Y}_i(x) \longrightarrow \mathfrak{q}_i\frac{\mathcal{Y}_{i-1}\left(x-\frac{\mathtt{m}}{r+1}\right)\mathcal{Y}_{i+1}\left(x+\frac{\mathtt{m}}{r+1}\right)}{\mathcal{Y}_i(x)}. \tag{6.8}$$

7 The iWeyl group

The transformations (6.7) and (6.8) generate a group, which we shall call the *instanton Weyl group*, or *iWeyl group*, ${}^i\mathcal{W}$, for short. This group can be defined for a much larger class of $\mathcal{N}=2$ theories, not necessarily of the superconformal quiver type we study in this work.

It is clear that the transformations r_i are reflections $r_i\circ r_i = \mathrm{Id}$, so the iWeyl group is the group, generated by reflections.

Now, by comparing equation (6.7) and equations (C.16), (C.26), we see that for the class I theories the iWeyl group is the finite Weyl group $W(\mathfrak{g})$. Similarly, for the class II theories the iWeyl group coincides with the affine Weyl group $W(\widehat{\mathfrak{g}})$. For the class II* theory the iWeyl group is the Weyl group $W\left(\widehat{\mathfrak{gl}}_\infty\right)$ of the group $\widehat{\mathrm{GL}}_\infty$.

The groups $\mathbf{G}$, $\widehat{\mathbf{G}}$, $\widehat{\mathrm{GL}}_\infty$ and their Weyl groups are discussed in Appendix C.

8 Moduli of vacua and mass parameters

After all the redefinitions (6.4) the original mass parameters m_e, the moduli $\left(\mathfrak{a}^{\mathcal{I}}\right)_{\mathcal{I}=1}^{\mathbf{r}}$ of the vacua $u\in\mathfrak{M}$, and the derivatives of the prepotential $\left(\partial\mathcal{F}/\partial\mathfrak{a}^{\mathcal{I}}\right)$ are recovered from the study of periods of certain differentials on the curve $\mathcal{C}_u$ defined as follows.

8.1 The first glimpses of the cameral curve

The functions $\mathcal{Y}_i(x)$, $i\in\mathrm{Vert}_\gamma$, after the maximal analytic continuation through the cuts $I_{i,\mathbf{a}}$ form a local ${}^i\mathcal{W}$-system. It is easy to see that, as long as $|\mathfrak{q}_j|\ll 1$ for all $j\in\mathrm{Vert}_\gamma$, there is exactly one branch of $\mathcal{Y}_i(x)$ as $x\to\infty$ which behaves as

$$\mathcal{Y}_i^{\mathrm{phys}}(x)\sim x^{\mathbf{v}_i}+\cdots.$$

The other branches behave as

$$\mathcal{Y}_i^{\mathrm{unphys}}(x)\sim\left(\prod_{j\in\mathrm{Vert}_\gamma}\mathfrak{q}_j^{n_{ji}}\right)x^{\mathbf{v}_i}+\cdots$$

with $n_{ji}\geq 0$, and

$$\sum_j n_{ji}>0.$$

Now, the branches meet at the cuts

$$\bigcup_{i\in\mathrm{Vert}_\gamma;\,\mathbf{a}=1,\ldots,\mathbf{v}_i} I_{i,\mathbf{a}}$$

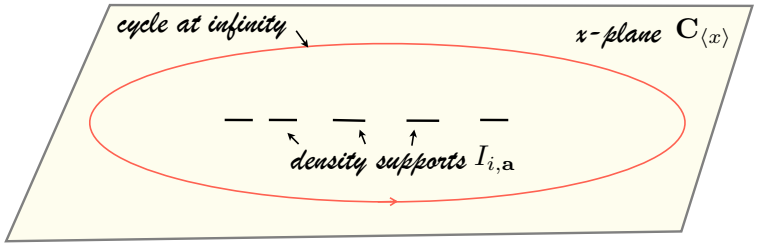


Figure 8.1. The cycle at $x = \infty$ surrounding the branch cuts of $\mathcal{Y}_i^{\rm phys}(x)$.

for the class I and II theories, and at the cuts

$$\bigcup_{i\in \mathrm{Vert}_\gamma;\, \mathbf{a}=1,\dots,N} I_{i,\mathbf{a}} + \frac{\mathfrak{m}}{r+1}\mathbb{Z}$$

for the class II* theories.

The collection of these branches defines a curve $\mathcal{C}_u$ which we shall describe explicitly in the next section. The curve $\mathcal{C}_u$ is a ${}^i\mathcal{W}$-cover of the x-plane $\mathbf{C}_{\langle x\rangle}$, with the branch points at the ends of the cuts $I_{i,\mathbf{a}}$. Because of the ${}^i\mathcal{W}$-action on $\mathcal{C}_u$ and the relation to the Weyl groups which permute Weyl cameras, the curve $\mathcal{C}_u$ will be called the *cameral curve*, following [30].

8.2 The special coordinates and the mass parameters

Take the physical branch and expand it at $x = \infty$. Then the next-to-leading term gives μ_i, the mass shift which determines (or partially determines, in the II* case) the bi-fundamental masses:

$$\mathcal{Y}_i^{\rm phys}(x) \sim x^{\mathbf{v}_i} + \mathbf{v}_i\mu_i x^{\mathbf{v}_i-1} + \cdots . \tag{8.1}$$

The equation (5.5) is modified by the μ_i-shift:

$$\mathfrak{a}_{i,\mathbf{a}} + \mu_i = \frac{1}{2\pi \mathrm{i}} \oint_{A_{i\mathbf{a}}} x \,\mathrm{d}\log \mathcal{Y}_i(x), \tag{8.2}$$

where $A_{i\mathbf{a}}$ is a loop on the physical sheet of $\mathcal{C}_u$ which surrounds the cut $I_{i,\mathbf{a}}$. Note that the only singularities of $\mathcal{Y}_i(x)$ on the physical sheet are at $x = \infty$ and at the cuts. Therefore (see Figure 8.1),

$$\sum_{\mathbf{a}=1}^{\mathbf{v}_i} \frac{1}{2\pi \mathrm{i}} \oint_{A_{i\mathbf{a}}} x \,\mathrm{d}\log \mathcal{Y}_i(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\infty} x \,\mathrm{d}\log \mathcal{Y}_i(x) = \mathbf{v}_i\mu_i,$$

which is consistent with equations (8.1), (8.2) thanks to (2.1), i.e.,

$$\sum_{\mathbf{a}=1}^{\mathbf{v}_i} \mathfrak{a}_{i,\mathbf{a}} = 0. \tag{8.3}$$

Remark 8.1. The residues

$$U_{ji} = \frac{1}{2\pi \mathrm{i}} \oint_{\infty} x^j \,\mathrm{d}\log \mathcal{Y}_i(x)$$

determine the "mirror map", the change of variables $(T_i(x))_{i\in \mathrm{Vert}_\gamma} \longrightarrow (\mathcal{T}_i(x))_{i\in \mathrm{Vert}_\gamma}$ we talked about earlier.

8.3 The dual special coordinates

Now let us discuss the dual coordinates $\mathfrak{a}^D_{i,\mathbf{a}}$. First of all, the equation (6.2) does not quite make sense in view of (8.3). Suppose we relax (8.3) by absorbing μ_i into the definition of $\mathfrak{a}_{i,\mathbf{a}}$. We can then analyze the limit shape problem in the usual fashion. We should keep in mind, however, that only the $\mathrm{SU}(\mathbf{v}_i)$ part of the gauge group is dynamical. The trace part of the dual special coordinates $\mathfrak{a}^D_{i,\mathbf{a}}$,

$$\mu^D_i = \sum_{\mathbf{a}=1}^{\mathbf{v}_i} \mathfrak{a}^D_{i,\mathbf{a}}$$

is ambiguous. The traceless part, i.e.,

$$\sum_{\mathbf{a}=1}^{\mathbf{v}_i} \lambda_{\mathbf{a}} \mathfrak{a}^D_{i,\mathbf{a}}$$

for any weight vector $(\lambda_{\mathbf{a}})$,

$$\sum_{\mathbf{a}} \lambda_{\mathbf{a}} = 0,$$

should be well-defined. Let us now see how this works in detail.

By differentiating (6.1) with respect to x, we find

$$-2\int_{\mathbb{R}} \rho_i(\tilde{x}) \mathcal{K}'(x-\tilde{x})\,\mathrm{d}\tilde{x} + \sum_{e:\, t(e)=i} \int_{\mathbb{R}} \rho_{s(e)}(\tilde{x}) \mathcal{K}'(x-\tilde{x}+m_e)\,\mathrm{d}\tilde{x} \tag{8.4}$$

$$+ \sum_{e:\, s(e)=i} \int_{\mathbb{R}} \rho_{t(e)}(\tilde{x}) \mathcal{K}'(x-\tilde{x}-m_e)\,\mathrm{d}\tilde{x} + \sum_{\mathfrak{f}=1}^{\mathbf{w}_i} \mathcal{K}'(x-m_{i,\mathfrak{f}}) + x\log\mathfrak{q}_i = \mathfrak{a}^D_{i,\mathbf{a}}, \qquad x \in I_{i,\mathbf{a}},$$

where

$$\mathcal{K}'(x) = x\log\left(\frac{x}{\tilde{\Lambda}_{\mathrm{UV}}}\right) \equiv \int_{\tilde{\Lambda}_{\mathrm{UV}}}^{x} \mathcal{K}''(\tilde{x})\,\mathrm{d}\tilde{x}$$

with

$$\mathcal{K}''(x) = \log\left(\frac{x}{\Lambda_{\mathrm{UV}}}\right)$$

and $\tilde{\Lambda}_{\mathrm{UV}} = \exp(1)\Lambda_{\mathrm{UV}}$. Using the definition (5.3) of $\mathcal{Y}_i(x)$ functions, we find

$$\mathfrak{a}^D_{i,\mathbf{a}} = \tilde{\Lambda}_{\mathrm{UV}}\log\mathfrak{q}_i - \int_{\tilde{\Lambda}_{\mathrm{UV}}}^{x} \mathrm{d}\tilde{x}\log\mathcal{Y}^{\mathrm{phys}}_i(\tilde{x}) + \int_{\tilde{\Lambda}_{\mathrm{UV}}}^{x} \mathrm{d}\tilde{x}\Bigg(-\log\mathcal{Y}^{\mathrm{phys}}_i(\tilde{x})$$

$$+ \sum_{e:\, t(e)=i} \log\mathcal{Y}^{\mathrm{phys}}_{s(e)}(\tilde{x}+m_e) + \sum_{e:\, s(e)=i} \log\mathcal{Y}^{\mathrm{phys}}_{t(e)}(\tilde{x}-m_e) + \log\mathcal{P}_i(\tilde{x})\Bigg). \tag{8.5}$$

The integration contour in the above formula runs over a physical sheet from a marked point $p_* \in \mathcal{C}_u$ which sits over the point $\tilde{\Lambda}_{\mathrm{UV}} \in \mathbf{C}_{\langle x\rangle}$ to a point $x \in I_{i,\mathbf{a}}$ which we view as sitting on $\mathcal{C}_u$. The choice of x is irrelevant[2] as long as $x \in I_{i,\mathbf{a}}$ precisely due to the critical point equations (6.3).

[2]Physically the meaning of the integrand in the effective electrostatic problem is the force acting on elementary charge, and the integral is the chemical potential for the charge, or the energy required to move an elementary charge from the density support to infinity. The force vanishes on the support of the charge in the stationary charge distribution.

The above expression for $\mathfrak{a}^D_{i,\mathbf{a}}$ can be converted into much nicer form by noting that the second integral in (8.5) is in fact

$$\int_{\tilde\Lambda_{\rm UV}}^{x} \log\left({}^{r_i}\mathcal{Y}_i^{\rm phys}(\tilde x)\right)\mathrm{d}\tilde x, \tag{8.6}$$

where r_i is the i-th reflection (6.7). In other words, (8.6) is the integral of the analytic continuation of the function $\mathcal{Y}_i(\tilde x)$ onto the mirror sheet of $\mathcal{C}_u$ obtained from the physical sheet by the ${}^i\mathcal{W}$-reflection r_i, i.e., by continuing across any of the cuts $I_{i,\mathbf{a}}$, $\mathbf{a}=1,\dots,\mathbf{v}_i$, supporting the density $\rho_i(x)$. Thus the integral of the expression in the brackets on the last two lines in (8.5) is equal to the integral

$$\int_{r_i(p_*)}^{x} \log\mathcal{Y}_i(\tilde x)\,\mathrm{d}\tilde x.$$

Thus we conclude

$$\mathfrak{a}^D_{i,\mathbf{a}} = \tilde\Lambda_{\rm UV}\log\mathfrak{q}_i - \int_{B_{i\mathbf{a}}} \log\mathcal{Y}_i(\tilde x)\,\mathrm{d}\tilde x, \tag{8.7}$$

where the contour $B_{i\mathbf{a}}$ starts at the point p_* which sits over $x=\tilde\Lambda_{\rm UV}$ on the physical sheet, runs through the cut $I_{i,\mathbf{a}}$ to the mirror sheet $r_i(\text{phys})$ and terminates at the point $r_i(p_*)$, which sits over $x=\tilde\Lambda_{\rm UV}$ on the mirror sheet.

It is tempting to send $\Lambda_{\rm UV}$ to infinity. However, there is a subtlety which we already discussed in the beginning of this section. The integral (8.7) diverges for $\Lambda_{\rm UV}\to\infty$. The linear divergence is canceled by the constant term $\tilde\Lambda_{\rm UV}\log\mathfrak{q}_i$ due to

$$\log\left({}^{r_i}\mathcal{Y}_i^{\rm phys}(\tilde x)\right) - \log\left(\mathcal{Y}_i^{\rm phys}(\tilde x)\right) = \log\mathfrak{q}_i, \qquad \tilde x\to\infty.$$

However the subleading logarithmic divergence does not, in general, cancel. The simplest way to calculate it is to compute the logarithmic derivative $\Lambda_{\rm UV}\mathrm{d}\mathfrak{a}^D_{i,\mathbf{a}}/\mathrm{d}\Lambda_{\rm UV}$ and then send $\Lambda_{\rm UV}\to\infty$:

$$\Lambda_{\rm UV}\frac{\mathrm{d}\mathfrak{a}^D_{i,\mathbf{a}}}{\mathrm{d}\Lambda_{\rm UV}} = -\sum_{\mathfrak{f}=1}^{\mathbf{w}_i} m_{i,\mathfrak{f}} + \sum_{e:\,t(e)=i}\mathbf{v}_{s(e)}m_e - \sum_{e:\,s(e)=i}\mathbf{v}_{t(e)}m_e. \tag{8.8}$$

Luckily the right-hand side of (8.8) does not depend on $\mathbf{a}$.

We can use the formal expression

$$\mathfrak{a}^D_{i,\mathbf{a}} = \int_{B_{i\mathbf{a}}} x\,\mathrm{d}\log\mathcal{Y}_i,$$

where the contour

$$B_{i\mathbf{a}}:\ \infty_{\rm phys}\to{}^{I_{i,\mathbf{a}}}\to r_i(\infty_{\rm phys})$$

starts at the point $x=\infty_{\rm phys}$ on the physical sheet, then runs through the cut $I_{i,\mathbf{a}}$ to the mirror sheet $r_i(\text{phys})$ and finishes at the point $r_i(\infty_{\rm phys})$ on this mirror sheet.

The canonical contour $B_{i\mathbf{a}}$ computing $\mathfrak{a}^D_{i\mathbf{a}}$ is an open contour, and, as we said above, the integral of $x\,\mathrm{d}\log\mathcal{Y}_i$ is divergent. However the variation of Coulomb parameters in $\mathrm{SU}(\mathbf{v}_i)$ concerns only the differences

$$\mathfrak{a}^D_{i,\mathbf{a}'} - \mathfrak{a}^D_{i,\mathbf{a}''} = \int_{B^{\mathbf{a}''}_{i;\mathbf{a}'}} x\,\mathrm{d}\log\mathcal{Y}_i \tag{8.9}$$

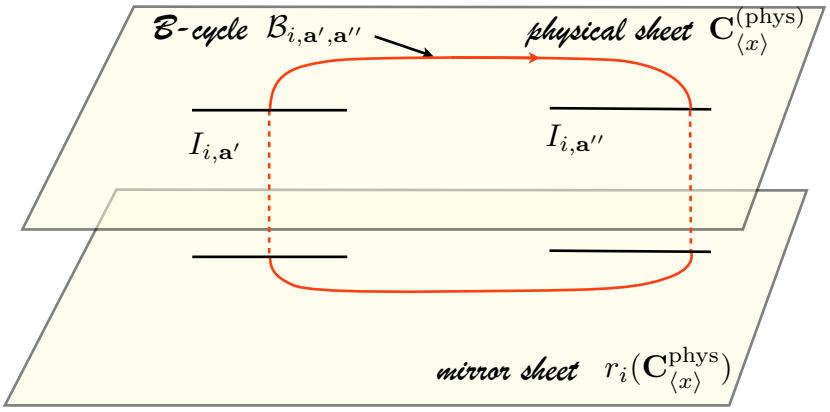


Figure 8.2. The cycles $B^{\mathbf{a}''}_{i;\mathbf{a}'}$.

computed by the closed contour $B^{\mathbf{a}''}_{i;\mathbf{a}'} = B_{i\mathbf{a}'} - B_{i\mathbf{a}''}$ running on physical sheet through the cut $I_{i\mathbf{a}'}$ to the mirror sheet $r_i(\mathrm{phys})$ and then through the cut $I_{i\mathbf{a}''}$ back to the physical sheet. The divergence (8.8) cancels in the integration over $B_{i\mathbf{a}'} - B_{i\mathbf{a}''}$.

In fact, the difference (8.9) is represented as the integral over the closed contour $B^{\mathbf{a}''}_{i;\mathbf{a}'}$ without any divergent quantities, as follows immediately from the (8.4), by connecting two points $x' \in I_{i,\mathbf{a}'}$ and $x'' \in I_{i,\mathbf{a}''}$ on the physical sheet and replacing the integrand as in the second integral of (8.5) over the physical sheet by an integral of $-\log \mathcal{Y}(x')dx'$ over the return segment from $x'' \in I_{i,\mathbf{a}''}$ to $x' \in I_{i,\mathbf{a}'}$ on the mirror sheet $r_i(\mathrm{phys})$, see Figure 8.2

In the weakly coupled regime we have the following BPS particles in the gauge theory: for each gauge group factor i the W-bosons associated with the breaking $\mathrm{SU}(\mathbf{v}_i) \to U(1)^{\mathbf{v}_i-1}$, which correspond to the roots of the $\mathrm{SU}(\mathbf{v}_i)$, and magnetic monopoles, which correspond to the fundamental weights. Accordingly, it seems natural to define the following cycles on the cameral curve: the $\mathcal{A}$- and $\mathcal{B}$-cycles, more precisely $\mathcal{A}_{\mathcal{I}}$, $\mathcal{B}^{\mathcal{I}}$, labelled by $\mathcal{I} = (i, \mathbf{a})$, with $i \in \mathrm{Vert}_\gamma$, and $\mathbf{a} = 1, \dots, \mathbf{v}_i - 1$:

$$\mathcal{A}_{\mathcal{I}} = A_{i\mathbf{a}} - A_{i(\mathbf{a}+1)}, \qquad \mathcal{B}^{\mathcal{I}} = \sum_{\mathbf{a}'=1}^{\mathbf{a}} B_{i\mathbf{a}'} - \frac{\mathbf{a}}{\mathbf{v}_i} \sum_{\mathbf{a}''=1}^{\mathbf{v}_i} B_{i\mathbf{a}''}.$$

The cycles $\mathcal{A}_{\mathcal{I}}$, $\mathcal{B}^{\mathcal{I}}$ determine the special coordinates

$$\mathfrak{a}^{\mathcal{I}} = \oint_{\mathcal{A}_{\mathcal{I}}} x \, \mathrm{d}\log \mathcal{Y}_i, \qquad \mathfrak{a}^D_{\mathcal{I}} = \oint_{\mathcal{B}^{\mathcal{I}}} x \, \mathrm{d}\log \mathcal{Y}_i. \tag{8.10}$$

In the weak coupling regime the pairing (8.10) between the cycles and the differentials is non-zero only for $\mathcal{I} = (i, \mathbf{a})$, for some $\mathbf{a} = 1, \dots, \mathbf{v}_i - 1$. The main property of these cycles is the vanishing of the following two-form on the space $\mathfrak{M}$ of u-parameters:

$$0 = \sum_{\mathcal{I}} \mathrm{d}\mathfrak{a}^{\mathcal{I}} \wedge \mathrm{d}\mathfrak{a}^D_{\mathcal{I}},$$

which follows simply from the equation (6.2).

9 Solution of the limit shape equations

In this section, we solve the equations (6.3), (6.5), (6.6), and get an explicit formula for the curve $\mathcal{C}_u$.

9.1 From invariants to the curve

Our strategy is to define a set of basic invariants $\mathcal{X}_i(\mathcal{Y}(x))$ of the ${}^i\mathcal{W}$ group. We shall find the basic invariants which are power series in $\mathfrak{q}_j$'s, and which are normalized in such a way that, for the class I and II theories:

$$\begin{aligned}&\mathcal{X}_i(\mathcal{Y}(x)) = \mathcal{Y}_i(x) + \sum_{\nu,\,|\nu|>0}\prod_{j\in \mathrm{Vert}_\gamma} \mathcal{P}_j^{\nu_j}(x)\Psi_{i\nu}(\mathcal{Y}_1(x),\dots,\mathcal{Y}_r(x)),\\ &\nu=(\nu_1,\dots,\nu_r), \qquad |\nu| = \sum_{j\in \mathrm{Vert}_\gamma}\nu_j,\end{aligned} \tag{9.1}$$

where $\Psi_\nu(\mathcal{Y}(x)) \in \mathbb{C}\big[\mathcal{Y}_j(x), \mathcal{Y}_j(x)^{-1}\big]$ are quasi-homogeneous Laurent polynomials

$$\sum_{j\in \mathrm{Vert}_\gamma}\left(\mathbf{v}_j\mathcal{Y}_j\frac{\partial}{\partial \mathcal{Y}_j} + \mathbf{w}_j\nu_j\right)\Psi_{i\nu} = \mathbf{v}_i\Psi_{i\nu}.$$

For the class II* theories there is one modification

$$\begin{aligned}&\mathcal{X}_i(\mathcal{Y}(x)) = \mathcal{Y}_i(x) + \sum_{\nu,\,|\nu|>0}\tilde{\mathfrak{q}}^\nu\tilde{\Psi}_{i\nu}(\mathcal{Y}(x)),\\ &\nu=(\nu_0,\dots,\nu_r), \qquad |\nu| = \sum_{j=0}^{r}\nu_j,\end{aligned} \tag{9.2}$$

where

$$\tilde{\mathfrak{q}}^\nu = \prod_{j=0}^{r}\mathfrak{q}_j^{\nu_j}$$

and $\tilde{\Psi}_{i\nu}(\mathcal{Y}(x)) \in \mathbb{C}\Big[\mathcal{Y}_j(x+\frac{n}{r+1}\mathfrak{m}), \mathcal{Y}_j^{-1}(x+\frac{n}{r+1}\mathfrak{m})\Big]$ with $-|\nu|\le n\le|\nu|$. The functions $\tilde{\Psi}_{i\nu}$ are quasi-homogeneous:

$$\sum_{j\in \mathrm{Vert}_\gamma} a_j\mathcal{Y}_j\frac{\partial}{\partial \mathcal{Y}_j}\tilde{\Psi}_{i\nu} = a_i\tilde{\Psi}_{i\nu}.$$

9.2 Master equations

Now, the ${}^i\mathcal{W}$-invariance of $\mathcal{X}_j(\mathcal{Y}(x))$ implies that they are continuous across all the cuts, that is they are single-valued analytic functions of x. Given their large x asymptotics, they are polynomials in x:

$$\mathcal{X}_j(\mathcal{Y}(x)) = T_j(x), \tag{9.3}$$

where

$$T_j(x) = T_{j,0}(\tilde{\mathfrak{q}})x^{\mathbf{v}_j} + T_{j,1}(\tilde{\mathfrak{q}};m)x^{\mathbf{v}_j-1} + \cdots + T_{j,\mathbf{v}_j}. \tag{9.4}$$

The coefficients $T_{j,0}(\tilde{\mathfrak{q}})$ are determined by the gauge couplings $\mathfrak{q}_k$

$$T_{j,0}(\tilde{\mathfrak{q}}) = 1 + \sum_{\nu,\,|\nu|>0}\tilde{\mathfrak{q}}^\nu\Psi_{j\nu}(1,\dots,1).$$

The coefficients $T_{j,1}(\tilde{\mathfrak{q}};m)$ are determined by the masses m_e from (8.1), (9.1), (9.2).

The rest of the coefficients

$$(T_{j,\mathbf{a}}(\tilde{\mathfrak{q}};m,u))_{\mathbf{a}=2,\dots,\mathbf{v}_j}^{j\in\mathrm{Vert}_\gamma}$$

is determined by $u=(u_{i,\mathbf{a}})_{i\in\mathrm{Vert}_\gamma}^{\mathbf{a}=2,\dots,\mathbf{v}_i}$. The equations (9.3) and (9.4) define an analytic curve

$$\mathcal{C}_u^\circ\subset \mathbf{C}_{\langle x\rangle}\times(\mathbb{C}^\times)^{\mathrm{Vert}_\gamma},$$

which can be compactified to the *cameral curve* $\mathcal{C}_u$ which is the ${}^i\mathcal{W}$-cover of $\mathbb{CP}^1_{\langle x\rangle}=\mathbf{C}_{\langle x\rangle}\cup\{\infty\}$:

$$\mathcal{C}_u\longrightarrow \mathbb{CP}^1_{\langle x\rangle}$$

unramified (for generic $\tilde{\mathfrak{q}}$) over $x=\infty$. The details of the compactification of $\mathbf{C}_{\langle x\rangle}\times(\mathbb{C}^\times)^{\mathrm{Vert}_\gamma}$ and $\mathcal{C}_u^\circ\subset\mathcal{C}_u$ will be discussed elsewhere. In what follows we drop the superscript $\circ$ in the definition of $\mathcal{C}_u$.

9.3 The periods

The cameral curve $\mathcal{C}_u$ depends on u, forming a family $\mathcal{C}$ of curves parametrized by the "u-plane" $\mathfrak{M}$. The family $\mathcal{C}$ depends on the microscopic couplings $\mathfrak{q}_i$ and on the mass parameters. Let us keep the masses fixed. When $|\mathfrak{q}_i|\ll 1$ for all $i\in\mathrm{Vert}_\gamma$ we have a well-separated system of cycles $\mathcal{A}_\mathcal{I}$ and $\mathcal{B}^\mathcal{I}$, which we defined in Section 8. We transport this system of cycles throughout the moduli space of gauge couplings via Gauß–Manin connection.

9.4 Vector-valued Seiberg–Witten differential

Let us introduce the following vector-valued 1-differential which is schematically given by

$$\mathrm{d}\mathbb{S}=x\sum_{i\in\mathrm{Vert}_\gamma}\big(\mathrm{d}\log\mathcal{Y}_i\alpha_i^\vee-\mathrm{d}\log\mathcal{P}_i(x)\lambda_i^\vee\big),$$

where $\alpha_i^\vee$ and $\lambda_i^\vee$ are the simple coroots and the fundamental coweights of $\mathfrak{g}_\mathrm{q}$. We shall have more specific notations for each class.

The differential $\mathrm{d}\mathbb{S}$ takes values in the vector space $\mathbb{C}^{\mathrm{Vert}_\gamma}$ which is acted upon by the ${}^i\mathcal{W}$-group. The group ${}^i\mathcal{W}$ also acts on the curve $\mathcal{C}_u$. It is clear from our construction that $\mathrm{d}\mathbb{S}$ is ${}^i\mathcal{W}$-equivariant:

$$w^*\mathrm{d}\mathbb{S}=w\cdot\mathrm{d}\mathbb{S}$$

for any $w\in{}^i\mathcal{W}$.

9.5 Degeneration and filtration

In this section we consider the theories of class I and class II, and the extended Coulomb moduli space $\mathfrak{M}^{\mathrm{ext}}$ (which includes the masses of bifundamental hypermultiplets, recall equation (4.4)). Consider conformal quiver with assigned dimensions $(\gamma,\mathbf{v},\mathbf{w})$ at vertices and recall that they satisfy $\mathbf{w}=C\mathbf{v}$, $\mathbf{w}_i\geq 0$, $\mathbf{v}_i\geq 0$ where C is the Cartan matrix of γ. We say that the theory $(\gamma,\mathbf{v},\mathbf{w})$ strictly *contains* the theory $(\gamma,\mathbf{v}',\mathbf{w}')$ if $0\leq\mathbf{v}'\leq\mathbf{v}$, $0\leq\mathbf{w}'\leq\mathbf{w}$ and $C\mathbf{v}'=\mathbf{w}'$, and $(\gamma,\mathbf{v}',\mathbf{w}')\neq(\gamma,\mathbf{v},\mathbf{w})$. The extended Coulomb moduli space $\mathfrak{M}^{\mathrm{ext}}$ of theory $(\gamma,\mathbf{v},\mathbf{w})$ contains a locus related to the Coulomb moduli $\mathfrak{M}^{\mathrm{ext}\prime}$ space of $(\gamma,\mathbf{v}',\mathbf{w}')$ as follows. Suppose that given a point $\mathfrak{m}_f\in\mathbf{C}_{\langle x\rangle}$ the polynomials $T_i(x)$, $\mathcal{P}_i(x)$, $i\in\mathrm{Vert}_\gamma$ factorize as follows:

$$T_i(x)=T_i'(x)\big(x-\mathfrak{m}_f\big)^{\mathbf{v}_i-\mathbf{v}_i'},\qquad \mathcal{P}_i(x)=\mathcal{P}_i'(x)\big(x-\mathfrak{m}_f\big)^{\mathbf{w}_i-\mathbf{w}_i'}.\tag{9.5}$$

Then it is clear that the character equations (9.3) factorize as well, and the $\mathcal{Y}_i(x)$ functions solving the theory $(\gamma, \mathbf{v}, \mathbf{w})$ are expressed in terms of $\mathcal{Y}'_i(x)$ functions solving the theory $(\gamma, \mathbf{v}', \mathbf{w}')$ as

$$\mathcal{Y}_i(x) = \big(x - \mathfrak{m}_f\big)^{(\mathbf{v}_i - \mathbf{v}'_i)} \mathcal{Y}'_i(x).$$

Suppose that the degeneration equation (9.5) is minimal, i.e., there is no intermediate and different $(\gamma, \mathbf{v}'', \mathbf{w}'')$ such that $\mathbf{v}' \le \mathbf{v}'' \le \mathbf{v}$ and $\mathbf{w}' \le \mathbf{w}'' \le \mathbf{w}$. Then we see that $\mathfrak{M}^{\mathrm{ext}}$ includes the loci $\mathfrak{M}^{\mathrm{ext}'} \times \mathbf{C}_{\langle x \rangle}$

$$\mathfrak{M}^{\mathrm{ext}'} \times \mathbf{C}_{\langle x \rangle} \subset \mathfrak{M}^{\mathrm{ext}},$$

where $\mathbf{C}_{\langle x \rangle}$ parameterizes the location of $\mathfrak{m}_f$.

In the monopole picture of Section 11.1 such degeneration corresponds to the complete screening of the point-like non-abelian monopoles by several Dirac monopoles.

Recall that dimensions $\mathbf{v}$ for the theories of class II parametrized by a single integer N such that $\mathbf{v}_i = Na_i$ where a_i are Dynkin marks. Therefore, the above inclusion is

$$\mathfrak{M}^{\mathrm{ext}}_{N-1} \times \mathbf{C}_{\langle x \rangle} \subset \mathfrak{M}^{\mathrm{ext}}_{N}.$$

Geometrically, such inclusion for theories of class II is associated with freckled (point) instantons described in more details after equation (9.13).

9.6 The cameral curve as a modular object

In this section we give the modular interpretation of the curve $\mathcal{C}_u$.

9.6.1 The class I theories

Let $\mathbf{G} = \mathbf{G}_{\mathrm{q}}$ be the simple complex Lie group corresponding to the quiver of the class I theory. Let Z be its center, and let $\mathrm{C}\mathbf{G}$ be the conformal extension of $\mathbf{G}$. Let $\check{\lambda}_i^\vee$, $\check{\alpha}_i^\vee$, $i = 1, \dots, r$ be the fundamental coweights and the simple coroots in $\mathrm{C}\mathfrak{h} \subset \mathrm{C}\mathfrak{g}$. Let

$$g(x) = \prod_{i=1}^{r} \mathcal{P}_i(x)^{-\check{\lambda}_i^\vee} \mathcal{Y}_i(x)^{\check{\alpha}_i^\vee} \in \mathrm{C}\mathbf{T} \subset \mathrm{C}\mathbf{G}. \tag{9.6}$$

In the notations of (C.25),

$$g(x) = \mathbf{g}_{\mathcal{P}(x), \mathcal{Y}_1(x), \dots, \mathcal{Y}_r(x)}.$$

We also use

$$g_\infty(x) = \mathbf{g}_{\mathcal{P}(x), 1, \dots, 1}.$$

The importance of $g(x)$ is that it transforms by the reflection in the Weyl group $W(\mathfrak{g})$ when crossing the cuts $I_{i,\mathbf{a}}$, cf. (C.26),

$$g_+(x) \longrightarrow g_-(x) = {}^{r_i} g_+(x),$$

which implies that for the class I theories the iWeyl group ${}^i\mathcal{W}$ is the Weyl group $W(\mathfrak{g})$ of the corresponding simple Lie group $\mathbf{G}$. In order to construct the ${}^i\mathcal{W}$-invariants one could take any $\mathrm{C}\mathbf{G}$-invariant function on $\mathrm{C}\mathbf{G}$. In fact, cf. (C.25),

$$\mathcal{X}_i(\mathcal{Y}(x)) = g_\infty(x)^{-\lambda_i} \chi_i(g(x)).$$

Using the formulae (H.1) and (H.2), we write

$$\mathcal{X}_i(\mathcal{Y}(x)) = \mathcal{Y}_i \sum_{\nu=(\nu_1,\ldots,\nu_r)} c_\nu^i \prod_{j=1}^r \left(\mathcal{P}_j \prod_{k=1}^r \mathcal{Y}_k^{-C_{kj}^{\mathfrak{g}_\mathrm{q}}} \right)^{\nu_j}, \tag{9.7}$$

where

$$c_\nu^i = \chi_{R_i, \lambda_i - \sum_{j=1}^r \nu_j \alpha_j}, \qquad \nu \in \mathbb{Z}_+^r.$$

In fact, the sum in (9.7) is finite, i.e., only for a finite number of vectors ν's the multiplicity c_ν^i is non-zero.

We thus obtain the following geometric picture. The solution of the class I theory is a $\mathbf{C}_{\langle x \rangle}$-parametrized family $[g(x)]$ of conjugacy classes in C$\mathbf{G}$, which vary with x polynomially, in the appropriate sense, and such that the value of the D-homomorphism on $[g(x)]$ is fixed for the theory:

$$D(g(x)) = \left(\prod_{j=1}^r \mathcal{P}_j(x)^{-l_{j\xi}} \right)_{\xi=1}^{z_\mathfrak{g}}.$$

Actually, as we explain in Section C.3.1, the coweights $\check{\lambda}_i^\vee$ are not uniquely specified. The group element $g(x)$ in (9.6) defines a well-defined conjugacy class $[g(x)] \in B^{\mathrm{ad}}(\mathfrak{g}_\mathrm{q})$ in $\mathbf{G}_\mathrm{q}/Z$. Its lift to C$\mathbf{G}$ can be twisted by any C-valued (meromorphic) function of x. We shall use this freedom in our manipulations with spectral curves.

The cameral curve $\mathcal{C}_u$ can be viewed, geometrically, as the lift to $C\mathbf{T}$ of the parametrized rational curve in $C\mathbf{T}/W(\mathfrak{g}) \approx (\mathbb{C}^\times)^{z_\mathfrak{g}} \times \mathbb{C}^r$:

$$x \mapsto D(g(x)) \times (T_1(x), \ldots, T_r(x)).$$

9.6.2 The class II theories

As we mentioned above, the quivers of the class II theories correspond to the simply laced affine Kac–Moody algebras, i.e., $\mathfrak{g}_\mathrm{q} = \widehat{\mathfrak{g}}$. Let $\widehat{\mathbf{G}}$ be the corresponding Kac–Moody group. Let $\widehat{\lambda}_i^\vee$, $\widehat{\alpha}_i^\vee$ be the corresponding affine coweights and coroots, $i = 0, 1, \ldots, r$ (see Appendix C.4). Define

$$g(x) = \prod_{i=0}^r \mathfrak{q}_i^{-\widehat{\lambda}_i^\vee} \mathcal{Y}_i(x)^{\widehat{\alpha}_i^\vee} \in \widehat{\mathbf{T}} \subset \widehat{\mathbf{G}}. \tag{9.8}$$

Again, strictly speaking $g(x)$ takes values in $\widehat{\mathbf{G}}/Z$ and so we should consider the modification of $\widehat{\mathbf{G}}$ corresponding to the conformal extension $C\widehat{\mathbf{G}}$, but since the subtlety with the center $Z \subset \mathbf{G}$ only involves the x-independent factor

$$\widehat{g}_\infty = \prod_{i=0}^r \mathfrak{q}_i^{-\widehat{\lambda}_i^\vee}$$

it will not affect the x-dependence of the invariants. The limit shape equations, as in the class I case, translate to the jump conditions

$$g_+(x) \longrightarrow g_-(x) = {}^{r_i} g_+(x)$$

for $x \in I_{i,\mathbf{a}}$, with r_i being the simple reflections generating the affine Weyl group $W(\widehat{\mathfrak{g}})$, which is the ${}^i\mathcal{W}$ group for the class II theories.

The invariants of $W(\widehat{\mathfrak{g}})$ are constructed using the characters $\widehat{\chi}_i$ of the fundamental representations $\widehat{R}_i$ of $\widehat{\mathbf{G}}$:

$$\mathcal{X}_i(\mathcal{Y}(x)) = \big(\widehat{g}_\infty^{\widehat{\lambda}_i}\big)^{-1}\widehat{\chi}_i(g(x)). \tag{9.9}$$

They can also be obtained by starting with $\mathcal{Y}_i(x)$ and averaging with respect to the $W(\widehat{\mathfrak{g}})$-action. The $W(\widehat{\mathfrak{g}})$-action consists of the translations by the coroot lattice $\mathrm{Q}^\vee$ and the $W(\mathfrak{g})$-transformations. The $\mathrm{Q}^\vee$-averaging produces the lattice theta-functions of various characteristics, of the schematic form (the details are given in Appendix I):

$$\Theta(\xi,\mathfrak{q}) = \sum_{\nu\in\mathrm{Q}^\vee} \mathfrak{q}^{\frac{1}{2}\langle\nu,\nu\rangle}\mathrm{e}^{i\langle\nu,\xi\rangle},$$

where

$$\mathfrak{q} = \prod_{i=0}^{r}\mathfrak{q}_i^{a_i}. \tag{9.10}$$

The affine analogue of the formula (9.7) is an infinite sum, however, it is a power series in $\mathfrak{q}$. Using the fact that the weights $\widehat{\lambda}$ of the fundamental representation $\widehat{R}_i$ differ from the highest weight $\widehat{\lambda}_i$ by a positive linear combination of simple roots, $\widehat{\lambda} = \widehat{\lambda}_i - \widehat{\nu}$,

$$\widehat{\nu} = \sum_{j=0}^{r}\nu_j\widehat{\alpha}_j, \qquad \nu_j\in\mathbb{Z}_+,$$

we can write, with

$$\begin{aligned}
&\tilde{\mathfrak{q}}^{\widehat{\nu}} = \prod_{j=0}^{r}\mathfrak{q}_j^{\nu_j},\\
&\mathcal{X}_i(\mathcal{Y}(x);\tilde{\mathfrak{q}}) = \mathcal{Y}_i\sum_{\widehat{\nu}}\widehat{C}^i_{\widehat{\nu}}\tilde{\mathfrak{q}}^{\widehat{\nu}}\prod_{k,j=0}^{r}\mathcal{Y}_k(x)^{-C^{\widehat{\mathfrak{g}}}_{kj}\nu_j},
\end{aligned} \tag{9.11}$$

where we made the $\tilde{\mathfrak{q}} = (\mathfrak{q}_0,\dots,\mathfrak{q}_r)$ dependence explicit, and

$$\widehat{C}^i_{\widehat{\nu}} = \chi_{\widehat{R}_i,\widehat{\lambda}_i-\widehat{\nu}}.$$

Write $\widehat{\nu} = n\delta+\nu$, where $n\in\mathbb{Z}_+$, and $\nu\in\mathrm{Q}$ belongs to the root lattice of $\mathfrak{g}$. Notice that the factor $\tilde{\mathfrak{q}}^{\widehat{\nu}}$ in (9.11) depends on n only via the $\mathfrak{q}^n$ factor. For fixed n the number of $\nu\in\mathrm{Q}$ such that $\widehat{C}^i_{n\delta+\nu}\neq 0$ is finite.

The characters of $\widehat{\mathbf{G}}$ are well-studied [93]. Physically they are the torus $\mathcal{E} = \mathbb{C}^\times/\mathfrak{q}^{\mathbb{Z}}$ conformal blocks of the WZW theories with the group G, and levels $k=a_i$, $i=0,1,\dots,r$ (see [29] for recent developments). The argument of the characters can be viewed as the background $\mathbf{G}$ $(0,1)$-gauge field $\bar{\mathbf{A}}$, which couples to the holomorphic current $\mathbf{J} = g^{-1}\partial g$:

$$Z_k(\tau;\bar{\mathbf{A}}) = \int Dg\exp k\left(S_{\mathrm{WZW}}(g) + \int_{\mathcal{E}}\langle\mathbf{J},\bar{\mathbf{A}}\rangle\right) = \sum_{\widehat{\lambda}\text{ at level }k} c_{\widehat{\lambda}}\cdot\widehat{\chi}_{\widehat{\lambda}}(\widehat{t};\mathfrak{q}).$$

The background gauge field has only r moduli. In practice, one chooses the gauge $\bar{\mathbf{A}} = \frac{\pi}{\operatorname{Im}\tau}\xi$, where $\xi = \mathrm{const}\in\mathfrak{h}$.

Technically, it is more convenient to build the characters using the free fermion theory, at least for the A_r, D_r cases, and for the groups E_6, E_7, E_8 at level 1. We review this approach in Appendix K.

The master equations (9.3) $\mathcal{X}_i(\mathcal{Y}(x);\tilde{\mathfrak{q}}) = T_i(x)$ describe a curve $\mathcal{C}_u \subset \mathbf{C}_{\langle x\rangle} \times (\mathbb{C}^\times)^{r+1}$ which is a $W(\widehat{\mathfrak{g}})$-cover of the x-parametrized rational curve Σ_u in $\mathbb{C}^{r+1} = \mathrm{Spec}\mathbb{C}[\widehat{\chi}_0,\dots,\widehat{\chi}_r]$, cf. (9.9):

$$\widehat{\chi}_i = \prod_j \mathfrak{q}_j^{-\widehat{\lambda}_i(\widehat{\lambda}_j^\vee)} T_i(x), \qquad i = 0,\dots,r. \tag{9.12}$$

Now, as we recall in Section F, the characters $\widehat{\chi}_i$, $i = 0,\dots,r$ are the sections of the line (orbi)bundle $\mathcal{O}(1)$ over the coarse moduli space $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ of holomorphic principal semi-stable $\mathbf{G}$-bundles over the elliptic curve $\mathcal{E}$. Therefore, (9.3) and (9.12) define for each u a quasimap U of the compactified x-plane $\mathbb{CP}^1_{\langle x\rangle}$ to $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$, which is actually an honest map near $x = \infty$, whose image approaches the fixed $\mathbf{G}$-bundle $\mathcal{P}_{\tilde{\mathfrak{q}}}$. This bundle can be described, e.g., by the transition function g_∞, which is one of the $\mathbf{T}$ lifts of

$$\tilde{g}_\infty = \prod_{i=1}^r \mathfrak{q}_i^{-\lambda_i^\vee} \in \mathbf{T}/Z.$$

By definition, the local holomorphic sections of $\mathcal{P}_{\tilde{\mathfrak{q}}}$ are the $\mathbf{G}$-valued functions $\Psi(z)$, defined in some domain in $\mathbb{C}^\times$ such that

$$\Psi(\mathfrak{q}z) = g_\infty \Psi(z).$$

The complex dimension of the space of quasimaps U with fixed $U(\infty)$ is the number of coefficients in the polynomials $(T_i(x))_{i\in \mathrm{Vert}_\gamma}$ excluding the highest coefficients, that is (cf. equation (4.4)),

$$\dim_{\mathbb{C}} \mathfrak{M}^{\mathrm{ext}} = \sum_{i\in \mathrm{Vert}_\gamma} \mathbf{v}_i = Nh.$$

We say that U is a quasimap, and not just a holomorphic map $\mathbb{CP}^1_{\langle x\rangle} \to \mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ for two reasons. Technically, a collection of $\widehat{\chi}_i$ in (9.12) defines a point in $\mathbb{WP}^{a_0,a_1,\dots,a_r}$ only if the polynomials $T_i(x)$ don't have common weighted factors. If, however, for some $\mathfrak{m}_f \in \mathbf{C}_{\langle x\rangle}$:

$$T_i(x) = \tilde{T}_i(x)(x-\mathfrak{m}_f)^{a_i}, \qquad \text{for all } i = 0,\dots,r, \tag{9.13}$$

then the map $\mathbb{CP}^1_{\langle x\rangle} \to \mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ is not well-defined at $x = \mathfrak{m}_f$. It is trivial to extend the map there by removing the $(x-\mathfrak{m}_f)^{a_i}$ factors. This operation lowers $N \to N-1$. In a way, the point $\mathfrak{m}_f$ carries a unit of the instanton charge. Such a configuration is called a freckled instanton [121]. Thus, the extended moduli space equation (4.4) of vacua $\mathfrak{M}^{\mathrm{ext}}{}_N$ of the gauge theory with $G_{\mathrm{g}} = \times_i \mathrm{SU}(Na_i)$, contains the locus $\mathfrak{M}^{\mathrm{ext}}{}_{N-1} \times \mathbf{C}_{\langle x\rangle}$. Allowing for several freckles at the unordered points $\mathfrak{m}_{f1}, \mathfrak{m}_{f2}, \dots, \mathfrak{m}_{fi}$ we arrive at the hierarchy of embeddings of the moduli spaces of vacua of the gauge theories with different gauge groups G_{g}:

$$\begin{aligned}\mathfrak{M}^{\mathrm{ext}}{}_N = \mathring{\mathfrak{M}}^{\mathrm{ext}}{}_N \cup \mathring{\mathfrak{M}}^{\mathrm{ext}}{}_{N-1} \times \mathbf{C}_{\langle x\rangle} \cup \mathring{\mathfrak{M}}^{\mathrm{ext}}{}_{N-2} \times \mathrm{Sym}^2 \mathbf{C}_{\langle x\rangle} \cup \cdots \\ \cup \mathring{\mathfrak{M}}^{\mathrm{ext}}{}_{N-i} \times \mathrm{Sym}^i \mathbf{C}_{\langle x\rangle} \cup \cdots \cup \mathrm{Sym}^N \mathbf{C}_{\langle x\rangle},\end{aligned}$$

where $\mathring{\mathfrak{M}}^{\mathrm{ext}}{}_N$ stands for the space of degree N rational maps $U\colon \mathbb{CP}^1_{\langle x\rangle} \to \mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$.

This hierarchy of gauge theories is more familiar in the context of class I theories. Presently, the freckled instantons to $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ correspond to the loci in $\mathfrak{M}$ where a Higgs branch of the gauge theory can open. Indeed, if (9.13) holds, then we can solve the master equation (9.3) by writing

$$\mathcal{Y}_j(x) = (x-\mathfrak{m}_f)^{a_j} \tilde{\mathcal{Y}}_j(x)$$

with $\tilde{\mathcal{Y}}_j(x)$ solving the master equation (9.3) of the

$$\times_{i\in \mathrm{Vert}_\gamma} \mathrm{SU}((N-1)a_i)$$

gauge theory. In the IIB string theory picture (see Appendix A.3) the full collection of fractional branes in the amount of a_i for the i-th type recombine, and detach themselves from the fixed locus, moving away at the position $\mathfrak{m}_f$ on the transverse $\mathbb{R}^2 = \mathbf{C}_{\langle x\rangle}$.

Now let us take $u \in \mathring{\mathfrak{M}}^{\mathrm{ext}}{}_N$. The corresponding map U defines a rational curve Σ_u in $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ of degree N.

Remark 9.1. Actually, there is another compactification of $\mathring{\mathfrak{M}}^{\mathrm{ext}}{}_N$, via genus zero Kontsevich stable maps of bi-degree $(1,N)$ to $\mathbb{CP}^1 \times \mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ (see [71], where the space of quasimaps is called the toric map spaces). It would be interesting to study its gauge theoretic meaning.

Remark 9.2. The highest-order coefficients $T_{i,0}(\tilde{\mathfrak{q}})$ of the polynomials $T_i(x)$ depend only on the gauge coupling constants, and determine the limit $U(x)$, $x \to \infty$

$$U(\infty) = [\mathcal{P}_{\tilde{\mathfrak{q}}}] \in \mathrm{Bun}_{\mathbf{G}}(\mathcal{E}).$$

The next-to-leading terms $T_{i,1}(\tilde{\mathfrak{q}}, m)$ depend only on the gauge couplings and the bi-fundamental masses. These define the first jet $\mathcal{T}_{[\mathcal{P}_{\tilde{\mathfrak{q}}}]}\Sigma_u$ of the rational curve Σ_u at $x=\infty$.

Summarizing, *the moduli space $\mathfrak{M}_N$ of vacua of the class II theory with the gauge group*

$$G_g = \times_{i\in \mathrm{Vert}_\gamma} \mathrm{SU}(Na_i)$$

is the moduli space of degree N finely framed at infinity quasimaps

$$U\colon\ \mathbb{CP}^1_{\langle x\rangle} \to \mathrm{Bun}_{\mathbf{G}}(\mathcal{E}) \approx \mathbb{WP}^{a_0,a_1,\dots,a_r},$$

where the fine framing is the condition that U is the honest map near $x=\infty$, and the first jet (the value and the tangent vector) at $x=\infty$ are fixed:

$$\big(U(\infty), U'(\infty)\big) \leftrightarrow (\tilde{\mathfrak{q}}, m).$$

We also have the identification of the extended moduli space $\mathfrak{M}^{\mathrm{ext}}$ with the space of framed quasimaps

9.6.3 The class II* theories

The theories with the affine quiver of the $\widehat{A}_r$ type can be solved uniformly in both class II and class II* cases. This is related to the fact that the current algebra $\widehat{u(r+1)}$, the affine Kac–Moody algebra based on $U(r+1)$ is a subalgebra of $\widehat{\mathfrak{gl}}_\infty$, consisting of the $(r+1)$-periodic infinite matrices.

Let γ be the affine Dynkin graph of the $\widehat{A}_r$ algebra. We have, $\mathrm{Vert}_\gamma = \mathrm{Edge}_\gamma = \{0,1,\dots,r\}$. Choose such an orientation of the graph γ that for any $e \in \mathrm{Edge}_\gamma$: $s(e) = e$, $t(e) = (e+1) \bmod r+1$. Let m_e, $e=0,\dots,r$ be the corresponding bi-fundamental multiplet masses, and

$$\mathfrak{m} = \sum_{e=0}^{r} m_e.$$

We are in the class II* theory iff $\mathfrak{m} \neq 0$.

It is convenient to extend the definition of m_e to the universal cover of γ. Thus, we define

$$\mathfrak{m}_i = m_{i \bmod(r+1)}, \qquad Y_i(x) = \mathcal{Y}_{i \bmod(r+1)}(x - \mathfrak{m}_{(i)}), \qquad i \in \mathbb{Z}. \tag{9.14}$$

The extended amplitudes $Y_i(x)$ obey

$$Y_{i+r+1}(x) = Y_i(x - \mathfrak{m}). \tag{9.15}$$

Define

$$t_j(x) = \check{t}_j \frac{Y_j(x)}{Y_{j-1}(x)}, \tag{9.16}$$

where

$$\check{t}_{j+1} = \mathfrak{q}_{j \bmod (r+1)} \check{t}_j, \qquad \prod_{j=0}^{r} \check{t}_j = 1, \qquad \check{t}_{j+r+1} = \mathfrak{q} \check{t}_j. \tag{9.17}$$

Then

$$t_{j+r+1}(x) = \mathfrak{q} t_j(x - \mathfrak{m}),$$

where for the $\widehat{A}_r$-series,

$$\mathfrak{q} = \prod_{j=0}^{r} \mathfrak{q}_j.$$

Now, consider the following element of $\widehat{\mathrm{GL}}_\infty$:

$$g(x) = \mathcal{Y}_0(x)^K \times \prod_{i \in \mathbb{Z}} t_i(x)^{E_{i,i}} \tag{9.18}$$

with $t_i(x)$ from (9.16), and $E_{i,j}$ denoting the matrix with all entries zero except 1 at the i-th row and j-th column. A closer inspection shows (9.18) is the direct generalization of (9.8) with the $(r+1)$-periodic matrix g_∞, and $(\mathcal{Y}_i(x))_{i \in \mathrm{Vert}_\gamma}$ replaced by the infinite array $(Y_i(x))_{i \in \mathbb{Z}}$. Indeed, the simple coroots of $\widehat{\mathrm{GL}}_\infty$ are the diagonal matrices, shifted in the central direction

$$\alpha_i^\vee = K \delta_{i,0} + E_{i,i} - E_{i+1,i+1}, \qquad i \in \mathbb{Z}$$

so that the analogue of (C.31) holds

$$K = \sum_{i \in \mathbb{Z}} \alpha_i^\vee$$

if we drop the telescopic sum $\sum_{i \in \mathbb{Z}} E_{i,i} - E_{i+1,i+1} \sim 0$.

We do not need to deal with all the coweights of $\widehat{\mathrm{GL}}_\infty$, only with the $(r+1)$-periodic ones, defined via:

$$\prod_{j=0}^{r} \mathfrak{q}_j^{-\tilde{\lambda}_j^\vee} = \prod_{b \in \mathbb{Z}} \prod_{j=1}^{r+1} \left(\mathfrak{q}^b \check{t}_j \right)^{E_{i+b(r+1),i+b(r+1)}}.$$

These coweights are the coweights of the $\widehat{A}_r$ Kac–Moody algebra, embedded into $\mathfrak{gl}_\infty$ as the subalgebra of $(r+1)$-periodic matrices

$$\sum_{i,j \in \mathbb{Z}} a_{i,j} E_{i,j}, \qquad a_{i+r+1,j+r+1} = a_{i,j}.$$

We shall describe the solution of this theory in detail in the next section.

9.7 Spectral curves

The cameral curve captures all the information about the limit shape, the special coordinates, the vevs of the chiral operators, and the prepotential. Its definition is universal.

However, the cameral curve is not very convenient to work with. In many cases one can extract the same information from a "smaller" curve, the so-called *spectral curve*. In fact, there are several notions of the spectral curve in the literature.

Suppose $\lambda \in \operatorname{Hom}\left((\mathbb{C}^\times)^{\mathrm{Vert}_\gamma}, \mathbb{C}^\times\right)$ is a dominant weight, i.e., $\lambda(\alpha_i^\vee) \geq 0$ for all $i \in \mathrm{Vert}_\gamma$. Let R_λ be the irreducible highest weight module of $\mathbf{G}_\mathrm{q}$ with the highest weight λ, and $\pi_\lambda \colon \mathbf{G}_\mathrm{q} \longrightarrow \mathrm{End}(R_\lambda)$ the corresponding homomorphism. Then the spectral curve $C_u^{R_\lambda}$ in $\mathbf{C}_{\langle x\rangle} \times \mathbf{C}_{\langle t\rangle}$ is

$$\det_{R_\lambda}\left(1 - t^{-1}\zeta(x)^{-1}\pi_\lambda(g(x))\right) = 0, \tag{9.19}$$

where

1. for the class I theories we introduce the factor

   $$\zeta(x) = g_\infty(x)^\lambda \times \text{a rational function of } x,$$

   having to do with the lift of the conjugacy class $[g(x)]$ from $\mathbf{G}^{\mathrm{ad}}$ to $\mathrm{C}\mathbf{G}$. The rational function is chosen so as to minimize the degree of the curve $C_u^{R_\lambda}$, as we explain in the examples below.
2. for the class II, II* theories the factor $\zeta(x)$ is a constant.

Generally, the curve $C_u^{R_\lambda}$ defined by (9.19) is not irreducible. The equation (9.19) factorizes into a product of components, one component for each Weyl orbit in the set of weights Λ_{R_λ} for the module R_λ. Each Weyl orbit intersects dominant chamber at one point and therefore can be parametrized by dominant weights μ. Therefore,

$$C_u^{R_\lambda} = \bigcup_{\mu \in \Lambda_{R_\lambda} \cap \Lambda^+} \mathrm{mult}(\lambda : \mu) \cdot (C_u^\mu),$$

where $\mathrm{mult}(\lambda : \mu)$ denotes multiplicity of weight μ in the module R_λ. If R_λ is minuscule module, then, by definition, the curve $C_u^{R_\lambda}$ is irreducible.

Example 9.3. Consider the A_1 theory and take $\lambda = 3\lambda_1$, i.e., the spin $\frac{3}{2}$ representation. If $T_1(x) = \mathrm{tr}_{R_1} g(x) = t(x) + t(x)^{-1}$ one finds that

$$C^{R_{\lambda_1}}\colon\ 0 = 1 - T_1(x)t + t^2,$$
$$C^{R_{3\lambda_1}}\colon\ 0 = \left(1 - T_1(x)t + t^2\right)\left(1 + 3T_1(x)t - T_1(x)^3 t + t^2\right).$$

Let ${}^i\mathcal{W}_\mu \subset {}^i\mathcal{W}$ be the stabilizer of μ in ${}^i\mathcal{W}$, a subgroup of ${}^i\mathcal{W}$. Consider the map

$$p_\mu\colon\ \mathbf{C}_{\langle x\rangle} \times \left(\mathbb{C}^\times\right)^{\mathrm{Vert}_\gamma} \longrightarrow \mathbf{C}_{\langle x\rangle} \times \mathbf{C}_{\langle t\rangle}$$

given by

$$p_\mu\colon\ \left(x, (\mathcal{Y}_i)_{i\in\mathrm{Vert}_\gamma}\right) \mapsto (x, t(x)),$$
$$t(x) = g(x)^\mu / g_\infty(x)^\mu = \prod_{i\in\mathrm{Vert}_\gamma} \mathcal{Y}_i^{\mu(\alpha_i^\vee)}.$$

Under the map p_μ the curve $\mathcal{C}_u$ maps to $C_u^\mu = \mathcal{C}_u/{}^i\mathcal{W}_\mu \subset \mathbf{C}_{\langle x\rangle} \times \mathbf{C}_{\langle t\rangle}$, the irreducible μ-component of the spectral curve. This curve comes with the canonical differential, which is the restriction of the differential on $\mathbf{C}_{\langle x\rangle} \times \mathbf{C}_{\langle t\rangle}{}^\times$:

$$\mathrm{d}S = x\frac{\mathrm{d}t}{t}.$$

Actually, in the case of the class II, II* theories the commonly used notion of the spectral curve differs from the one in (9.19).

Although we suspect the study of spectral curves associated with the integrable highest weight representations of affine Kac–Moody algebras may be quite interesting, in this paper for the analysis of the class II and II* theories we use the conventional notion of the spectral curve used for the study of families of $\mathbf{G}$-bundles.

To define it, let us fix an irreducible representation R of $\mathbf{G}$, $\pi_R\colon \mathbf{G} \to \mathrm{End}(R)$, and let us study the theory of a complex chiral fermion valued in R, more precisely, an $(1,0)$ bc system in the representations (R^*, R):

$$\mathcal{L}_{bc} = \sum_{i=1}^{\dim R} \int b_i \bar\partial c^i$$

coupled to a background $\mathbf{G}\times\mathbb{C}^\times$ gauge field $\bar{\mathbf{A}}\oplus\bar A$, and compute its partition function on the torus $\mathcal{E}$:

$$Z(\mathbf{t},t,q) = \mathrm{Tr}_{\mathcal{H}_R}\left((-t)_0^J \mathbf{t}^{\mathbf{J}_0} q^{L_0}\right).$$

Mathematically, we consider the space

$$H_R = R\big[z,z^{-1}\big] = H_R^+ \oplus H_R^- \tag{9.20}$$

of R-valued functions on the circle $\mathbb{S}^1$. In (9.20), we took Laurent polynomials in $z\in\mathbb{C}^\times$, which correspond to Fourier polynomials on the circle. We may take some completion of H_R but we shall not do this in the definition of the spectral determinant below. Consider an element $\widehat g \in \widehat{\mathbf{G}}$ of the affine Kac–Moody group, i.e., the central extension of $\widetilde{\mathrm{L}\mathbf{G}} = \mathrm{L}\mathbf{G}\ltimes\mathbb{C}^\times$, the loop group $\mathrm{L}\mathbf{G}$ extended by the $\mathbb{C}^\times$ acting by loop rotations. We have the canonical homomorphism-projection $f\colon \widehat{\mathbf{G}} \longrightarrow \widetilde{\mathrm{L}\mathbf{G}}$ with the fiber $\mathbb{C}^\times$, the center of the central extension

$$f\colon\ \widehat g \mapsto g(z) q^{z\partial_z}. \tag{9.21}$$

The projection is topologically non-trivial.

Now, $\widetilde{\mathrm{L}\mathbf{G}}$ acts in H_R via rotation and evaluation, and so does $\widehat{\mathbf{G}}$ thanks to (9.21): for $\Psi\in H_R$:

$$\left(f(\widehat g)\cdot\Psi\right)(z) = \pi_R(g(z))\cdot\Psi(qz).$$

We would like to define the spectral determinant of $f(\widehat g)$ in the representation H_R. The eigenvalues of $f(\widehat g)$ are easy to compute

$$\mathrm{Eigen}(f(\widehat g)) = \left\{\mathbf{t}^\mu q^n \mid \mu\in\Lambda_R,\ n\in\mathbb{Z}\right\}, \tag{9.22}$$

where we transformed $g(z)$ to a constant $\mathbf{t}\in\mathbf{T}$ by means of a z-dependent $\mathbf{G}$-gauge transformation:

$$g(z)\mapsto h^{-1}(z) g(z) h(qz) = \mathbf{t}.$$

The fibration $f\colon \widehat{\mathbf{G}} \to \widetilde{\mathrm{L}\mathbf{G}}$, restricted onto $\mathbb{C}_q^\times\times\mathbf{T}\subset\widetilde{\mathrm{L}\mathbf{G}}$ becomes trivial, $f^{-1}\left(\mathbb{C}_q^\times\times\mathbf{T}\right)\approx \mathbb{C}_c^\times\times\mathbb{C}_q^\times\times\mathbf{T}$. Let us denote by c the coordinate on the first factor.

The eigenvalues (9.22) concentrate both near 0 and ∞, so we define

$$\begin{aligned}\det_{H_R}\left(1-t^{-1}\widehat g\right) &:= \det_{H_R^+}\left(1-t^{-1}\widehat g\right)\det_{H_R^-}\left(1-t\widehat g^{-1}\right)\\ &= c^{\kappa_R}\prod_{\mu\in\Lambda_R}\prod_{n=0}^\infty \left(1-q^n t^{-1}\mathbf{t}^\mu\right)\left(1-q^{n+1}t\mathbf{t}^{-\mu}\right).\end{aligned} \tag{9.23}$$

The expression (9.23) is $W(\widehat{\mathfrak{g}})$-invariant. The shifts by Q act as follows, cf. (E.5):

$$(\mathbf{t}, c) \mapsto \big(q^{\beta} \cdot \mathbf{t}, \mathbf{t}^{\beta} q^{\frac{1}{2}\langle\beta,\beta\rangle} \cdot c\big),$$

where we view $\beta \in \mathrm{Q}$ both as a vector in the root lattice and as a vector in the coroot lattice, and $\langle\,,\,\rangle$ is the Killing metric. The level κ_R in (9.23) is defined as follows:

$$\sum_{\mu\in\Lambda_R} \mu\,\langle\mu,\beta\rangle = \kappa_R \beta \tag{9.24}$$

for any vector $\beta \in \mathrm{Q}$. Geometrically the spectral curve corresponding to R is obtained as follows: consider the universal principal $\mathbf{G}$-bundle $\mathcal{U}$ over $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E}) \times \mathcal{E}$, and associate the vector bundle $\mathcal{R}$ with the fiber R:

$$\mathcal{R} = \mathcal{U} \times_{\mathbf{G}} R.$$

Now restrict it onto the rational curve $\Sigma_u \subset \mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$. We get the R-bundle over $\Sigma_u \times \mathcal{E}$.

For generic point $x \in \mathbb{CP}^1_{\langle x\rangle}$ over the corresponding point $U(x) \in \Sigma_u$ we get the vector bundle $\mathcal{R}_x$ over $\mathcal{E}$, which is semi-stable, and splits as a direct sum of line bundles

$$\mathcal{R}_x = \bigoplus_{\mu\in\Lambda_R} \mathcal{L}_{\mu,x},$$

where the summands are the degree zero line bundles on $\mathcal{E}$. Under the identification $\mathrm{Pic}_0(\mathcal{E})$ with $\mathcal{E}$ the line bundle $\mathcal{L}_{\mu,x}$ corresponds to the point $\mathbf{t}(x)^{\mu} \bmod \mathfrak{q}^{\mathbb{Z}}$ for some $\mathbf{t}(x) \in \mathbf{T}/\mathfrak{q}^{\mathrm{Q}^\vee}$. The closure of the union

$$\bigcup_{x\in\mathbb{CP}^1_{\langle x\rangle}} \big\{\mathbf{t}(x)^{\mu} \mid \mu \in \Lambda_R\big\} \subset \mathbb{CP}^1_{\langle x\rangle} \times \mathcal{E}$$

is the spectral curve $C_u^R \subset \mathbb{CP}^1_{\langle x\rangle} \times \mathcal{E}$. It is given by the vanishing locus of the regularized determinant (9.23):

$$c(x)^{\kappa_R} \prod_{\mu\in\Lambda_R} \theta\big(t^{-1}\mathbf{t}(x)^{\mu}; q\big) = 0$$

the choice of the x-dependence of $c(x)$ seems immaterial at this point, as long as $c(x) \in \mathbb{C}^{\times}$.

Degree of the spectral curve

The x-degree of the spectral curve for class II theories in representation R is $N\kappa_R$ where κ_R is given by (9.24). The κ_R is the proportionality constant for the second Casimir in representation R $\mathrm{tr}_R(\cdot,\cdot) = \kappa_R(\cdot,\cdot)_2$ where the $(\cdot,\cdot)_2$ is the canonical Killing form in which the long roots have length square equal to 2. The standard computations leads to

$$\kappa_R = \frac{\dim_R}{\dim_{\mathfrak{g}}}(\lambda_R, \lambda_R + 2\rho)_2,$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the Weyl vector. For fundamental representations R_1 we find for all cases

$$\kappa_{R_1}(A_r) = 1, \qquad \kappa_{R_1}(D_r) = 2, \qquad \kappa_{R_1}(E_6) = 6, \qquad \kappa_{R_1}(E_7) = 12, \qquad \kappa_{R_1}(E_8) = 60.$$

9.8 Obscured curve

In the previous construction, in view of the identification $\mathcal{L}_{\mu,x} \leftrightarrow \mathbf{t}(x)^\mu$ we can decompose, for each weight

$$\mu = \sum_{i=1}^{r} \mu_i \lambda_i \in \Lambda_R, \qquad \mathcal{L}_{\mu,x} = \bigotimes_{i=1}^{r} \mathbb{L}_{i,x}^{\otimes \mu_i}$$

for some "basic" line bundles $\mathbb{L}_{i,x}$ corresponding to the fundamental weights. These basic line bundles are ordered, so they define a point

$$\{\mathbb{L}_{1,x}, \dots, \mathbb{L}_{r,x}\} \in \mathrm{Pic}_0(\mathcal{E})^r \approx \mathcal{E}^r,$$

the Cartesian product of r copies of the elliptic curve. Taking the whole family and including the parametrization we obtain the *obscured curve* $\mathcal{C}_u$:

$$\mathcal{C}_u = \left\{(x; \mathbb{L}_{1,x}, \dots, \mathbb{L}_{r,x}) \mid x \in \mathbb{CP}^1_{\langle x \rangle}\right\} \in \mathbb{CP}^1_{\langle x \rangle} \times \mathcal{E}^r.$$

Let us present another simple construction of $\mathcal{C}_u$. Namely, let us use the fact [31, 55, 56, 59, 60], that

$$\mathrm{Bun}_{\mathbf{G}}(\mathcal{E}) = (\mathcal{E} \otimes \mathrm{Q})/W(\mathfrak{g}), \tag{9.25}$$

where the tensor product is understood in the category of abelian groups. At the level of manifolds, (9.25) simply says that

$$\mathrm{Bun}_{\mathbf{G}}(\mathcal{E}) = \mathcal{E}^r/W(\mathfrak{g})$$

for some natural action of the Weyl group $W(\mathfrak{g})$ on the Cartesian product of r copies of $\mathcal{E}$. Let us denote by π_W the projection

$$\pi_W : \ \mathcal{E}^r \to \mathrm{Bun}_{\mathbf{G}}(\mathcal{E}) = \mathcal{E}^r/W(\mathfrak{g}).$$

The rational curve Σ_u in $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ lifts to a curve in $\mathcal{E}^r$, and the graph of the parametrized curve $\Sigma_u \in \mathbb{CP}^1_{\langle x \rangle} \times \mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ lifts to the graph in $\mathbb{CP}^1_{\langle x \rangle} \times \mathcal{E}^r$ which is our friend *obscured curve* $\mathcal{C}_u$. It is the quotient of the cameral curve by the lattice $\mathrm{Q}^\vee$:

$$\mathcal{C}_u = \mathcal{C}_u/\mathrm{Q}^\vee.$$

In Section 11.2, we shall present yet another construction of $\mathbb{L}_{i,x}$'s, using gauge theory.

There is the so-called determinant line bundle L over the moduli space $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$, whose sections are the fundamental characters $\widehat{\chi}_i$, $i = 0, 1, \dots, r$. In Loojienga's identification $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E}) \approx \mathbb{WP}^{a_0,a_1,\dots,a_r}$ this line bundle is just the $\mathcal{O}(1)$ orbibundle over the weighted projective space.

We have then the line bundle $\mathcal{L}$ over $\mathcal{E}^r$:

$$\mathcal{L} = \pi_W^* L.$$

Let us call this line bundle *the abelianized determinant line bundle*.

10 The Seiberg–Witten curves in some detail

In this section, we shall discuss the geometry of curves describing the limit shape configurations and the special geometry of the gauge theories under consideration. When possible we identify the cameral or the spectral curves with the analogous curves of some algebraic integrable systems,

namely the Hitchin systems on the genus zero (i.e., Gaudin model) or genus one (i.e., spin elliptic Calogero–Moser system) curves with punctures. These identifications are less universal than the identification with the spectral curves of the spin chains based on the Yangian algebra built on $\mathfrak{g}$, $\widehat{\mathfrak{g}}$, or $\widehat{\mathrm{GL}}_\infty$, respectively. The latter identification is a subject of a separate venue of research which touches upon various advances in geometric representation theory, study of the symplectic geometry of moduli spaces of instantons and monopoles, quantum cohomology of quiver varieties, to name just a few. We shall only mention the relation to spin chains in a few examples, in this work.

Throughout this section we shall use the notation

$$g_\lambda(x) = \zeta(x)^{-1}\pi_\lambda(g(x))$$

for the projectively modified operator in the representation (R_λ, π_λ) of $\mathbf{G}_\mathrm{q}$, corresponding to the group element $g(x) \in \mathbf{G}_\mathrm{q}$.

10.1 Class I theories of A type

This is the so-called linear quiver theory. The set of vertices $\mathrm{Vert}_\gamma = \{1, \dots, r\}$, the set of edges $\mathrm{Edge}_\gamma = \{1, \dots, r-1\}$, the maps s, t for a particular orientation are given by $s(e) = e$, $t(e) = e + 1$. The bi-fundamental masses are a trivial cocycle:

$$m_e = \mu_{e+1} - \mu_e.$$

The corresponding conformal group $\mathrm{C}\mathbf{G} = \mathrm{GL}(r+1, \mathbb{C})$, the fundamental characters χ_i are the characters of the representations $\Lambda^i\mathbb{C}^{r+1}$. We shall now describe the spectral curve in the representation $R_{\lambda_1} \approx \mathbb{C}^{r+1}$. The corresponding group element $g_{\lambda_1}(x)$ in (9.6) is the diagonal matrix

$$g_{\lambda_1}(x) = \mathrm{diag}(t_1(x), \dots, t_{r+1}(x))$$

with

$$\begin{aligned} &t_1(x) = \zeta(x)\mathcal{Y}_1(x), \qquad t_{r+1}(x) = \zeta(x)\mathcal{P}^{[r]}(x)\mathcal{Y}_r(x)^{-1}, \\ &t_i(x) = \zeta(x)\mathcal{P}^{[i-1]}(x)\mathcal{Y}_i(x)\mathcal{Y}_{i-1}(x)^{-1}, \qquad i = 2, \dots, r, \end{aligned} \tag{10.1}$$

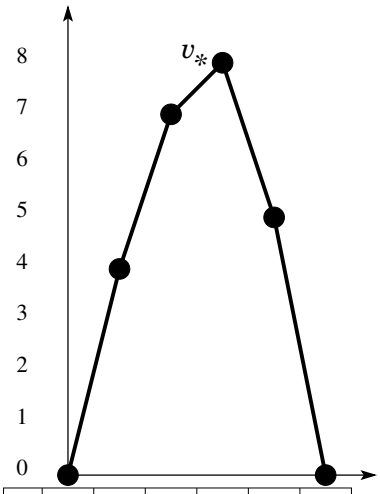
with some normalization factor $\zeta(x)$ which we choose shortly, and the explicit formula for the invariants $\mathcal{X}_i(\mathcal{Y}(x))$ is (we omit the x-dependence in the right-hand side):

$$\mathcal{X}_i(\mathcal{Y}(x)) = \prod_{j=1}^{i-1} \mathcal{P}_j^{j-i} e_i\big(\mathcal{Y}_1, \mathcal{Y}_2\mathcal{Y}_1^{-1}\mathcal{P}^{[1]}, \dots, \mathcal{Y}_i\mathcal{Y}_{i-1}^{-1}\mathcal{P}^{[i-1]}, \dots, \mathcal{Y}_r^{-1}\mathcal{P}^{[r]}\big), \tag{10.2}$$

where e_i are the elementary symmetric polynomials in $r+1$ variables. Our master equations (9.3) equate the right-hand side of (10.2) with the degree $\mathbf{v}_i$ polynomial $T_i(x)$ in x, cf. (9.4).

It is convenient to organize the invariants (10.2) into a generating polynomial, which is nothing but the characteristic polynomial of the group element $g(x)$ in some representation of $\mathrm{C}\mathbf{G}$. The most economical is, of course, the defining fundamental representation $\mathbb{C}^{r+1}$ with the highest weight λ_1:

$$\begin{aligned} \det\left(t \cdot 1_{r+1} - g_{\lambda_1}(x)\right) = t^{r+1} + \sum_{i=1}^{r} (-1)^i t^{r+1-i} \zeta(x)^i \prod_{j=1}^{i-1} \mathcal{P}_j^{i-j}(x) \mathcal{X}_i(\mathcal{Y}(x)) \\ + (-\zeta(x))^{r+1} \prod_{j=1}^{r} \mathcal{P}_j^{r+1-j}(x). \end{aligned} \tag{10.3}$$



i	0	1	2	3	4	5
v	0	4	7	8	5	0
w	0	1	2	4	2	0

Figure 10.1. Degree profile example for A_4 theory and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = (4, 7, 8, 5)$. For convenience one can set boundary conditions $\mathbf{v}_0 = \mathbf{v}_{r+1} = \mathbf{w}_0 = \mathbf{w}_{r+1} = 0$.

The group ${}^i\mathcal{W}$ is the symmetric group $\mathcal{S}_{r+1}$, which acts by permuting the eigenvalues of $g(x)$ in (10.1). The cameral curve $\mathcal{C}_u$ is the $(r+1)!$-fold ramified cover of the compactified x-plane $\mathbb{CP}^1_{\langle x\rangle}$. The points in the fiber are the *ordered* sets of roots $(t_1(x), \dots, t_{r+1}(x))$ of the polynomial (10.3).

The curve $\mathcal{C}_u$ covers the *spectral curve* C_u. The latter is defined as the zero locus of the characteristic polynomial (10.3). The cover $\mathcal{C}_u \to C_u$ is $r! : 1$, it sends the ordered $(r+1)$-tuple of roots $(t_1, \dots, t_{r+1})$ to the first root t_1. The cover $C_u \to \mathbf{C}_{\langle x\rangle}$ is $(r+1) : 1$.

Explicitly, the curve C_u is given by

$$0 = \mathcal{P}(t, x) = \sum_{i=0}^{r+1} (-1)^i t^{r+1-i} \zeta(x)^i \prod_{j=1}^{i-1} \mathcal{P}_j(x)^{i-j} T_i(x). \tag{10.4}$$

10.2 Relation to Gaudin model

It is easy to see, using the equations (3.3), (3.4) and Figure 10.1 that $\mathbf{w}_{i_*} = w_+ + w_-$, where

$$w_+ = \mathbf{v}_{i_*} - \mathbf{v}_{i_*+1} \geq 0, \qquad w_- = \mathbf{v}_{i_*} - \mathbf{v}_{i_*-1} \geq 0$$

and it useful to record

$$\mathbf{v}_1 = \mathbf{w}_1 + \dots + \mathbf{w}_{i_*-1} + \mathbf{w}_-,$$
$$\mathbf{v}_r = \mathbf{w}_r + \dots + \mathbf{w}_{i_*+1} + \mathbf{w}_+,$$
$$\mathbf{v}_* = \sum_{i=1}^{i_*-1} i\mathbf{w}_i + i_*\mathbf{w}_-,$$
$$\mathbf{v}_* = \sum_{i=i_*+1}^{r} (r+1-i)\mathbf{w}_i + (r+1-i_*)\mathbf{w}_+.$$

Accordingly, we can factorize the polynomial $\mathcal{P}_{i_*}(x)$ as

$$\mathcal{P}_{i_*}(x) = \mathfrak{q}_{i_*} \mathcal{P}^+(x) \mathcal{P}^-(x),$$

where $\mathcal{P}^{\pm}$ are monic polynomials of degrees

$$\deg \mathcal{P}^{\pm} = w_{\pm}.$$

We can actually transform (10.4) into something nice, by adjusting $\zeta(x)$:

$$\zeta(x)^{-1} = \mathcal{P}^{-}(x)\mathcal{P}^{[i_*-1]}(x). \tag{10.5}$$

Then $D(g(x))$ is given by

$$\begin{aligned} D(g(x)) &= \frac{P_0(x)}{P_\infty(x)}, \\ P_0(x) &= \mathcal{P}^{+}(x)^{r+1-i_*} \prod_{j=i_*+1}^{r} \mathcal{P}_j(x)^{r+1-j}, \\ P_\infty(x) &= \mathcal{P}^{-}(x)^{i_*} \prod_{j=1}^{i_*-1} \mathcal{P}_j(x)^{j}. \end{aligned}$$

Then $\mathcal{P}(t,x)$ can be written as

$$\mathcal{P}(t,x) = \prod_{j=1}^{i_*-1} \mathfrak{q}_j^j \cdot \frac{P(t,x)}{P_\infty(x)},$$

where $P(t,x)$ is a degree $N = \mathbf{v}_{i_*}$ polynomial in x, and the degree $r+1$ polynomial in t, which is straightforward to calculate

$$\begin{aligned} (-1)^{i_*} \prod_{j=1}^{i_*} \mathfrak{q}_j^j P(t,x) = (-\mathfrak{q}_{i_*})^{i_*} t^{r+1} P_\infty(x) &+ \sum_{i=1}^{i_*} t^{r+1-i} T_i(x) \mathfrak{q}_{i_*}^{i_*-i} (-\mathcal{P}_*^{-}(x))^{i_*-i} \prod_{j=i}^{i_*-1} \mathcal{P}_j^{j-i}(x) \\ &+ \sum_{i=i_*+1}^{r} t^{r+1-i} T_i(x) (-\mathcal{P}_*^{+}(x))^{i-i_*} \prod_{j=i_*+1}^{i-1} \mathcal{P}_j^{i-j}(x) \\ &+ (-1)^{r+1-i_*} P_0(x). \end{aligned} \tag{10.6}$$

Now, recall that $T_{j,0}$ is fixed by the couplings $\mathfrak{q}$:

$$T_{j,0}(\mathfrak{q}) = \prod_{j=1}^{i-1} \mathfrak{q}_j^{j-i} e_i(1, \mathfrak{q}_1, \mathfrak{q}_1\mathfrak{q}_2, \ldots, \mathfrak{q}_1\mathfrak{q}_2 \ldots \mathfrak{q}_i, \ldots, \mathfrak{q}_1 \ldots \mathfrak{q}_r)$$

and the coefficient $T_{j,1}$ is fixed by the masses $m_{i,\mathfrak{f}}$ and m_e.

Therefore, the coefficient of x^N in $P(t,x)$ can be computed explicitly

$$\sum_{i=0}^{r+1} (-1)^i t^{r+1-i} \prod_{j=1}^{i_*} \mathfrak{q}_j^{-i} e_i(1, \mathfrak{q}_1, \mathfrak{q}_1\mathfrak{q}_2, \ldots, \mathfrak{q}_1\mathfrak{q}_2 \ldots \mathfrak{q}_i, \ldots, \mathfrak{q}_1 \ldots \mathfrak{q}_r) = \prod_{i=0}^{r} (t - \check{t}_i),$$

where

$$\check{t}_i = \frac{\prod_{j=1}^{i} \mathfrak{q}_j}{\prod_{j=1}^{i_*} \mathfrak{q}_j}, \qquad i = 0, \ldots, r.$$

We thus rewrite the curve C_u in the (x,t)-space, defined by the equation

$$0 = \mathcal{R}_{A_r}(t,x) = \frac{P(t,x)}{\prod_{i=0}^{r} (t - \check{t}_i)} = \prod_{l=1}^{N} (x - x_l(t)) = x^N + \frac{1}{\prod_{i=0}^{r} (t - \check{t}_i)} \sum_{j=1}^{N} p_j(t) x^{N-j}, \tag{10.7}$$

where

$$N = \mathbf{v}_{i_*}.$$

It is clear from the equation (10.7) that as $t \to \check{t}_i$ one of the roots $x_l(t)$ has a pole, while the other $N-1$ roots are finite. Near $t=0$ the polynomial $P(t,x)$ approaches:

$$P(0,x) = (-1)^{r+1-i_*} P_0(x),$$

while near $t=\infty$

$$P(t,x) t^{-r-1} \to (-\mathfrak{q}_{i_*})^{i_*} P_\infty(x).$$

Let

$$\mathrm{d}S = x \frac{\mathrm{d}t}{t}.$$

Then our discussion above implies that the differential dS has the first-order poles on C_u: at one of the N preimages of the points $\check{t}_i$, $i = 0, 1, \ldots, r$, and at all preimages of the points $t = 0$ and $t = \infty$. The residues of $\mathrm{d}S$ are linear combinations of the masses of the hypermultiplets, in agreement with the observations in [38, 167].

Remarkably, we can identify C_u with the spectral curve of the meromorphic Higgs field Φ:

$$\Phi = \Phi(t)\mathrm{d}t = \sum_{j=-1}^{r+1} \Phi_j \frac{\mathrm{d}t}{t - \check{t}_j}, \tag{10.8}$$

where $\check{t}_{-1} = 0$, $\check{t}_{r+1} = \infty$, and Φ_j are $N \times N$ matrices, which have rank one for $j = 0, 1, \ldots, r$, and have the maximal rank for $j = -1, r+1$. Moreover, the eigenvalues of Φ_j are all fixed in terms of the masses. The spectra of Φ_j, $j = -1, \ldots, r+1$ have specified multiplicity:

1. The matrix Φ_{-1} has w_+ eigenvalues of multiplicity $r+1-i_*$, and $\mathbf{w}_{r+1-j}$ eigenvalues of multiplicity j, for $j = 1, \ldots, r-i_*$; the eigenvalues are fixed by the masses.
2. The matrices Φ_j, $j = 0, 1, \ldots, r$ has one non-vanishing eigenvalue each, and $N-1$ vanishing eigenvalues. We can write

   $$(\Phi_j)^b_a = u^j_a v^b_j, \qquad a, b = 1, \ldots, N$$

   for some vectors $u^j, v_j \in \mathbb{C}^N$, obeying

   $$\sum_{a=1}^{N} u^j_a v^a_j = M_j \tag{10.9}$$

   and considered up to an obvious $\mathbb{C}^\times$-action, for some M_j which is linear in the bi-fundamental and fundamental masses.
3. The matrix Φ_{r+1} has w_- eigenvalues of multiplicity i_*, and $\mathbf{w}_j$ eigenvalues of multiplicity j, for $j = 1, \ldots, i_* - 1$.

Then

$$\left(\frac{\mathrm{d}t}{t}\right)^N \mathcal{R}_{A_r}(t,x) = \det\left(x\frac{\mathrm{d}t}{t} - \Phi\right). \tag{10.10}$$

We can make an $\mathrm{SL}(N)$ Higgs field out of Φ by shifting it by the scalar meromorphic one-form $\frac{1}{N}\operatorname{Tr}_N \Phi$, which is independent of the moduli u of the curve C_u.

The moduli space of $(r+3)$-ples of matrices Φ_j, obeying

$$\sum_{j=-1}^{r+1} \Phi_j = 0 \tag{10.11}$$

with fixed eigenvalues of the above mentioned multiplicity, considered up to the simultaneous $\mathrm{SL}(N)$-similarity transformation, is the phase space $\mathfrak{P}^H_{0,r+3}$ of the genus zero version of $\mathrm{SL}(N)$ Hitchin system, the classical Gaudin model on $r+3$ sites. The general Gaudin model has the residues Φ_j belonging to arbitrary conjugacy classes.

See [109, 110] for the geometry of complex coadjoint orbits. The Hitchin system with singularities was studied in [35, 75, 79, 80, 137, 178]. In [61, 131, 132, 133, 134] this Hitchin system with singularities was discussed from the point of view of brane constructions such as [61, 175].

Remark 10.1. The curve C_u is much more economical then $\mathcal{C}_u$. However, the price we pay is the complexity of the relation between the special coordinates $\mathfrak{a}_{i\mathbf{a}}$, $\mathfrak{a}^D_{i\mathbf{a}}$ and the moduli u of the curve C_u. Roughly speaking, all special coordinates are linear combinations of the periods of the differential

$$x\frac{\mathrm{d}t}{t}$$

and the masses. The coordinates $\mathfrak{a}_{1\mathbf{a}}$ come from the periods

$$\oint x\,\mathrm{d}\log g_1(x) \sim \oint x\,\mathrm{d}t/t,$$

the coordinates $\mathfrak{a}_{2\mathbf{a}}$ come from the periods

$$\oint x\,\mathrm{d}\log(g_1(x)g_2(x)) \sim \oint x\,\mathrm{d}t/t + \oint x\,\mathrm{d}t/t,$$

the coordinates $\mathfrak{a}_{i\mathbf{a}}$ come from the periods

$$\oint x\,\mathrm{d}\log(g_1(x)\cdots g_i(x)) \sim \oint x\,\mathrm{d}t/t + \cdots + \oint x\,\mathrm{d}t/t,$$

etc.

Remark 10.2. In the A_2 case our solution matches the one found in [169].

Remark 10.3. We can connect the cameral curve $\mathcal{C}_u$ to the spectral curve C_u via a tower of ramified covers:

$$\mathcal{C}_u \to C_u^{(r)} \to C_u^{(r-1)} \to \cdots \to C_u^{(1)} = C_u \to \mathbb{CP}^1_{\langle x\rangle},$$

which we can call the *Gelfand–Zeitlin* tower of curves. The curve $C_u^{(i)}$ is the quotient of $\mathcal{C}_u$ by the subgroup $W(A_{r-i})$ of the Weyl group $W(A_r)$, which acts on the amplitudes $(\mathcal{Y}_{i+1},\ldots,\mathcal{Y}_r)$ while preserving $(\mathcal{Y}_1,\ldots,\mathcal{Y}_i)$.

Remark 10.4. We should warn the reader that our cameral curves need not be the cameral curves of Hitchin systems [30]. We mapped the spectral curve of the family of conjugacy classes $[g(x)]$ corresponding to the fundamental representation R_1 to the spectral curve of the $\mathrm{GL}(N)$-Gaudin system, i.e., the genus zero Hitchin system, corresponding to the N-dimensional representation. One could then build the cameral curve for the $\mathrm{GL}(N)$-Gaudin system. This curve has all the reasons to differ from our cameral curve $\mathcal{C}_u$.

However, the identification of $\mathfrak{M}$ with the moduli spaces of curves describing the spectrum of the transfer matrix in the quasi classical limit of the $Y(A_r)$ spin chain is more natural, and carries over to the level of cameral curves.

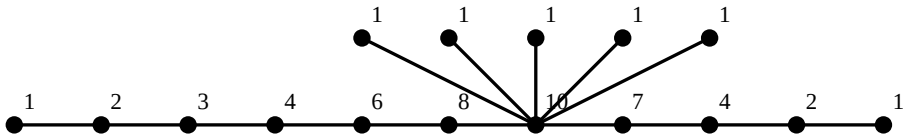


Figure 10.2. The example quiver variety for A_4 quiver at $\mathbf{v} = (7, 10, 8, 5)$ and $\mathbf{w} = (4, 5, 1, 2)$ with $i_* = 2$ and $\mathbf{w}_* = \mathbf{w}_- + \mathbf{w}_+$ with $\mathbf{w}_- = 3$ and $\mathbf{w}_+ = 2$. The labels at vertices denote the dimensions in the pattern as explained.

Remark 10.5. In view of [61], it is natural to identify the space of couplings $\tilde{\mathfrak{q}} = (\mathfrak{q}_1, \dots, \mathfrak{q}_r)$ with a coordinate patch in the moduli space $\overline{\mathcal{M}}_{0,r+3}$ of stable genus zero curves with $r+3$ punctures. In this fashion the linear quiver theories (the class I type A_r theories) can be analytically continued to other weakly coupled regions (weak coupling corresponds to the maximal degeneration of the stable curve). Most of these regions do not have a satisfactory Lagrangian description. Nevertheless, it would be interesting to try to generalize the limit shape equations even without knowing their microscopic origin. What would the iWeyl group look like in this case?

10.3 Quiver description

We have thus found that a particular subset of Gaudin–Hitchin models, with all but two residues of the minimal type, are the Seiberg–Witten integrable systems of the class I A_r type theories. As a check, let us compute the dimension of the moduli space $\mathfrak{P}^H_{0,r+3}$ of solutions to the (traceless part of the) moment map equation (10.11) divided by the $\mathrm{SL}(N, \mathbb{C})$-action is equal to

$$2(r+1)(N-1) - 2(N^2-1) + \left(N^2 - \sum_{j=1}^{i_*-1} j^2 \mathbf{w}_j - i_*^2 w_-\right) + \left(N^2 - \sum_{j=i_*+1}^{r} (r+1-j)^2 \mathbf{w}_j - (r+1-i_*)^2 w_+\right) = 2\sum_{i=1}^{r}(\mathbf{v}_i - 1) = 2\dim \mathfrak{M}.$$

Actually, the moduli space $\mathfrak{P}^H_{0,r+3}$ can be described as a quiver variety. Its graph is an $(r+3)$-pointed star, with $r+1$ legs of length 1, and two long legs, of the lengths $l_{-1} = \mathbf{v}_r - 1$ and $l_{r+1} = \mathbf{v}_1 - 1$, respectively. The dimensions of the vector spaces assigned to vertices are: the $(r+3)$-valent vertex (the star) has dimension N, the tails of the short legs all have dimension 1, the dimensions along the long legs start at 1 at the tails, then grow with the step 1 for the first $\mathbf{w}_1$ (respectively, $\mathbf{w}_r$) vertices, then grow with the step 2 for the next $\mathbf{w}_2$ (respectively, $\mathbf{w}_{r-1}$) and so on. (See example in Figure 10.2.)

The extended phase space $\mathfrak{P}^{\mathrm{ext}}$ for the class I A_r type theories is easy to describe. One just need to relax the $\mathbb{C}^\times$ moment map constraints (10.9) as well as the analogous $\mathbb{C}^\times$ constraints for the Φ_{-1}, Φ_{r+1} residues. In the quiver description we make the quiver gauge group the product of the special unitary groups as opposed to the product of unitary groups.

10.4 Reduction to the spin chain

The simplest example of the class I theory of the A type is, of course, the A_1 theory. This is the celebrated $N_f = 2N_c$ theory, with $\mathbf{w}_1 = N_f$, $\mathbf{v}_1 = N_c = N$, in our notation. Let $\mathfrak{q} = \mathfrak{q}_1$ and let $T(x) = T_{1,0}^{-1} T_1(x)$ denote the monic degree N polynomial.

The reduced curve (10.6) assumes a very simple form

$$\mathfrak{q}\mathcal{P}^-(x)t + \mathcal{P}^+(x)t^{-1} = (1+\mathfrak{q})T(x). \tag{10.12}$$

It is not difficult to recognize in this formula the quasiclassical limit of Baxter's T-Q equation [10] for the XXX $\mathfrak{sl}_2$ spin chain. In fact, it was observed already in [73, 74] that the Seiberg–Witten curve of the $\mathcal{N}=2$ supersymmetric QCD can be interpreted using the integrable spin chain, albeit in a somewhat different fashion. Note that a possible lift of $[g(x)]$ to $C\mathbf{G}=\mathrm{GL}(2,\mathbb{C})$ in this case is given by the diagonal matrix

$$g(x)=\begin{pmatrix}\mathfrak{q}t\mathcal{P}^-(x) & 0\\ 0 & t^{-1}\mathcal{P}^+(x)\end{pmatrix},$$

where t solves (10.12). However this choice of $g(x)$ is not continuous in x. As we cross the cuts $I_{1,\mathbf{a}}$ the matrix $g(x)$ will have its diagonal entries exchanged. We can conjugate $g(x)\to h^{-1}(x)g(x)h(x)$ into a form, e.g.,

$$\mathbf{g}(x)=\begin{pmatrix}\mathfrak{q}T(x) & 1\\ \mathfrak{q}\left(T^2(x)-\mathcal{P}^+(x)\mathcal{P}^-(x)\right) & T(x)\end{pmatrix}, \tag{10.13}$$

whose entries are polynomials. This is a particular case of a general statement [174], lifting a family of conjugacy classes in $\mathbf{G}_\mathrm{q}$ to $\mathbf{G}_\mathrm{q}$ itself (slightly adapted for the conformal extension $C\mathbf{G}$). The lift (10.13) does not depend on the split $\mathcal{P}(x)$ into the product of $\mathcal{P}^\pm$ factors.

There is yet another lift of $[g(x)]$ to $C\mathbf{G}$, which does depend on the factorization, and makes closer contact with spin chains. We shall discuss it in the section devoted to the study of the phase spaces of the integrable systems corresponding to our gauge theories.

10.5 Duality

In the mapping to the Gaudin–Hitchin system we employed a particular lift $g(x)$ of the conjugacy class $[g(x)]$ in $\mathrm{SL}(r+1,\mathbb{Z})/\mathbb{Z}_{r+1}$ to the conjugacy class in $\mathrm{GL}(r+1,\mathbb{C})$ by a judicious choice of the normalization factor $\zeta(x)$. More importantly, the spectral one-form describing the eigenvalues of the Higgs field, is equal to $x\mathrm{d}t/t$ where x is the argument of the amplitude function, and t is the spectral variable describing the eigenvalues of $g(x)$. For the group $\mathrm{GL}(r+1,\mathbb{C})$ the eigenvalues of $g(x)$ in some representation take values in $\mathbf{C}_{\langle t\rangle}=\mathbb{C}^\times$ which gets naturally compactified to $\mathbb{CP}^1$ to allow the degenerations.

To summarize, the Lax operator of Gaudin–Hitchin system, the Higgs field $\Phi(t)\mathrm{d}t$ lives on the curve $\mathbf{C}_{\langle t\rangle}$ of the eigenvalues of the "Lax operator" $g(x)$ of the gauge theory. Vice versa, the "Lax operator" $g(x)$ of the gauge theory lives on the curve $\mathbf{C}_{\langle x\rangle}$ of the eigenvalues of the Higgs field of Hitchin system.

We shall encounter some versions of this "eigenvalue – spectral parameter" duality in other sections of this work.

10.6 Class I theories of D type

These are the $\mathrm{SU}(\mathbf{v}_1)\times\cdots\times\mathrm{SU}(\mathbf{v}_r)$ theories whose quiver contains a trivalent vertex which connects two one-vertex legs to a leg of the length $r-3$. The corresponding group $\mathbf{G}_\mathrm{q}$ is $\mathrm{Spin}(2r,\mathbb{C})$, its conformal version $C\mathbf{G}$ is the extension of $\mathbf{G}$ by $\mathbb{C}^\times$ or $\mathbb{C}^\times\times\mathbb{C}^\times$, depending on the parity of r.

Passing from the A type theories to the D type theories we encounter new phenomenon. In addition to the exterior powers $\wedge^i V$ of the vector representation $V=\mathbb{C}^{2r}$ of $\mathrm{Spin}(2r)$ the fundamental representations of the group $\mathbf{G}_\mathrm{q}$ come also from spin representations $S_\pm$. We should use the cameral curve $\mathcal{C}_u$ to get the special coordinates and the prepotential, however a lot of information is contained in the spectral curve C_u^R in some fundamental representation R, which we shall take to be the vector $2r$-dimensional representation $V=R_{\lambda_1}=\mathbb{C}^{2r}$. In order to describe the spectral curve we need to know the characters of the group element $g(x)$ (9.6) in

the representations $\wedge^i V$, for $i = 1, \dots, 2r$. When we deal with V and its exterior powers only, we do not see the full conformal version of $\mathbf{G}$, only its one-dimensional extension (which we shall denote simply by C$\mathbf{G}$) which consists of the matrices $g \in \mathrm{GL}(2r, \mathbb{C})$, such that $gg^t = D(g) \cdot \mathbf{1}_{2r}$, with $D(g) \in \mathbb{C}^\times$ a scalar.

The spectral curve $C_u = C_u^V$ in the vector representation can be modified by the transformation similar to (10.5) to get the curve of minimal degree in x. Let us label the vertices of the D_r Dynkin diagram in such a way, that the trivalent vertex is $r-2$, the tails are $r-1$, r, and the end vertex of the "long leg" has the label 1, see Appendix A. Then the product of the matter polynomials $\mathcal{P}_{r-1}$ and $\mathcal{P}_r$ has degree

$$\deg(\mathcal{P}_{r-1}\mathcal{P}_r) = 2(\mathbf{v}_{r-1} + \mathbf{v}_r - \mathbf{v}_{r-2}).$$

Now we shall factorize $\mathcal{P}_{r-1}\mathcal{P}_r$ into a product of two factors of equal degrees

$$\mathcal{P}_{r-1}\mathcal{P}_r = \mathcal{P}^+\mathcal{P}^-, \qquad \deg \mathcal{P}^+ = \deg \mathcal{P}^- = \mathbf{v}_{r-1} + \mathbf{v}_r - \mathbf{v}_{r-2}. \tag{10.14}$$

There are many possible factorizations. For example, if $\mathbf{w}_{r-1} \le \mathbf{w}_r$, then we can take: $\mathcal{P}_r(x) = \mathcal{P}^+(x)S(x)$, $\mathcal{P}^-(x) = S(x)\mathcal{P}_{r-1}(x)$ for any degree $\mathbf{v}_r + \mathbf{v}_{r-1} - \mathbf{v}_{r-2} \le \mathbf{w}_r = 2\mathbf{v}_r - \mathbf{v}_{r-2}$ subfactor $\mathcal{P}^+(x)$ in $\mathcal{P}_r(x)$. We shall normalize $\mathcal{P}^\pm(x)$ so that the highest coefficient in both polynomials equals

$$\sqrt{\mathfrak{q}_{r-1}\mathfrak{q}_r}.$$

That there exist different decompositions (10.14) is a generalization of S-duality of the $\mathcal{S}$-class $\mathcal{N} = 2$ theories of the A_r type studied in [61]. The spectral curve C_u corresponding to the $2r$-dimensional vector representation of $C\mathrm{Spin}(2r, \mathbb{C})$ is mapped to the curve $P^C_{D_r}(t, x) = 0$ in the (t, x)-space, where

$$P^C_{D_r}(t, x) = t^{-r} P_\infty(x) \det_{R_1}(t \cdot 1_{2r} - g(x)) \tag{10.15}$$

with some polynomial $P_\infty(x)$ to be determined below. The group element $g(x)$ in the vector representation $\mathbb{C}^{2r}$ of $C\mathbf{G}_\mathrm{q}$ is given by

$$g(x) = E^{-81}\,\mathrm{diag}(g_1(x), \dots, g_{2r}(x))E$$

with E being any matrix such that

$$\left(EE^t\right)_{ij} = \delta_{i,2r+1-j}$$

represents the symmetric bilinear form on $\mathbb{C}^{2r}$ and

$$\begin{aligned}
g_1(x) &= \zeta(x)\mathcal{Y}_1(x),\\
g_i(x) &= \zeta(x)\mathcal{P}^{[i-1]}(x)\frac{\mathcal{Y}_i(x)}{\mathcal{Y}_{i-1}(x)}, \qquad i = 2, \dots, r-2,\\
g_{r-1}(x) &= \zeta(x)\mathcal{P}^{[r-2]}(x)\frac{\mathcal{Y}_{r-1}(x)\mathcal{Y}_r(x)}{\mathcal{Y}_{r-2}(x)},\\
g_r(x) &= \zeta(x)\mathcal{P}^{[r-2]}(x)\mathcal{P}_{r-1}(x)\frac{\mathcal{Y}_r(x)}{\mathcal{Y}_{r-1}(x)},\\
g_{r+1}(x) &= \zeta(x)\mathcal{P}^{[r-2]}(x)\mathcal{P}_r(x)\mathcal{Y}_{r-1}(x)/\mathcal{Y}_r(x),\\
g_{r+2}(x) &= \zeta(x)\mathcal{P}^{[r]}(x)\mathcal{Y}_{r-2}(x)/(\mathcal{Y}_{r-1}(x)\mathcal{Y}_r(x)),\\
g_{2r+1-i}(x) &= \zeta(x)\frac{\mathcal{P}^{[r]}(x)\mathcal{P}^{[r-2]}(x)}{\mathcal{P}^{[i-1]}(x)}\frac{\mathcal{Y}_{i-1}(x)}{\mathcal{Y}_i(x)}, \qquad i = 2, \dots, r-2,\\
g_{2r} &= \zeta(x)\mathcal{P}^{[r]}(x)\mathcal{P}^{[r-2]}(x)\frac{1}{\mathcal{Y}_1(x)}.
\end{aligned}$$

The factor $\zeta(x)$ which likely gives the minimal degree curve is

$$\zeta(x)^{-1} = \mathcal{P}^+(x)\mathcal{P}^{[r-2]}(x).$$

Thus, the scalar $D(g(x))$ is equal to

$$D(g(x)) = \frac{\mathcal{P}^-(x)}{\mathcal{P}^+(x)}$$

and the prefactor in (10.15) is

$$P_\infty(x) = \mathcal{P}^+(x)^r \prod_{j=1}^{r-2} \mathcal{P}_j(x)^j. \tag{10.16}$$

After some manipulations, we find

$$\begin{aligned}(-1)^r P^C_{D_r}(t,x) = {} & T_r^2\mathcal{P}_{r-1} + T_{r-1}^2\mathcal{P}_r - \eta T_{r-1}T_r + \sum_{l=1}^{\left[\frac{r}{2}\right]} T_{r-2l}\Bigg(\prod_{j=r+1-2l}^{r-2} \mathcal{P}_j^{j-r+2l}\Bigg)\xi_l^2 \\ & - \sum_{l=1}^{\left[\frac{r-1}{2}\right]} T_{r-2l-1}\Bigg(\prod_{j=r-2l}^{r-2} \mathcal{P}_j^{j-r+2l+1}\Bigg)\xi_l\xi_{l+1}, \\ \xi_l = {} & (\mathcal{P}^+t)^l - \big(\mathcal{P}^-t^{-1}\big)^l, \qquad \eta = \mathcal{P}^+t + \mathcal{P}^-t^{-1}.\end{aligned} \tag{10.17}$$

This equation has degree $N = 2(\mathbf{v}_r + \mathbf{v}_{r-1}) - \mathbf{v}_{r-2}$ in the x variable. Note

$$\mathbf{v}_{r-2} \le N \le 2\mathbf{v}_{r-2}.$$

As in the A_r case, the curve C_u has branches going off to infinity in the x-direction, over $2r$ points $\check{t}_i$, $\check{t}_i^{-1}$, $i = 1, \dots, r$ in the t-line $\mathbb{CP}^1_t$ which correspond to the weights of R_1

$$\check{t}_i = \frac{1}{\sqrt{\mathfrak{q}_{r-1}\mathfrak{q}_r}} \frac{\mathfrak{q}^{[i-1]}}{\mathfrak{q}^{[r-2]}}.$$

In addition, there are special points $t = 0, \infty$. Over these points the curve C_u has N branches, where x approaches one of the roots of the polynomial $P_0(x)$

$$P_0(x) = \mathcal{P}^-(x)^r \prod_{j=1}^{r-2} \mathcal{P}_j(x)^j.$$

and $P_\infty(x)$, cf. (10.16), respectively.

The curve C_u is invariant under the involution

$$t \mapsto \frac{\mathcal{P}_-(x)}{\mathcal{P}_+(x)} t^{-1}. \tag{10.18}$$

The fixed points of (10.18) are the points of intersection of the curve C_u and the curve

$$\mathcal{P}^+(x)t - \mathcal{P}^-(x)t^{-1} = 0. \tag{10.19}$$

The equations $\mathcal{R}_{D_r}(t,x) = 0$ (10.17) and (10.19) imply

$$T_r^2(x)\mathcal{P}_{r-1}(x) + T_{r-1}^2(x)\mathcal{P}_r(x) = T_{r-1}(x)T_r(x)\big(\mathcal{P}^+(x)t + \mathcal{P}^-(x)t^{-1}\big)$$

and

$$T_r^2(x)\mathcal{P}_{r-1}(x) = T_{r-1}^2(x)\mathcal{P}_r(x).$$

Again, the curve C_u is more economical then the full cameral curve $\mathcal{C}_u$. Again, the special coordinates $\mathfrak{a}_{i,\mathbf{a}}$ and the duals $\mathfrak{a}_{i,\mathbf{a}}^D$ are the linear combinations of the periods of the differential $x\,\mathrm{d}t/t$ and the masses.

Let us map the curve C_u to the curve Σ_u in the space S which is a $\mathbb{Z}_2$-quotient of the (blowup of the) $\mathbf{C}_{\langle x\rangle}\times\mathbb{CP}^1_t$ space, parametrized by (x,s), where

$$s = \frac{\mathcal{P}^+(x)}{\mathcal{P}^-(x)}t^2.$$

The curve Σ_u is described by the equations $s+s^{-1}=2c$ and

$$P^\Sigma_{D_r}(x,c) \equiv \mathbf{A}(x,c)^2 - 2\mathcal{P}_r(x)\mathcal{P}_{r-1}(x)(c+1)\mathbf{B}(x,c)^2 = 0,$$

where $\mathbf{A}$, $\mathbf{B}$ are the polynomials in x and c of bi-degrees $\left(N,\left[\frac{r}{2}\right]\right)$ and $\left(\mathbf{v}_{r-1}+\mathbf{v}_r,\left[\frac{r-1}{2}\right]\right)$, respectively,

$$\mathbf{A}(x,c) = T_r^2\mathcal{P}_{r-1} + T_{r-1}^2\mathcal{P}_r + 2\sum_{l=1}^{\left[\frac{r}{2}\right]}\mathbf{C}_l(c)T_{r-2l}\mathcal{P}_{r-1}^l\mathcal{P}_r^l\prod_{j=r+1-2l}^{r-2}\mathcal{P}_j^{j-r+2l},$$

$$\mathbf{B}(x,c) = T_{r-1}T_r + 2\sum_{l=1}^{\left[\frac{r-1}{2}\right]}\mathbf{D}_l(c)T_{r-2l-1}\mathcal{P}_{r-1}^l\mathcal{P}_r^l\prod_{j=r-2l}^{r-2}\mathcal{P}_j^{j-r+2l+1},$$

where the degree l polynomials $\mathbf{C}_l(c)$, $\mathbf{D}_l(c)$ are defined as follows:

$$\mathbf{C}_l(c) = \frac{1}{2}\left(s^l+s^{-l}\right)-1, \qquad s+s^{-1}=2c,$$

$$\mathbf{D}_l(c) = \frac{\left(s^l-1\right)\left(s^{l+1}-1\right)}{2s^l(s+1)} = \sum_{j=0}^{l-1}(-1)^j\mathbf{C}_{l-j}(c).$$

Over the points $c=1$ and $c=-1$ the equation for Σ_u becomes reducible at $c=1$:

$$P^\Sigma_{D_r}(x,1) = \left(\mathcal{P}_{r-1}T_r^2 - \mathcal{P}_rT_{r-1}^2\right)^2 \tag{10.20}$$

and at $c=-1$:

$$P^\Sigma_{D_r}(x,-1) = \mathbf{A}(x,-1)^2. \tag{10.21}$$

It is easy to see that the curve Σ_u has double points at (x,s) where either $s=1$ and x being any of the N roots of (10.20) or $s=-1$ and x is any of the N roots of (10.21). The locations of these roots are not fixed by the masses of the matter fields.

Let us normalize the equation of Σ_u by dividing $P^\Sigma_{D_r}$ by the coefficient at x^{2N}:

$$\mathcal{R}_{D_r}(x,c) = \frac{P^\Sigma_{D_r}(x,c)}{\prod_{i=1}^r\left(s-\check{t}_i^2\right)\left(1-s^{-1}\check{t}_i^{-2}\right)} \tag{10.22}$$

times a constant such that $\mathcal{R}_{D_r}(x,c)$ is monic in x and a rational function of c.

We thus arrive at the following interpretation of the curve Σ_u. It is the spectral curve

$$\mathcal{R}_{D_r}\left(x,\frac{s^2+1}{2s}\right)\left(\frac{\mathrm{d}s}{s}\right)^{2N} = \mathrm{Det}_{2N}\left(x\frac{\mathrm{d}s}{s}-\Phi(s)\right)$$

of the genus zero Higgs field

$$\Phi(s)=\sum_{s_j\in J}\Phi_j\frac{\mathrm{d}s}{s-s_j},$$

where $J\subset\mathbb{CP}^1_s$ is the set of $2r+2$ singularities

$$J=\{0,\infty\}\cup\big\{\check{t}_i^2,\,\check{t}_i^{-2}\mid i=1,\dots,r\big\}.$$

Let $\sigma\colon\mathbb{CP}^1_s\to\mathbb{CP}^1_s$ be the involution $\sigma(s)=s^{-1}$. The Higgs field must obey

$$\sigma^*\Phi=\Omega\Phi^t\Omega^{-1},\tag{10.23}$$

where Ω is a constant anti-symmetric matrix (cf. [95]), which defines the symplectic structure on $V=\mathbb{C}^{2N}$. If we expand

$$\Phi(s)=\Phi_0\frac{\mathrm{d}s}{s}+\sum_{i=1}^r\Phi_i^+\frac{\mathrm{d}s}{s-\check{t}_i^2}+\sum_{i=1}^r\Phi_i^-\frac{\mathrm{d}s}{s-\check{t}_i^{-2}},$$
$$\Phi_\infty=-\Phi_0-\sum_{i=1}^r\big(\Phi_i^++\Phi_i^-\big),$$

then (10.23) implies

$$\Phi_\infty=\Omega\Phi_0^t\Omega^{-1},\qquad\Phi_i^+=\Omega(\Phi_i^-)^t\Omega^{-1},\qquad i=1,\dots,r.$$

Also, the matrices Φ_i^+, Φ_i^-, $i=1,\dots,r$, must have rank one, while the matrices $\Phi_{0,\infty.\pm1}$ have rank $2N$. We can interpret

$$\mu=\Phi_0+\Phi_\infty+\sum_{i=1}^r\big(\Phi_i^++\Phi_i^-\big),\qquad\mu^t=\Omega^{-1}\mu\Omega$$

as the moment map for the $\mathrm{Sp}(2N)$ group action on the product of some orbits

$$\mathcal{O}_0\times\mathcal{O}_{-1}\times\mathcal{O}_1\times_{i=1}^r\mathcal{O}_i,$$

which generates the action $\Phi_j\mapsto g^{-1}\Phi_j g$ of $g\in\mathrm{Sp}(2N)$, such that

$$g\Omega g^t=\Omega.$$

It would be nice to develop further the theory of these orbifold Hitchin–Gaudin systems. We shall encounter a genus one version of such theory in the class II D_r section below.

The differential whose periods determine the special coordinates is equal to

$$\mathrm{d}S=x\frac{\mathrm{d}s}{s}.$$

10.7 Freezing example

Here we will illustrate how the D_4 theory with $v_1=v_3=v_4=v$, $v_2=2v$ and $w_1=w_3=w_4=0$, $w_2=v$ reduces to A_3 with $v_1=v_3=v$, $v_2=2v$ and $w_2=2v$ when the node 4 freezes under $\mathfrak{q}_4\to0$. Keeping in mind unfreezing to the affine $\widehat{D}_4$, let polynomial Y_0 of degree v denote the fundamental matter polynomial attached to the node "2".

The D_4 spectral curve for the node "1" from (10.17) in terms of variable η

$$\eta = t + \frac{\mathfrak{q}_1^2\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4}{t}, \tag{10.24}$$

where $\mathcal{Y}_2 = Y_0 t$ is

$$\begin{aligned}\mathcal{R}_{D_4}(\eta, x) = {}& \eta^4 Y_0^2 - \eta^3 T_1 Y_0 + \eta^2\big(\mathfrak{q}_1 T_2 - 4\mathfrak{q}_1^2\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4 Y_0^2\big) + \eta\big(-\mathfrak{q}_1^2\mathfrak{q}_2 T_3 T_4 + 4\mathfrak{q}_1^2\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4 T_1 Y_0\big)\\ & - 4\mathfrak{q}_1^3\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4 T_2 + \mathfrak{q}_1^3\mathfrak{q}_2^2\mathfrak{q}_4 T_3^2 + \mathfrak{q}_1^3\mathfrak{q}_2^2\mathfrak{q}_3 T_4^2.\end{aligned} \tag{10.25}$$

Notice that the curve is polynomial of degree 4 in η with polynomial coefficients in x of degree $2v$. In the limit $x \to \infty$ we find the limiting values of η are

$$1 + \mathfrak{q}_1^2\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4, \qquad \mathfrak{q}_1 + \mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4, \qquad \mathfrak{q}_1\mathfrak{q}_2 + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4, \qquad \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 + \mathfrak{q}_1\mathfrak{q}_3\mathfrak{q}_4.$$

Notice that the differential is

$$\lambda = x\frac{\mathrm{d}t}{t} = x\frac{\mathrm{d}\eta}{\big(\eta^2 - 4\mathfrak{q}_1^2\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4\big)^{\frac{1}{2}}}.$$

Also notice that at $\eta = \pm 2\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3^{\frac{1}{2}}\mathfrak{q}_4^{\frac{1}{2}}$ the curve factorizes as

$$\mathcal{R}_{D_4}\big(\pm 2\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3^{\frac{1}{2}}\mathfrak{q}_4^{\frac{1}{2}}, x\big) = \mathfrak{q}_1^3\mathfrak{q}_2^2(\mathfrak{q}_3 T_4(x) \mp \mathfrak{q}_4 T_3(x))^2 \tag{10.26}$$

as well as it factorizes at $\eta = \infty$

$$\mathcal{R}_{D_4}(\eta = \infty, x) = Y_0(x)^2. \tag{10.27}$$

We can interpret the multi-valued nature of λ on the η-plane as the deformation of the punctured sphere underlying the A_r-type theories to the curve describing the D_r-type theories, by opening punctures into cuts. Perhaps one can elevate this observation to the corresponding deformation of the Liouville theory coupled to some conformal matter, along the lines of [66, 106].

We see that in the decoupling limit $\mathfrak{q}_4 = 0$ the above curve reduces to

$$\mathcal{R}_{A_3}(\eta, x) = \eta^4 Y_0^2 - \eta^3 T_1 Y_0 + \eta^2\mathfrak{q}_1 T_2 - \eta\mathfrak{q}_1^2\mathfrak{q}_2 T_3 Y_4 + \mathfrak{q}_1^3\mathfrak{q}_2^2\mathfrak{q}_3 Y_4^2, \tag{10.28}$$

where we just set that $\mathcal{Y}_4$ freezes and converts to a factor of degree v contributing to the fundamental matter polynomial for the node "2"; we denote this factor by $Y_4 \equiv \mathcal{Y}_4 = T_4$. The curve (10.28) is precisely the A_3 curve for the node "1" (10.6) in terms of the variable $\mathcal{Y}_1 = Y_0\eta$. This curve corresponds to the GL(2) Hitchin system with punctures at four punctures

$$1, \quad \mathfrak{q}_1, \quad \mathfrak{q}_1\mathfrak{q}_2, \quad \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3.$$

Moreover, from the discussion after (10.8) (we have $\mathbf{w}_0 = 0$, $\mathbf{w}_2 = 2$, $\mathbf{w}_3 = 0$ and $i_* = 2$ and $\mathbf{w}_+ = \mathbf{w}_{-1} = 1$) it is clear the eigenvalues of the Higgs field residues at $\eta = 0$ and at $\eta = \infty$ are doubly degenerate which effectively means that $\mathrm{SL}(2,\mathbb{C})$ part of the Higgs field does not have punctures at $\eta = 0$ and $\eta = \infty$. We can continue the freezing reduction and now we shall set $\mathfrak{q}_3 = 0$ declaring the function $\mathcal{Y}_3$ as contributing to the fundamental matter at the node "2", we denote $\mathcal{Y}_3 = T_3 = Y_3$. After factoring out η, the curve (10.28) reduces to the A_2 curve

$$\mathcal{R}_{A_2}(\eta, x) = \eta^3 Y_0^2 - \eta^2 T_1 Y_0 + \eta\mathfrak{q}_1 T_2 - \mathfrak{q}_1^2\mathfrak{q}_2 Y_3 Y_4. \tag{10.29}$$

The corresponding Gaudin system has punctures at $\eta = 0$ and $\eta = \infty$ and at

$$1, \quad \mathfrak{q}_1, \quad \mathfrak{q}_1\mathfrak{q}_2.$$

Finally, we can freeze the node "1" by sending $\mathfrak{q}_1$ to zero and rescaling $\eta = \tilde{\eta}\mathfrak{q}_1$ so that the former punctures $\mathfrak{q}_1$, $\mathfrak{q}_1\mathfrak{q}_2$ on the $\tilde{\eta}$-plane in terms of $\tilde{\eta}$ become

$$1, \quad \mathfrak{q}_2$$

while the puncture $\eta = 1$ is send away to $\tilde{\eta} = \infty$. We set $Y_1 \equiv \mathcal{Y}_1 = T_1$ and find that (10.29) reduces to the familiar A_1 curve with gauge polynomial T_2 of degree $2v$ and four factors (Y_0, Y_1, Y_3, Y_4) of degree v which make fundamental polynomial of degree $4v$

$$\mathcal{R}_{A_2}(\eta, x) = -\tilde{\eta}^2 Y_1 Y_0 + \tilde{\eta} T_2 - \mathfrak{q}_2 Y_3 Y_4.$$

The punctures of the corresponding Gaudin model in $\tilde{\eta}$ plane are at $(0, \mathfrak{q}_2, 1, \infty)$.

10.8 Class I theories of E type

We are using Bourbaki conventions to label the nodes on the Dynkin graph of E_r series, see figures in Appendix A. One can construct the analogues of the spectral curves C_u or Σ_u using the minuscule representations in the E_6 and E_7 cases. For E_8 one can construct the spectral curve using the adjoint representation **248**. However it seems more advantageous to use the degenerate version of del Pezzo/E-bundle correspondence, which we review below in the discussion of class II theories of E type. For the standard conformal E_r quivers, which are obtained by freezing of the node "0" in the affine E_r quivers with ranks $\mathbf{v}_i = Na_i$ where a_i are Dynkin marks, we find spectral curves of (t, x)-degree equal to $(27, 6N)$ for E_6, $(56, 12N)$ for E_7 and $(240, 60N)$ for E_8. These degrees can be understood from the degeneration of $\widehat{E}_r$ spectral curves computed in Section 9.7.

10.9 The E_6 theory

The spectral curve in the fundamental representation $R_6 = \mathbf{27}$ associated with the node "6", in which the group element of the conformal extension of E_6 is $g(x) = (\mathcal{Y}_6(x), \dots)$ has the form

$$\mathcal{R}_{E_6}(t, x) = 0,$$

where the explicit expression is of the form[3]

$$\begin{aligned}\mathcal{R}_{E_6}(t,x) = \det_{R_6}(t\cdot 1_{27} - g(x)) = t^{27} - t^{26}T_6 + t^{25}\mathcal{P}_6 T_5 - t^{24}\mathcal{P}_5\mathcal{P}_6^2 T_4 \\ + t^{23}\left(-\mathcal{P}_2^2\mathcal{P}_3^2\mathcal{P}_4^4\mathcal{P}_5^4\mathcal{P}_6^4 T_1^2 + \mathcal{P}_1\mathcal{P}_2^2\mathcal{P}_3^2\mathcal{P}_4^4\mathcal{P}_5^4\mathcal{P}_6^4 T_3 + \mathcal{P}_4\mathcal{P}_5^2\mathcal{P}_6^3 T_2 T_3\right. \\ \left. -\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4^2\mathcal{P}_5^2\mathcal{P}_6^3 T_1 T_5 + \mathcal{P}_1^2\mathcal{P}_2^3\mathcal{P}_3^4\mathcal{P}_4^6\mathcal{P}_5^5\mathcal{P}_6^4 T_6\right) + \dots - \mathcal{P}_1^{18}\mathcal{P}_2^{27}\mathcal{P}_3^{36}\mathcal{P}_4^{54}\mathcal{P}_5^{45}\mathcal{P}_6^{36},\end{aligned} \tag{10.30}$$

where we have omitted the explicit expressions for the terms from t^{24} to t^1, and we omitted the dependence on x in the notations for the polynomial coefficients so that $\mathcal{P}_i \equiv \mathcal{P}_i(x)$ and $T_i \equiv T_i(x)$. The curve (10.30) has x-degree $27v_6$, and, of course, is not the most economical. By rescaling $g(x) \to \zeta(x)g(x)$ with a suitably chosen $\zeta(x)$ of degree $-v_6$ made of some powers of the factors in fundamental polynomials we can reduce the degree of (10.30).

The most standard conformal E_6 quiver, which arises from the degenerate limit $\mathfrak{q}_0 \to 0$ in the node "0" of the affine $\widehat{E}_6$ quiver, has matter polynomial $\mathcal{P}_2 = \mathfrak{q}_2 Y_0$ of degree N only at the node "2" to which the affine node "0" was attached, while the degrees of the gauge polynomials are fixed by the Dynkin marks $\mathbf{v}_i = Na_i$, that is $(\mathbf{v}_1, \dots, \mathbf{v}_6) = (N, 2N, 2N, 3N, 2N, N)$. For such conformal E_6 quiver, the curve (10.30) has canonical reduced form under the choice

[3] The explicit expression, which we do not list here, is available upon a request; it is computed by the straightforward expansion of the exterior powers $\bigwedge^\bullet R_6$ in the representation ring $\mathrm{Rep}(E_6)$ over the fundamental representations $R_1, \dots, R_6$.

$\zeta^{-1}(x) = Y_0(x)$ and the degree of the reduced curve is $6N = 2\mathbf{v}_*$ where $\mathbf{v}_* \equiv \mathbf{v}_4 = 3N$ denotes the rank in the trivalent node "4". The reduced curve of such special conformal E_6 quiver is $\mathcal{R}_{E_6}(t,x)$, with $\mathcal{P}_i = \mathfrak{q}_i$, $i \neq 2$; $\mathcal{P}_2 = \mathfrak{q}_2 Y_0$ we find

$$\begin{aligned}\mathcal{R}_{E_6}(t,x) = {} & t^{27}Y_0^6 - t^{26}Y_0^5 T_6 + t^{25}\mathfrak{q}_6 Y_0^4 T_5 - t^{24}\mathfrak{q}_5\mathfrak{q}_6^2 Y_0^3 T_4 + t^{23}\left(-\mathfrak{q}_2^2\mathfrak{q}_3^2\mathfrak{q}_4^4\mathfrak{q}_5^4\mathfrak{q}_6^4 Y_0^4 T_1^2\right. \\ & + \mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_3^2\mathfrak{q}_4^4\mathfrak{q}_5^4\mathfrak{q}_6^4 Y_0^4 T_3 + \mathfrak{q}_4\mathfrak{q}_5^2\mathfrak{q}_6^3 Y_0^2 T_2 T_3 - \mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4^2\mathfrak{q}_5^2\mathfrak{q}_6^3 Y_0^3 T_1 T_5 \\ & \left. + \mathfrak{q}_1^2\mathfrak{q}_2^3\mathfrak{q}_3^4\mathfrak{q}_4^6\mathfrak{q}_5^5\mathfrak{q}_6^4 Y_0^5 T_6\right) + \cdots - t^2\mathfrak{q}_1^{15}\mathfrak{q}_2^{23}\mathfrak{q}_3^{30}\mathfrak{q}_4^{46}\mathfrak{q}_5^{39}\mathfrak{q}_6^{32} Y_0^4 T_3 \\ & + t\mathfrak{q}_1^{16}\mathfrak{q}_2^{25}\mathfrak{q}_3^{33}\mathfrak{q}_4^{50}\mathfrak{q}_5^{42}\mathfrak{q}_6^{34} Y_0^5 T_1 - \mathfrak{q}_1^{18}\mathfrak{q}_2^{27}\mathfrak{q}_3^{36}\mathfrak{q}_4^{54}\mathfrak{q}_5^{45}\mathfrak{q}_6^{36} Y_0^6, \end{aligned} \tag{10.31}$$

where again we only indicated the middle terms but skipped the explicit expressions. Indeed, one sees that the curve (10.31) of the E_6 quiver with the standard rank assignments $\mathbf{v}_i = Na_i$ has degree $6N$. At the limit $x \to \infty$ the 27 roots of $\mathcal{R}_{E_6}(t,x)$ in (10.31) approach the set of points in the t-plane labeled by the weights λ in the **27** representation of E_6 and given explicitly by $\prod_{i=1}^6 \mathfrak{q}_i^{(\lambda_i, \lambda-\lambda_i)}$, or

$$\left\{ \prod_{i=1}^6 \mathfrak{q}_i^{n_i} \;\middle|\; \sum_{i=1}^6 n_i\alpha_i = \lambda_6 - \lambda,\ \lambda \in \text{weights}(R_6) \right\}, \tag{10.32}$$

where n_i are the coefficients of the expansion in the basis of simple roots of the difference between a given weight in **27** and the highest weight. One can associate a Higgs field to the spectral curve (10.31) with poles in the 27 punctures (10.32) with certain relations. In other words, the curve (10.31) realizes a certain embedding of the standard conformal E_6 quiver theory with gauge group ranks $\mathbf{v}_i = (N, 2N, 2N, 3N, 2N, N)$ to some specialization of the A_{26} theory with ranks $(6N, 6N, \ldots, 6N)$, and this embedding can be lifted to the Higgs field spectral curve representation of (10.31).

For non-standard assignments of $\mathbf{w}_i$ and $\mathbf{v}_i$ for the conformal E_6 quiver we did not find a simple choice of $\zeta(x)$ reducing the curve (10.30) to the minimal degree. For small ranks $\mathbf{v}_i$, $\mathbf{w}_i$ we can find the reduced curve using the brute search minimization problem on the total degree of the reduced curve under $g(x) \to \zeta(x)g(x)$. We have found different chambers in the space of parameters $\mathbf{w}_i$, $\mathbf{v}_i$ with piece-wise linear dependence of the reduced degree of $\mathbf{w}_i$ or $\mathbf{v}_i$'s but not a simple expression. For example, in several examples we find

(w_i)	(v_i)	reduced curve x-degree
$(0,4,0,0,0,0)$	$(4,8,8,12,8,4)$	24
$(3,0,0,0,0,3)$	$(6,6,9,12,9,6)$	33
$(6,0,0,0,0,0)$	$(8,6,10,12,8,4)$	40
$(4,0,0,0,0,1)$	$(6,5,8,10,7,4)$	31
$(6,0,0,0,0,3)$	$(10,9,14,18,13,8)$	53

where the first three lines list different conformal E_6 quivers sharing the same $\mathbf{v}_* = 12$, and one can see that the curve of the minimal degree $2\mathbf{v}_*$ is obtained in the standard assignment $\mathbf{w}_i = 0$, $i \neq 2$ associated to the degenerate limit of the affine E_6.

10.10 E_7 theory

We write the spectral curve in, for example, the **56** representation of E_7 similar to the E_6 case. If $(\mathbf{v}_0, \ldots, \mathbf{v}_7) = Na_i$ where a_i are Dynkin marks of E_7 quiver, again, similar to E_6 quiver we find that the reduced curve of the standard conformal E_7 quiver obtained from the degenerate limit of the affine theory has x-degree $12N = 3\mathbf{v}_*$ where $\mathbf{v}_* = \mathbf{v}_4 = 4N$ is rank at the trivalent node. The standard E_7 quiver spectral curve hence is realized as a specialization of the spectral curve for A_{55} quiver with ranks $(12N, 12N, \ldots, 12N)$, or Hitchin system with 56 punctures on t-plane associated to the weights in **56**.

10.11 E_8 theory

For E_8 the minimal representation is adjoint **248**. The reduced curve in the adjoint representation for the standard conformal E_8 quiver obtained from the degenerate limit of the affine theory has x-degree $60N = 10\mathbf{v}_*$ where $\mathbf{v}_* = 6N$ is rank at the trivalent node. Hence the standard conformal E_8 quiver spectral curve is realized as a specialization of the spectral curve for A_{247} quiver with ranks $60(N, N, \dots, N)$, or Hitchin system with 240 punctures on t-plane associated to the non-zero adjoint weights in **248**.

10.12 Class II theories of A type and class II* theories

Let us start with the simplest nontrivial examples, and then pass onto a general case.

10.13 Class II $\widehat{A}_1$ theory

For the class II theory we shift the arguments of $\mathcal{Y}_i(x)$ by μ_i to get rid of the bi-fundamental masses.

Let $g(x) \in \widehat{\mathrm{SL}_2}$:

$$g(x) = \mathfrak{q}_0^{-\widehat{\lambda}_0^\vee} \mathfrak{q}_1^{-\widehat{\lambda}_1^\vee} \mathcal{Y}_0(x)^{\widehat{\alpha}_0^\vee} \mathcal{Y}_1(x)^{\widehat{\alpha}_1^\vee}.$$

We have: $\mathfrak{q} = \mathfrak{q}_0\mathfrak{q}_1$,

$$g(x)^{\alpha_1} = \frac{\mathcal{Y}_1^2}{\mathfrak{q}_1\mathcal{Y}_0^2}, \qquad g(x)^{-\delta} = \mathfrak{q}, \qquad g(x)^{\widehat{\lambda}_0} = \mathcal{Y}_0(x).$$

The normalized $\widehat{\mathfrak{sl}_2}$ characters (9.9) of the fundamental representations $\widehat{R}_0$, $\widehat{R}_1$ are equal to

$$\begin{aligned}
\mathcal{X}_0(\mathcal{Y}(x), \mathfrak{q}) &= \frac{\mathcal{Y}_0(x)}{\phi(\mathfrak{q})}\theta_3\left(\frac{\mathcal{Y}_1(x)^2}{\mathfrak{q}_1\mathcal{Y}_0(x)^2}; \mathfrak{q}^2\right), \\
\mathcal{X}_1(\mathcal{Y}(x), \mathfrak{q}) &= \left(\frac{\mathfrak{q}_1}{\mathfrak{q}_0}\right)^{\frac{1}{4}} \frac{\mathcal{Y}_0(x)}{\phi(\mathfrak{q})}\theta_2\left(\frac{\mathcal{Y}_1(x)^2}{\mathfrak{q}_1\mathcal{Y}_0(x)^2}; \mathfrak{q}^2\right)
\end{aligned} \tag{10.33}$$

(see Appendix O for our conventions on elliptic functions). The characters (10.33) are invariant under the Weyl transformations

$$\mathcal{Y}_0 \to \mathfrak{q}_0\mathcal{Y}_0^{-1}\mathcal{Y}_1^2, \qquad \mathcal{Y}_1 \to \mathfrak{q}_1\mathcal{Y}_1^{-1}\mathcal{Y}_0^2$$

and therefore we can equate them to the polynomials

$$\begin{aligned}
\mathcal{X}_0(\mathcal{Y}(x), \mathfrak{q}) &= T_0(x), \qquad T_{0,0} = \frac{\theta_3\left(\mathfrak{q}_1^{-1}; \mathfrak{q}^2\right)}{\phi(\mathfrak{q})}, \\
\mathcal{X}_1(\mathcal{Y}(x), \mathfrak{q}) &= T_1(x), \qquad T_{1,0} = \left(\frac{\mathfrak{q}_1}{\mathfrak{q}_0}\right)^{\frac{1}{4}} \frac{\theta_2\left(\mathfrak{q}_1^{-1}; \mathfrak{q}^2\right)}{\phi(\mathfrak{q})}.
\end{aligned}$$

The values of characters (10.33) and $\mathfrak{q}_0$, $\mathfrak{q}_1$ define $\mathcal{Y}_0$ and $\mathcal{Y}_1$ up to an affine Weyl transformation. To recover $\mathcal{Y}_0$ and $\mathcal{Y}_1$ we invert the relations (10.33):

$$\mathcal{Y}_1(x) = \mathfrak{q}_1^{\frac{1}{2}}\mathcal{Y}_0(x)t, \qquad \mathcal{Y}_0(x) = \frac{\phi(\mathfrak{q})}{\theta_3\left(t^2; \mathfrak{q}^2\right)}T_0(x)$$

and express

$$\left(\frac{\mathfrak{q}_0}{\mathfrak{q}_1}\right)^{\frac{1}{4}} \frac{\theta_3\left(t^2; \mathfrak{q}^2\right)}{\theta_2\left(t^2; \mathfrak{q}^2\right)} = \frac{T_0(x)}{T_1(x)}.$$

Actually, the ratio

$$\xi = \left(\frac{\mathfrak{q}_0}{\mathfrak{q}_1}\right)^{\frac{1}{4}} \frac{\theta_3\big(t^2;\mathfrak{q}^2\big)}{\theta_2\big(t^2;\mathfrak{q}^2\big)}$$

is a meromorphic function on $\mathcal{E}$ with two first-order poles at $t = \pm \mathrm{i}$ and two simple zeroes at $t = \pm \mathrm{i}\mathfrak{q}$. Therefore

$$\xi = \xi_\infty \frac{X(t,\mathfrak{q}) - X_0}{X(t,\mathfrak{q}) - X_1}, \qquad X_0 := X(\mathrm{i}\mathfrak{q},\mathfrak{q}), \qquad X_1 := X(\mathrm{i},\mathfrak{q})$$

with

$$\xi_\infty = \left(\frac{\mathfrak{q}_0}{\mathfrak{q}_1}\right)^{\frac{1}{4}} \frac{\theta_3\big(1,\mathfrak{q}^2\big)}{\theta_2\big(1,\mathfrak{q}^2\big)}$$

and the explicit $\mathfrak{q}$-series for $X(t,\mathfrak{q})$ is given in (O.2) and (O.2). Hence, the algebraic Seiberg–Witten curve C_u describing the $\widehat{A}_1$ theory is a two-fold cover of the rational curve Σ_u

$$(\xi_\infty T_1(x) - T_0(x))X - (\xi_\infty T_1(x) X_0 - T_0(x) X_1) = 0$$

defined by the Weierstraß cubic (O.3). There are $4N$ branch points of the $2:1$ cover $C_u \to \Sigma_u$:

$$\begin{aligned}
&\xi_\infty T_1(x_{\infty,\mathbf{a}}) - T_0(x_{\infty,\mathbf{a}}) = 0,\\
&(\xi_\infty T_1(x_{\alpha,\mathbf{a}}) - T_0(x_{\alpha,\mathbf{a}}))e_\alpha - (\xi_\infty T_1(x_{\alpha,\mathbf{a}}) X_0 - T_0(x_{\alpha,\mathbf{a}}) X_1) = 0,\\
&\alpha = 1,2,3, \qquad \alpha = 1,\dots,N,
\end{aligned}$$

which can be split into 2 groups of N pairs, corresponding to the cycles $A_{i\mathbf{a}}$ with $i = 0,1$, e.g., $A_{0,\mathbf{a}}$ is a small circle around the cut which connects $x_{1,\mathbf{a}}$ to $x_{2,\mathbf{a}}$, while $A_{1,\mathbf{a}}$ is a small circle around the cut which connects $x_{3,\mathbf{a}}$ to $x_{\infty,\mathbf{a}}$. The special coordinates are computed by the periods of

$$\mathrm{d}S_- = x\,\mathrm{d}\log(t) = x\frac{\mathrm{d}X}{Y}.$$

The curve C_u is the spectral curve. The cameral curve $\mathcal{C}_u$ is a $\mathbb{Z}$-cover of spectral curve C_u, which is given by the same equations but now with $t \in \mathbb{C}^\times$ as opposed to $t \in \mathcal{E}$. On cameral curve $\mathcal{C}_u$ we have the second differential

$$\mathrm{d}S_+ = x\,\mathrm{d}\log\theta_3\big(t^2;\mathfrak{q}\big),$$

which would be a multi-valued differential on spectral curve C_u whose periods are defined up to the periods of $\mathrm{d}S_-$, similar to the polylogarithm motives [15].

10.14 Class II* $\widehat{A}_0$ theory

This is a (noncommutative) $U(1)$ $\mathcal{N} = 2^*$ theory. This theory was solved in [149] by the similar method. There is only one amplitude $\mathcal{Y}(x) = \mathcal{Y}_0(x)$, with the single interval I as its branch cut, the single function

$$t(x) \equiv t_0(x) = \frac{\mathcal{Y}(x)}{\mathcal{Y}(x+\mathfrak{m})}$$

with two branch cuts I and $I - \mathfrak{m}$. Crossing the I cut maps $t(x) \mapsto \mathfrak{q}t(x-\mathfrak{m})$. Crossing the cut $I - \mathfrak{m}$ has the opposite effect: $t(x) \mapsto \mathfrak{q}^{-1}t(x+\mathfrak{m})$. The extended functions

$$t_j(x) = \mathfrak{q}^j t(x - j\mathfrak{m}).$$

The analytically continued function $t(x)$ has cuts at $I+\mathfrak{m}\mathbb{Z}$. The sheets of the Riemann surface of $t(x)$ are labeled by $j\in\mathbb{Z}$, so that on the sheet j the cuts are at $I-j\mathfrak{m}$, and $I-(j+1)\mathfrak{m}$. Upon crossing $I+j\mathfrak{m}$ the $t_j(x)$ function transforms to $t_{j+1}(x)$ function. As $x\to\infty$ on this sheet the corresponding branch of $t(x)$ approaches $\mathfrak{q}^j$. These conditions uniquely fix the inverse function to be the logarithmic derivative of θ_1:

$$x=a+\mathfrak{m}t\frac{\mathrm{d}}{\mathrm{d}t}\log\theta_1(t;\mathfrak{q}).$$

10.15 Class II A_r theories

In order to solve the general rank r theory, it is convenient to form a linear combination of fundamental characters of $\widehat{A}_r$. Ultimately we would like to define a regularized version of the characteristic polynomial of $g(x)$, where, as in the general case, after the shift of the arguments of $\mathcal{Y}_i(x)\to\mathcal{Y}_i(x+\mu_i)$:

$$g(x)=\prod_{i=0}^{r}\mathfrak{q}_i^{-\widehat{\lambda}_i^\vee}\mathcal{Y}_i(x)^{\widehat{\alpha}_i^\vee}.$$

Using $t_i(x)=g(x)^{e_i}$ (see the appendix), we compute

$$t_i(x)=\check{t}_i\frac{Y_i(x)}{Y_{i-1}(x)},\qquad i=1,\dots,r+1,\tag{10.34}$$

where we extended the amplitude functions $\mathcal{Y}_j(x)$ defined for $j=0,\dots,r$ to be defined for all $j\in\mathbb{Z}$ by periodicity $Y_j(x)=\mathcal{Y}_{j+(r+1)}(x)$ and where

$$\mathbf{t}(x)=(t_1(x),t_2(x),\dots,t_{r+1}(x))$$

represents an element of the maximal torus of $\mathrm{SL}(r+1,\mathbb{C})$, i.e.,

$$\prod_{i=1}^{r+1}t_i(x)=1.$$

The $\check{t}_i$ are the asymptotic values at $x\to\infty$ of $t_i(x)$ and are given by

$$\check{t}_i=(\mathfrak{q}_i\dots\mathfrak{q}_r)^{-1}\big(\mathfrak{q}_1\mathfrak{q}_2^2\dots\mathfrak{q}_r^r\big)^{\frac{1}{r+1}},\qquad i=1,\dots,r+1,$$

and

$$g(x)^{-\delta}=\mathfrak{q},\qquad g(x)^{\widehat{\lambda}_0}=\mathcal{Y}_0(x).$$

Now we shall explore the relation between the conjugacy classes in Kac–Moody group and the holomorphic bundles on elliptic curve $\mathcal{E}$. We will consider a family of bundles on $\mathcal{E}$ parametrized by the $\mathbf{C}_{\langle x\rangle}$-plane, e.g., as in [60]. We start with individual bundles.

Let V be a rank $r+1$ polystable vector bundle of degree zero over the elliptic curve $\mathcal{E}=\mathbb{C}^\times/\mathfrak{q}^{\mathbb{Z}}$, with trivial determinant,

$$\det V\approx\mathcal{O}_{\mathcal{E}}.$$

Such bundle always splits as a direct sum of line bundles

$$V=\bigoplus_{i=1}^{r+1}L_i.$$

Each summand is a degree zero line bundle L_i which can be represented as $L_i = \mathcal{O}(p_0)^{-1}\mathcal{O}(t_i)$ where $\mathcal{O}(p)$ is the degree one line bundle whose divisor is a single point $p \in E$ and p_0 denotes the point $t = 1$ corresponding to the identity in the abelian group law on the elliptic curve $\mathcal{E}$. A meromorphic section s_i of L_i with a simple pole at $t = 1$ and zero at $t = t_i$ can be written explicitly using the theta-functions:

$$s_i(t) = \frac{\theta(t/t_i;\mathfrak{q})}{\theta(t;\mathfrak{q})}$$

and is unique up to a multiplicative constant. To each degree zero vector bundle V with the divisor

$$D_V = -(r+1)p_0 + t_1 + \cdots + t_{r+1}$$

of $\det V$ we associate a projectively unique section s of its determinant $\det V$ which has zeroes at $t_1, \ldots, t_{r+1}$ and a pole of the order not greater than $r+1$ at $t = 1$:

$$s(t;\mathfrak{t}) = \prod_{i=1}^{r+1} \frac{\theta(t/t_i;\mathfrak{q})}{\theta(t;\mathfrak{q})}, \tag{10.35}$$

where we explicitly indicate the $\mathfrak{t}$ dependence of the section s. Now set $t_i = t_i(x)$ given by (10.34). The meromorphic sections $s(t;\mathfrak{t}(x);\mathfrak{q})$ can be expanded in terms of the theta-functions $\Theta_j(\mathcal{Y}_0(x);\mathfrak{t};\mathfrak{q})$ and characters of $\widehat{A}_r$ (see (J.4) and (J.3)) as follows

$$\begin{aligned}\mathcal{Y}_0(x)\prod_{i=1}^{r+1}\frac{\theta(t/t_i(x);\mathfrak{q})}{\theta(t,\mathfrak{q})} &= \sum_{i=0}^{r}\mathfrak{q}^{-\frac{i}{2}}\mathfrak{q}^{\frac{i^2}{2(r+1)}}\Theta_i(\mathcal{Y}_0(x);\mathfrak{t}(x);\mathfrak{q})\phi_i(t;\mathfrak{q})\\ &= \phi(\mathfrak{q})^r\sum_{i=0}^{r}\chi_i(\mathcal{Y}_0(x);\mathfrak{t}(x);\mathfrak{q})\phi_i(t;\mathfrak{q}),\end{aligned}$$

where the functions $\phi_i(t;\mathfrak{q})$ are normalized meromorphic elliptic functions defined in Appendix O.3. Hence we find from (9.16) and (O.6) that the section $s(t,x)$ (10.35) obeys

$$\mathcal{Y}_0(x)s(t,x) = \phi(\mathfrak{q})^r\sum_{i=0}^{r}\chi_i(\mathcal{Y}_0(x);\mathfrak{t}(x);\mathfrak{q})M_{i\tilde{j}}(\mathfrak{q})\tilde{\phi}_{\tilde{j}}(t;\mathfrak{q}),$$

where $\tilde{\phi}_{\tilde{j}}(t;\mathfrak{q})$ denotes the Weierstraß monomials of Weierstraß elliptic functions $X(t,\mathfrak{q})$ and $Y(t,\mathfrak{q})$; and $M_{i\tilde{j}}$ is a certain modular matrix as defined in Appendix O.3. Recalling (9.3) that the characters $(\chi_i(\mathcal{Y}_0(x);\mathfrak{t}(x);\mathfrak{q}))$ evaluated on the solutions $(\mathcal{Y}_i(x))$ are polynomials in x, from (9.9) and (9.12) we get

$$\frac{\mathcal{Y}_0(x)s(t,x)}{\phi(\mathfrak{q})^r} = \sum_{i=0}^{r}\left(\prod_{j=0}^{r}\mathfrak{q}_j^{-\widehat{\lambda}_i(\widehat{\lambda}_j^\vee)}\right)T_i(x)\sum_{\tilde{j}}M_{i\tilde{j}}(\mathfrak{q})\tilde{\phi}_{\tilde{j}}(t;\mathfrak{q}). \tag{10.36}$$

The section $s(t,x)$ vanishes at the $r+1$ points $t_1(x), \ldots, t_{r+1}(x)$ for each $x \in \mathbf{C}_{\langle x\rangle}$, and hence defines the $(r+1)$-folded spectral cover of $\mathbf{C}_{\langle x\rangle}$ plane by the equation

$$R(t,x) = 0, \tag{10.37}$$

where $R(t,x)$ is the right-hand side of (10.36). The curve (10.37) coincides with the curve in [175] constructed from by lifting to M-theory the IIA brane arrangement realizing the elliptic model with $\mathfrak{m} = 0$.

10.16 Class II* theory

Recall that in (9.14) we defined an infinite set of functions $Y_i(x)$, $i \in \mathbb{Z}$. The analogue of the formula (1.1) is the matrix $g(x) \in \widehat{\mathrm{GL}}_\infty$, (cf. (9.16)):

$$g(x) = Y_0(x)^K \times \mathrm{diag}(t_i(x))_{i\in\mathbb{Z}}, \qquad t_i(x) = \check{t}_i \frac{Y_i(x)}{Y_{i-1}(x)}, \tag{10.38}$$

where $\check{t}_i$, $i \in \mathbb{Z}$ solve

$$\check{t}_{i+1} = \mathfrak{q}_{i \bmod (r+1)} \check{t}_i,$$

and are normalized as in (9.17)

$$\prod_{j=1}^{r+1} \check{t}_j = 1$$

so that for $i = 1, \dots, r+1$ the $\check{t}_i$ coincide with those in (9.17), and

$$\check{t}_{i+b(r+1)} = \check{t}_i \mathfrak{q}^b, \qquad \check{t}^{[i+b(r+1)]} = \check{t}^{[i]} \left(\mathfrak{q}^{r+1}\right)^{\frac{b(b-1)}{2}}. \tag{10.39}$$

The fundamental characters of $\widehat{\mathrm{GL}}_\infty$ evaluated on $g(x)$, $\chi_i(g(x))$ are associated with representations $\mathcal{R}_i$ of $\widehat{\mathrm{GL}}_\infty$ with the highest weight taking value (cf. (J.13)):

$$g(x)^{\check{\lambda}_i} = Y_i(x)\check{t}^{[i]} = Y_0(x)t(x)^{[i]}.$$

The characters are given by the infinite sums over all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0)$ and so are the normalized invariants

$$\begin{aligned}\mathcal{X}_i(\{Y_j(x)\}, \mathfrak{q}) &= \frac{1}{\check{t}^{[i]}}\chi_i(g(x)) = \sum_\lambda \prod_{j=1}^{\ell(\lambda)} \left(\mathfrak{q}_{i-j+1}^{[\lambda_j]} \frac{Y_{i+\lambda_j-j+1}(x)}{Y_{i+\lambda_j-j}(x)}\right) Y_{i-\ell(\lambda)}(x) \\ &= Y_i(x) + \mathfrak{q}_i \frac{Y_{i+1}(x)Y_{i-1}(x)}{Y_i(x)} + \cdots,\end{aligned} \tag{10.40}$$

where we use the notation Section 1.2.

The invariant $\mathcal{X}_i$ in (10.40) is a convergent series for $|\mathfrak{q}_i| < 1$ like the theta-series, if $t_i(x)$ is uniformly bounded. In fact, for the periodic chain of arguments, i.e., for $Y_i(x) = Y_{i+r+1}(x)$ the $\mathfrak{gl}_\infty$ character (10.40) reduces to the usual affine character of $\widehat{\mathfrak{gl}}_r$. The convergence of $\mathcal{X}_i$ in the class II* case is more subtle. We shall comment on this below. For the moment let us view the invariants as the formal power series in $\mathfrak{q}$ with coefficients in Laurent polynomials in $Y_i(x)$.

For the class II* theory the extended amplitudes $Y_i(x)$ are quasi-periodic in i, cf. (9.15), so

$$\mathcal{X}_{i+r+1}(\{Y_j(x)\}, \mathfrak{q}) = \mathcal{X}_i(\{Y_j(x - (r+1)\mathfrak{m})\}, \mathfrak{q}). \tag{10.41}$$

The cameral curve $\mathcal{C}_u$ for the class II* A_r theory is defined by the system of $r+1$ functional equations

$$\mathcal{X}_i(\{Y_j(x)\}, \mathfrak{q}) = T_i(x), \qquad i = 0, \dots, r,$$

with

$$T_i(x) = T_{i,0}x^N + T_{i,1}x^{N-1} + \sum_{\mathbf{a}=2}^{N} u_{i,\mathbf{a}} x^{N-\mathbf{a}}, \qquad T_{i,0} = \sum_\lambda \prod_{j=1}^{\ell(\lambda)} \mathfrak{q}_{i-j+1}^{[\lambda_j]}.$$

Let us now describe the II* analogue of the spectral curve, and find its realization in terms of some version of the Hitchin's system. Along the way we shall get an alternative derivation of (10.36) with the benefit of getting its Hitchin's form as well.

We form the generating function of $\mathcal{X}_i$'s and study its automorphic properties. The idea is to regularize the infinite product

$$\prod_{i\in\mathbb{Z}}(1-t_i(x)/t)/(1-\check{t}_i/t),$$

while keeping the same set of zeroes and poles. Thus, we define

$$R(t,x)=\frac{Y_0(x)}{D_0(t;\mathfrak{q})}\prod_{k=1}^{\infty}\bigl(1-t_k(x)t^{-1}\bigr)\bigl(1-tt_{1-k}(x)^{-1}\bigr), \tag{10.42}$$

where

$$D_0(t;\mathfrak{q})=\prod_{k=1}^{\infty}\bigl(1-\check{t}_kt^{-1}\bigr)\bigl(1-t\check{t}_{1-k}^{-1}\bigr)=\prod_{i=1}^{r+1}\frac{\theta\bigl(t/\check{t}_i;\mathfrak{q}\bigr)}{\phi(\mathfrak{q})}.$$

First of all, given that at large x the eigenvalues $t_k(x)$ approach $\check{t}_k$ which, in turn, behave as $\mathfrak{q}^{\frac{k}{r+1}}$, we expect (10.42) to define the converging product, at least for large enough x.

Secondly, let us check that (10.42) is ${}^i\mathcal{W}$-invariant. Let $i=0,\dots,r$, $\mathbf{a}=1,\dots,N$. While crossing the $I_{i,\mathbf{a}}$ cut the "eigenvalue" $t_i(x)$ maps to $t_{i+1}(x)$, which, in case $i\geq 1$ or $i<0$, leaves (10.42) manifestly invariant. For $i=0$ several factors in $\Delta(t,x)$ transform, altogether conspiring to make it invariant

$$\begin{aligned}&Y_0(x)\mapsto \mathfrak{q}_0Y_{-1}(x)Y_1(x)/Y_0(x)=t_1(x)/t_0(x),\\&\bigl(1-t_1(x)t^{-1}\bigr)\bigl(1-tt_0(x)^{-1}\bigr)\\&\qquad\mapsto\bigl(1-t_0(x)t^{-1}\bigr)\bigl(1-tt_1(x)^{-1}\bigr)=\frac{t_0(x)}{t_1(x)}\bigl(1-t_1(x)t^{-1}\bigr)\bigl(1-tt_0(x)^{-1}\bigr).\end{aligned}$$

Thirdly, let us introduce the analogues of the spectral determinants for all fundamental representations $\mathcal{R}_i$:

$$\begin{aligned}\Delta_i(t,x)&=\frac{Y_i(x)}{D_i(t;\mathfrak{q})}\prod_{k=i+1}^{\infty}\bigl(1-t_k(x)t^{-1}\bigr)\bigl(1-tt_{2i+1-k}(x)^{-1}\bigr),\\D_i(t;\mathfrak{q})&=\prod_{k=i+1}^{\infty}\bigl(1-\check{t}_kt^{-1}\bigr)\bigl(1-t\check{t}_{2i+1-k}^{-1}\bigr).\end{aligned}$$

Using $D_{i+1}(t;\mathfrak{q})=-t\check{t}_{i+1}^{-1}D_i(t;\mathfrak{q})$, $Y_{i+1}(x)=t_{i+1}(x)\check{t}_{i+1}^{-1}Y_i(x)$ we derive: $\Delta_i(t,x)=R(t,x)$ for all $i\in\mathbb{Z}$.

Then, the quasi-periodicity (10.39) and (10.41) implies

$$R(\mathfrak{q}t,x+\mathfrak{m})=\Delta_{r+1}(t,x)=R(t,x). \tag{10.43}$$

Given the large x asymptotics of $Y_0(x)$ and $t_i(x)$, we conclude

$$R(t,x)=x^N+\sum_{k=1}^{N}\delta_k(t)x^{N-k},$$

where $\delta_k(t)$ are the quasi-elliptic functions, which have the first-order poles at $t = \check{t}_i$, $i = 0, \dots, r$ on the elliptic curve $\mathcal{E} = \mathbb{C}^\times/\mathfrak{q}^{\mathbb{Z}}$. Indeed, the poles come from the $D_0(t;\mathfrak{q})$ denominator, while the quasi-ellipticity of $\delta_k(t)$ follows from (10.43):

$$\delta_i(\mathfrak{q}t) - \delta_i(t) = \mathfrak{m}^i + \text{polynomial in } \mathfrak{m} \text{ linear in } \delta_k(\mathfrak{q}t), \qquad k < i.$$

Now use (J.11) and (J.12) to rewrite $R(t,x)$ as

$$R(t,x) = \frac{\sum_{i\in\mathbb{Z}}(-t)^i \check{t}^{[i]} \mathcal{X}_i\left(\{Y_j(x)\},\mathfrak{q}\right)}{D_0(t;\mathfrak{q})} = \frac{1}{D_0(t;\mathfrak{q})}\sum_{i\in\mathbb{Z}}(-t)^i\check{t}^{[i]}T_i(x), \tag{10.44}$$

where we extended the definition of gauge polynomials $T_i(x)$ to $i \in \mathbb{Z}$ by quasi-periodicity implied by (10.41):

$$T_{i+r+1}(x) = T_i(x - \mathfrak{m}). \tag{10.45}$$

Armed with (10.39) and (10.45), we reduce (10.44) to a finite sum: let

$$r(t,x) = \sum_{i=0}^{r}(-t)^i\check{t}^{[i]}T_i(x),$$

then (cf. (O.5))

$$\begin{aligned} R(t,x) &= \frac{1}{D_0(t;\mathfrak{q})}\sum_{b\in\mathbb{Z}} r(t,x-b\mathfrak{m})\Big((-t)^b\mathfrak{q}^{\frac{b(b-1)}{2}}\Big)^{r+1} \\ &= \frac{1}{D_0(t;\mathfrak{q})}\big(\theta\big(-(-t)^{r+1};\mathfrak{q}^{r+1}\big) *_{\mathfrak{m}} r(t,x)\big), \end{aligned}$$

where the $*_\hbar$-product is defined by the usual Moyal formula:

$$(f *_\hbar g)(t,x) = \mathrm{e}^{\hbar\frac{\partial^2}{\partial\xi_1\partial\eta_2} - \hbar\frac{\partial^2}{\partial\xi_2\partial\eta_1}}|_{\xi=\eta=0} f(t+\eta_1, x+\xi_2)g(t+\eta_2,x+\xi_2).$$

The appearance of the $*$-product is the first hint that the class II* theory has something to do with the noncommutative geometry. We shall indeed soon see that a natural interpretation of the solution to the limit shape equations of the class II* theory involves instantons on the noncommutative four-manifold $\mathbb{R}^2 \times \mathbb{T}^2$, where the noncommutativity is "between" the $\mathbb{R}^2$ and the $\mathbb{T}^2$ components.

10.17 Hitchin system on T^2

The above solution can be represented by the affine $\mathrm{GL}(N)$ Hitchin system on $\mathcal{E}$:

$$\Phi(\mathfrak{q}t) = \Phi(t) + N\mathfrak{m}\cdot\mathbf{1}_N$$

with $r+1$ rank 1 punctures $\check{t}_j$:

$$\begin{aligned} &\Phi(t) \sim \Phi_j\frac{\mathrm{d}t}{t-\check{t}_j}, \qquad j = 1,\dots,r+1, \\ &\Phi_j = \mathbf{u}_j\otimes\mathbf{v}_j^t, \qquad \mathbf{u}_j,\mathbf{v}_j\in\mathbb{C}^N, \end{aligned}$$

whose eigenvalues are fixed in terms of masses

$$\mathbf{v}_j^t\mathbf{u}_j = \operatorname{tr}\Phi_j = Nm_j. \tag{10.46}$$

Actually, the vectors and covectors $\mathbf{v}_j$, $\mathbf{u}_j$ are defined up to the $\mathbb{C}^\times$-action

$$(\mathbf{v}_j, \mathbf{u}_j) \mapsto \big(z_j \mathbf{v}_j, z_j^{-1} \mathbf{u}_j\big), \qquad z_j \in \mathbb{C}^\times$$

and (10.46) is the corresponding moment map equation, defining the coadjoint orbit $\mathcal{O}_j$ of $\mathrm{SL}(N,\mathbb{C})$. We can shift $\Phi(t)$ by the meromorphic scalar matrix

$$\mathbf{\Phi}(t) = \Phi(t) - \sum_{j=1}^{r+1} m_j \xi\big(t/\check{t}_j\big) \frac{\mathrm{d}t}{t} \mathbf{1}_N,$$

which gives the following traceless meromorphic Higgs field (see [137]):

$$\mathbf{\Phi}(t) = \left\| p_a \delta_a^b + \sum_{j=0}^{r} u_j^b v_a^j \big(1 - \delta_a^b\big) \frac{\theta_1(t/t_j w_b/w_a)\theta_1'(1)}{\theta_1(t/t_j)\theta_1(w_b/w_a)} \right\|_{a,b=1}^{N},$$

which depends, in addition to the $\mathrm{SL}(N,\mathbb{C})$-orbits $\mathcal{O}_1, \dots, \mathcal{O}_{r+1}$ on the choice $(w_1, \dots, w_N)$ of a holomorphic $\mathrm{SL}(N,\mathbb{C})$ bundle on $\mathcal{E}$, and the dual variables $(p_1, \dots, p_N)$, subject to

$$\sum_{a=1}^{N} p_a = 0, \qquad \prod_{a=1}^{N} w_a = 1.$$

There are additional constraints

$$\sum_{j=1}^{r+1} u_j^a v_a^j = \mathfrak{m},$$

which generate the action of the residual gauge transformations in the maximal torus $\mathbf{T} = (\mathbb{C}^\times)^{N-1}$ of $\mathrm{SL}(N,\mathbb{C})$. The dimension of the corresponding phase space $\mathfrak{P}$, whose open subset $\mathfrak{P}^\circ$ is isomorphic to

$$\mathfrak{P}^\circ \approx \big(T^* \mathrm{Bun}_{\mathrm{SL}(N,\mathbb{C})}(\mathcal{E}) \times \times_{j=1}^{r+1} \mathcal{O}_j\big) /\!/ \mathbf{T}$$

is equal to

$$\dim \mathfrak{P} = 2(N-1) + (r+1)(2(N-1)) - 2(N-1) = 2(r+1)(N-1) = 2\mathbf{r},$$

which is twice the dimension of the moduli space $\mathfrak{M}$ of vacua of the class II* A_r theory with the gauge group $G_{\mathrm{g}} = \mathrm{SU}(N)^{r+1}$. The remaining $r+1$ mass parameters are encoded in the symplectic moduli of the coadjoint orbits $\mathcal{O}_j$, as expected.

The relation to our solution is in the equality of two spectral determinants

$$R(t,x) = \mathrm{Det}_N \left[\left(x - \sum_j m_j \xi(t/t_j) \right) \cdot \mathbf{1}_N - \mathbf{\Phi}(t) \right] = 0, \tag{10.47}$$

which is established by comparing the modular properties and the residues of the left and the right-hand sides.

Note the duality of the twisted periodicities of the gauge theory and Hitchin's system Lax operators

$$\Phi(\mathfrak{q}t) = w^{-1} \Phi(t) w + \mathfrak{m} \cdot \mathbf{1}_N \in \mathfrak{sl}(N,\mathbb{C}),$$
$$\mathfrak{q} \cdot g(x - \mathfrak{m}) = S^{-1} g(x) S \in \widehat{\mathrm{GL}}_\infty,$$

where S is the shift operator $S = \sum_{i \in \mathbb{Z}} E_{i,i+r+1}$, and $w = \mathrm{diag}(w_1, \dots, w_N)$. The equation (10.47) can be suggestively written as

$$\mathrm{Det}_N(x - \Phi(t)) \approx \mathrm{Det}_H(t - g(x)),$$

where H is the single-particle Hilbert space of a free fermion ψ.

10.18 Relation to many-body systems and spin chains

The parameters of the spectral curve (10.47) are holomorphic functions on $\mathfrak{P}^\circ$, which Poisson-commute, and define the integrable system. One way of enumerating the Hamiltonians of the integrable system is to mimic the construction of Hamiltonians (4.14) of the higher genus Hitchin system. For example, the quadratic Casimir is a meromorphic 2-differential on $\mathcal{E}$ with the fixed second-order poles at $t=\check{t}_j$

$$\operatorname{tr}\mathbf{\Phi}(t)^2=\sum_{j=1}^{r+1}N^2m_j^2\wp\big(t/\check{t}_j\big)\mathrm{d}t^2+\sum_{j=1}^{r+1}U_{2,1,j}\xi\big(t/\check{t}_j\big)+U_{2,0}.$$

The Hamiltonians $U_{2,0}$, $U_{2,1,j}$ are computed explicitly in [137]. They describe the motion of N particles on $\mathcal{E}$ with the coordinates $w_1,\dots,w_N$, which have additional $\mathrm{GL}(r+1,\mathbb{C})$-spin degrees of freedom. However, in view of our gauge theory analysis, it seems more natural to view this system as the $\widehat{\mathrm{GL}}_\infty$-spin chain. We conjecture that the deformation quantization of the properly compactified phase space $\mathfrak{P}$ will contain the subalgebra $\mathcal{A}_\mathfrak{m}$ of the Yangian $Y(\widehat{\mathrm{GL}}_\infty)$ algebra, which is a deformation of the Yangian of the affine $\widehat{A}_r$.

The relation of many-body systems and spin chains based on finite-dimensional symmetry groups was discussed in the context of Hecke symplectic correspondences in [114, 162]. One can also interpret the results of [50] as the quantum version of this correspondence.

10.19 Class II theories of D type

In this section $\mathfrak{g}_\mathrm{q}=\widehat{D}_r$.

The fundamental weights of $\widehat{D}_r$ are λ_0, $\widehat{\lambda}_i=a_i^\vee\lambda_0+\lambda_i$, $i=1,\dots,r$ where λ_i are fundamental weights of D_r, and Dynkin labels are $(a_0,\dots,a_r)=(1,1,2,\dots,2,1,1)$ (see Appendix D.2). Correspondingly,

$$\begin{aligned}
&t_1(x)=\check{t}_1\mathcal{Y}_1(x)/\mathcal{Y}_0(x),\\
&t_2(x)=\check{t}_2\mathcal{Y}_2(x)/(\mathcal{Y}_1(x)\mathcal{Y}_0(x)),\\
&t_i(x)=\check{t}_i\mathcal{Y}_i(x)/\mathcal{Y}_{i-1}(x),\qquad i=3,\dots,r-2,\\
&t_{r-1}(x)=\check{t}_{r-1}\mathcal{Y}_{r-1}(x)\mathcal{Y}_r(x)/\mathcal{Y}_{r-2}(x),\\
&t_r(x)=\check{t}_r\mathcal{Y}_r(x)/\mathcal{Y}_{r-1}(x)
\end{aligned}$$

with

$$\begin{aligned}
&\check{t}_i=(\mathfrak{q}_i\mathfrak{q}_{i+1}\dots\mathfrak{q}_{r-2})^{-1}(\mathfrak{q}_{r-1}\mathfrak{q}_r)^{-\frac{1}{2}},\qquad i=1,\dots,r-2,\\
&\check{t}_{r-1}=(\mathfrak{q}_{r-1}\mathfrak{q}_r)^{-\frac{1}{2}},\qquad \check{t}_r=(\mathfrak{q}_{r-1}/\mathfrak{q}_r)^{\frac{1}{2}}.
\end{aligned}\tag{10.48}$$

There are 4 irreducible $\widehat{D}_r$ highest weight modules $\widehat{R}_0$, $\widehat{R}_1$, $\widehat{R}_{r-1}$, $\widehat{R}_r$ at level 1, and $r-3$ irreducible $\widehat{D}_r$ highest weight modules $\widehat{R}_2,\dots,\widehat{R}_{r-2}$ at level 2. In this section, to shorten formulae, we are using not the characters of $\widehat{R}_i$ themselves but the closely related affine Weyl invariant functions ${}_2\tilde{\mathcal{X}}_j^{\widehat{D}}$ at level 2 and ${}_1\tilde{\mathcal{X}}_j^{\widehat{D}}$ at level 1 expressed terms of theta-functions explicitly as given in (J.10). Such functions ${}_1\tilde{\mathcal{X}}_j^{\widehat{D}}$ and ${}_2\tilde{\mathcal{X}}_j^{\widehat{D}}$ differ from the actual characters by a simple power of Euler function $\phi(\mathfrak{q})$ and some $\mathfrak{q}$-dependent constant, also ${}_1\tilde{\mathcal{X}}_0^{\widehat{D}}$, ${}_1\tilde{\mathcal{X}}_1^{\widehat{D}}$ appear as a linear combination of $\widehat{R}_0$ and $\widehat{R}_1$ characters, while ${}_1\tilde{\mathcal{X}}_r^{\widehat{D}}$, ${}_1\tilde{\mathcal{X}}_{r-1}^{\widehat{D}}$ appear as linear combination of $\widehat{R}_{r-1}$ and $\widehat{R}_r$ characters (see (J.10))

$${}_1\tilde{\mathcal{X}}_j^{\widehat{D}}(\mathcal{Y}_0(x),\mathfrak{t}(x);\mathfrak{q})=T_j(x),\qquad {}_2\tilde{\mathcal{X}}_j^{\widehat{D}}(\mathcal{Y}_0(x),\mathfrak{t}(x);\mathfrak{q})=T_j(x),\tag{10.49}$$

where polynomials $T_j(x)$ are of degree N for $j=0,1,r-1,r$ and of degree $2N$ for $j=2,\dots,r-2$. so that ${}_1\tilde{\mathcal{X}}^{\widehat{D}}$ is of degree 1 in $\mathcal{Y}_0$ for $j=0,1,r-1,r$ and ${}_2\tilde{\mathcal{X}}^{\widehat{D}}$ is of degree 2 in $\mathcal{Y}_0$ for $j=2,\dots,r-2$. Also, in this section the highest coefficient of the polynomial $T_j(x)$ is normalized differently then in (9.9); one can find it as theta-series evaluating (J.10) on $\check{t}_i$.

Using the standard embedding $\mathfrak{so}(2r)\subset\mathfrak{sl}(2r)$, we construct the algebraic equation of the spectral curve of the $\widehat{D}_r$ theory as the specialization of the spectral curve for $\widehat{A}_{2r-1}$ theory. Indeed, a vector bundle V associated to the vector representation of $\mathrm{SO}(2r)$ splits as the sum of r pairs of line bundles

$$L_{t_i}\oplus L_{t_i^{-1}}$$

with the degree zero line bundle L_t being

$$L_t=\mathcal{O}(p_0)^{-1}\mathcal{O}(t)$$

and $p_0\in\mathcal{E}$ is our friend $t=1$. Then we proceed as in (O.5), (O.7), (10.35) by considering a meromorphic section of the determinant bundle $\det V\approx\mathcal{O}_{\mathcal{E}}$

$$s(t,x)=\prod_{i=1}^{r}\frac{\theta(t/t_i(x);\mathfrak{q})}{\theta(t;\mathfrak{q})}\frac{\theta\big(t/t_i(x)^{-1};\mathfrak{q}\big)}{\theta(t;\mathfrak{q})}. \tag{10.50}$$

From Section O.5, we find

$$\mathcal{Y}_0^2 s(t,x)=\sum_{i=0}^{r}\Xi_i(\mathcal{Y}_0;\mathfrak{t}(x);\mathfrak{q})M_{ij}(\mathfrak{q})X^j(t;\mathfrak{q}), \tag{10.51}$$

where $X^j(t,\mathfrak{q})$ are powers of Weierstraß monomials forming a basis in the space $H^0_{\mathrm{even}}(\mathcal{E},\mathcal{O}(2rp_0))$ of meromorphic functions on elliptic curve symmetric under the reflection $t\to t^{-1}$ and with a pole of order no greater then $2r$ at the origin, and $M_{ij}(\mathfrak{q})$ is a certain modular matrix.

The linear relations (J.9) allow to express Ξ_i in terms of

$$\begin{aligned}
&\tilde{\Xi}_i={}_2\tilde{\mathcal{X}}_i^{\widehat{D}}, \qquad i=2,\dots,r-2,\\
&\tilde{\Xi}_i=\big({}_1\tilde{\mathcal{X}}_i^{\widehat{D}}\big)^2, \qquad i=0,1,r-1,r,
\end{aligned}$$

as

$$\Xi_i=\sum_{\tilde{i}=0}^{r}\tilde{\Xi}_{\tilde{i}}\tilde{M}_{\tilde{i}i}(\mathfrak{q}),$$

where $\tilde{M}_{i\tilde{i}}(\mathfrak{q})$ is a certain modular transformation matrix. Using the character equations (10.49), the spectral curve (10.51) turns into

$$\mathcal{Y}_0^2 s(t,x)=\sum_{\tilde{i},j}\tilde{T}_{\tilde{i}}(x)\tilde{\tilde{M}}_{\tilde{i}j}(\mathfrak{q})X^j(t,\mathfrak{q}), \tag{10.52}$$

where $\tilde{\tilde{M}}_{\tilde{i}j}(\mathfrak{q})=\tilde{M}_{\tilde{i}i}(\mathfrak{q})M_{ij}(\mathfrak{q})$ and

$$\begin{aligned}
&\tilde{T}_{\tilde{i}}(x)=T_{\tilde{i}}(x), \qquad \tilde{i}=2,\dots,r-1,\\
&\tilde{T}_{\tilde{i}}(x)=(T_{\tilde{i}}(x))^2, \qquad \tilde{i}=0,1,r-1,r,
\end{aligned}$$

The spectral curve of the $\widehat{D}_r$ theory is the algebraic equation $R(t,x)=0$ where $R(t,x)$ is the right-hand side of (10.52) combined with the Weierstraß cubic equation (O.3). The $\widehat{D}_r$ curve

is the specialization of the $\widehat{A}_{2r-1}$ curve in two ways. First, there are no odd in Y monomials in (10.52), and, second, the polynomial coefficients $\tilde{T}_{\tilde{i}}(x)$ of degree $2N$ in x satisfy factorization condition: they are full squares for $\tilde{i}=0,1,r-1,r$.

To interpret the curve in Hitchin–Gaudin formalism, we will rewrite it in a slightly different form. First, notice that[4]

$$\prod_{i=1}^{r}\frac{\theta_1\big(t/\check{t}_i;\mathfrak{q}\big)}{\theta_1(t;\mathfrak{q})}\frac{\theta_1\big(t/\check{t}_i^{-1};\mathfrak{q}\big)}{\theta_1(t;\mathfrak{q})}=\prod_{i=1}^{r}\theta_1\big(\check{t}_i;\mathfrak{q}\big)\theta_1\big(\check{t}_i^{-1};\mathfrak{q}\big)\big(X-\check{X}_i\big). \tag{10.53}$$

We used here the notations (O.2) and (O.4) for the Weierstraß functions and

$$\check{X}_i=X\big(\check{t}_i;\mathfrak{q}\big),\qquad \check{Y}_i^2=4\prod_{\alpha=1}^{3}\big(\check{X}_i-e_\alpha\big).$$

Then, if we divide (10.50) by (10.53), we find[5]

$$\begin{aligned}&\mathcal{Y}_0^2(x)\prod_{i=1}^{r}\theta_1\big(\check{t}_i;\mathfrak{q}\big)\theta_1\big(\check{t}_i^{-1};\mathfrak{q}\big)\prod_{i=1}^{r}\frac{\theta_1\big(t/t_i(x);\mathfrak{q}\big)}{\theta_1\big(t/\check{t}_i;\mathfrak{q}\big)}\frac{\theta_1\big(t/t_i^{-1}(x);\mathfrak{q}\big)}{\theta_1\big(t/\check{t}_i^{-1};\mathfrak{q}\big)}=R(x,X(t,\mathfrak{q})),\\ &R(x,X):=\frac{\sum_{\tilde{i},j=0}^{r}\tilde{T}_{\tilde{i}}(x)\tilde{\tilde{M}}_{\tilde{i}j}(\mathfrak{q})X^j}{\prod_{i=1}^{r}\big(X-\check{X}_i\big)}.\end{aligned} \tag{10.54}$$

Now, at the order two points on $\mathcal{E}$,[6] the value of the section $R(x,X)$ can be expressed in terms of the weight 1 invariants ${}_1\tilde{\mathfrak{X}}^{\widehat{D}}_{r-1}$, ${}_1\tilde{\mathfrak{X}}^{\widehat{D}}_{r}$, ${}_1\tilde{\mathfrak{X}}^{\widehat{D}}_{0}$, ${}_1\tilde{\mathfrak{X}}^{\widehat{D}}_{1}$ (compare with (J.10) and (J.5), (J.6)), and it factorizes as

$$\begin{aligned}&R(x,X)|_{X\to\infty}=(T_{r-1}(x))^2,\\ &R(x,X)|_{X\to e_1}=c_2(\tilde{\mathfrak{q}})(T_r(x))^2,\\ &R(x,X)|_{X\to e_2}=c_3(\tilde{\mathfrak{q}})(T_0(x))^2,\\ &R(x,X)|_{X\to e_3}=c_4(\tilde{\mathfrak{q}})(T_1(x))^2,\end{aligned} \tag{10.55}$$

where

$$c_k(\tilde{\mathfrak{q}})=\prod_{i=1}^{r}\frac{\theta_1\big(\check{t}_i;\mathfrak{q}\big)\theta_1\big(\check{t}_i^{-1};\mathfrak{q}\big)}{\theta_k\big(\check{t}_i;\mathfrak{q}\big)\theta_k\big(\check{t}_i^{-1};\mathfrak{q}\big)},\qquad k=2,3,4. \tag{10.56}$$

The Seiberg–Witten differential is given by

$$\lambda=x\frac{\mathrm{d}X}{Y}.$$

It is defined on the two fold cover C_u of the curve $R(x,X)=0$, which is a curve in the product $\mathbb{CP}^2_{(X:Y:Z)}\times\mathbf{C}_{\langle x\rangle}$, given by the equations

$$Y^2Z=4(X-e_1Z)(X-e_2Z)(X-e_3Z),\qquad F(x,Z,X)=Z^rR(x,X/Z)=0. \tag{10.57}$$

[4]Indeed, the left-hand or right-hand sides is the meromorphic elliptic function with $2r$ zeroes at points X_i, Y and X_i, $-Y$ and the pole of order $2r$ at $t=1$, or $X=\infty$. Such function is unique up to a normalization which is fixed by the asymptotics at $t=1$.

[5]And use $\theta_1(t,\mathfrak{q})$ in lieu of $\theta(t,\mathfrak{q})$ as the basic function, so that strictly speaking there is slightly different transformation matrix $\tilde{\tilde{M}}_{\tilde{i}j}$ compared to (10.50) and (O.7).

[6]For example, the points $\big(1,-1,-\mathfrak{q}^{-1/2},\mathfrak{q}^{1/2}\big)$ in the t-parametrization, where vanish the respective theta functions $\theta_1(t;\mathfrak{q})$, $\theta_2(t;\mathfrak{q})$, $\theta_3(t;\mathfrak{q})$, $\theta_4(t;\mathfrak{q})$, or, equivalently, at the four branch points in the X plane: (∞,e_1,e_2,e_3).

The curve C_u can be interpreted at the spectral curve of GL($2N$) Hitchin–Gaudin system on the orbifold $\mathcal{E}/\mathbb{Z}_2$, such that at the fixed point $X=\infty, e_1, e_2, e_3$ the GL($2N$) system reduces to the Sp($2N$) system. For more details on the Hitchin system, Nahm transform and the brane construction of the spectral curve for the $\widehat{D}_r$ quiver see [95, 96]. Our main result is the rigorous derivation of the spectral curve and its periods from the gauge theory considerations.

10.20 Deforming the $N_f = 4$ SU(2) theory

The $\widehat{D}_4$ theory can be interpreted as the theory obtained from gauging the flavor group of the D_4 theory with $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = (N, 2N, N, N)$ theory, and with $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) = (0, N, 0, 0)$ matter multiplets. In the limit $\mathfrak{q}_0 \to 0$ the elliptic curve $\mathcal{E}$ degenerates to the cylinder $\mathbb{C}^\times_{\langle t\rangle}$, while Seiberg–Witten curve (10.57) degenerates to the Seiberg–Witten curve of the D_4 theory (10.22).

Let us consider the case $N=1$. Let us parametrize the polynomials T_0, T_1, T_3, T_4 as

$$T_i(x) = T_{i,0}(\tilde{\mathfrak{q}})(x - m_i), \qquad i = 0, 1, 3, 4, \tag{10.58}$$

and

$$T_2(x) = T_{2,0}(\tilde{\mathfrak{q}})\big(x^2 - m_2 x + u\big),$$

where parameters q_i, m_i and u are related to the microscopic couplings $\mathfrak{q}_i$ and the $U(1)^4 \times \mathrm{SU}(2)$ Coulomb moduli

$$\begin{aligned}
&T_{3,0}(\tilde{\mathfrak{q}}) = \prod_{i=1}^4 \theta_1(\check{t}_i), \qquad T_{4,0}(\tilde{\mathfrak{q}}) = \prod_{i=1}^4 \theta_2(\check{t}_i),\\
&T_{0,0}(\tilde{\mathfrak{q}}) = \prod_{i=1}^4 \theta_3(\check{t}_i), \qquad T_{1,0}(\tilde{\mathfrak{q}}) = \prod_{i=1}^4 \theta_4(\check{t}_i),\\
&T_{2,0}(\tilde{\mathfrak{q}}) = \Xi_2\big(1, \check{\mathfrak{t}}, \mathfrak{q}\big),
\end{aligned} \tag{10.59}$$

where $\check{t}_i$ are defined in (10.48). Then the spectral curve of the $\widehat{D}_4$ theory (10.54) and (10.55) has the generic form

$$R(x, X) = T_3^2(x) + \sum_{i=1}^4 \frac{b_i(x)}{X - \check{X}_i}, \tag{10.60}$$

where $b_i(x)$ are some polynomials of degree 2 that we want to relate to the coupling constants and Coulomb parameters. The first thing to notice is that $R(x, X)$ in (10.60) obtained from (10.54) does not have poles at $X = \check{X}_i$ at $x \to \infty$ in the leading order x^2. Therefore, the polynomials $b_i(x)$ are actually degree 1 polynomials containing 8 coefficients. There are 6 linear equations on these coefficients coming from 3 factorization equations (10.55) viewed as coefficients at x^1 and x^0 (and notice that the equations at x^2 are identically satisfied because of (10.56) and (10.59))

$$\begin{aligned}
&\sum_{i=1}^4 \frac{b_i(x)}{e_1 - \check{X}_i} = c_2 T_4^2(x) - T_3^2(x),\\
&\sum_{i=1}^4 \frac{b_i(x)}{e_2 - \check{X}_i} = c_3 T_0^2(x) - T_3^2(x),\\
&\sum_{i=1}^4 \frac{b_i(x)}{e_3 - \check{X}_i} = c_4 T_1^2(x) - T_3^2(x).
\end{aligned} \tag{10.61}$$

The above three equations determine four linear functions $b_i(x)$ up to a single linear function, which depends on two parameters $\tilde{m}_2$, $\tilde{u}$:

$$\tilde{b}_j(x) = (-1)^j(-\tilde{m}_2 x + \tilde{u})\,\mathrm{Det}\left\|\frac{1}{e_a - \check{X}_b}\right\|_{a=1,\dots,3}^{b=1,\dots,4,b\neq j}. \tag{10.62}$$

From (10.54), it is clear that $\tilde{m}_2$, $\tilde{u}$ are proportional to m_2, u_2. To summarize, we can describe the spectral curve (10.60) of $\widehat{D}_4$ theory by the coupling constants $\mathfrak{q}_i$, $i=0,\dots,4$, which define the elliptic curve $\mathcal{E}(\mathfrak{q})$ with modulus $\mathfrak{q} = \mathfrak{q}_0\mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4$ and positions of 4 punctures $\check{X}_i$ in the $\mathbb{C}_{\langle X\rangle}$ plane for Weierstraß cubic using (10.48), the 4 parameters m_i, $i=0,1,3,4$ entering into relations (10.61) through (10.58) and 2 parameters $\tilde{m}_2$, $\tilde{u}$ in (10.62).

Now consider the limit $\mathfrak{q}_0 \to 0$ which turns the $\widehat{D}_4$ class II quiver theory to the D_4 class I quiver theory. In this limit the Weierstraß cubic degenerates: $e_1 = -2e_3$, $e_2 = e_3 = 1/12$,

$$Y^2 = 4\,(X-e_3)^2\,(X+2e_3)^2$$

with

$$X = \frac{t}{(1-t)^2} + \frac{1}{12}, \qquad Y = \frac{t(1+t)}{(1-t)^3}.$$

The Seiberg–Witten differential $x\frac{\mathrm{d}X}{Y}$ becomes $x\frac{\mathrm{d}t}{t}$. The elliptic curve $\mathcal{E}$ degenerates to the rational curve which is the double cover $t \mapsto X$ of the complex projective line $\mathbb{CP}^1_X$. To make contact with the Seiberg–Witten curve of the D_4 quiver theory it is convenient to work in the coordinate which is related to the coordinate X by rational transformation

$$\eta = 2 + \frac{1}{X-e_3} = t + t^{-1}.$$

The function $\eta(X)$ is degree two meromorphic function on $\mathcal{E}$ with values at the four $\mathbb{Z}_2$ invariant points given by

$$\eta(e_2) = \eta(e_3) = \infty, \qquad \eta(e_1) = -2, \qquad \eta(\infty) = 2.$$

Rewriting (10.54) in terms of $\eta(x)$, we find the equation of spectral curve $\tilde{\mathcal{R}}^{D_4}(\eta,x)=0$ for

$$\tilde{\mathcal{R}}^{D_4}(\eta,x) = \sum_{i=0}^{4}\eta^i p_i(x),$$

where $p_i(x)$ are some polynomials of degree 2 in x. Moreover, the factorization conditions (10.55) translates to the statement that $\tilde{\mathcal{R}}^{D_4}(\eta,x)$ is full square at $\eta=\infty$ and at $\eta=\pm 2$ in the polynomial ring of x. Notice that this is precisely the factorization conditions (10.26) and (10.27) of the curve (10.25) for the D_4 quiver. (The variables t and η in the equations (10.25) and (10.24) correspond to t and η of this section multiplied by a factor $\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3^{\frac{1}{2}}\mathfrak{q}_4^{\frac{1}{2}}$.)

Given the above discussion and Section 10.7, let us summarize the freezing hierarchy $\widehat{D}_4 \to D_4 \to A_3 \to A_2 \to A_1$. For $\widehat{D}_4$ theory, we start with elliptic curve $\mathcal{E}(\mathfrak{q})$ with $\mathbb{Z}_2$ reflection symmetry $t \to t^{-1}$ (or $Y \to -Y$) and 8 $\mathbb{Z}_2$-symmetrically located punctures in 4 pairs $\left(\check{t}_i, \check{t}_i^{-1}\right)$. As we freeze $\mathfrak{q}_0 \to 0$, the elliptic curve $\mathcal{E}(\mathfrak{q})$ degenerates to a $\mathbb{Z}_2$-symmetrical cylinder $\mathbb{C}_t^\times$ with 4-fold pairs $(\check{t}_i, \check{t}_i^{-1})$ of punctures. The cylinder $\mathbb{C}_t^\times$ double covers its $\mathbb{Z}_2$-quotient $\mathbb{CP}^\times_\eta$. This is the situation of D_4 quiver theory (10.25). As we freeze $\mathfrak{q}_4 \to 0$ the second sheet of the double cover $\mathbb{C}_t^\times \to \mathbb{C}_\eta^\times$ is removed to infinity and we are left with 4 punctures of A_3 quiver at $\left(\mathfrak{q}_1^{-1}, 1, \mathfrak{q}_2, \mathfrak{q}_2\mathfrak{q}_3\right)$.[7] Notice, that as discussed after (10.28) the $\mathrm{SL}(2,\mathbb{C})$ residues of the Higgs

[7] Keeping in mind the ultimate configuration of the A_1 quiver dynamical at node "2" we have rescaled the position of punctures by a factor of $\mathfrak{q}_1$.

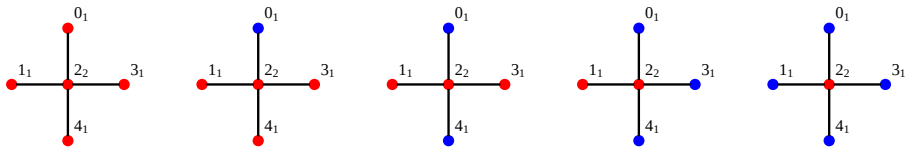


Figure 10.3. The freezing $\widehat{D}_4 \to D_4 \to A_3 \to A_2 \to A_1$. The live nodes are denoted by red, the frozen nodes are denoted by blue. The nodes are labeled as $i_{\mathbf{v}_i}$.

field vanish at the punctures in 0 and ∞. As we freeze $\mathfrak{q}_3 \to 0$ the puncture at $\mathfrak{q}_2\mathfrak{q}_3$ (with non-trivial $\mathrm{SL}(2,\mathbb{C})$ residue of Higgs field) merges with the puncture 0 and we are in the situation of the A_2 quiver with $\mathrm{SL}(2,\mathbb{C})$ punctures at $\left(\mathfrak{q}_1^{-1}, 1, \mathfrak{q}_2, 0\right)$ and $\mathrm{GL}(1,\mathbb{C})$ puncture at ∞. Finally as we freeze $\mathfrak{q}_1 \to 0$ the puncture at $\mathfrak{q}_1^{-1}$ (with non-trivial residue of the $\mathrm{SL}(2,\mathbb{C})$ Higgs field) is merged with the puncture at ∞ and we are left with $\mathbb{CP}^1$ with $\mathrm{SL}(2,\mathbb{C})$ punctures at $(\infty, 1, \mathfrak{q}_2, 0)$ for the A_1 quiver theory defined at the dynamical node "2". See Figure 10.3.

10.21 Class II theories of E type

The main technical tool is the natural isomorphism between the moduli space of the E_k-bundles on elliptic curve $\mathcal{E}$ and the moduli space of del Pezzo surfaces $\mathcal{S}_k$, which are obtained by blowing up k points in $\mathbb{CP}^2$, and have the fixed elliptic curve $\mathcal{E}$ as the anticanonical divisor. The spectral curve is found using the "cylinder map" [94], and see [25, 36, 37] for applications.

Another way of encoding the geometry of the moduli space the E_k-bundles is in the unfolding of the parabolic unimodular singularities [3] $\widehat{T}_{a,b,c}$ with

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1,$$

which are

$$\begin{aligned}
&\tilde{E}_6 = P_8 = \widehat{T}_{3,3,3}\colon \ x^3 + y^3 + z^3 + mxyz, && m^3 + 27 \neq 0,\\
&\tilde{E}_7 = X_9 = \widehat{T}_{2,4,4}\colon \ x^4 + y^4 + z^2 + mxyz, && m^4 - 64 \neq 0,\\
&\tilde{E}_8 = J_{10} = \widehat{T}_{2,3,6}\colon \ x^6 + y^3 + z^2 + mxyz, && 4m^6 - 432 \neq 0.
\end{aligned}$$

We shall not pursue this direction in this work.

Remark 10.6. Another important question left for future work is the connection between our description of the special geometry via the periods of $\mathrm{d}\mathbb{S}$ and the periods of non-compact Calabi–Yau threefolds of [101].

10.22 Del Pezzo and E_6 bundles

The Del Pezzo surface $\mathcal{S}_6 \subset \mathbb{WP}^{1,1,1,1} = \mathbb{CP}^3$ is a zero locus of a homogeneous degree 3 polynomial

$$\Gamma_3(X_0, X_1, X_2, X_3) = \sum_{i=0}^{3} X_0^{3-i} \mathcal{G}_i(X_1, X_2, X_3),$$

where $\mathcal{G}_i$ is the degree i homogeneous polynomial in X_1, X_2, X_3. In particular,

$$\mathcal{G}_3(X_1, X_2, X_3) = -X_1 X_3^2 + 4X_2^3 - g_2 X_1^2 X_2 - g_3 X_1^3$$

defines the elliptic curve $\mathcal{E}$, which determines the gauge coupling $\mathfrak{q} = \exp(2\pi \mathrm{i}\tau)$, cf. (O.3),

$$\tau = \frac{\oint_B \mathrm{d}X/Y}{\oint_A \mathrm{d}X/Y}, \qquad X = X_2/X_1, \qquad Y = X_3/X_1.$$

The rest of the coefficient functions $\mathcal{G}_{0,1,2}$ is parametrized as follows:

$$\mathcal{G}_2(X_1,X_2,X_3) = p_0X_1^2 + p_1X_1X_2 + p_6X_2X_3,$$
$$\mathcal{G}_1(X_1,X_2,X_3) = p_2X_1 + p_3X_2 + p_5X_3,$$
$$\mathcal{G}_0(X_1,X_2,X_3) = p_4.$$

The isomorphism classes of $\mathcal{S}_6$ surfaces containing the fixed elliptic curve $\mathcal{E}$ are in one-to-one correspondence with the points

$$[p] = (p_0 : p_1 : p_6 : p_2 : p_3 : p_5 : p_4) \in \mathcal{M}$$

in the weighted projective space $\mathcal{M} = \mathbb{WP}^{1,1,1,2,2,2,3}$, which is also isomorphic, by E. Loojienga's theorem [118], to the moduli space $\mathrm{Bun}^{ss}_{E_6(\mathbb{C})}(\mathcal{E})$ of holomorphic semi-stable principal E_6-bundles on $\mathcal{E}$. We label the projective coordinates p_i in such a way that the projective weight of p_i equals Dynkin mark a_i in our conventions Appendix A. The correspondence between the E_6-bundles on $\mathcal{E}$ and the del Pezzo surfaces $\mathcal{S}_6$ is geometric: there are precisely 27 degree 1 rational curves ("the (-1)-lines") C_a on $\mathcal{S}_6$, $a = 1,\dots,27$, which are the divisors of the line bundles $\mathcal{L}_a$ on $\mathcal{S}_6$. The direct sum

$$\mathcal{U} = \bigoplus_{a=1}^{27} \mathcal{L}_a$$

has no infinitesimal deformations as a bundle on $\mathcal{S}_6$. The mapping class group of $\mathcal{S}_6$ acts on the (-1)-lines by the E_6 Weyl transformations. As a result, the bundle $\mathcal{U}$ is a vector bundle associated to a canonical principal $E_6(\mathbb{C})$-bundle $\mathcal{P}_{\mathcal{S}_6}$ over $\mathcal{S}_6$ with the help of a **27** representation:

$$\mathcal{U} = \mathcal{P}_{\mathcal{S}_6} \times_{E_6(\mathbb{C})} \mathbf{27}.$$

The restriction of $\mathcal{P}_{\mathcal{S}_6}|_E$ is the holomorphic principal $E_6(\mathbb{C})$ bundle over E which corresponds to the point $[s]$ in Loojienga's theorem. Again, the associated rank 27 vector bundle $\mathcal{U}|_E$ splits

$$\mathcal{U}|_{\mathcal{E}} = \bigoplus_{a=1}^{27} \mathcal{L}_a.$$

The line subbundles $\mathcal{L}_a$ can be expressed as

$$\mathcal{L}_a = \bigotimes_{i=1}^{6} \mathbb{L}_i^{w_{a,i}},$$

where $w_{a,i}$, $i = 1,\dots,6$, $a = 1,\dots,27$ are the components of the weight vector. The line bundles $\mathbb{L}_i$, $i = 1,\dots,6$ are defined up to the action of the E_6 Weyl group. Let us now compute the $\mathcal{L}_a$'s. The rational curve of degree one in $\mathcal{S}_6$ is a rational curve of degree one in $\mathbb{CP}^3$ which is contained in $\mathcal{S}_6$. A parametrized rational curve of degree one in $\mathbb{CP}^3$ is a collection of 4 linear functions: $\zeta \mapsto \mathbf{X}(\zeta)$,

$$\mathbf{X}(\zeta) = (X_0 + \zeta v_0, X_1 + \zeta v_1, X_2 + \zeta v_2, X_3 + \zeta v_3).$$

The two quadruples

$$\mathbf{X}(\zeta) \qquad \text{and} \qquad (c\zeta + d)\mathbf{X}\left(\frac{a\zeta + b}{c\zeta + d}\right)$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

define identical curves in $\mathbb{CP}^3$. We can fix the $\mathrm{GL}_2(\mathbb{C})$ gauge by choosing the parameter ζ so that

$$\mathbf{X}(\zeta) = (\zeta, 1, X + \zeta v_X, Y + \zeta v_Y).$$

The requirement that the curve lands in $\mathcal{S}_6 \subset \mathbb{CP}^3$ reads as

$$\Gamma_3\left(\zeta, 1, X + \zeta v_X, Y + \zeta v_Y\right) = \sum_{i=0}^{3} \zeta^i \Xi_i(X, Y; v_X, v_Y) \equiv 0,$$

which is a system of 4 equations

$$\Xi_i(X, Y; v_X, v_Y) = 0, \qquad i = 0, \dots, 3$$

on 4 unknowns X, Y, v_X, v_Y:

$$\begin{aligned}
\Xi_0 &= -Y^2 + 4X^3 - g_2 X - g_3, \\
\Xi_1 &= -g_2 v_X + p_6 XY + p_1 X + p_0 + 12X^2 v_X - 2Y v_Y, \\
\Xi_2 &= p_6 Y v_X + p_1 v_X + p_6 X v_Y + p_3 X + p_5 Y + p_2 + 12X v_X^2 - v_Y^2, \\
\Xi_3 &= p_6 v_X v_Y + p_3 v_X + p_5 v_Y + p_4 + 4v_X^3.
\end{aligned}$$

The equation $\Xi_0 = 0$ in the above system is the equation of the elliptic curve $\mathcal{E}$. To find the equation of the spectral cover associated with the vector bundle $\mathcal{U}|_{\mathcal{E}}$ in the **27** representation we can express v_Y from the equation $\Xi_1 = 0$, then plug it into the remaining equations $\Xi_2 = 0$ and $\Xi_3 = 0$, compute the resultant of these two polynomials with respect to the variable v_X, reduce modulo the equation $\Xi_0 = 0$ defining the elliptic curve $\mathcal{E}$, arriving at

$$C^{E_6}(X, Y; g_2, g_3, p_0, \dots, p_6) = -4Y^4 \mathrm{res}_{v_X}(\Xi_2|_{v_Y:\Xi_1=0}, \Xi_3|_{v_Y:\Xi_1=0}) \mod \Xi_0. \tag{10.63}$$

The resultant $C^{E_6}(X, Y; g_2, g_3, p_0, \dots, p_6)$ is a polynomial in X, Y with polynomial coefficients in $g_2, g_3, p_0, \dots, p_6$ of the form

$$\begin{aligned}
C^{E_6}(X, Y; g_2, g_3, p_0, \dots, p_6) = {} & \left(p_0^6 + \cdots\right) + \left(6p_0^5 p_1 + \cdots\right) X + \cdots + \left(-256 p_3^3 + \cdots\right) X^{12} \\
& + \left(12 g_3 p_0^4 p_5 + \cdots\right) Y + \left(32 g_3 p_0^4 p_5 + \cdots\right) XY + \cdots \\
& + \left(-256 p_5^3 + \cdots\right) X^{12} Y.
\end{aligned}$$

(A short `Mathematica` version of this formula is given in Appendix P.1.1.) Now let us imagine having a family $\mathcal{U}$ of the E_6-bundles on E.

In our solution the vacuum u of the gauge theory is identified with the degree N quasimap:

$$p\colon\ \mathbb{CP}^1_{\langle x \rangle} \to \mathrm{Bun}_{E_6(\mathbb{C})}(\mathcal{E}) \simeq \mathbb{WP}^{1,1,1,2,2,2,3}$$

given by the polynomials $p_i(x)$ of degree Na_i

$$p_i = p_i(x), \qquad i = 0, \dots, 6.$$

Together with the equation of the Weierstraß cubic $\Xi_0(X, Y, g_2, g_3) = 0$, the equation

$$C^{E_6}(X, Y; g_2, g_3, p_0(x), \dots, p_6(x)) = 0$$

defines the Seiberg–Witten curve of the affine E_6 theory as an algebraic curve. Given that the degree of X on $\mathcal{E}$ is 2 and the degree Y on $\mathcal{E}$ is 3 the polynomial C^{E_6} is of degree 27, i.e., the equation $C^{E_6}=0$ defines 27 points on the elliptic curve $\mathcal{E}$.

The top degree coefficients of the polynomials $p_i(x)$ are determined explicitly in terms of the coupling constants $\mathfrak{q}_i$. Indeed, the E_6 characters, or more conveniently in the present case, the E_6 theta-functions, $c_i(\mathcal{Y}_0,\mathbf{t},\mathfrak{q})$ as set in (I.3), define a set of projective coordinates on $\mathbb{WP}^{1,1,1,2,2,2,3}$ (which differs slightly from the set $(\widehat{\chi}_i)_{i=0}^6$):

$$(c_0:c_1:c_6:c_2:c_3:c_5:c_4).$$

In these coordinates the solution of the theory has the canonical form (9.12)

$$c_i(\mathcal{Y}_0,\mathbf{t},\mathfrak{q})=\left(\prod_{j=0}^{r}\mathfrak{q}_j^{-\lambda_i(\lambda_j^\vee)}\right)T_i(x).$$

The "del Pezzo projective coordinates" $(p_i)_{i=0}^6$ are related to the theta-function coordinates (c_i) on $\mathrm{Bun}_{E_6(\mathbb{C})}^{ss}(\mathcal{E})$ by a polynomial map of the form

$$p_i=\sum_{j_1\le j_2\le j_3\cdots}M_{i,\{j_1,j_2,j_3,\dots\}}(\mathfrak{q})c_{j_1}c_{j_2}c_{j_3}\cdots,$$

where $M_{i,\{j_1,j_2,\dots\}}(\mathfrak{q})$ is certain modular transformation matrix. This matrix can be explicitly computed by comparing the spectral curve (10.63) and the $\widehat{A}_{26}$ spectral curve (10.36) specialized to the the embedding $\widehat{E}_6\subset\widehat{A}_{26}$ by fundamental representation. The coefficients $M_{i,\{j_1,j_2,j_3,\dots\}}$ are modular forms for modular group $\Gamma(6)$ with a certain modular weights that can computed by observation that the weights of variables (X_0,X_1,X_2,X_3) under the modular transformation $\tau\to-\tau^{-1}$ on $\mathcal{E}(\mathfrak{q})$ for $\mathfrak{q}=\exp(2\pi\mathrm{i}\tau)$ are

$$\begin{pmatrix}X_0&X_1&X_2&X_3\\6&1&2&3\end{pmatrix}.$$

This implies that the modular weights of p_i are

$$\begin{pmatrix}p_0&p_1&p_6&p_2&p_3&p_5&p_4\\0&-2&-5&-6&-8&-9&-12\end{pmatrix}.$$

The (c_i) have modular weight 3 because they are rank 6 lattice theta-functions. From this assignment of weights one finds the modular weights of all coefficients $M_{i,\{j_1,\dots,j_k\}}(\mathfrak{q})$; for example $M_{4,\{4\}}$ has modular weight 15. The space of modular forms for $\Gamma(N)$ of a given weight k is a finite-dimensional vector space. (For any integer $k>0$ the dimension is $k+1$ for $\Gamma(3)$ and $6k$ for $\Gamma(6)$). Matching a finite number of the coefficients in the $\mathfrak{q}$ expansion one finds explicitly the modular coefficients $M_{i,\{j_1,\dots\}}(\mathfrak{q})$, see Appendix P.2.

10.23 Del Pezzo and E_7 bundles

The story for the E_7-bundles is similar. There is a family of del Pezzo surfaces $\mathcal{S}_7\subset\mathbb{WP}^{1,1,1,2}$, described by the degree 4 equation

$$\Gamma_4(X_0,X_1,X_2,X_3)=\sum_{i=0}^{4}X_0^i\mathcal{G}_{4-i}(X_1,X_2,X_3)$$

with

$$\mathcal{G}_4(X_1,X_2,X_3)=-X_3^2+4X_1X_2^3-g_2X_1^3X_2-g_3X_1^4\big(X_3^2-4X_1X_2^3+g_2X_2X_1^3+g_3X_1^4\big),$$

$$\mathcal{G}_3(X_1,X_2,X_3) = p_0X_1^3 + p_7X_1^2X_2\big(a_1X_1^3 + a_2X_2X_1^2\big),$$
$$\mathcal{G}_2(X_1,X_2,X_3) = p_1X_1^2 + p_2X_1X_2 + p_6X_2^2\big(b_1X_1^2 + b_2X_1X_2 + b_3X_2^2\big),$$
$$\mathcal{G}_1(X_1,X_2,X_3) = p_3X_1 + p_5X_2(c_1X_1 + c_2X_2),$$
$$\mathcal{G}_0 = p_4.$$

Again, the divisor $X_0 = 0$ is the elliptic curve $\mathcal{E}$, which is realized as a zero locus of the degree 4 polynomial equation $\mathcal{G}_4(X_1,X_2,X_3) = 0$ in $\mathbb{WP}^{1,1,2}$.

The isomorphism classes of $\mathcal{S}_7$, containing the fixed elliptic curve $\mathcal{E}$, are in one-to-one correspondence with the points

$$[p] = (p_0 : p_7 : p_1 : p_2 : p_5 : p_3 : p_5 : p_4) \in \mathbb{WP}^{1,1,2,2,2,3,3,4}.$$

Again, we study the "(-1)-curves", which in a particular gauge look like

$$\mathbf{x}(\zeta) = \left(\zeta, 1, X + \zeta v_X, Y + \zeta v_Y + \frac{1}{2}\zeta^2 w_Y\right),$$

where $(X,Y;v_X,v_Y,w_Y)$ obey a system of 5 equations $\Xi_i(X,Y;v_X,v_Y,w_Y) = 0$, $i = 0,\dots,4$:

$$\Gamma_4\left(\zeta, 1, X + \zeta v_X, Y + \zeta v_Y + \frac{1}{2}\zeta^2 w_Y\right) = \sum_{i=0}^{4}\zeta^i\Xi_i(X,Y;v_X,v_Y,w_Y),$$

where

$$\Xi_0 = -Y^2 + 4X^3 - g_2Y - g_3,$$
$$\Xi_1 = p_0 + Xp_7 + 12X^2v_X - g_2v_X - 2Yv_Y,$$
$$\Xi_2 = p_1 + Xp_2 + X^2p_6 + p_7v_X + 12Xv_X^2 - v_Y^2 - Yw_Y,$$
$$\Xi_3 = p_3 + Xp_5 + p_2v_X + 2Xp_6v_X + 4v_X^3 - v_Yw_Y,$$
$$\Xi_4 = p_4 + p_5v_X + p_6v_X^2 - \frac{w_Y^2}{4}.$$

We proceed similarly to the E_6 case: we solve for v_Y and w_Y from the equations Ξ_1 and Ξ_2, plug the solution into the polynomial Ξ_3 and Ξ_4 and compute the resultant

$$C^{E_7}(X,Y;g_2,g_3,p_0,\dots,p_7) = -2^{10}Y^{18}\mathrm{res}_{v_X}(\Xi_3|_{v_Y,w_Y:\Xi_{1,2}=0},\Xi_4|_{v_Y,w_Y:\Xi_{1,2}=0}) \mod \Xi_0,$$

which has the structure of the degree 28 polynomial in X with the coefficients polynomial in $(p_0,\dots,p_7)$ of total degree 12:

$$C^{E_7}(X,Y;g_2,g_3,p_0,\dots,p_7) = \big(p_0^{12} + 24g_3p_0^{10}p_1 + \cdots\big) + \big(24g_2p_0^{10}p_1 + \cdots\big)X + \cdots$$
$$+ \big(2^{16}p_5^4 - 2^{19}p_4p_5^2p_6 + 2^{20}p_4^2p_6^2\big)X^{28}. \tag{10.64}$$

(A short `Mathematica` version of this formula is given in Appendix P.1.2.) The polynomial $C^{E_7}(X,Y;g_2,g_3,p_0(x),\dots,p_7(x))$ together with the Weierstraß cubic $\Xi_0(X,Y;g_2,g_3)$ defines the algebraic Seiberg–Witten curve for E_7 quiver theory. Since on the elliptic curve $\mathcal{E}$ the degree of X is 2, at each $x \in \mathbf{C}_{\langle x\rangle}$ the spectral curve (10.64) defines $56 = 2\times 28$ points on $\mathcal{E}$ encoding the vector bundle in the **56** representation of E_7.

The relation between the del Pezzo parametrization (p_i) and the theta-function parametrization (c_i) of $\mathrm{Bun}^{ss}_{E_7(\mathbb{C})}(\mathcal{E})$ can be in principle written in terms of a certain modular matrix $M(\mathfrak{q})$, as in the the E_6 theory (see Appendix P.2). We do not record this transformation in this work.

10.24 Del Pezzo and E_8 bundles

To get an effective description of E_8-bundles we study the family of del Pezzo surfaces $\mathcal{S}_8 \subset \mathbb{WP}^{1,1,2,3}$:

$$\Gamma_6(X_0, X_1, X_2, X_3) = -X_3^2 + 4X_2^3 - X_2 G_2(X_0, X_1) - G_3(X_0, X_1) = 0,$$
$$G_2(X_0, X_1) = g_2 X_1^4 + \sum_{j=1}^{4} a_j X_0^j X_1^{4-j},$$
$$G_3(X_0, X_1) = g_3 X_1^6 + \sum_{j=2}^{6} b_j X_0^j X_1^{6-j}.$$

The isomorphism classes of the del Pezzo surfaces $\mathcal{S}_8$ containing the fixed elliptic curve

$$Y^2 = 4X^3 - g_2 X - g_3$$

are parametrized by

$$[s] = (a_1 : a_2 : b_2 : a_3 : b_3 : a_4 : b_4 : b_5 : b_6) \in \mathbb{WP}^{1,2,2,3,3,4,4,5,6}.$$

The "-1"-curves in $\mathcal{S}_8$ are described by the parametrizations $\mathbf{x}(\zeta) = \big(\zeta, 1, X + \zeta v_X + \frac{1}{2}\zeta^2 w_X, Y + \zeta v_Y + \frac{1}{2}\zeta^2 w_Y + \frac{1}{6}\zeta^3 u_Y\big)$, where $(X, Y, v_X, v_Y, w_X, w_Y, u_Y)$ to be found from the equations

$$\Xi_i(Y, Y, v_X, v_Y, w_X, w_Y, u_Y) = 0, \qquad i = 0, \dots, 6,$$

where

$$\Gamma_6\left(\zeta, 1, X + \zeta v_X + \frac{1}{2}\zeta^2 w_X, Y + \zeta v_Y + \frac{1}{2}\zeta^2 w_Y + \frac{1}{6}\zeta^3 u_Y\right)$$
$$= \sum_{i=0}^{6} \zeta^i \Xi_i(X, Y, v_X, v_Y, w_X, w_Y, u_Y).$$

To find explicitly the equation of affine E_8 spectral curve, one shall proceed in spirit similarly to the E_6, E_7 cases considered above. However the explicit computation becomes much more tedious as the minimal representation of E_8 is **248**, and the expected x-degree of the curve is $60N$ (see below). We leave this task for future investigation.

11 The integrable systems of monopoles and instantons

As we reviewed above, the $\mathcal{N} = 2$ gauge theory compactified on a circle $\mathbb{S}^1$ becomes, at low energy, the $\mathcal{N} = 4$ supersymmetric sigma model with the hyperkäbler target space $\mathfrak{P}$. The triplet of complex structures on $\mathfrak{P}$ is in correspondence with the choices of a supercharge $\mathcal{Q}$, which is nilpotent up to an infinitesimal translation along $\mathbb{S}^1$. The one supercharge which is nilpotent even in the decompactified theory (it corresponds to the topological supercharge of the Donaldson theory, for pure $\mathcal{N} = 2$ super-Yang–Mills theory) corresponds to the complex structure $\mathbf{I}$. In this complex structure $\mathfrak{P}$ has the structure of an algebraic integrable system $(\mathfrak{P}, \Omega, h)$:

$$h\colon\ \mathfrak{P} \longrightarrow \mathfrak{M}, \qquad \Omega|_{h^{-1}(u)} = 0, \qquad u \in \mathfrak{M}.$$

The $\mathbf{I}$-holomorphic $(2,0)$ form Ω is the form which we previously denoted by $\Omega_{\mathbf{I}}$.

We shall now describe these systems for the class I, II and II* theories we studied so far. For some theories several presentations of the same integrable system are possible.

In all cases we study the phase spaces $\mathfrak{P}$ have parameters corresponding to the masses m of the matter field in the gauge theory. The cohomology class of $[\Omega]$ is linear in m. An explanation of this fact in the symplectic geometry is the existence of a "larger" symplectic manifold $\mathfrak{P}^{\rm ext}$ with the holomorphic Hamiltonian torus $\mathbb{T} = (\mathbb{C}^\times)^M$ action, whose holomorphic symplectic quotient of $\mathfrak{P}^{\rm ext}$ at some level m of the moment map produces $\mathfrak{P}$.

The explanation in gauge theory is the three-dimensional mirror symmetry. Our phase space $\mathfrak{P}$ is the Coulomb branch of the three-dimensional $\mathcal{N} = 4$ gauge theory, obtained by the $\mathbb{S}^1$ compactification of the four-dimensional $\mathcal{N} = 2$ theory. The masses of the matter fields are the vacuum expectation values of the scalars in the vector multiplet. Under the three-dimensional mirror symmetry [86] these are exchanged with the Fayet–Iliopoulos terms, which are the levels of the three moment maps in the hyperkähler quotient construction [84] of the Higgs branch of the mirror theory.

It is amusing to identify $\mathfrak{P}^{\rm ext}$ and the action of the torus in the examples below. We shall treat the case of the class II theories in some detail, leaving other examples to the interested reader.

11.1 Periodic Monopoles and the phase space of class I theories

We shall now demonstrate that for the class I theories the phase space $\mathfrak{P}$ is the moduli space of the charge $\mathbf{v}$ G-monopoles on $\mathbb{R}^2\times\mathbb{S}^1$ with Dirac singularities, whose location and the embedding of the Dirac $U(1)$-monopoles into G is parametrized by $\mathbf{w}$ and the masses $m_{i,\mathfrak{f}}$.

Let us discuss the monopole moduli space in more detail.

The ordinary G-monopoles are the solutions of Bogomolny equation on $\mathbb{R}^3$

$$D_A\phi + \star F_A = 0 \tag{11.1}$$

with finite L^2-energy

$$\mathcal{E}(A,\phi) = \int_{\mathbb{R}^3} \langle F_A, \star F_A\rangle + \langle D_A\phi, \star D_A\phi\rangle. \tag{11.2}$$

One shows that as $\vec{x}\to\infty$, the conjugacy class of $\phi(\vec{x})$ approaches a fixed value. Equivalently, $\phi(\vec{x}) \longrightarrow g^{-1}(\vec{x})\phi_\infty g(\vec{x})$, for some fixed $\phi_\infty \in \mathfrak{h}_\mathbb{R}$, the Cartan subalgebra of the maximal compact subgroup G. Actually, $\phi_\infty \in \mathfrak{h}_\mathbb{R}/W(\mathfrak{g})$, but, since $\mathbb{S}^2_\infty$ is simply connected, one can choose a uniform representative $\phi_\infty \in \mathfrak{h}_\mathbb{R}$. This lift from $\mathfrak{h}_\mathbb{R}/W(\mathfrak{g})$ to $\mathfrak{h}_\mathbb{R}$ is going to be trickier in the case of periodic monopoles we shall study below.

Suppose ϕ_∞ is generic, i.e., the only gauge transformations which commute with it belong to the normalizer $N(T)$ of a maximal torus T. The restriction of ϕ onto a two-sphere $\mathbb{S}^2_\infty$ of a very large radius defines a map

$$\varphi\colon\ \mathbb{S}^2_\infty \longrightarrow G/T$$

and a T-subbundle $\mathbb{T}$ of the trivial G-bundle $P = G\times\mathbb{S}^2_\infty$. The latter is characterized by its Chern classes, which can be also identified with the class $[\varphi]$ of φ in

$$\pi_2(G/T) \approx \pi_1(T) \approx \mathrm{Q}^\vee,$$

also known as the magnetic charges of the monopole solution. The magnetic charge can also be read off the solution of (11.1) by projecting the curvature F_A to the Cartan subalgebra $\mathfrak{h}$ defined by $\phi|_{\mathbb{S}^2_\infty}$, and by taking the corresponding integrals

$$\mathbf{m}_\varphi = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{S}^2_\infty} F_A^\mathfrak{h}.$$

Now let us compactify one of the spatial directions, i.e., replace $\mathbb{R}^3$ by $M^3=\mathbb{S}^1\times\mathbb{R}^2$. Let $\psi\in[0,2\pi)$ be the angular coordinate on $\mathbb{S}^1$ and let $x=x_1+\mathrm{i}x_2$ be the complex coordinate on $\mathbb{R}^2$. Let us normalize the metric on M^3 so that the circumference of $\mathbb{S}^1$ is equal to one.

Consider the complex connection in the $\mathbb{S}^1$ direction (we use the physical convention where A is represented by Hermitian matrices, i.e., by a real-valued one-form for the $U(1)$ gauge group):

$$\nabla=\partial_\psi+\mathrm{i}A_\psi-\phi.$$

The equation (11.1) implies that the $\bar{x}$-variation of ∇ is an infinitesimal gauge transformation

$$\bar{\partial}_{\bar{x}}\nabla+[A_{\bar{x}},\nabla]=0.$$

Therefore, the conjugacy class of the holonomy $g(x,\bar{x})$ of ∇ around $\mathbb{S}^1$ varies holomorphically with x, and, in the gauge where $g(x,\bar{x})\in\mathbf{T}$, it is locally holomorphic

$$[g(x)]=\left[P\exp\left(\oint_0^{2\pi}\mathrm{i}\,\mathrm{d}\psi(A_\psi(\psi,x,\bar{x})+\mathrm{i}\phi(\psi,x,\bar{x}))\right)\right]\in B(\mathfrak{g}). \tag{11.3}$$

As we shall clarify later, when $x\to\infty$,

$$[g(x)]\to\left[\prod_{i=1}^{r}\mathfrak{q}_i^{-\lambda_i^\vee}\right]=b_\infty\in B^{\mathrm{ad}}(\mathfrak{g}).$$

One is left with the quasimap $u\colon\mathbb{CP}^1_{\langle x\rangle}\to B^{\mathrm{ad}}(\mathfrak{g})$. It is instructive to calculate (11.3) for the Dirac monopole on M^3.

Recall that the Dirac monopole at $\vec{p}\in\mathbb{R}^3$ is the connection in the $U(1)$ bundle over $\mathbb{R}^3\backslash\vec{p}$, which is a pullback of the constant curvature connection on the Hopf bundle over $\mathbb{S}^2$ via the projection map

$$\pi_{\vec{p}}\colon\ \mathbb{R}^3\backslash\vec{p}\longrightarrow\mathbb{S}^2,\qquad \pi_{\vec{p}}(\vec{r})=\frac{\vec{r}-\vec{p}}{|\vec{r}-\vec{p}|}.$$

The corresponding curvature two-form F is given by

$$F=2\pi\mathrm{i}\pi_{\vec{p}}^*\varpi_2=\frac{\mathrm{i}}{2}\frac{(\vec{x}-\vec{p})\cdot\mathrm{d}\vec{x}\times\mathrm{d}\vec{x}}{|\vec{x}-\vec{p}|^3},$$

where

$$\int_{\mathbb{S}^2}\varpi_2=1.$$

The fact that up to the $|\vec{r}-\vec{p}|^2$ rescaling the two-form ϖ_2 coincides with the volume form on $\mathbb{S}^2$ obtained from the flat metric on $\mathbb{R}^3$ implies that F solves Maxwell equations in $\mathbb{R}^3\backslash\vec{p}$ and moreover there is a magnetic potential ϕ, such that

$$\mathrm{d}\phi+\star_3F=0.$$

Moreover, if ϕ is normalized to approach zero at infinity, then

$$\phi=-\frac{1}{2|\vec{r}-\vec{p}|}. \tag{11.4}$$

The periodic Dirac monopole, i.e., the solution of Maxwell equations on $M^3\backslash p=(\psi_0,x_0,\bar{x}_0)$ can be obtained from the basic monopole in $\mathbb{R}^3$ by taking the superposition of the fields of an

infinite periodic array of monopoles, living on the universal cover $\widetilde{M^3\backslash p} = \mathbb{R}^3\backslash(\psi_0+2\pi\mathbb{Z}, x_0, \bar{x}_0)$. The magnetic potential is given by the regularized sum of potentials (11.4)

$$\phi(\psi, x, \bar{x}; p) = \phi_\infty + \frac{\log(\pi)-\gamma}{2\pi} + \sum_{n\in\mathbb{Z}}\left(\varphi(\psi-\psi_0-2\pi n, x-x_0) + \frac{1-\delta_{n,0}}{4\pi|n|}\right),$$

$$\varphi(\psi, x) = -\frac{1}{2\sqrt{\psi^2+x\bar{x}}}.$$

We calculate

$$\frac{1}{2\pi}\int_0^{2\pi}\phi(\psi, x, \bar{x}; p)\mathrm{d}\psi = \phi_\infty + \frac{1}{2\pi}\log|x-x_0|.$$

The calculation of

$$\int_0^{2\pi} A_\psi(\psi, x, \bar{x}; p)\,\mathrm{d}\psi$$

is a bit tricky. Fortunately, its derivative is easy to compute

$$\mathrm{d}\int_0^{2\pi} A_\psi(\psi, x, \bar{x}; p)\,\mathrm{d}\psi = \oint_{\mathbb{S}^1} F = \frac{\mathrm{i}}{4}\sum_{n\in\mathbb{Z}}\int_0^{2\pi}\mathrm{d}\psi\wedge\frac{(x-x_0)\mathrm{d}\bar{x}-(\bar{x}-\bar{x}_0)\mathrm{d}x}{\left(|x-x_0|^2+(\psi-\psi_0+2\pi n)^2\right)^{3/2}}$$
$$= \frac{\mathrm{i}}{2}\,\mathrm{d}\log\left(\frac{x-x_0}{\bar{x}-\bar{x}_0}\right).$$

Thus

$$\mathrm{d}\oint(A_\psi+\mathrm{i}\phi)\mathrm{d}\psi = \mathrm{i}\,\mathrm{d}\log(x-x_0)$$

and the monodromy is equal to

$$g(x) = (x-x_0)^{-1}$$

up to some multiplicative constant. Now, if we have a superposition of several Dirac monopoles, in the theory with the gauge group T, with the monopoles of the type i, i.e., corresponding to the coweight $\lambda_i^\vee \in \mathrm{Hom}(U(1), T)$ located at the points $(\psi_{i,\mathfrak{f}}, m_{i,\mathfrak{f}}, \bar{m}_{i,\mathfrak{f}})$, then the monodromy of the corresponding complexified connection $A+\mathrm{i}\phi\,\mathrm{d}\psi$ is equal to

$$g(x) \propto \prod_{i=1}^{r}\prod_{\mathfrak{f}=1}^{\mathbf{w}_i}(x-m_{i,\mathfrak{f}})^{-\lambda_i^\vee}.$$

Now let us consider the nonabelian Bogomolny equation on M^3. Instead of solving the equation (11.1) modulo G-gauge transformations, let us solve two out of three equations in (11.1), namely the equation

$$[D_{\bar{x}}, \nabla] = 0 \tag{11.5}$$

modulo the action of the group $\mathcal{G}^{\mathbb{C}}$ of $\mathbf{G}$-valued (complex) gauge transformations. In fact, (11.5) can be viewed as the complex moment map for $\mathcal{G}^{\mathbb{C}}$, acting on the space of (A, Φ), endowed with the $\mathcal{G}^{\mathbb{C}}$-invariant holomorphic symplectic form:

$$\Omega = \frac{1}{2\pi}\int_{M^3}\langle\delta\nabla\wedge\delta A_{\bar{x}}\rangle\,\mathrm{d}\psi\mathrm{d}x\mathrm{d}\bar{x}. \tag{11.6}$$

Let us now try to analyze the solutions to (11.5) in some domain $D\times\mathbb{S}^1\subset M^3$ over $D\subset\mathbf{C}_{\langle x\rangle}$. We fix the $\mathbf{G}$-gauge where $A_\psi+\mathrm{i}\phi=\xi(x,\bar{x})$, $\partial_\psi\xi=0$. This gauge leaves some residual gauge freedom. Passing to

$$g(x)=\exp(2\pi\mathrm{i}\xi(x))$$

partially reduces the residual gauge invariance. The equation (11.5) implies that $\bar{\partial}_{\bar{x}}\xi=0$, and $A_{\bar{x}}=a_{\bar{x}}\in\mathfrak{h}$. We then proceed with constructing the cameral curve $\mathcal{C}_u$ which is the union of the $W(\mathfrak{g})$-orbits of $g(x)$ over all $x\in\mathbb{CP}^1_{\langle x\rangle}=\mathbf{C}_{\langle x\rangle}\cup\{\infty\}$. The fiber $\mathcal{A}_u$ of the projection $h\colon\mathfrak{P}\to\mathfrak{M}$ is the space of $W(\mathfrak{g})$-equivariant $\mathbf{T}$-bundles over $\mathcal{C}_u$ of fixed multi-degree. We shall discuss in more detail the analogous situation for the class II theories in the next section.

The asymptotics of the solution to (11.1) is characterized by a vector of magnetic charges. Namely, over $\mathbb{T}^2_\infty=\mathbb{S}^1\times\mathbb{S}^1_\infty$ where $\mathbb{S}^1_\infty$ is a large radius circle in $\mathbb{R}^2$, the gauge group G is broken down to T. The gauge bundle is therefore characterized by the vector of the first Chern classes:

$$\mathbf{m}\in H^2\big(\mathbb{T}^2_\infty,\pi_1(\mathbf{T})\big)=\mathrm{Q}^\vee.$$

One can compute $\mathbf{m}$ by analyzing the behavior of the conjugacy class of the holonomy $g(x)$ of the complexified connection ∇. One can show that the finite energy (11.2) solutions with non-trivial magnetic charge do not exist. Indeed, macroscopically the system looks two-dimensional, and asymptotically it looks like a charged vortex, whose energy diverges logarithmically at large distances.

However, infinite 3-dimensional energy solutions may correspond to the finite tension higher-dimensional objects. For example, the noncommutative $U(1)$ monopole describes a finite tension string, which is attached to the worldvolume of the gauge theory [78]. Similarly, the infinite energy periodic monopole solutions have interpretation in the higher-dimensional theory, e.g., in the brane realization [175] of the pure $\mathcal{N}=2$ $\mathrm{SU}(N)$ gauge theory in four dimensions [18].

One can actually make the energy finite in the infrared by allowing point-like singularities in M^3. The idea is to screen the asymptotic magnetic charge of the non-abelian solution by the opposite charge of the Dirac monopole singularities.

Let us study the general situation. Suppose $\gamma\subset\mathbf{C}_{\langle x\rangle}$ is a closed contour. For each point $x\in\gamma$ compute the holonomy $g(x,\bar{x})$ of the complexified connection ∇, e.g., starting at the point $\psi=0$ on the fiber $\mathbb{S}^1$. It is a functional of the gauge field A and the Higgs field ϕ: $(A,\phi)\mapsto g(x,\bar{x})$. The gauge transformed (A,ϕ) leads to the similarity-transformed function $g(x,\bar{x})$: $(A^h,\phi^h)\mapsto h^{-1}(0,x,\bar{x})g(x,\bar{x})h(0,x,\bar{x})$. We have a well-defined map

$$[g_\gamma]\colon\ \gamma\longrightarrow,B(\mathfrak{g}),$$

which is a restriction on γ of the holomorphic (cf. (11.5)) map $U\colon\mathbb{CP}^1_{\langle x\rangle}\to B^{\mathrm{ad}}(\mathfrak{g})$, $U\colon x\mapsto[g(x)]$.

Now let $\Xi(\mathfrak{g})\subset B^{\mathrm{ad}}(\mathfrak{g})$ denote the set of irregular orbits of $W(\mathfrak{g})$ in $\mathbf{T}/Z$. Generically the image Σ_u of $[g(x)]$ crosses $\Xi(\mathfrak{g})$ at some isolated points, which are the branch points of the cameral curve $\mathcal{C}_u$:

$$\Xi_x=U^{-1}(\Sigma_u\cap\Xi(\mathfrak{g})).$$

Let us consider the subvariety $B(\mathfrak{g})^{\mathrm{reg}}=B^{\mathrm{ad}}(\mathfrak{g})\backslash\Xi(\mathfrak{g})$. The fundamental group $\pi_1(B(\mathfrak{g})^{\mathrm{reg}})$ is related to Artin braid group associated with the $\mathfrak{g}$ root system. Let us also define $\mathbf{T}^{\mathrm{reg}}$ to be the subvariety in $\mathbf{T}$ consisting of the regular elements, i.e., the elements of the maximal torus whose stabilizer in $\mathbf{G}$ is the maximal torus $\mathbf{T}$. It is invariant under the translations by Z. We have a map: $\pi_*\colon\pi_1(\mathbf{T}^{\mathrm{reg}})\to\pi_1(B(\mathfrak{g})^{\mathrm{reg}})$, induced by the projection $\pi\colon\mathbf{T}^{\mathrm{reg}}\to\mathbf{T}^{\mathrm{reg}}/\left(Z\times W(\mathfrak{g})\right)$.

Now, let us go back to the loop γ. We would like to define a $\mathbf{T}$-bundle over $\mathbb{S}^1\times\gamma$, by choosing a gauge where $A_\psi+\mathrm{i}\phi$ is ψ-independent element of the Lie algebra $\mathfrak{h}\subset\mathfrak{g}$. Then Bogomolny

equations imply that $A_{\bar{x}}$ also belongs to $\mathfrak{h}$, assuming that $A_\psi + \mathrm{i}\phi$ is regular, i.e., its stabilizer in $\mathbf{G}$ is the maximal torus $\mathbf{T}$.

There is an obstruction for such a gauge being possible throughout γ. Namely the class of γ in $\pi_1(B(\mathfrak{g})^{\mathrm{reg}})$ should lie in the image of π_*. This is related to our solution of the four-dimensional $\mathcal{N}=2$ theory of the class I as follows.

For the asymptotically conformal theories, with the assignments of dimensions $\mathbf{v}$, $\mathbf{w}$, we have defined a $\mathbf{G}/Z$-valued function on $\mathbf{C}_{\langle x\rangle}$ minus a finite number of points:

$$g(x) = \prod_{i=1}^{r} \mathcal{P}_i(x)^{-\lambda_i^\vee} \mathcal{Y}_i(x)^{\alpha_i^\vee}. \tag{11.7}$$

We identify $g(x)$ in (11.7) with the holonomy in (11.3). The factor

$$g_\infty(x) = \prod_{i=1}^{r} \mathcal{P}_i(x)^{-\lambda_i^\vee}$$

clearly corresponds to the Dirac monopoles sitting at some points $(\psi_{i,\mathfrak{f}}, m_{i,\mathfrak{f}}, \bar{m}_{i,\mathfrak{f}})$ with the charges $\lambda_i^\vee$. The remaining factor has to do with the nonabelian monopoles.

Recall that the map $U\colon \mathbf{C}_{\langle x\rangle}\backslash\Xi_x \to B(\mathfrak{g})$ is determined by the collection of gauge polynomials $T_i(x)$, $i=1,\dots,r$. The singular locus Ξ_x of U are at the zeroes and poles of the discriminant

$$\Delta(x) = g(x)^{-2\rho} \prod_{\alpha\in R_+} \bigl(g(x)^\alpha - 1\bigr)^2,$$

where R_+ is the set of positive roots of $\mathfrak{g}$, and

$$\rho = \frac{1}{2}\sum_{\alpha\in R_+} \alpha.$$

The discriminant $\Delta(x)$ is a rational function in $T_i(x)$'s and $\mathcal{P}_i(x)$'s. Now, given a loop γ in $\mathbf{C}_{\langle x\rangle}\backslash\Xi_x$, when can we lift $[g(x)]|_\gamma$ to the $\mathbf{T}$-valued loop?

For the simple root α_i let us denote by $\Xi_{x,i}$ the set of solutions to the equation $g(x)^{\alpha_i}=1$ on the physical sheet of $\mathcal{C}_u$.

$$\Xi_{x,i} = \bigl\{x \mid g(x)^{\alpha_i} = 1\bigr\}$$

so that

$$\Xi_x = \bigcup_{i=1}^{r} \Xi_{x,i}.$$

Actually the points of $\Xi_{x,i}$ are the endpoints of the cuts $I_{i,\mathbf{a}}$.

Our claim is that for the loops $\gamma = A_{i\mathbf{a}}$ which encircle the individual cuts $I_{i,\mathbf{a}}$ the class $[g(x)] \in B(\mathfrak{g})^{\mathrm{reg}}$ lifts to $\mathbf{T}^{\mathrm{reg}}$, and defines a $\mathbf{T}$-bundle over $\mathbb{S}^1\times\gamma$. Its characteristic class is equal to $\alpha_i^\vee \in \mathrm{Q}^\vee$.

Thus the magnetic monopoles which correspond to the limit shape of the $\mathcal{N}=2$ theory have the Dirac monopole charges $\mathbf{q}_{\mathrm{Dir}} = -\sum_{i=1}^r \mathbf{w}_i\lambda_i^\vee$ which are distributed at the points $(\psi_{i,\mathfrak{f}}, m_{i,\mathfrak{f}}, \bar{m}_{i,\mathfrak{f}})$, and the nonabelian monopole charges $\mathbf{q}_{\prime\mathrm{tHP}} = \sum_{i=1}^r \mathbf{v}_i\alpha_i^\vee$ which are located over the cuts $I_{i,\mathbf{a}}$. The net charge at infinity is equal to

$$\mathbf{q}_{\mathrm{Dir}} + \mathbf{q}_{\prime\mathrm{t\,HP}} = 0$$

for the asymptotically conformal theories.

For the asymptotically free theories the net charge at infinity is equal to (cf. (3.1)):

$$\mathbf{q}_{\text{Dir}} + \mathbf{q}_{\text{'t HP}} = -\sum_{i=1}^{r} \beta_i \lambda_i{}^{\vee}.$$

The fact that $\beta_i \leq 0$ should follow from the positivity of energy (as it does in the A_1 case) but we could not find a simple proof for general $\mathbf{G}$.

The relation of the monopole picture of $\mathfrak{P}$ to the Hitchin system picture we had in Section 10.2 goes via the Nahm transform, or, since we are ultimately working only in a particular complex structure of the moduli space, via a version of Fourier–Mukai transform. The $U(k)$ monopoles on $\mathbb{R}^3$ are mapped via Nahm's transform to the solutions of a one-dimensional system of Nahm's equations. The $U(k)$ monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$ are mapped via Nahm's transform to the solutions of a two-dimensional system of Hitchin's equations, with singularities. Indeed, our spectral curve in the form (10.47) captures the solutions to two, $\bar{\partial}_A \Phi = 0$, $\partial_A \bar{\Phi}$ out of three Hitchin's equations. The remaining equation $F_A + [\Phi, \bar{\Phi}] = 0$ away from the punctures would fix the hyperkähler metric on $\mathfrak{P}$. Unfortunately we are not in the position to discuss the metric on $\mathfrak{P}$ as long as we stay within the realm of the four-dimensional gauge theory. See [14, 22, 82, 129].

We do need to discuss the holomorphic symplectic geometry of $\mathfrak{P}$. The symplectic form on $\mathfrak{P}$ descends from the two-form (11.6) via the Hamiltonian reduction with (11.5) being the moment map. The textbook construction of the action-angle variables of the integrable system produces the special coordinates $\mathfrak{a}^{\mathcal{I}}$, $\mathfrak{a}^{D}_{\mathcal{I}}$ of the gauge theory. We claim this construction is equivalent to the one using the periods of the differentials $x \, \mathrm{d} \log \mathcal{Y}_i$ on the cameral curve. The essential points of the demonstration are identical for the class I and for the class II theories.

We thus return to this question in Section 11.2.

Now let us study the A_1 case in some more detail. We wish to present yet another perspective on the phase space $\mathfrak{P}$.

Consider the product of N A_1 surfaces $\mathcal{O}_a$, $a = 1, \ldots, N$, the complex coadjoint orbits of $\mathrm{SL}(2, \mathbb{C})$. Each surface $\mathcal{O}_i$ is a quadric in $\mathbb{C}^3$, given by the equation

$$\xi_a^2 + \eta_a^2 + \zeta_a^2 = s_a^2$$

with some fixed constant $s_a \in \mathbb{C}$. The surface $\mathcal{O}_a$ has the holomorphic symplectic form

$$\varpi_a = \frac{\mathrm{d}\xi_a \wedge \mathrm{d}\eta_a}{\zeta_a},$$

which has the period s_a along a non-contractible two-sphere in $\mathcal{O}_a$. The moment map for the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathcal{O}_a$ is a 2×2 traceless matrix. Let us extend it into a general 2×2 matrix (which corresponds to the conformal extension of the group)

$$L_a(x) = \begin{pmatrix} x + \xi_a & \eta_a + \mathrm{i}\zeta_a \\ -\eta_a + \mathrm{i}\zeta_a & x - \xi_a \end{pmatrix}$$

and define the "monodromy matrix"

$$L(x) = \begin{pmatrix} \mathfrak{q} & 0 \\ 0 & 1 \end{pmatrix} \times L_N(x - \mu_N) L_{N-1}(x - \mu_{N-1}) \cdots L_1(x - \mu_1).$$

It is actually better to work with the somewhat differently normalized "local Lax operators"

$$g_a(x) = \mathbf{1}_2 + \frac{1}{x - s_a} \begin{pmatrix} u_a^+ & v_a^+ \\ v_a^- & u_a^- \end{pmatrix}, \tag{11.8}$$

where

$$u_a^{\pm} = s_a \pm \xi_a, \qquad v_a^{\pm} = \mathrm{i}\zeta_a \pm \eta_a$$

obey

$$u_a^+ u_a^- - v_a^+ v_a^- = 0$$

and define

$$g(x) = \begin{pmatrix} \mathfrak{q} & 0 \\ 0 & 1 \end{pmatrix} \times g_N(x-\mu_N) g_{N-1}(x-\mu_{N-1}) \cdots g_1(x-\mu_1). \tag{11.9}$$

Then, cf. [74],

$$\mathrm{Det}(g(x)) = \mathfrak{q} \frac{\mathcal{P}^+(x)}{\mathcal{P}^-(x)},$$

where we defined

$$\mathcal{P}^{\pm}(x) = \prod_{a=1}^{N} (x - \mu_a \pm s_a).$$

Now, define the phase space to be the complex symplectic quotient of the product of the A_1 surfaces by the diagonal action of the $\mathbb{C}^\times$,

$$\mathfrak{P} = \times_{a=1}^{N} \mathcal{O}_a // \mathbb{C}^\times \tag{11.10}$$

generated by

$$H_1 = \sum_{a=1}^{N} \xi_a.$$

The variety (11.10), defined by fixing the level ξ of H_1 and dividing the corresponding level set $H_1^{-1}(\xi)$ by $\mathbb{C}^\times$, carries the induced holomorphic symplectic structure. The functions h_k defined as

$$\mathrm{tr}_2\, g(x) = (1+\mathfrak{q}) \left(1 + \sum_{k=1}^{\infty} x^{-k} h_k \right).$$

Poisson-commute with respect to the induced Poisson structure. Moreover,

$$h_1 = H_1 + \frac{\mathfrak{q}-1}{\mathfrak{q}+1} \sum_a s_a$$

and the next $N-1$ h_k's are independent. The rest of the expansion coefficients can be expressed in terms of the first N.

We argue that the normalized Lax operator $g(x)$ is the complexified monodromy field $g(x)$ in the corresponding periodic singular monopole problem, with the monopole group $U(2)$ (the compact form of C$\mathbf{G}$) and $\mu_a \pm s_a$ are the locations of $2N$ Dirac monopoles. *Of course, our methods do not allow to establish the correspondence with monopoles outside the identification of complex symplectic manifolds, for complex structure I and $(2,0)$-form Ω_I.*

It is not difficult to convince oneself that the "local Lax operator" (11.8) is indeed the complexified monodromy of a single $U(2)$ monopole screened by two Dirac monopoles of the opposite $U(1)$ charges, located at $\pm s_a$.

What is amusing is that the equation (11.9) suggests that the complexified monodromy of the charge N $U(2)$ monopole screened by $2N$ Dirac monopoles factorizes as the product of N elementary monodromy matrices.

Note in passing that if we do not perform the reduction with respect to $\mathbb{C}^\times$ generated by H_1, i.e., work with the $2N$-dimensional phase space $\tilde{\mathfrak{P}}$ (this is a first step towards the extended phase space $\mathfrak{P}^{\rm ext}$), then Sklyanin's separation of variables [171, 172] identifies its open subset $\tilde{\mathfrak{P}}^\circ$ with the N-th symmetric product of (which is most likely [76] resolved into the Hilbert scheme of N points on) $\mathbf{C}_{\langle x\rangle}\times\mathbf{C}_{\langle t\rangle}{}^\times$. Incidentally [39], this manifold is symplectomorphic to the moduli space of regular charge N SU(2) monopoles on $\mathbb{R}^3$.

In recent years the connection between the gauge theories and the spin chains, and the inspired duality between the Gaudin-like integrable systems and the Heisenberg spin chain was discussed in [17, 41, 126, 128], building on the earlier work in [68, 69, 127, 155, 156].

Before concluding this section, note that the masses of the bi-fundamental matter hypermultiplets are encoded in the next-to-leading terms in the asymptotics of the complexified connection $A_\psi+\mathrm{i}\phi$ near $x\to\infty$. We shall explain this in more detail in the similar context in Section 11.2.

The moduli space of singular $\mathbf{G}$-monopoles with fixed conjugacy class of the monodromy of $A_\psi+\mathrm{i}\phi$ at $x\to\infty$, with unspecified location of Dirac singularities of specified charges defines our extended phase space $\mathfrak{P}^{\rm ext}$. It is acted upon by the torus $T\times T_M$, where $T_M\subset G_M$ is the maximal torus of the flavor group. The action of T is via the constant gauge transformations preserving the gauge field and the Higgs field at infinity, while T_M acts by changing the glueing data for the grafted Dirac monopoles. Fixing the level of the corresponding moment maps and reducing with respect to the complexification of the $T\times T_M$ action produces $\mathfrak{P}$.

Summarizing, we conclude with the conjecture: *the moduli space of singular* $\mathbf{G}$*-monopoles on* $\mathbb{R}^2\times\mathbb{S}^1$ *is acted upon by the Poisson–Lie group, which is a quasi-classical limit of the Yangian* $Y(\mathfrak{g}_{\rm q})$. *The deformation quantization of* $\mathfrak{P}^{\rm ext}$ (*in the complex structure* I) *produces the Yangian* $Y(\mathfrak{g})$. *Fixing the asymptotics of the complexified gauge field at infinity as well as the locations of the Dirac singularities would specify.* It should be interesting to explore the holomorphic symplectic geometry of $\mathfrak{P}$ in all of its complex structures and find the analogue of the variety of opers (cf. [152, 159]), which in the A_r case is the variety of local systems on the genus zero curve coming from the N-th order differential operators with regular singularities at $r+3$ punctures with the monodromies whose eigenvalues coincide with our description of the residues of the Higgs field just above the equation (10.10) (cf. [53]).

Note that in [67] a connection between the representation theory of Yangians and the moduli spaces of monopoles on $\mathbb{R}^3$ was found. Also, in [6, 7] the relations between the monopole solutions and the solutions to the Yang–Baxter equations were discussed. It remains to be seen, whether any of these connections carries over to the $\mathbb{R}^2\times\mathbb{S}^1$ case and whether it is the one we need.

What happens if one tries to study periodic $\widehat{\mathbf{G}}$-monopoles where $\widehat{\mathbf{G}}$ is Kac–Moody affine group? One naturally finds double periodic instantons similar to relation between $\widehat{\mathbf{G}}$-monopoles and periodic $\mathbf{G}$ instantons [65].

11.2 Double-periodic instantons and class II theories

Let us now discuss the class II theories. Recall from Section 9.6.2 that an elliptic curve $\mathcal{E}=\mathbb{C}^\times/\mathfrak{q}^{\mathbb{Z}}$, with $\mathfrak{q}$ given by (9.10), is associated with the class II gauge theory. Also recall that the gauge group is the product of special unitary groups

$$G_{\rm g}=\times_{i=0}^{r}\mathrm{SU}(Na_i)$$

for some number N. Using (9.12), we have identified the extended moduli space $\mathfrak{M}_N^{\rm ext}$ of vacua of the theory (4.4), $\dim_{\mathbb{C}}\mathfrak{M}_N^{\rm ext}=Nh^\vee$, with the moduli space of degree N framed quasimaps of $\left(\mathbb{CP}^1_{\langle x\rangle},\infty\right)$ to the moduli space $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ of holomorphic $\mathbf{G}$-bundles on $\mathcal{E}$, sending ∞ to

a particular bundle $[\mathcal{P}_{\tilde{\mathfrak{q}}}]$. This space of quasimaps is a natural base of the projection from the moduli space

$$\mathfrak{P}^{\text{ext}}{}_N \approx \text{Bun}^{ss}_{\mathbf{G};N}(S_{\mathcal{E}})^{\mathcal{P}_{\tilde{\mathfrak{q}}}}_{\infty}$$

of framed semi-stable holomorphic principal $\mathbf{G}$-bundles $\mathcal{P}$, $c_2(\mathcal{P}) = N$, on the surface $S_{\mathcal{E}} \equiv \mathbb{CP}^1_{\langle x\rangle} \times \mathcal{E}$, where the framing is the identification of the restriction of $\mathcal{P}$ at the fiber at infinity

$$\mathcal{P}\big|_{\{\infty\}\times\mathcal{E}} \approx \mathcal{P}_{\tilde{\mathfrak{q}}}.$$

In what follows we drop the subscript N.

The projection $\mathfrak{P}^{\text{ext}} \to \mathfrak{M}^{\text{ext}}$ is defined on the open dense subset of $\mathfrak{P}^{\text{ext}}$ by restricting the bundle $\mathcal{P} \in \mathfrak{P}^{\text{ext}}$ to the fiber $\mathcal{E}_x$ and taking its equivalence class $\mathbf{t}(x) = [\mathcal{P}|_{\mathcal{E}_x}]$ in $\text{Bun}_{\mathbf{G}}(\mathcal{E})$. This gives the desired quasimap $U\colon x \mapsto \mathbf{t}(x)$. This quasimap is a map near $x = \infty$, approaching a particular holomorphic bundle $[\mathcal{P}_{\tilde{\mathfrak{q}}}] \in \text{Bun}_{\mathbf{G}}(\mathcal{E})$ over $\mathcal{E}_\infty = \{\infty\} \times \mathcal{E}$. The reason U is not, in general, a map, has to do with the usual difference between the stability condition in two complex dimensions and the fiberwise stability condition, cf. [121].

The semi-stable framed holomorphic principal bundles $\mathcal{P}$ on $S_{\mathcal{E}}$, are in one-to-one correspondence with the G-instantons on $\mathbb{R}^2 \times \mathbb{T}^2$ with the flat metric (cf. [40]), i.e., connections on a principal G-bundle P over $\mathbb{S}^2 \times \mathbb{T}^2$, endowed with some metric, which is conformally flat on $\mathbb{R}^2 \times \mathbb{T}^2$, which obey

$$F_A = - \star F_A$$

and have finite action,

$$\int_{\mathbb{R}^2\times\mathbb{T}^2} \langle F_A \wedge \star F_A\rangle < \infty$$

(as we explain later, these instantons correspond to the M2 branes and the instanton action is the tension of the stack of M2 branes) and this forces the curvature F_A of the G-gauge field tend to zero as $|\vec{x}| \to \infty$, $\vec{x} \in \mathbb{R}^2$. We fix the instanton charge:

$$N = -\frac{1}{8\pi^2 h}\int_{\mathbb{R}^2\times\mathbb{T}^2} \langle F_A \wedge F_A\rangle$$

(remember that $\langle\,,\,\rangle$ is the Killing form, which is the trace in the adjoint representation). The real dimension of the moduli space $\mathfrak{P}^{\text{ext}}$ of G-instantons of finite action on $\mathbb{R}^2 \times \mathbb{T}^2$ with fixed framing at infinity $\mathbb{T}^2_\infty$ is equal to $4Nh$.

Actually, there is a subtlety here. The moduli space of charge N G-instantons on $\mathbb{R}^2 \times \mathbb{T}^2$ may have several components. This has to do with the fact that the moduli space of flat G-connections on $\mathbb{T}^3 = \mathbb{S}^1_\infty \times \mathbb{T}^2$, *which parametrize the asymptotics of instantons on* $\mathbb{R}^2 \times \mathbb{T}^2$, may have several components [177]. We shall assume we are always in the component of the trivial connection.

Note that $\mathfrak{P}^{\text{ext}}$ is acted upon by the maximal torus T of G, the symmetry group of the flat connection at infinity. This action lifts to the action of the algebraic torus $\mathbf{T}$ on the moduli space of framed holomorphic bundles $\mathfrak{P}^{\text{ext}}$. This is entirely parallel to the action of the group $\mathbf{G}$ on the moduli space $\mathfrak{M}^{\text{framed}}_G(\mathbb{R}^4)$ of framed G-instantons on $\mathbb{R}^4$.

A holomorphic principal $\mathbf{G}$-bundle $\mathcal{P}$ on $S_{\mathcal{E}}$ can be described using the transition functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbf{G}$ defined on the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ of the open sets in the appropriate open cover $(U_\alpha)_{\alpha\in A}$ of $S_{\mathcal{E}}$. The transition functions are holomorphic $\bar\partial g_{\alpha\beta} = 0$, must obey the cocycle condition $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, and the cocycles differing by the holomorphic coboundaries define equivalent bundles

$$g_{\alpha\beta} \sim g_\alpha g_{\alpha\beta} g_\beta^{-1},$$

where $g_\alpha\colon U_\alpha \to \mathbf{G}$ are holomorphic $\mathbf{G}$-valued functions on the open sets U_α themselves.

Another way to describe the holomorphic bundle is to introduce a connection $\nabla = \mathrm{d} + A$ on a smooth bundle, such that its $(0,1)$-part is $(0,2)$-flat

$$\nabla_{\bar\partial}^2 = 0. \tag{11.11}$$

The local holomorphic sections of $\mathcal{P}_\alpha = \mathcal{P}|_{U_\alpha}$ are the solutions to $\nabla_{\bar\partial} s_\alpha = 0$. Over an intersection $U_{\alpha\beta}$ these solutions must differ by a holomorphic $\mathbf{G}$-valued gauge transformation:

$$s_\alpha\big(\zeta,\bar\zeta\big) = g_{\alpha\beta}(\zeta) s_\beta\big(\zeta,\bar\zeta\big),$$

where ζ stand for local holomorphic coordinates. This is sometimes expressed by saying that locally $\bar A$ is a pure $\mathbf{G}$-gauge

$$\bar A\big|_{U_\alpha} = -s_\alpha^{-1}\bar\partial s_\alpha.$$

In the case at hand $\zeta = (x,z)$, where $x \in \mathbf{C}_{\langle x\rangle} \subset \mathbb{CP}^1_{\langle x\rangle}$ and z is the additive coordinate on $\mathcal{E}$. The equation (11.11) reads

$$\bar\partial_{\bar x} A_{\bar z} - \bar\partial_{\bar z} A_{\bar x} + [A_{\bar x}, A_{\bar z}] = 0. \tag{11.12}$$

This equation can be viewed as the complex moment map equation for the action of the group $\mathcal{G}$ of the $\mathbf{G}$-gauge transformations on the space $\mathcal{A}$ of connections on a given smooth principal $\mathbf{G}$-bundle over $S_\mathcal{E}$, endowed with the holomorphic symplectic form

$$\Omega = \int_{S_\mathcal{E}} \mathrm{d}x \wedge \mathrm{d}z \wedge \big\langle \delta\bar A \wedge \delta\bar A\big\rangle. \tag{11.13}$$

We have a little subtlety here. The two-form $\mathrm{d}x\wedge\mathrm{d}z$ was perfectly good on $S_\mathcal{E}^\circ$ but on $S_\mathcal{E}$ is has a pole along the divisor $\mathcal{E}_\infty = \{\infty\}\times\mathcal{E}$. We impose the condition that $A_{\bar z}$ approaches a specific value as $x\to\infty$. More specifically, recall (F.2) that generic $A_{\bar z}$ can be $\mathcal{G}$-transformed to the normal form $A_{\bar z} \to \frac{2\pi\mathrm{i}}{\tau-\bar\tau}\xi \in \mathfrak{h}$, $\partial_z\xi = \partial_{\bar z}\xi = 0$. We impose the boundary conditions:

$$A_{\bar z}(x,\bar x) \to \frac{2\pi\mathrm{i}}{\tau-\bar\tau}\xi_\infty + o\big(|x|^{-2}\big), \qquad |x|\to\infty \tag{11.14}$$

for fixed ξ_∞, which we relate to the gauge couplings $\mathfrak{q}_i$, $i=0,1,\dots,r$, via

$$\xi_\infty = -\frac{1}{2\pi\mathrm{i}}\sum_{i=1}^r \log\mathfrak{q}_i \lambda_i^\vee.$$

The limiting gauge field $\bar\partial_{\bar z} + \frac{2\pi\mathrm{i}}{\tau-\bar\tau}\xi_\infty$ corresponds to the holomorphic bundle $\mathcal{P}_{\tilde{\mathfrak{q}}}$ on $\mathcal{E}$. The decay rate (11.14) makes the integral (11.13) convergent. One can impose weaker conditions, allowing even the $O\big(x^{-1}\big)$ (but no $x^{-1}\bar x^{-1}$ terms!) decay, which would make (11.13) convergent with the principal value prescription. In fact, these subleading terms correspond to the bi-fundamental masses of the G_g-theory. Let us interpret them using the $\mathbf{T}$-action on $\mathfrak{P}^\mathrm{ext}$.

The constant $\mathbf{T}$-gauge transformation with the parameter $\varepsilon\in\mathfrak{h}$ acts on the $(0,1)$-gauge field $\bar A$ as follows:

$$\delta\bar A = [\varepsilon,\bar A] + \nabla_{\bar\partial}\eta_\varepsilon, \tag{11.15}$$

where $\eta_\varepsilon(x,\bar x,z,\bar z)$ is the compensating gauge transformation which sufficiently fast decays at $x\to\infty$. Contracting the vector field (11.15) with Ω in (11.13) gives us a closed one-form $\delta\mathbf{m}_\varepsilon$ on the space of connections obeying (11.12), where $\mathbf{m}_\varepsilon$ is linear in ε, $\mathbf{m}_\varepsilon = \langle\varepsilon,\mathbf{m}\rangle$ for some $\mathbf{m}\in\mathfrak{h}$

$$\delta\mathbf{m}_\varepsilon = \int_{S_\mathcal{E}} \mathrm{d}x\wedge\mathrm{d}z\wedge\big\langle([\varepsilon,\bar A] + \nabla_{\bar\partial}\eta_\varepsilon)\delta\bar A\big\rangle = \int_{S_\mathcal{E}} \mathrm{d}x\wedge\mathrm{d}z\wedge\bar\partial\big\langle\varepsilon,\delta\bar A\big\rangle$$

$$= \delta \oint_{x=\infty} \mathrm{d}x \left\langle \varepsilon, \int_{\mathcal{E}_x} \mathrm{d}^2 z A_{\bar{z}} \right\rangle . \tag{11.16}$$

Thus the moment map $\mathbf{m}$ for the $\mathbf{T}$ action is the residue at $x = \infty$ of the zero mode of the $\mathfrak{h}$-projection of the $A_{\bar{z}}$ gauge field. The analogous statement holds for the monopoles of Section 11.1.

Now let us solve (11.12) locally over some domain D in $\mathbf{C}_{\langle x \rangle}$. We choose the gauge (F.2) over each point $x \in D$:

$$A_{\bar{z}}(z, \bar{z}; x, \bar{x}) \longrightarrow \frac{2\pi \mathrm{i}}{\tau - \bar{\tau}} \xi(x, \bar{x}) \in \mathfrak{h}. \tag{11.17}$$

If $\infty \in D$, then, using (11.16), in the gauge (11.17)

$$\xi(x, \bar{x}) = \xi_\infty + \frac{\mathbf{m}}{x} + \cdots , \qquad x \to \infty,$$

where $\mathbf{m} \in \mathfrak{h}$ is a linear function of the bi-fundamental masses $(\mathbf{m}_e)_{e \in \mathrm{Edge}_\gamma}$.

Recall that $\mathbf{G}$ is a simple simply-connected Lie group. Therefore, the restriction $\mathcal{P}|_{\mathcal{E}_x}$ is trivial as a smooth $\mathbf{G}$-bundle. Therefore $\bar{A}$ is just a $\mathfrak{g}$-valued $(0,1)$-form on $D \times \mathcal{E}$. Now let us decompose $A_{\bar{x}} = a_{\bar{x}} + W_{\bar{x}}$, with $a_{\bar{x}} \in \mathfrak{h}$ and $W_{\bar{x}} \in \mathfrak{h}^\perp \subset \mathfrak{g}$, the orthogonal decomposition being provided by the Killing form $\langle \cdot, \cdot \rangle$. Then (11.12) implies: $\bar{\partial}_{\bar{x}} \xi(x, \bar{x}) = 0$, $\bar{\partial}_{\bar{z}} a_{\bar{x}} = 0$, $W_{\bar{x}} = 0$, the latter equation being valid for generic $\xi(x)$, corresponding to the irreducible bundles on $\mathcal{E}$. Indeed, the $F^{0,2} = 0$ equation (11.12) splits in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as follows:

$$\begin{aligned} &F^{0,2}\big|_{\mathfrak{h}} = \bar{\partial}_{\bar{x}} \xi - \bar{\partial}_{\bar{z}} a_{\bar{x}} = 0, \\ &F^{0,2}\big|_{\mathfrak{h}^\perp} = \bar{\partial}_{\bar{z}} W_{\bar{x}} + \frac{2\pi \mathrm{i}}{\tau - \bar{\tau}} [\xi, W_{\bar{x}}] = 0. \end{aligned} \tag{11.18}$$

For generic $\xi \in \mathfrak{h}$ (irreducible $\mathrm{ad}(\mathcal{P})^\perp$) the $\mathfrak{h}^\perp$-equation (11.18) does not have non-zero solutions, hence $W_{\bar{x}} = 0$. As for the $\mathfrak{h}$-equation (11.18) in our gauge its solution is

$$a_{\bar{x}}(x, \bar{x}, z, \bar{z}) = a^{(0)}_{\bar{x}}(x, \bar{x}, z) + \bar{z} \bar{\partial}_{\bar{x}} \xi(x, \bar{x}).$$

Since on $D \times \mathcal{E}$ the connection form $A_{\bar{z}} \, \mathrm{d}\bar{z} + a_{\bar{x}} \, \mathrm{d}\bar{x}$ is simply an $\mathfrak{h}$-valued $(0,1)$-form the component $a_{\bar{x}}(x, \bar{x}, z, \bar{z})$ must be z-periodic. This is only possible if $\bar{\partial}_{\bar{x}} \xi(x, \bar{x}) = 0$, which also implies $a_{\bar{x}}(x, \bar{x}, z, \bar{z})$ is z-independent.

Of course, (11.17) is not the complete gauge fixing: there remain the z-independent $\mathbf{T}$-valued gauge transformations and the $W(\mathfrak{g})$-transformations (F.3), which combine into the locally z-constant $N(\mathbf{T})$-gauge transformations. There are also the shifts (F.4) by the lattice $\mathrm{Q}^\vee \oplus \tau \mathrm{Q}^\vee$, generated by the $(z, \bar{z})$-dependent $\mathbf{T}$-valued gauge transformations with discrete, as far as the $(x, \bar{x})$-dependence is concerned, parameters $\beta_1, \beta_2 \in \mathrm{Q}^\vee$. Let us expand:

$$\xi(x) = \sum_{i=1}^{r} \xi_i(x) \alpha_i^\vee .$$

The residual shifts act on the components $\xi_i(x)$ by

$$\xi_i(x) \mapsto \xi_i(x) + \beta_{1,i} + \beta_{2,i} \tau, \qquad i = 1, \ldots, r+1,$$

where $\beta_{1,i}, \beta_{2,i} \in \mathbb{Z}$, for $i = 1, \ldots, r$ are the expansion coefficients:

$$\beta_A = \sum_{i=1}^{r} \beta_{A,i} \alpha_i^\vee , \qquad A = 1, 2.$$

Let $\mathbf{t}(x) = \big(t_i(x) = \mathrm{e}^{2\pi\mathrm{i}\xi_i(x)}\big)_{i=1}^r \in (\mathbb{C}^\times)^r$ and $[\mathbf{t}(x)] \in \mathcal{E}^r$ be the equivalence classes for the actions of the lattices $\mathrm{Q}^\vee$ and $\mathrm{Q}^\vee \oplus \tau\mathrm{Q}^\vee$, respectively.

Thus, dividing by all but the locally z-independent $N(\mathbf{T})$-gauge transformations we arrive at the collection of r points on $\mathcal{E}$, or, in a more sophisticated fashion, a point $[\mathbf{t}(x)]$ in $\mathcal{E}\otimes\mathrm{Q}$, in addition to the $\mathfrak{h}$-valued gauge field $a_{\bar{x}}(x,\bar{x})\mathrm{d}\bar{x}$. This is all done over the generic point $x \in \mathbb{CP}^1_{\langle x\rangle}$. We haven't completely fixed the gauge, though.

Let us forget for a moment about the gauge field

$$\bar{a} = a_{\bar{x}}(x,\bar{x})\,\mathrm{d}\bar{x}. \tag{11.19}$$

Then we'd divide by the action $W(\mathfrak{g})$, giving as back the point $[\mathcal{P}|_{\mathcal{E}_x}]$ in the orbispace $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E}) = \mathcal{E}\otimes\mathrm{Q}/W(\mathfrak{g})$, the holomorphic $\mathbf{G}$-bundle on $\mathcal{E}$ whose holomorphic structure is given by $\bar\partial_{\bar z} + \frac{2\pi\mathrm{i}}{\tau-\bar\tau}\xi(x)$. The (quasi)map $U\colon x \mapsto U(x) = [\mathcal{P}_x]$ is point $u \in \mathfrak{M}^{\mathrm{ext}}$ in the extended moduli space of the four-dimensional class II gauge theory.

The considerations similar to those in [60] show that the instanton charge $c_2(\mathcal{P}) = N$ bundles on $S_\mathcal{E}$ correspond to the degree N quasimaps $u\colon \mathbb{CP}^1_{\langle x\rangle} \to \mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$. This is of course in line with the original observations on the relations between the sigma model and gauge instantons [5].

By following the whole $W(\mathfrak{g})$-orbit of $[\mathbf{t}(x)]$ in $\mathcal{E}^r$ as x varies, we obtain the curve $\mathcal{C}_u$, which is a ramified $W(\mathfrak{g})$-cover of $\mathbb{CP}^1_{\langle x\rangle}$. The fiber of the projection $\mathcal{C}_u \to \mathbb{CP}^1_{\langle x\rangle}$ over a point x is the orbit

$$W(\mathfrak{g})\cdot[\mathbf{t}(x)] \subset \mathcal{E}\otimes\mathrm{Q}.$$

This leads us to the *obscured curve* which we have encountered earlier in our solution of the gauge theory using the limit shape equations.

Now, as in our prior discussion of Hitchin systems, let us recall the gauge field $\bar a$ in (11.19). It looks like the $\mathfrak{h}$-valued one-form on $\mathbb{CP}^1_{\langle x\rangle}$, since it is locally z-independent, and therefore might be a $(0,1)$ part of a $\mathbf{T}$-connection, defining a holomorphic $\mathbf{T}$-bundle over $\mathbb{CP}^1_{\langle x\rangle}$. However, we have $|W(\mathfrak{g})|$ worth of choices for $\bar\partial + \bar a$ at any particular point x since the residual $W(\mathfrak{g})$-symmetry acts both on $[\mathbf{t}(x)]$ and $\bar a$. This means that $\bar a$ becomes well-defined when lifted to $\mathcal{C}_u$. The way it transforms under the $W(\mathfrak{g})$-action permuting the sheets of the cover $\mathcal{C}_u \to \mathbb{CP}^1_{\langle x\rangle}$ makes it into the $W(\mathfrak{g})$-equivariant gauge field on $\mathcal{C}_u$, defining a $W(\mathfrak{g})$-equivariant holomorphic $\mathbf{T}$-bundle $\mathcal{T}$ over $\mathcal{C}_u$:

$$\mathcal{T} \in \mathcal{A}_u = \mathrm{Bun}_{\mathbf{T}}(\mathcal{C}_u)^{W(\mathfrak{g})} \approx \mathrm{Hom}_{W(\mathfrak{g})}(\Lambda, \mathrm{Pic}(\mathcal{C}_u)), \tag{11.20}$$

the idea of the last equality is that every weight $\lambda \in \Lambda = \mathrm{Hom}(\mathbf{T},\mathbb{C}^\times)$ defines a $\mathbb{C}^\times$-bundle $\mathcal{T}^\lambda$ on $\mathcal{C}_u$, in the $W(\mathfrak{g})$-compatible fashion.

Of course this discussion is not adequate at the branch points, yet hopefully it can be extended similarly to other constructions of spectral covers [30, 83]. In particular, the analysis near the branch points should demonstrate that the bundle $\mathcal{T}$ has a fixed topological type, which we shall not attempt to determine in this paper.

We conjecture that (11.20) is the fiber of the projection $\pi\colon \mathfrak{P}^{\mathrm{ext}} \to \mathfrak{M}^{\mathrm{ext}}$, sending the extended moduli space of vacua of the class II gauge theory on $\mathbb{R}^3\times\mathbb{S}^1$ to that of the infinite volume four-dimensional gauge theory. Since the projection $\mathcal{P} \mapsto U$ is roughly of the form $(A_{\bar x}, A_{\bar z}) \mapsto A_{\bar z}$, it is Lagrangian with respect to (11.13). This is in agreement with $\dim_{\mathbb{C}}\mathfrak{P}^{\mathrm{ext}} = 2\dim_{\mathbb{C}}\mathfrak{M}^{\mathrm{ext}} = 2Nh$. From now on we fix the masses, divide by the global $\mathbf{T}$-action and work with $\mathfrak{P}$ and $\mathfrak{M}$.

Thus the study of holomorphic bundles on $S_\mathcal{E}$ brought us the following picture: over each point $x \in \mathbb{CP}^1_{\langle x\rangle}$ we hang the $W(\mathfrak{g})$-orbit of $[\mathbf{t}(x)]$ in $\mathcal{E}\otimes\mathrm{Q}$. As x-varies, so does the orbit, spanning a curve $\mathcal{C}_u$ in $\mathbb{CP}^1_{\langle x\rangle}\times\mathcal{E}\otimes\mathrm{Q}$, which projects down to $\mathbb{CP}^1_{\langle x\rangle}$. However $\mathcal{C}_u$ is not yet the cameral curve $\mathcal{C}_u$. It is however relatively straightforward to lift $\mathcal{C}_u$ to $\mathcal{C}_u$ using the abelianized determinant line bundle over $\mathcal{E}^r$ we discussed in Section 9.8.

Now let us discuss the period map. The moduli space $\mathfrak{P}$ carries the holomorphic symplectic structure Ω which descends from (11.13). From the complexified textbooks on classical mechanics we learn, as we reviewed in Section 4.4, that choosing some basis $A_{\mathcal{I}}$, $B^{\mathcal{I}}$, in the integral homology $H_1(\mathcal{A}_u, \mathbb{Z})$ lattice, such that the cup product of the basis vectors obeys (4.11) and then defining $\mathfrak{a}^{\mathcal{I}}$, $\mathfrak{a}^D_{\mathcal{I}}$ using (4.9), (4.10), one verifies (4.12). It remains to compare this definition of the special coordinates with the periods of the differentials $x\,\mathrm{d}\log y_i(x)$.

Let us work in the domain $|\mathfrak{q}_i| \ll 1$. The cycles $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{B}^{\mathcal{I}}$ which we defined (cf. Figures 5.1 and 8.2) on the cameral curve $\mathcal{C}_u$ define the corresponding one-cycles on $\mathcal{C}_u = \mathcal{C}_u/\mathrm{Q}^\vee$ and consequently on $\mathcal{A}_u$.

Now let us compute the symplectic form Ω on the reduced phase space $\mathfrak{P}$. Using that in our gauge

$$\int_{\mathcal{E}} \mathrm{d}^2 z A_{\bar z}(x, \bar x) = \xi(x),$$

we obtain

$$\Omega = \int \mathrm{d}^2 x \, \langle \delta a_{\bar x} \wedge \delta \xi \rangle = \frac{1}{|W(\mathfrak{g}|} \int_{\mathcal{C}_u} \langle \delta \bar a \wedge \delta \varphi \rangle \,,$$

where we denoted by $\bar a$ the $W(\mathfrak{g}$-equivariant $\mathbf{T}$-gauge field on $\mathcal{C}_u$ and by $\delta\varphi$ an $\mathfrak{h}$-valued, $W(\mathfrak{g})$-equivariant $(1,0)$-form on $\mathcal{C}_u$, given by

$$\delta\varphi = x\delta\xi.$$

It is now easy to pin down the periods of $\mathrm{d}\mathbb{S}$ among the periods of $\delta\varphi$. One uses the cycles in $\mathcal{A}_u$, where the monodromy of the gauge field $\bar a$ along the cycle $\mathcal{A}_{\mathcal{I}}$ or $\mathcal{B}^{\mathcal{I}}$ changes by 2π.

To conclude this section, in parallel with the class I story we conjecture, that *the theories of class II lead to the representation theory of the Yangians built on the Kac–Moody algebras, i.e., the toroidal algebras.* It would be nice to see whether the Kac–Moody symmetry of instanton moduli spaces [28, 117] leads to the quasi-classical limit of the Yangian action, at the level of the phase space $\mathfrak{P}^{\mathrm{ext}}$ symmetry.

11.3 Noncommutative instantons and class II* theories

In this subsection, we show that the data $t_i(x)$ with the cross-cut transformations (6.8) correspond naturally to the charge N instantons on the noncommutative space $\mathbb{R}^2 \times \mathbb{T}^2$, with the gauge group $U(r+1)$. We shall use the constructions analogous to the constructions [139] of instantons on the noncommutative $\mathbb{R}^4$ adapted to the periodic case. The noncommutative solitons on a cylinder were studied in [27].

The idea of the construction is the following. Consider first the commutative situation. Let us denote the coordinates on $\mathbb{R}^2$ by x_1, x_2. Let $(u_1, u_2) \in \mathbb{T}^2 \subset \mathbb{C}^\times \times \mathbb{C}^\times$, $|u_1| = |u_2| = 1$. Let us denote by z, x the holomorphic local coordinates on $\mathbf{C}_{\langle x\rangle} \times \mathcal{E}$,

$$x = x_1 + \mathrm{i}x_2, \qquad z = \frac{1}{2\pi\mathrm{i}}(\log(u_1) + \tau \log(u_2)).$$

Consider the $(0,1)$-component of $U(r+1)$ gauge field $\bar A = A_{\bar z}\,\mathrm{d}\bar z + A_{\bar x}\,\mathrm{d}\bar x$, where $A_{\bar z}$, $A_{\bar x}$ are the $(r+1)\times(r+1)$ complex matrices. They obey (11.12) and in the gauge (11.21). we have

$$A_{\bar z} = \frac{2\pi\mathrm{i}}{\bar\tau - \tau}\,\mathrm{diag}(\xi_i(x,\bar x))_{i=1}^{r+1}, \tag{11.21}$$

where, by the considerations analogous to those around (11.18), we conclude that $\bar\partial_{\bar x}\xi_i = 0$. Again, there are the residual gauge transformations

$$A_{\bar z} \mapsto G^{-1} A_{\bar z} G + G^{-1}\partial_{\bar z} G,$$

which permute the eigenvalues $\xi_i(x)$: $\{\xi_i(x)\} \mapsto \{\xi_{\sigma(i)}(x)\}$, for some $\sigma \in \mathcal{S}_{r+1}$, and shift them

$$\xi_i(x) \mapsto \xi_i(x) + l_i - \tau n_i. \tag{11.22}$$

The latter are generated by the diagonal gauge transformations

$$G = \operatorname{diag}\left(u_1^{n_i} u_2^{l_i}\right)_{i=1}^{r+1}.$$

We can partially reduce the ambiguity (11.22) by passing to the exponential variables

$$t_j(x) = \mathrm{e}^{2\pi \mathrm{i} \xi_j(x)}, \qquad j = 1, \dots, r+1.$$

The residual gauge transformations (11.22) become

$$t_j(x) \mapsto \mathfrak{q}^{n_j} t_j(x), \qquad t_j(x) \mapsto t_{\sigma(j)}(x)$$

for some integers $n_i \in \mathbb{Z}$ and permutations $\sigma \in \mathcal{S}_{r+1}$.

Let us now consider the noncommutative case. Let us replace the algebra of functions of u_1, u_2, x_1, x_2 by the noncommutative algebra, with the generators $\widehat{u}_1$, $\widehat{u}_2$, $\widehat{x}_1$, $\widehat{x}_2$, obeying

$$\widehat{u}_1^{-1} \widehat{x}_i \widehat{u}_1 = \widehat{x}_i + \hbar_i \cdot 1, \qquad \widehat{u}_2^{-1} \widehat{x}_i \widehat{u}_2 = \widehat{x}_i, \qquad \widehat{u}_1 \widehat{u}_2 = \widehat{u}_2 \widehat{u}_1, \qquad [\widehat{x}_1, \widehat{x}_2] = 0, \tag{11.23}$$

where

$$\frac{\mathfrak{m}}{r+1} = \hbar = \hbar_1 + \mathrm{i}\hbar_2.$$

Then the analogue of (11.22) for

$$G = \operatorname{diag}\left(\widehat{u}_1^{n_i} \widehat{u}_2^{l_i}\right)_{i=1}^{r+1}$$

gives

$$t_i(x) \mapsto q^{n_i} t_i\Big(x - \frac{n_i}{r+1}\mathfrak{m}\Big), \qquad t_i(x) \mapsto t_{\sigma(i)}(x).$$

These are precisely the ${}^i\mathcal{W}$-transformations of the class II* type $\widehat{A}_r$ gauge theory, with $t_i(x)$ given in (6.6) and (10.38).

Instead of giving more systematic discussion along the lines of [4, 21, 43, 139, 153, 165] let us comment on the relation to the group $\widehat{\mathrm{GL}}_\infty$. We claim that the $U(r+1)$ instantons on the noncommutative $\mathbb{R}^2 \times \mathbb{T}^2$ with both $\mathbb{R}^2$ and $\mathbb{T}^2$ separately commutative, can be interpreted as the commutative periodic monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$ with the gauge group $\widehat{\mathrm{GL}}_\infty$.

The idea is to interpret the noncommutative $U(r+1)$ gauge fields on $\mathbb{R}^2 \times \mathbb{T}^2$ as the $\widehat{\mathrm{GL}}_\infty$-gauge fields on $\mathbb{R}^2_{x_1,x_2} \times \mathbb{S}^1_\psi$, using the relation of the group $\widehat{\mathrm{GL}}_\infty$ to the quantization of the volume-preserving diffeomorphisms of a cylinder $\mathbb{R}^1_p \times \mathbb{S}^1_q$, i.e., the pseudo-differential operators ΨDO on $\mathbb{S}^1_q$. Here the generators $\widehat{u}_{1,2}$, $\widehat{x}_{1,2}$ of the algebra (11.23) are related to the commutative coordinates (x_1, x_2, ψ) of the monopole theory and the generators $(p, \mathrm{e}^{\mathrm{i}q})$ of $\widehat{\mathrm{GL}}_\infty$ in the via

$$\widehat{u}_2 = \mathrm{e}^{\mathrm{i}\psi}, \qquad \widehat{u}_1 = \mathrm{e}^{\mathrm{i}q}, \qquad \widehat{x}_1 = x_1 - \mathrm{i}\hbar_1 \partial_q, \qquad \widehat{x}_2 = x_2 - \mathrm{i}\hbar_2 \partial_q.$$

12 Higher-dimensional theories

In this section, we briefly go over the higher-dimensional generalizations. We shall consider the lifts of all our theories to five dimensions, and the lifts of some of our theories to six dimensions. The latter restriction comes from the possibility of encountering gauge and mixed anomalies in six dimensions, which would prohibit the decoupling of the gauge sector from the supergravity and ultimately string theory modes.

The classical $\mathcal{N}=2$ theories in four dimensions admit a canonical lift to the $\mathcal{N}=1$ theories in six dimensions. Under this lift the vector multiplets become the vector multiplets, and the hypermultiplets become the hypermultiplets. The structure of the hypermultiplet does not change, while the structure of the vector multiplet does change, for the complex scalar Φ in the adjoint representation becomes the remaining component $A_5+\mathrm{i}A_6$ of the gauge field.

We then compactify the theory on a two-torus $\mathbb{T}^2$. In addition to the metric on the torus we shall also fix a background B-field, a constant two-form. The combined metric and the two-form moduli are described by a 2×2 matrix G, with the positive definite symmetric part g. It is convenient to parametrize G by two complex numbers T, U, with $\operatorname{Im}U, \operatorname{Im}T \geq 0$. The parameter T encodes the complex structure of $\mathbb{T}^2$, while the parameter U is the complexified Kähler class:

$$U=\int_{\mathbb{T}^2} B+\mathrm{i}\int_{\mathbb{T}^2}\sqrt{\det(g)}, \qquad T=\frac{g_{12}+\mathrm{i}\sqrt{\det(g)}}{g_{11}}.$$

Five(six-)-dimensional supersymmetric gauge theory compactified on a circle (two-torus) would look four-dimensional at low energy. The microscopic gauge coupling τ in four dimensions is proportional to U while T determines the complex geometry of the x-plane $\mathbf{C}_{\langle x\rangle}$. One can study the corresponding Seiberg–Witten geometry. Its key feature compared to special geometry of more traditional four-dimensional models is the periodicity in the x-variable [138]. In the five-dimensional theory the x-plane becomes the cylinder $\mathbf{C}_{\langle x\rangle}{}^{\times}$, while in the six-dimensional theory the x-plane becomes the two-torus $\mathbf{C}_{\langle x\rangle}/2\pi\mathrm{i}(\mathbb{Z}\oplus T\mathbb{Z})$. It has to do with the large gauge transformations. The result of these additional symmetries is the relativistic nature of the corresponding integrable systems. For example, the periodic Toda chain describing the pure $\mathcal{N}=2$ theory in four dimensions becomes the relativistic Toda chain. The Hamiltonians of the relativistic systems have periodic dependence on momenta, which are the rapidities of the particles. The resulting quantized Hamiltonians are the difference operators.

Our discussion modifies in the case of five-dimensional gauge theories in two aspects. First, the notion of the amplitude function accommodates the large gauge transformations:

$$\mathcal{Y}_i(x)=\exp\left\langle \operatorname{tr}_{\mathbf{v}_i}\log\left(\mathrm{e}^{\mathrm{i}\beta(x-\Phi_i)}-1\right)\right\rangle_u,$$

where the dimension length β characterizes the circumference of the compactification circle. The limit shape integral equations (6.1) generalize straightforwardly, with the kernel

$$\mathcal{K}_\beta(\mathrm{x})=\frac{\mathrm{i}\beta}{12}\mathrm{x}^3-\frac{\log(\mathrm{i}\beta\Lambda_{\mathrm{UV}})}{2}\mathrm{x}^2-\frac{1}{\mathrm{i}\beta^2}\operatorname{Li}_3\left(\mathrm{e}^{-\mathrm{i}\beta\mathrm{x}}\right). \tag{12.1}$$

Secondly, there are additional couplings in five dimensions: the levels k_i of the Chern–Simons interactions $\mathrm{CS}_5(A^i)$, $i\in\mathrm{Vert}_\gamma$. Effectively the Chern–Simons term changes the gauge coupling $\mathfrak{q}_i$ to the x-dependent quantity

$$\mathfrak{q}_i\longrightarrow\mathfrak{q}_i\mathrm{e}^{\mathrm{i}\beta k_i x}.$$

In the six-dimensional case the amplitude and the kernel (12.1) modify to

$$\mathcal{Y}_i(x)=\exp\left\langle \operatorname{tr}_{\mathbf{v}_i}\log\theta\left(\mathrm{e}^{\mathrm{i}\beta(x-\Phi_i)};Q\right)\right\rangle_u,$$

$$\mathcal{K}_\beta(\mathrm{x}) = \frac{\mathrm{i}\beta}{12}\mathrm{x}^3 - \frac{\log(\mathrm{i}\beta\Lambda_{\mathrm{UV}})}{2}\mathrm{x}^2 - \frac{1}{\mathrm{i}\beta^2}\operatorname{Li}_3\left(\mathrm{e}^{-\mathrm{i}\beta\mathrm{x}}\right) - \sum_{n=1}^{\infty}\left(\operatorname{Li}_3\left(\mathrm{e}^{\mathrm{i}\beta\mathrm{x}}Q^n\right) + \operatorname{Li}_3\left(\mathrm{e}^{-\mathrm{i}\beta\mathrm{x}}Q^n\right)\right),$$

where $Q = \exp(2\pi\mathrm{i}T)$, and now β^2 is the scale of the area of the compactification torus $\mathbb{T}^2$.

A McKay correspondence, D-branes, and M-theory

A.1 From finite groups to Lie groups

McKay correspondence states [125] that the affine ADE graphs γ can be constructed from the representation theory of finite subgroups Γ of SU(2).

The vertices $i \in \mathrm{Vert}_\gamma$ in this case are the irreducible representations of Γ, $i \mapsto \mathcal{R}_i$. One usually enumerates them, $\mathrm{Vert}_\gamma = \{0, 1, \ldots, r\}$, so that 0 corresponds to the trivial representation $\mathcal{R}_0 = \mathbb{C}$. By r in this section we denote the number of nodes in the finite graph γ_{fin} obtained by discarding the node "0" from affine γ. In all cases r is the rank of the Cartan matrix associated to Dynkin graph. Moreover, the Dynkin marks a_i are the dimensions of V_i, and the numbers of colors in affine quivers is

$$\mathbf{v}_i = Na_i,$$

where $a_i = \dim \mathcal{R}_i$, and N is some non-negative integer. As we said above, $i = 0$ corresponds to the trivial representation, so $a_0 = 1$. In the table of McKay correspondence we write "$i(a_i)$" by each node to denote its label i and Dynkin mark a_i. We label the nodes in Bourbaki conventions.

The number of edges I_{ij} in the McKay graph between the node i and the node j is the multiplicity of representation $\mathcal{R}_j$ in the tensor product of $\mathcal{R}_i$ with the defining representation $\mathbb{C}^2$

$$\mathbb{C}^2 \otimes \mathcal{R}_i = \bigoplus_j \mathbb{C}^{I_{ij}} \otimes \mathcal{R}_j.$$

The equation $\beta_i = 0$ is verified by computing the dimensions of the left and the right-hand sides.

The order of Γ agrees with the dimensions of the irreducible representations / Dynkin marks of the McKay/Dynkin affine graph computed using the standard relation from the orthogonality of characters

$$|\Gamma| = \sum_i \dim \mathcal{R}_i^2 = \sum_i a_i^2.$$

McKay's observation is that γ is affine ADE Dynkin diagram, with the trivial representation $\mathcal{R}_0$ associated with the affine node "0". The finite Dynkin graph γ_{fin} is always a tri-star graph $T_{a,b,c}$ with one trivalent vertex and three legs containing a, b, c vertices (where in the counting we included the center trivalent vertex). Hence the rank of the finite quiver γ_{fin} is $a+b+c-2$. In fact a, b, c have simple interpretation: the group Γ is always a Coxeter group $\mathrm{Cox}(a,b,c)$ defined on three generators (x,y,z) subject to relation

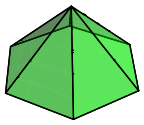
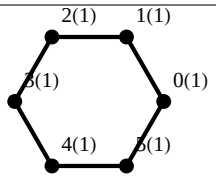
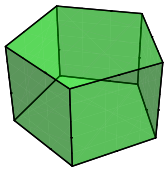
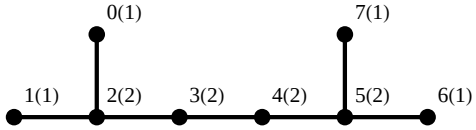
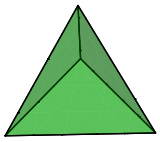
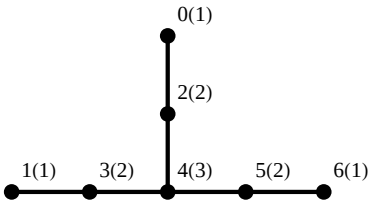
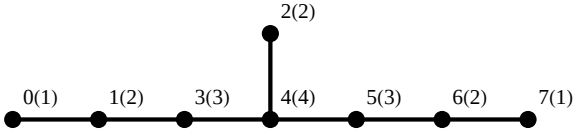
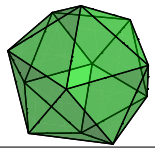
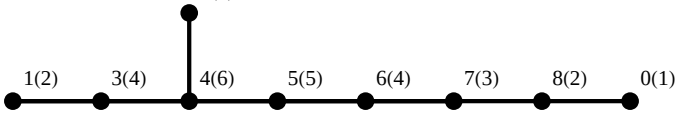
$$x^a = y^b = z^c = xyz.$$

The affine graph of $\widehat{E}_r$ series is also a trivalent graph denoted by $\left(\widehat{T}_{a,b,c}\right)$ in the table of McKay correspondence. Note that for each of the three cases $\widehat{E}_6$, $\widehat{E}_7$, $\widehat{E}_8$ the identity

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

holds, which makes contact with the unimodular parabolic singularities.

Now we shall list explicitly the discrete subgroups of SU(2), the classification going back to Plato and Klein [105].

Polyhedron	Γ	$T_{a,b,c}$	Affine Dynkin graph γ with labels $i(a_i)$	Lie $\widehat{\mathfrak{g}}$
	$\mathbb{Z}_{r+1}$ (r=5)	$T_{r,1,1}$		$\widehat{A}_r$ (r=5)
	$\mathbb{BD}_{r-2}$ (r=7)	$T_{r-2,2,2}$		$\widehat{D}_r$ (r=7)
	$\mathbb{BT}$	$T_{3,3,2}$ ($\widehat{T}_{3,3,3}$)		$\widehat{E}_6$
	$\mathbb{BO}$	$T_{4,3,2}$ ($\widehat{T}_{4,4,2}$)		$\widehat{E}_7$
	$\mathbb{BI}$	$T_{5,3,2}$ ($\widehat{T}_{6,3,2}$)		$\widehat{E}_8$

A.2 Platonic symmetries

A.2.1 Cyclic group

Let us present cyclic group $\mathbb{Z}_{r+1}$ in the (a,b,c) notation: consider group generated by x, y, z subject to

$$x^r = y = z = xyz.$$

Then $z = y$ and $xy = 1$ and $y = x^r$, hence

$$x^{r+1} = 1$$

hence $\mathbb{Z}_{r+1} = \mathrm{Cox}(r,1,1) = \mathrm{McKay}(A_r)$. The order of Γ is

$$|\mathbb{Z}_{r+1}| = r+1 = \sum_{i\in\gamma} a_i^2 = (r+1)\cdot 1.$$

A.2.2 Dihedral group

The dihedral group $\mathbb{D}_{r-2} \simeq \mathbb{Z}_{r-2} \rtimes \mathbb{Z}_2$ of order $2(r-2)$ is the symmetry group of regular $(r-2)$-gon considered as a subgroup of SO(3). The lift of dihedral group $\mathbb{Z}_2 \to \mathbb{BD}_{r-2} \to \mathbb{D}_{r-2}$ to a subgroup of SU(2) is called bidihedral group $\mathbb{BD}$, which has order $4(r-2)$. The $\mathbb{BD}_r$ is generated by (x,y,z) subject to relations

$$x^{r-2} = y^2 = z^2 = xyz$$

hence $\mathbb{BD}_{r-2} = \mathrm{Cox}(r-2,2,2) = \mathrm{McKay}(D_r)$. The order of Γ is

$$|\mathbb{BD}_{r-2}| = 4(r-2) = \sum_{i\in\gamma} a_i^2 = (r-3)\cdot 4 + 4\cdot 1.$$

A.2.3 Tetrahedral group

The tetrahedral group $\mathbb{T} \simeq A_4$, of order 12, is the subgroup of SO(3) realizing the symmetries of tetrahedron. The lift of tetrahedral group $\mathbb{Z}_2 \to \mathbb{BT} \to \mathbb{T}$ to a subgroup of SU(2) is called bitetrahedral group $\mathbb{BT}$, which has order 24 and is generated by (x, y, z) subject to the relations

$$x^3 = y^3 = z^2 = xyz.$$

Hence $\mathbb{BT} = \mathrm{Cox}(3,3,2) = \mathrm{McKay}(E_6)$. The order of Γ is

$$|\mathbb{BT}| = 24 = \sum_{i\in\gamma} a_i^2 = 3\cdot\left(1^2+2^2\right)+3^2 = 24.$$

A.2.4 Octahedral group

The octahedral group $\mathbb{O} \simeq S_4$, of order 24, is the subgroup of SO(3) realizing the symmetries of the cube/octahedron. The lift to SU(2) is called bioctahedral group $\mathbb{BO}$, of order 48; the $\mathbb{BO}$ can be generated by (x, y, z) subject to relations

$$x^4 = y^3 = z^2 = xyz$$

associated to the symmetry axes of the cube/octahedron of the order $4, 3$ and 2. Hence $\mathbb{BO} = \mathrm{Cox}(4,3,2) = \mathrm{McKay}(E_7)$. The order of Γ is

$$|\mathbb{BO}| = 48 = \sum_{i\in\gamma} a_i^2 = 2\cdot\left(1^2+2^2+3^2\right)+4^2+2^2.$$

A.2.5 Icosahedral group

The icosahedral group $\mathbb{I} \simeq A_5$, of order 60, is the subgroup of SO(3) realizing the symmetries of the icosahedron/dodecahedron The lift to SU(2) is called biicosahedral group $\mathbb{BO}$, of order 120; the biicosahedral group $\mathbb{BI}$ can be generated by (x, y, z) subject to relations

$$x^5 = y^3 = z^2 = xyz$$

associated to the symmetry axes of the icosahedron/dodecahedron of the order 5, 3 and 2. Hence $\mathbb{BO} = \mathrm{Cox}(5,3,2) = \mathrm{McKay}(E_8)$. The order of Γ is

$$|\mathbb{BI}| = 120 = \sum_{i\in\gamma} a_i^2 = \left(1^2+2^2+3^2+4^2+5^2+6^2\right)+2^2+4^2+3^2.$$

A.3 D-branes at singularities

The physical explanation of the relation between the $\mathbb{C}^2/\Gamma$-singularities [108] and the superconformal theories with $\mathcal{N}=2$ supersymmetry is the following. Consider the IIB string in the background $\mathbb{C}^2/\Gamma \times \mathbb{C}^1 \times \mathbb{R}^{1,3}$, where $\Gamma \subset \mathrm{SU}(2) \subset \mathrm{SO}(4)$ acts on $\mathbb{C}^2 \approx \mathbb{R}^4$ by the hyperkähler rotations. This background preserves half of the ten-dimensional IIB supersymmetry. Now add a stack of N regular D3 branes. At the orbifold singularity the regular D3 brane splits as a collection of $r+1$ types of fractional branes, which correspond to the irreducible representations of Γ [42, 92]. Moreover, the regular brane contains a_i fractional branes of the $\mathcal{R}_i$ type. The stack of N regular brane splits, therefore, as a collection of Na_i fractional branes of $\mathcal{R}_i$ type, for all $i = 0, \ldots, r$. Each cluster of fractional branes can be classically moved anywhere in the two-plane, transverse to the singularity and the D3 brane worldvolume. To summarize, the D3 branes are located at:

$$0 \times \left\{\mu_0^{Na_0}, \mu_1^{Na_1}, \ldots, \mu_r^{Na_r}\right\} \times \mathbb{R}^{1,3}.$$

Here $\{\mu_0^{v_0},\dots,\mu_r^{v_r}\}$ represents the positions of the v_i copies of the $\mathcal{R}_i$ type of the fractional branes in the two-dimensional plane $\mathbb{C}^1\approx\mathbb{R}^2$ transverse to the singular ALE space $\mathbb{C}^2/\Gamma$ and the worldvolume $\mathbb{R}^{1,3}$ of the branes. The positions μ_i are the ones which enter the equation (2.5). At the low energy the worldvolume theory on the D3 branes is the quiver gauge theory we are studying. If all μ_i parameters are scaled to zero (or at least all coincide), the theory has no scale except for the string scale which is absent in the low energy description. Therefore the theory has the scaling invariance, which is promoted to the full $\mathcal{N}=2$ superconformal invariance. The couplings τ_i, except for

$$\tau=\sum_{i=0}^{r}a_i\tau_i, \tag{A.1}$$

which corresponds to the IIB dilaton-axion field, are not of geometric origin. The gauge couplings τ_i, $i=1,\dots,r$, come from the twisted sector fields

$$\tau_i=\int_{\Sigma_i}B_{\mathrm{RR}}+\tau B_{\mathrm{NS}},$$

where B_{NS}, B_{RR} are the Neveu–Schwarz (NSNS) and the Ramond–Ramond two-form fields of the IIB supergravity, and Σ_i stand for the non-trivial two-cycles in the resolved $\widetilde{\mathbb{C}^2/\Gamma}$ geometry. These exceptional cycles are well-known to correspond to the simple roots of $\mathfrak{g}_{\mathrm{q}}$.

The 4d gauge theory on the stack of D3 branes in the above geometry is $\mathcal{N}=2$ supersymmetric theory of the affine ADE quiver type. To find the algebraic integrable system associated to the IR solution of such theory we compactify the worldvolume of the D3 branes (x_0,x_1,x_2,x_3) on a circle S^1, say, along x_3. Hence, our setup is N D3 branes along $\mathbb{R}^{1,2}\times\mathbb{S}^1$ inside the IIB string theory on $\mathbb{R}^{1,2}\times\mathbb{S}^1_{\langle x_3\rangle}\times\mathbf{C}_{\langle x\rangle}\times\mathbb{C}^2/\Gamma$. Here $\mathbf{C}_{\langle x\rangle}$ is the real-two-dimensional space associated with the scalars of $\mathcal{N}=2$ vector multiplet in 4 dimensions. For 4d theories this is affine complex line $\mathbf{C}_{\langle x\rangle}=\mathbb{C}$, for 5d theories on S^1 this is a cylinder $\mathbf{C}_{\langle x\rangle}=\mathbb{C}^\times$ and for 6d theories on T^2 this is a torus $\mathbf{C}_{\langle x\rangle}=\mathcal{E}_{\langle x\rangle}$.

Given this type IIB string theory realization of the affine ADE quiver theory, first we perform T-duality along $\mathbb{S}^1_{\langle x_3\rangle}$. The stack of N D3-branes converts to the stack of N D2-branes on $\mathbb{R}^{2,1}$ in type IIA string theory on $\mathbb{R}^{1,2}\times\mathbb{S}^1_{\langle x_3\rangle}\times\mathbf{C}_{\langle x\rangle}\times\mathbb{C}^2/\Gamma$. Next we lift the type IIA string to M-theory on $\mathbb{R}^{1,2}\times\mathbb{S}^1_{\langle x_3\rangle}\times\mathbf{C}_{\langle x\rangle}\times\mathbb{C}^2/\Gamma\times\mathbb{S}^1_{\langle x_{10}\rangle}$, where $\mathbb{S}^1_{\langle x_{10}\rangle}$ is the M-theory circle. The radius of M-theory circle is determined by IIB coupling constant (A.1). We treat the product of two circles along x_3 and along x_{10} as the elliptic curve $\mathcal{E}=\mathbb{S}^1_{\langle x_3\rangle}\times\mathbb{S}^1_{\langle x_{10}\rangle}$ which has elliptic modulus $\mathfrak{q}=\mathrm{e}^{2\pi\mathrm{i}\tau}$. The stack of N D2 branes converts to the stack of N M2 branes along $\mathbb{R}^{1,2}$. So finally we arrive to the M-theory picture.

That is, we consider the following configuration in M-theory: the space-time background is

$$\mathbb{R}^{1,2}\times X_4\times\mathbb{C}^2/\Gamma \tag{A.2}$$

to which we add a stack of N M2 branes whose worldvolume is $\mathbb{R}^{1,2}$ in (A.2) localized at the orbifold singularity in $\mathbb{C}^2/\Gamma$, and anywhere in $X_4=\mathbb{C}_{\langle x\rangle}\times\mathcal{E}_{\mathfrak{q}}$. The M-theory on $\mathbb{C}^2/\Gamma$ is believed to contain the seven-dimensional G-gauge theory with sixteen supersymmetries, localized at the singular locus. The maximal torus T part of the gauge bosons and their superpartners descend from the supergravity modes associated with the hyperkähler deformations of the metric on $\mathbb{R}^4/\Gamma$. In particular, the gauge fields are the modes of the three-form field C_3 integrated along the collapsed two-cycles Σ_i, $i=1,\dots,r$ in the deformed geometry $\widetilde{\mathbb{R}^4/\Gamma}$. The remaining W-bosons come from the M2-branes wrapped on various two-cycles in $H_2(\widetilde{\mathbb{R}^4/\Gamma},\mathbb{Z})\approx\mathrm{Q}$.

In the IR, the stack of N M2 branes stretched along $\mathbb{R}^{1,2}$ in this 7d G-gauge theory on $\mathbb{R}^{1,2}\times X_4$ dissolve into the charge N G-instantons on $X_4=\mathbf{C}_{\langle x\rangle}\times\mathcal{E}_{\mathfrak{q}}$ [13, 26, 86, 95, 96, 101, 176].

The moduli space of framed charge N G-instantons on X_4 is the phase space of the algebraic integrable system (see Section 4) we have found from the microscopic four-dimensional instanton counting in the quiver gauge theory.

B Partitions and free fermions

Recall that a partition λ is a non-increasing sequence of integers, stabilizing at zero,

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0 = 0 = 0 = \cdots).$$

The number $\ell(\lambda) \geq 0$ is called the *length* of the partition λ, while

$$|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i$$

is called the *size* of the partition λ.

Consider the theory of a single chiral complex fermion, which is a $(1/2, 0)$-differential, in two dimensions

$$\mathcal{L} = \int \tilde{\psi} \bar{\partial} \psi. \tag{B.1}$$

The theory (B.1) has a $U(1)$-symmetry

$$\left(\psi, \tilde{\psi}\right) \mapsto \left(\mathrm{e}^{\mathrm{i}\alpha}\psi, \mathrm{e}^{-\mathrm{i}\alpha}\tilde{\psi}\right). \tag{B.2}$$

One can couple the fermion to the background abelian gauge field corresponding to this symmetry:

$$\bar{\partial} \mapsto \bar{\partial} + \bar{A},$$

so that the Lagrangian (B.1) deforms to

$$\mathcal{L} \to \mathcal{L} + \int \bar{A} J,$$

where $J = \tilde{\psi}\psi$ is the $U(1)$ current, which we define as an operator below.

In studying the space of states $\mathcal{H}$ corresponding to the quantization of the theory (B.2) one can distinguish various sectors, corresponding to twisted boundary conditions. Expand

$$\psi(z) = \left(\frac{\mathrm{d}z}{z}\right)^{\frac{1}{2}} \sum_{r \in \mathbb{Z}+\alpha} \psi_r z^{-r}, \qquad \tilde{\psi}(z) = \left(\frac{\mathrm{d}z}{z}\right)^{\frac{1}{2}} \sum_{r \in \mathbb{Z}-\alpha} \tilde{\psi}_r z^{-r},$$

where $\alpha = \int_{\mathbb{S}^1} \bar{A}$. The fermion modes ψ_r, $\tilde{\psi}_s$ form the Clifford algebra

$$\left\{\psi_r, \tilde{\psi}_s\right\} = \delta_{r+s,0}, \qquad \left\{\psi_r, \psi_s\right\} = \left\{\tilde{\psi}_r, \tilde{\psi}_s\right\} = 0.$$

The vacuum state $|\varnothing; \alpha\rangle$ is annihilated by all the α-positive harmonics

$$\psi_r |\varnothing; \alpha\rangle = 0, \qquad r > \alpha,$$
$$\tilde{\psi}_r |\varnothing; \alpha\rangle = 0, \qquad r > -\alpha.$$

The space of states is built by acting on $|\varnothing;\alpha\rangle$ with creation operators ψ_r, $r \leq \alpha$, and $\tilde{\psi}_s$, $s \leq -\alpha$. The resulting Hilbert space $\mathcal{H}_\alpha$ has a basis labeled by partitions:

$$\mathcal{H}_\alpha = \bigoplus_\lambda \mathbb{C}|\lambda;\alpha\rangle,$$
$$|\lambda;\alpha\rangle = \psi_{-\lambda_1+\alpha}\psi_{-\lambda_2+1+\alpha}\cdots\psi_{-\lambda_{\ell(\lambda)}+\ell(\lambda)-1+\alpha},\tilde{\psi}_{-\alpha}\tilde{\psi}_{-\alpha-1}\cdots\tilde{\psi}_{-\alpha+1-\ell(\lambda)}|\varnothing;\alpha\rangle.$$

Define the normal ordering with respect to the $\alpha = 0$ vacuum:

$$:\psi_i\tilde{\psi}_j: = \begin{cases}\psi_i\tilde{\psi}_j, & j > 0,\\ -\tilde{\psi}_j\psi_i, & j \leq 0.\end{cases}$$

The $U(1)$ symmetry of the Lagrangian (B.1) is promoted to the $\widehat{u}(1)$ current algebra symmetry. It is generated by the operator

$$J(z) = :\tilde{\psi}(z)\psi(z): = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}\mathrm{d}z,$$
$$J_n = \sum_{r\in\mathbb{Z}+\alpha} :\tilde{\psi}_r\psi_{n-r}:.$$

The generating function

$$\sum_\lambda \mathfrak{q}^{|\lambda|}t^{\ell(\lambda)} = \prod_{n=1}^\infty \frac{1}{1-t\mathfrak{q}^n}$$

is a character of the fundamental $\widehat{u(1)}$ module, $\mathcal{H}_0$.

C Lie groups and Lie algebras

In this section we fix our notations for the notions from the Lie group and Lie algebra theory we are using in the work. In this section we work over the field $\mathbb{C}$ of complex numbers.

C.1 Finite-dimensional Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simply-laced simple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra. Let $\mathbf{G}$ be the corresponding *simply-connected* simple Lie group, and $\mathbf{T} \subset \mathbf{G}$ the corresponding to $\mathfrak{h}$ maximal torus. We have the exponential map $\exp_\mathfrak{h}\colon \mathfrak{h} \to \mathbf{T}$, which is a restriction on $\mathfrak{h}$ of the exponential map $\exp_\mathfrak{g}\colon \mathfrak{g} \to \mathbf{G}$. We shall only use the $\exp_\mathfrak{h}$ map in this work and will omit the $\mathfrak{h}$ subscript in what follows. We also use the notation

$$\mathbf{e}(\mathbf{x}) = \exp(2\pi\mathrm{i}\mathbf{x}) = \mathrm{e}^{2\pi\mathrm{i}\mathbf{x}} \in \mathbf{T}, \qquad \text{for } \mathbf{x} \in \mathfrak{h}. \tag{C.1}$$

C.1.1 The coroots

The kernel of the map (C.1), i.e., the set $\mathrm{Q}^\vee \subset \mathfrak{h}$ which is mapped to the identity element in $\mathbf{T}$, is called the coroot lattice (it is obviously an abelian group). The coroot lattice $\mathrm{Q}^\vee$ has the rank $r = \mathrm{rk}\mathbf{G} = \dim\mathfrak{h}$. Let us denote by $\alpha_i^\vee$, $i = 1,\ldots,r$ its integral basis

$$\mathrm{Q}^\vee = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\alpha_2^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_r^\vee.$$

The generators $\alpha_i^\vee$ are called the simple coroots.

Now, let $z \in \mathbb{C}^\times$, and $\alpha^\vee \in \mathrm{Q}^\vee$, then

$$z^{\alpha^\vee} \equiv \mathbf{e}\left(\frac{\log(z)}{2\pi\mathrm{i}}\alpha^\vee\right) \in \mathbf{T} \tag{C.2}$$

is independent of the choice of the branch of the $\log(z)$. This identifies the coroot lattice with the lattice of homomorphisms:

$$\mathrm{Q}^\vee = \mathrm{Hom}\big(\mathbb{C}^\times, \mathbf{T}\big). \tag{C.3}$$

Using equation (C.2), we can parametrize $\mathbf{T}$ by

$$\mathbf{z} = (z_1, \dots, z_r) \mapsto t_{\mathbf{z}} = \prod_{i=1}^{r} z_i^{\alpha_i^\vee},$$

where $z_i \in \mathbb{C}^\times$.

C.2 The weights

The dual lattice is called the lattice of weights

$$\Lambda = \mathrm{Hom}\big(\mathbf{T}, \mathbb{C}^\times\big).$$

Let us represent element $t \in \mathbf{T}$ as

$$t = \mathbf{e}(\mathbf{x}) \tag{C.4}$$

for some $\mathbf{x} \in \mathfrak{h}$, which is defined, as we recall, up to an a shift by an element of the coroot lattice $\mathrm{Q}^\vee$. Then for any $\lambda \in \Lambda$, the value of the homomorphism λ on t, which we denote by $t^\lambda \in \mathbb{C}^\times$, can be computed as

$$t^\lambda = \mathrm{e}^{2\pi\mathrm{i}\lambda(\mathbf{x})}, \tag{C.5}$$

where $\lambda(\mathbf{x})$ is a linear function of $\mathbf{x}$. In this way we view λ as an element of $\mathfrak{h}^*$, so that $\Lambda \subset \mathfrak{h}^*$. In order for (C.5) be independent of the choice of $\mathbf{x}$ in (C.4), the value of λ on any element of the coroot lattice must be an integer

$$\lambda\big(\mathrm{Q}^\vee\big) \subset \mathbb{Z},$$

which is another definition of the dual lattices. Let us fix the basis $(\lambda_i)_{i=1}^r$ of Λ:

$$\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \oplus \cdots \oplus \mathbb{Z}\lambda_r$$

dual to the basis $(\alpha_i^\vee)_{i=1}^r$ of simple coroots, so that

$$\lambda_i\big(\alpha_j^\vee\big) = \delta_{ij}.$$

C.2.1 The roots

The torus $\mathbf{T}$ acts on $\mathbf{G}$ via the adjoint action

$$\mathrm{Ad}\colon\ \mathbf{T} \times \mathbf{G} \to \mathbf{G}, \qquad (t, g) \mapsto t^{-1}gt.$$

Infinitesimally, it acts linearly on the Lie algebra

$$\mathrm{Ad}\colon\ \mathbf{T} \longrightarrow \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{Ad}_t(\xi) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} t^{-1}\exp(s\xi)t \in \mathfrak{g} \tag{C.6}$$

for $\xi \in \mathfrak{g}$, and finally, this defines an action of $\mathfrak{h}$ on $\mathfrak{g}$:

$$\mathrm{ad}\colon\ \mathfrak{h} \to \mathrm{End}(\mathfrak{g}), \qquad \mathrm{ad}_{\mathbf{x}}(\xi) = [\xi, \mathbf{x}] \in \mathfrak{g}$$

for $\mathbf{x} \in \mathfrak{h}$, $\xi \in \mathfrak{g}$.

This action gives us several structures: the root decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathbb{C} e_\alpha,$$

where $R \subset \Lambda$ is a set of non-vanishing weights of the adjoint representation

$$\mathrm{ad}_t(e_\alpha) = t^\alpha e_\alpha.$$

These weights are called roots, and the lattice $\mathrm{Q} \subset \Lambda$ they generate is a sublattice of Λ, called the *root lattice*. It has a basis $(\alpha_i)_{i=1}^r$ of *positive simple roots*

$$\mathrm{Q} = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \cdots \oplus \mathbb{Z}\alpha_r,$$

which allows us to define the *Cartan matrix* $C^{\mathfrak{g}}$ of $\mathfrak{g}$:

$$C^{\mathfrak{g}}_{ij} = \alpha_i\big(\alpha_j^\vee\big), \tag{C.7}$$

which is non-degenerate. The additional requirement we impose on $\mathfrak{g}$ is that it is *simply-laced*, i.e., by an appropriate choice of integral bases one can make $C^{\mathfrak{g}}$ symmetric:

$$C^{\mathfrak{g}}_{ij} = C^{\mathfrak{g}}_{ji}.$$

C.2.2 The center

The quotient

$$Z = \Lambda/\mathrm{Q}$$

is an abelian group, which is a subgroup both of $\mathbf{T}$ and $\mathbf{G}$. In fact, it is the center of $\mathbf{G}$. Clearly, the center does not act in the adjoint representation, so that in the equation (C.6) it is the quotient $\mathbf{T}/Z$ which acts faithfully. Hence the root lattice

$$\mathrm{Q} \subset \Lambda \subset \mathfrak{h}^*$$

can be also identified with the lattice of $\mathbf{T}/Z$ characters

$$\mathrm{Q} = \mathrm{Hom}\big(\mathbf{T}/Z, \mathbb{C}^\times\big).$$

Finally, the coweight lattice $\Lambda^\vee \subset \mathfrak{h}$ is both the integral dual to Q, and the lattice of $\mathbf{T}/Z$ cocharacters:

$$\Lambda^\vee = \mathrm{Hom}\big(\mathbb{C}^\times, \mathbf{T}/Z\big) \tag{C.8}$$

with the basis $\big(\lambda_i^\vee\big)_{i=1}^r$, dual to that of Q:

$$\alpha_i\big(\lambda_j^\vee\big) = \delta_{ij}.$$

The expression

$$w^{\lambda^\vee} = \mathbf{e}\left(\frac{\log(w)}{2\pi\mathrm{i}}\lambda^\vee\right) \in \mathbf{T}$$

$\mathbf{G}$	$z_{\mathfrak{g}}$	ℓ_ξ, $\xi=1,\dots,a_{\mathfrak{g}}$
A_r	1	$r+1$
D_{2s}	2	2, 2
D_{2s+1}	1	4
E_6	1	3
E_7	1	2
E_8	0	

Table C.1. The number of cyclic factors in Z with their orders.

does depend on the choice of the branch of the logarithm $\log(w)$, however, the ambiguity is in the multiplicative Z-valued factor, since for any $\lambda^\vee \in \mathrm{Q}^\vee$:

$$\mathbf{e}\big(\lambda^\vee\big) \in Z \subset \mathbf{T}.$$

Thus, for any $w \in \mathbb{C}^\times$, $w^{\lambda^\vee}$ is well-defined as an element of $\mathbf{T}/Z$, as is claimed by (C.8).

The center Z, being a finite abelian group, is isomorphic to a product of cyclic groups

$$Z \approx \bigotimes_{\xi=1}^{z_{\mathfrak{g}}} \mathbb{Z}/\ell_\xi\mathbb{Z}$$

for some $z_{\mathfrak{g}}$, which is equal to 0, 1, or 2.

Table C.1 shows the values of $z_{\mathfrak{g}}$'s and ℓ_ξ's for all simple simply-laced Lie groups.

Let $\varpi_\xi \in \mathbf{T}$ be the generator of the $\mathbb{Z}/\ell_\xi\mathbb{Z}$ factor in $Z \subset \mathbf{T}$. In other words,

$$\varpi_\xi^k \neq 1 \quad \text{for } k=1,2,\dots,\ell_\xi-1 \qquad \text{and} \qquad \varpi_\xi^{\ell_\xi} = 1 \in T. \tag{C.9}$$

Of course, the equation (C.9) does not characterize ϖ_ξ uniquely. Indeed, for any integer s_ξ, which is mutually prime with ℓ_ξ, i.e., $\big(s_\xi,\ell_\xi\big)=1$, the Z element $\widetilde{\varpi_\xi} = \varpi_\xi^{s_\xi}$ also generates $\mathbb{Z}/\ell_\xi\mathbb{Z}$. We write

$$\begin{aligned}
&\mathbf{e}\big(\lambda_i^\vee\big) = \prod_{\xi=1}^{z_{\mathfrak{g}}} \varpi_\xi^{l_{i\xi}}, \qquad i=1,\dots,r,\\
&\varpi_\xi = \prod_{j=1}^{r} \mathbf{e}\left(\frac{w_{\xi j}}{\ell_\xi}\alpha_j^\vee\right)
\end{aligned} \tag{C.10}$$

for some integers $l_{i\xi}, w_{\xi j} \in \mathbb{Z}$, which are normalized

$$0 \le l_{i\xi} < \ell_\xi, \qquad 0 \le w_{\xi j} < \ell_\xi \tag{C.11}$$

and

$$\big(w_{\xi j},\ell_\xi\big) = 1.$$

Note that

$$\lambda_i\big(\lambda_j^\vee\big) = \big(C^{\mathfrak{g}}\big)^{-1}_{ij} \in \mathbb{Q}. \tag{C.12}$$

By combining equations (C.12) with (C.10), we derive

$$\big(C^{\mathfrak{g}}\big)^{-1}_{ij} = \sum_{\xi=1}^{z_{\mathfrak{g}}} \frac{l_{i\xi}w_{\xi j}}{\ell_\xi} + \mathcal{L}^{\mathfrak{g}}_{ij}, \tag{C.13}$$

where $\mathcal{L}^{\mathfrak{g}}$ is some integral matrix. Note in passing that if we were to study the group G over the field $\mathbb{F}_{p^n}$ where p divides ℓ_y then $C^{\mathfrak{g}}_{ij}$ would correspond to an affine (or even double affine) root system.

C.2.3 Killing metric

Another structure we gain from the adjoint action of T on $\mathfrak{g}$ is the Killing metric on $\mathfrak{h}$,

$$\langle x, x\rangle = \frac{1}{h_{\mathfrak{g}}} \operatorname{tr}_{\mathfrak{g}} \operatorname{ad}_x^2, \tag{C.14}$$

which identifies $\mathfrak{h}$ with $\mathfrak{h}^*$, $\mathrm{Q}^\vee$ with Q, and $\Lambda^\vee$ with Λ, and the constant $h_{\mathfrak{g}}$ is chosen so that

$$\langle \lambda_i, \alpha_j\rangle = \delta_{ij}, \qquad \langle \alpha_i, \alpha_j\rangle = C^{\mathfrak{g}}_{ij}, \qquad \langle \lambda_i, \lambda_j\rangle = (C^{\mathfrak{g}}_{ij})^{-1}.$$

C.2.4 The Weyl group

The torus $\mathbf{T}$ has some symmetries within $\mathbf{G}$. Namely, for any $t \in \mathbf{T}$ there are transformations of the form:

$$t \mapsto g^{-1} t g \tag{C.15}$$

for some $g \in \mathbf{G}$, for which $g^{-1}tg \in \mathbf{T}$ also. Such transformations form a group, which is called the *normalizer* $N(\mathbf{T})$ *of* $\mathbf{T}$. This group obviously contains $\mathbf{T}$, since the transformation (C.15) with $g \in \mathbf{T}$ doesn't even move t. It turns out that there are additional nontrivial transformations. These additional transformations form *the Weyl group* $W(\mathfrak{g}) = N(\mathbf{T})/\mathbf{T}$.

For $t \in \mathbf{T}$ and $w \in W(\mathfrak{g})$ let us denote by ${}^w t$ the result of the application of g_w representing w in $N(\mathbf{T}) \subset \mathbf{G}$:

$${}^w t = g_w^{-1} t g_w.$$

By taking the limit $t \to 1$ we get the action of $W(\mathfrak{g})$ on $\mathfrak{h}$: for $\xi \in \mathfrak{h}$

$$\xi \mapsto {}^w\xi = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} {}^w\big(\exp s\xi\big).$$

It is clear that $W(\mathfrak{g})$ acts on $\mathfrak{h}$ by the orthogonal transformations preserving the metric (C.14). The action of $W(\mathfrak{g})$ on $\mathfrak{h}^*$ ($\mathfrak{h}$) preserves both the (co)root lattice and the (co)weight lattice.

The less trivial result is that the group $W(\mathfrak{g})$ is generated by reflections r_i at the simple roots. The corresponding transformations on $\mathfrak{h}$ and $\mathbf{T}$ are

$${}^{r_i}x = x - \alpha_i(x)\alpha_i^\vee, \qquad {}^{r_i}t = t\big(t^{-\alpha_i}\big)^{\alpha_i^\vee}.$$

The $W(\mathfrak{g})$-action on $\mathbf{T}$ can be also described in the z-coordinates

$${}^{r_i}t_{\mathbf{z}} = t_{\mathbf{z}}\big(t_{\mathbf{z}}^{-\alpha_i}\big)^{\alpha_i^\vee} = t_{\tilde{\mathbf{z}}},$$

where

$$\tilde{z}_j = z_j, \qquad j \neq i, \qquad \tilde{z}_i = z_i \prod_{k=1}^{r} z_k^{-C^{\mathfrak{g}}_{ik}}. \tag{C.16}$$

C.3 Weyl group and the center

Since $W(\mathfrak{g})$ acts on $\mathbf{T}$ by similarity transformations in $\mathbf{G}$, the center $Z \subset \mathbf{T} \subset \mathbf{G}$ is fixed by any $w \in W(\mathfrak{g})$. In particular, r_i preserves ϖ_ξ for any i and ξ, which is equivalent to

$$\varpi_\xi^{\alpha_i} = 1. \tag{C.17}$$

Substituting into equation (C.17) the representation (C.11) and recalling the definition (C.7) of $C^{\mathfrak{g}}$, we get

$$\frac{1}{\ell_\xi}\sum_{j=1}^{r} w_{\xi j} C^{\mathfrak{g}}_{ji} = \mathfrak{C}^{\mathfrak{g}}_{\xi i} \in \mathbb{Z}, \qquad \xi = 1,\dots,z_{\mathfrak{g}}, \quad i = 1,\dots,r.$$

Combining this relation with equation (C.13), we obtain

$$\delta_{ij} = \sum_{\xi=1}^{z_{\mathfrak{g}}} l_{i\xi}\mathfrak{C}^{\mathfrak{g}}_{\xi j} + \sum_{k=1}^{r} \mathcal{L}^{\mathfrak{g}}_{ik} C^{\mathfrak{g}}_{kj}.$$

On the other hand, using the equation (C.10), and the relations $\sum_{j=1}^{r} C^{\mathfrak{g}}_{ij}\lambda^\vee_j = \alpha^\vee_i$, and $\mathbf{e}(\alpha^\vee_i) = 1$ for any i, we derive

$$\prod_{j=1}^{r}\prod_{\xi=1}^{z_{\mathfrak{g}}} \varpi_\xi^{C^{\mathfrak{g}}_{ij} l_{j\xi}} = 1,$$

which implies

$$\frac{1}{\ell_\xi}\sum_{j=1}^{z_{\mathfrak{g}}} C^{\mathfrak{g}}_{ij} l_{j\xi} = \mathfrak{c}^{\mathfrak{g}}_{i\xi} \in \mathbb{Z}.$$

C.3.1 Langlands dual, adjoint, and conformal groups

The simple Lie group $\mathbf{G}/Z$ has a trivial center, but it is not simply-connected. This group is also denoted by $\mathbf{G}^{\mathrm{ad}}$, since it is represented faithfully in the adjoint representation $\mathfrak{g}$. The maximal torus $\mathbf{T}^{\mathrm{ad}}$ of $\mathbf{G}^{\mathrm{ad}}$, is the quotient $\mathbf{T}/Z = \mathfrak{h}/\mathrm{Q}$. Also, since the lattices of weights and roots of $\mathbf{G}$ and $\mathbf{G}^{\mathrm{ad}}$ are dual to each other, these groups are Langlands duals, ${}^{\mathrm{L}}\mathbf{G} = \mathbf{G}^{\mathrm{ad}}$.

The group $\mathbf{G}^{\mathrm{ad}}$ is not very convenient to work with. For one thing, the center Z looks differently for different groups. One defines the *conformal extension* $\mathrm{C}\mathbf{G}$ of $\mathbf{G}$ as the group (see [57, 115, 116] for recent applications in a related context),

$$\mathrm{C}\mathbf{G} = (\mathrm{C}\times\mathbf{G})/Z,$$

where

$$\mathrm{C} = (\mathbb{C}^\times)^{z_{\mathfrak{g}}},$$

where the center Z acts on $\mathbf{G}$ in the usual way, and on C via some character

$$\chi \in \mathrm{Hom}(Z, \mathrm{C}).$$

For example, we can choose

$$\chi(\varpi_x) = \left(\zeta_{\ell_y}^{\delta_{x,y}}\right)_{y=1}^{z_{\mathfrak{g}}},$$

where ζ_l is the primitive l-th root of unity

$$\zeta_l = \mathrm{e}^{2\pi\frac{\mathrm{i}}{l}}.$$

The elements of the group $\mathrm{C}\mathbf{G}$ are the classes of pairs $[(\mathfrak{b}; g)]$, where $\mathfrak{b}\in\mathrm{C}$ is the $z_{\mathfrak{g}}$-tuple of non-zero complex numbers, $\mathfrak{b} = (\mathfrak{b}_1,\dots,\mathfrak{b}_{z_{\mathfrak{g}}})$, and $g\in\mathbf{G}$, under the equivalence $(\mathfrak{b}; g) \sim ({}^x\mathfrak{b}; \varpi_x g)$, where

$${}^x\mathfrak{b} = \left(\mathfrak{b}_y \zeta_{\ell_x}^{-\delta_{x,y}}\right)_{y=1}^{z_{\mathfrak{g}}}$$

with the multiplication law

$$[(\mathfrak{b}_1; g_1)] \cdot [(\mathfrak{b}_2; g_2)] = [(\mathfrak{b}_1\mathfrak{b}_2; g_1 g_2)].$$

The maximal torus C$\mathbf{T}$ of C$\mathbf{G}$ has rank $r + z_{\mathfrak{g}}$. The center CZ of C$\mathbf{G}$ is the subgroup of C$\mathbf{G}$ which consists of the equivalence classes containing $(\mathfrak{b}; 1)$, with $\mathfrak{b} \in \mathrm{C}$. Clearly, C$Z \approx$ C. In this way, the centers of the conformal extensions look the same for all groups of equal $z_{\mathfrak{g}}$.

The lattice

$$\mathrm{CQ}^\vee = \mathrm{Hom}(\mathbb{C}^\times, \mathrm{C}\mathbf{T}) \tag{C.18}$$

is an extension of $\mathrm{Q}^\vee$ by the rank $z_{\mathfrak{g}}$ lattice with the generators $\beta_\xi^\vee$, $\xi = 1, \dots, z_{\mathfrak{g}}$:

$$\mathrm{CQ}^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i^\vee \oplus \bigoplus_{\xi=1}^{z_{\mathfrak{g}}} \mathbb{Z}\beta_\xi^\vee.$$

The root subspace decomposition of $\mathfrak{cg} = \mathrm{Lie}(\mathrm{C}\mathbf{G}) = \mathfrak{g} \oplus \mathbb{C}^{z_{\mathfrak{g}}}$ is easy to compute. There are r simple roots which we denote by α_i. They act on $\mathfrak{ch} = \mathrm{Lie}(\mathrm{C}\mathbf{T})$ as follows:

$$\alpha_i(\alpha_j^\vee) = C_{ji}^{\mathfrak{g}}, \qquad \alpha_i(\beta_\xi^\vee) = \mathfrak{C}_{\xi i}^{\mathfrak{g}}, \tag{C.19}$$

so that if we define

$$K_\xi = -\ell_\xi \beta_\xi^\vee + \sum_{j=1}^{r} w_{\xi j}\alpha_j^\vee, \tag{C.20}$$

then

$$\alpha_i(K_\xi) = 0 \tag{C.21}$$

for any i, ξ. The meaning of (C.20) is the following. The lattice of cocharacters of the maximal torus of $\mathrm{C} \times \mathbf{G}$ is the direct sum of the lattice $\mathrm{Q}^\vee$ of coroots of G and the lattice

$$\mathrm{Hom}(\mathbb{C}^\times, \mathrm{C} \times \mathbf{T}) = L \oplus \mathrm{Q}^\vee, \qquad L = \pi_1(\mathrm{C}) = \bigoplus_{\eta=1}^{z_{\mathfrak{g}}} \mathbb{Z}K_\eta,$$

so that a generic element of $\mathrm{C} \times \mathbf{T}$ can be represented by

$$\widehat{t}_{\mathfrak{b},\mathbf{g}} = \prod_{\xi=1}^{z_{\mathfrak{g}}} \mathfrak{b}_\xi^{K_\xi} \times \prod_{i=1}^{r} g_i^{\alpha_i^\vee} \tag{C.22}$$

(again, in writing (C.22) we assume normalization $\mathbf{e}(K_\xi) = 1 \in \mathrm{C}$.) The action of Z on $\mathrm{C} \times \mathbf{G}$ translates to the action on $\mathfrak{b}, \mathbf{g}$:

$$\varpi_\xi \colon\ \widehat{t}_{\mathfrak{b},\mathbf{g}} \mapsto \widehat{t}_{\mathfrak{b},\mathbf{g}} \zeta_{\ell_\xi}^{-K_\xi + \sum_{i=1}^{r} w_{\xi i}\alpha_i^\vee}.$$

The quotient C$\mathbf{T} = (\mathrm{C} \times \mathbf{T})/Z$ is coordinatized by

$$[(\mathfrak{b}; t_{\mathbf{z}})] = \check{t}_{\mathfrak{u},\mathbf{z}} = \prod_{\xi=1}^{z_{\mathfrak{g}}} \mathfrak{u}_\xi^{\beta_\xi^\vee} \prod_{i=1}^{r} z_i^{\check{\alpha}_i^\vee} \in \mathrm{C}\mathbf{T}, \qquad \mathfrak{u}_\xi = \mathfrak{b}_\xi^{-\ell_\xi}, \qquad z_i = g_i \prod_{\xi=1}^{z_{\mathfrak{g}}} \mathfrak{b}_\xi^{w_{\xi i}} \tag{C.23}$$

in agreement with (C.18). Simply put, $\beta_\xi^\vee \in \mathrm{Hom}(\mathbb{C}^\times, \mathrm{C}\mathbf{T})$, $\ell_\xi\beta_\xi^\vee \in \mathrm{Hom}(\mathbb{C}^\times, \mathrm{C} \times \mathbf{T})$, and $\beta_\xi^\vee \notin \mathrm{Hom}(\mathbb{C}^\times, \mathrm{C} \times \mathbf{T})$.

Clearly, there are other choices of the Z-invariant coordinates, which differ by multiplication of z_i by any function of $\mathfrak{u}_\xi$'s.

Comparing the equations (C.23) and (C.22), we arrive at (C.20). The equation (C.21) simply reflects the fact that C acts trivially in the adjoint representation. The lattice CQ generated by α_i is isomorphic to the root lattice Q.

The weight lattice $\widetilde{\mathrm{C}\Lambda}$ of $\mathrm{C}\times\mathbf{G}$ is the direct sum of the weight lattice of C and that of $\mathbf{G}$:

$$\widetilde{\mathrm{C}\Lambda} = \mathrm{Hom}(\mathrm{C}\times\mathbf{T},\mathbb{C}^\times) = L^\vee\oplus\Lambda.$$

The weight lattice $\mathrm{C}\Lambda$ of $\mathrm{C}\mathbf{G}$ is a sublattice of $\widetilde{\mathrm{C}\Lambda}$ which consists of the weights which are trivial on the elements of $\mathrm{C}\times\mathbf{T}$ of the form

$$\zeta_{\ell_\xi}^{-K_\xi+\sum_{i=1}^r w_{\xi i}\alpha_i^\vee}$$

for any ξ. In the $\widetilde{\mathrm{C}\Lambda}$-basis $(\mu_\xi;\lambda_i)$, $\xi=1,\dots,z_\mathfrak{g}$, $i=1,\dots,r$ such that

$$\begin{aligned}&\mu_\xi(K_\eta)=\delta_{\xi,\eta}, &&\mu_\xi\big(\alpha_i^\vee\big)=0,\\ &\lambda_i(K_\eta)=0, &&\lambda_i\big(\alpha_j^\vee\big)=\delta_{ij}\end{aligned}$$

the lattice $\mathrm{C}\Lambda$ is spanned by

$$\check\mu_\xi=-\ell_\xi\mu_\xi,\qquad \check\lambda_i=\lambda_i+\sum_\xi w_{\xi i}\mu_\xi.$$

An easy computation gives

$$\begin{aligned}&\check\mu_\xi\big(\beta_\eta^\vee\big)=\delta_{\xi\eta}, &&\check\lambda_i\big(\alpha_j^\vee\big)=\delta_{ij},\\ &\check\mu_\xi\big(\alpha_i^\vee\big)=0, &&\check\lambda_i\big(\beta_\eta^\vee\big)=0.\end{aligned}$$

Finally, the lattice $\mathrm{C}\Lambda^\vee$ of coweights of $\mathrm{C}\mathbf{G}$, which is dual to the root lattice CQ, is generated by the fundamental coweights $\check\lambda_i^\vee$, which obey

$$\alpha_i\big(\check\lambda_j^\vee\big)=\delta_{ij}. \tag{C.24}$$

The equations (C.24) define $\check\lambda_i^\vee$ up to the shifts by the integer multiples of K_ξ. We choose the representative

$$\check\lambda_i^\vee=\sum_{j=1}^r\mathcal{L}_{ij}^\mathfrak{g}\alpha_j^\vee+\sum_{\xi=1}^{z_\mathfrak{g}}l_{i\xi}\beta_\xi^\vee.$$

Shifting $\check\lambda_i^\vee$ by $\mathrm{d}_\xi K_\xi$ would change the coefficient $l_{i\xi}\mapsto l_{i\xi}+\ell_\xi\mathrm{d}_\xi$.

As we see, the coweights $\check\lambda_i^\vee$ are the integral linear combinations of the coroots $\alpha^\vee$ and $\beta^\vee$, as if $\mathrm{C}\mathbf{G}$ were the simple simply-connected group with the trivial center.

C.3.2 The D-homomorphism

By construction of the conformal group, there is a homomorphism

$$\begin{aligned}&D\colon\ \mathrm{C}\mathbf{G}\longrightarrow\mathrm{C}/Z\approx\mathrm{C},\\ &D[(\mathfrak{b};g)]=\big(\mathfrak{b}_\xi^{\ell_\xi}\big)_{\xi=1}^{z_\mathfrak{g}}.\end{aligned}$$

In terms of the weights, the restriction of the homomorphism D onto $C\mathbf{T}$ is given by

$$D\big(\check t_{\mathfrak{u},\mathbf{z}}\big)=\big(\check t_{\mathfrak{u},\mathbf{z}}^{\check\mu_\xi}\big)_{\xi=1}^{z_\mathfrak{g}}=(\mathfrak{u}_\xi)_{\xi=1}^{z_\mathfrak{g}}.$$

C.3.3 Twisted Weyl group action

Now we can define the twisted action of Weyl group $W(\mathfrak{g})$. It is nothing but the natural action of $W(\mathfrak{g})$ on $C\mathbf{T}$. However, we shall encounter a somewhat redundant parametrization of $C\mathbf{T}$, by $\mathbf{T}\times\mathbf{T}$:

$$\mathbf{g}_{\mathcal{P},\mathcal{Y}} = \prod_{i=1}^{r} \mathcal{P}_i^{-\check{\lambda}_i^\vee} \mathcal{Y}_i^{\check{\alpha}_i^\vee}, \tag{C.25}$$

which is equal to $\check{t}_{\mathbf{u},\mathbf{z}}$ with

$$\mathfrak{u}_\xi = \prod_j \mathcal{P}_j^{-l_{j\xi}}, \qquad z_i = \mathcal{Y}_i \prod_j \mathcal{P}_j^{-\mathcal{L}_{ji}}.$$

Using (C.19), we compute the Weyl group $W(\mathfrak{g})$ action on $C\mathbf{T}$: under the reflection r_i the group element $\mathbf{g}_{\mathcal{P},\mathcal{Y}}$ maps to ${}^{r_i}\mathbf{g}_{\mathcal{P},\mathcal{Y}} = \mathbf{g}_{\mathcal{P},\tilde{\mathcal{Y}}}$, where

$$\tilde{\mathcal{Y}}_j = \mathcal{Y}_j, \qquad j\neq i, \qquad \tilde{\mathcal{Y}}_i = \mathcal{P}_i\mathcal{Y}_i \prod_j \mathcal{Y}_j^{-C^{\mathfrak{g}}_{ij}}. \tag{C.26}$$

The homomorphism D is $W(\mathfrak{g})$-invariant

$$D(\mathbf{g}_{\mathcal{P},\mathcal{Y}}) = D(\mathbf{g}_{\mathcal{P},\tilde{\mathcal{Y}}}) = \left(\prod_{j=1}^{r} \mathcal{P}_j^{-l_{j\xi}}\right)_{\xi=1}^{z_{\mathfrak{g}}}.$$

C.4 Affine Lie algebras

The affine Lie algebras $\widehat{\mathfrak{g}}$ show up in the solution of the theories of the class II. In preparing this section we consulted with [9, 93]. Given a simple Lie group $\mathbf{G}$ with its Lie algebra $\mathfrak{g}$ one defines the loop group $\mathrm{L}\mathbf{G}$ and the loop algebra $\mathrm{L}\mathfrak{g}$ of (suitably defined, analytic, formal, polynomial) maps of the neighborhood of a circle in $\mathbb{C}^\times$ into $\mathbf{G}$ and $\mathfrak{g}$, respectively,

$$\mathrm{L}\mathbf{G} = \mathrm{Maps}(\mathbb{C}^\times, \mathbf{G}), \qquad \mathrm{L}\mathfrak{g} = \mathrm{Maps}(\mathbb{C}^\times, \mathfrak{g}), \tag{C.27}$$

with the point-wise product and Lie brackets, respectively. Then one defines the central extension $\widetilde{\mathrm{L}\mathfrak{g}} = \mathrm{L}\mathfrak{g}\oplus\mathbb{C}K$ by

$$[f_1(t)\oplus a_1K, f_2(t)\oplus a_2K] = [f_1(t), f_2(t)] \oplus \oint_{\mathbb{S}^1} \langle \mathrm{d}f_1(t), f_2(t)\rangle K$$

and the additional (non-central) extension $\widehat{\mathfrak{g}}$ of $\widetilde{\mathrm{L}\mathfrak{g}}$ by $\mathbb{C}$, which acts on $\mathrm{L}\mathfrak{g}$ by the infinitesimal rotation of $\mathbb{C}^\times$ in (C.27). The elements of $\widehat{\mathfrak{g}}$ can be represented as the $\mathfrak{g}$-valued first-order differential operators on $\mathbb{S}^1$ plus a constant:

$$\tau\mathrm{d} + f(t)\oplus aK \in \widehat{\mathfrak{g}}, \qquad \tau, a\in\mathbb{C}, f(t)\in\mathfrak{g}$$

with the commutation relations

$$\begin{aligned}&[\tau_1\mathrm{d} + f_1(t)\oplus a_1K, \tau_2\mathrm{d} + f_2(t)\oplus a_2K]\\ &\quad = 0\cdot\mathrm{d} + t\left(\tau_1 f_2'(t) - \tau_2 f_1'(t)\right) + [f_1(t), f_2(t)] \oplus \oint_{\mathbb{S}^1} \langle f_1(t), \mathrm{d}f_2(t)\rangle K.\end{aligned} \tag{C.28}$$

Analogously, one defines the central extension $\widetilde{\mathrm{L}\mathbf{G}}$ which is a non-trivial $\mathbb{C}^\times$-bundle over $\mathrm{L}\mathbf{G}$ (this is analogous to the construction of the conformal group $\mathrm{C}\mathbf{G}$ in the previous section), and then the extension $\widehat{\mathbf{G}}$ of $\widetilde{\mathrm{L}\mathbf{G}}$ by $\mathbb{C}^\times$ which acts by rotation of $\mathbb{C}^\times$ in (C.27).

The algebra $\widehat{\mathfrak{g}}$ can also be defined by the general construction of the generalized Kac–Moody algebras associated with the Cartan matrix $C_{ij}^{\widehat{\mathfrak{g}}}$. The affine Cartan matrix has exactly one eigenvector with the eigenvalue zero:

$$\sum_{j=0}^{r} C_{ij}^{\widehat{\mathfrak{g}}} a_j = 0. \tag{C.29}$$

The Cartan subalgebra $\widehat{\mathfrak{h}}$ of the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$ with the underlying finite-dimensional simply-laced Lie algebra $\mathfrak{g}$ of rank r is the complex vector space of dimension $r+2$. The dual space $\widehat{\mathfrak{h}}^*$ contains the root lattice $\widehat{\mathrm{Q}} \subset \widehat{\mathfrak{h}}^*$, which is generated by the simple roots $\widehat{\alpha}_i$, $i=0,\dots,r$, of which the simple roots $\widehat{\alpha}_i$ with $i>0$ generate the root lattice of $\mathfrak{g}$. Likewise, the Cartan subalgebra $\widehat{\mathfrak{h}}$ contains the coroot lattice $\widehat{\mathrm{Q}}^\vee \subset \widehat{\mathfrak{h}}$, generated by the simple coroots $\alpha_i^\vee$, which obey:

$$\widehat{\alpha}_i\big(\widehat{\alpha}_j^\vee\big) = C_{ij}^{\widehat{\mathfrak{g}}}.$$

The following linear combination of the simple roots:

$$\delta = \sum_{i=0}^{r-1} a_i \widehat{\alpha}_i \in \widehat{\mathfrak{h}}^* \tag{C.30}$$

is called the imaginary root. It annihilates the simple coroots, cf. (C.29),

$$\delta\big(\widehat{\alpha}_i^\vee\big) = 0, \qquad i = 0,\dots,r-1.$$

The analogous linear combination of coroots

$$K = \sum_{i=0}^{r-1} a_i \widehat{\alpha}_i^\vee \in \widehat{\mathfrak{h}} \tag{C.31}$$

obeys

$$\widehat{\alpha}_i(K) = 0, \qquad i = 0,\dots,r \tag{C.32}$$

and generates the center of the affine Kac–Moody algebra since (C.32) implies it commutes with everything in $\widehat{\mathfrak{g}}$. Then

$$\widehat{\mathrm{Q}}_{\mathbb{C}}^\vee = \mathfrak{h} \oplus \mathbb{C}K, \qquad \widehat{\mathrm{Q}}_{\mathbb{C}} = \mathfrak{h}^* \oplus \mathbb{C}\delta,$$

where $\mathfrak{h}$ and $\mathfrak{h}^*$ are the Cartan subalgebra and its dual space of the corresponding finite-dimensional simply-laced Lie algebra $\mathfrak{g}$, respectively. In order to generate $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$ as the vector spaces over $\mathbb{C}$ we need to add one more generator in addition to the simple roots and the simple coroots, respectively:

$$\widehat{\mathfrak{h}}^* = \widehat{\mathrm{Q}}_{\mathbb{C}} \oplus \mathbb{C}\lambda_0, \qquad \widehat{\mathfrak{h}} = \widehat{\mathrm{Q}}_{\mathbb{C}}^\vee \oplus \mathbb{C}\delta^\vee,$$

which obey:

$$\widehat{\lambda}_0\big(\widehat{\alpha}_i^\vee\big) = \widehat{\alpha}_i\big(\delta^\vee\big) = \delta_{i0}.$$

The generator $\delta^\vee$ is equal to the generator of the infinitesimal loop rotation d we used in (C.28).

The weight lattice $\widehat{\Lambda}$ of $\widehat{\mathfrak{g}}$ is generated by the fundamental weights $\lambda_i \in \widehat{\mathfrak{h}}^*$, $i = 0, 1, \dots, r$:

$$\widehat{\Lambda} = \Lambda \oplus \mathbb{Z}\lambda_0 \subset \widehat{\mathfrak{h}}^*,$$

which obey the following basic relations:

$$\widehat{\lambda}_i\big(\widehat{\alpha}_j^\vee\big) = \delta_{ij}, \qquad \widehat{\lambda}_i\big(\delta^\vee\big) = 0, \qquad \delta\big(\delta^\vee\big) = a_0 = 1.$$

The fundamental coweights $\widehat{\lambda}_i^\vee \in \widehat{\mathfrak{h}}$ obey

$$\widehat{\alpha}_j\big(\widehat{\lambda}_i^\vee\big) = \delta_{ij}$$

and form the coweight lattice

$$\widehat{\Lambda}^\vee = \bigoplus_{i=0}^{r} \mathbb{Z}\lambda_i^\vee \subset \mathfrak{h}.$$

The level of a weight $\widehat{\lambda} \in \widehat{\Lambda}$ is defined as $k = \widehat{\lambda}(K)$ so that the level of the i-th fundamental weight is equal to

$$\widehat{\lambda}_i(K) = a_i = \delta\big(\widehat{\lambda}_i^\vee\big).$$

Note that $\widehat{\mathrm{Q}}_{\mathbb{C}}^\vee$ is the Cartan subalgebra of the central extension $\widehat{\mathrm{L}\mathfrak{g}}$ of the loop algebra $L\mathfrak{g}$. Adding $\delta^\vee$ makes up the Cartan of the affine Kac–Moody algebra of $\mathfrak{g}$, which is an extension of $\widehat{\mathrm{L}\mathfrak{g}}$ by the operator of the infinitesimal loop rotation, the zero mode of the Virasoro generator. In physics literature the more common notation for $\delta^\vee$ is L_0.

Also we notice that

$$\lambda_i\big(\lambda_j^\vee\big) = (C^{\mathfrak{g}})^{-1}_{ij}, \qquad i, j = 1, \dots, r,$$

hence

$$\widehat{\lambda}_i\big(\widehat{\lambda}_j^\vee\big) = \begin{pmatrix} 0 & 0 \\ 0 & (C^{\mathfrak{g}})^{-1} \end{pmatrix}_{ij}$$

A useful relation obtained from identity operator $|\Lambda_i\rangle\langle\alpha_i^\vee| + |\delta\rangle\langle\delta^\vee|$ is

$$\sum_{i=0}^{r} \widehat{\lambda}_i\big(\widehat{\lambda}_j^\vee\big)\widehat{\alpha}_k\big(\widehat{\alpha}_i^\vee\big) = \widehat{\alpha}_k\big(\widehat{\lambda}_j^\vee\big) - \delta\big(\widehat{\lambda}_j^\vee\big)\alpha_k\big(\delta^\vee\big) = \delta_{jk} - a_j\delta_{k0}.$$

Let $\lambda \in \mathfrak{h}^*$ for $\widehat{\lambda} \in \widehat{\mathfrak{h}}^*$ denote the image of the projection $\widehat{\mathfrak{h}}^* \to \mathfrak{h}^*$, i.e., forgetting components spanned by δ and $\widehat{\lambda}_0$. Then

$$\widehat{\lambda}_i^\vee = a_i\delta^\vee + \lambda_i^\vee, \qquad \widehat{\lambda}_j = a_j\widehat{\lambda}_0 + \lambda_j.$$

D ADE Cartan matrices, roots and weights

Our ADE conventions are summarized in the table in Appendix A.

D.1 A_r series

The Cartan matrix $C_{ij}^{\widehat{A}_r}$ with $i,j=0,\dots,r$

$$C^{\widehat{A}_r}=\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & -1\\ -1 & 2 & -1 & \dots & \dots & 0\\ 0 & -1 & 2 & -1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \dots & 2 & -1\\ -1 & 0 & \dots & \dots & -1 & 2 \end{pmatrix}$$

The affine simple roots are

$$\widehat{\alpha}_0=\delta-\theta,\quad \widehat{\alpha}_1=e_1-e_2,\quad \widehat{\alpha}_2=e_2-e_3,\quad \dots,\quad \widehat{\alpha}_r=e_r-e_{r+1},$$

where $\theta=e_1-e_{r+1}$ is the highest root. The Dynkin marks are $a_i=1$, $i=0,\dots,r$. The fundamental weights are

$$\begin{aligned}
&\widehat{\lambda}_0,\\
&\widehat{\lambda}_1=\widehat{\lambda}_0+e_1-\frac{1}{r}\mathbf{e},\\
&\widehat{\lambda}_2=\widehat{\lambda}_0+e_1+e_2-\frac{2}{r}\mathbf{e},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\widehat{\lambda}_r=\widehat{\lambda}_0+e_1+e_2+\dots+e_r-\frac{r}{r+1}\mathbf{e},
\end{aligned}$$

where

$$\mathbf{e}=e_1+\dots+e_r.$$

The inverse Cartan matrix of the block $i,j=1\dots r$, i.e., the inverse Cartan matrix $\left(C^{A_r}\right)^{-1}$ of $\mathfrak{g}=A_r$ is

$$\lambda_i\left(\lambda_j^\vee\right)=\left(C^{A_r}\right)^{-1}_{ij}=\max(i,j)-\frac{ij}{r+1},\qquad i,j=1,\dots,r.$$

D.2 D_r series

In the standard basis $\{e_i\mid i=1,\dots,r\}$ the simple roots of D_r are

$$\alpha_i=e_i-e_{i+1},\qquad i=1,\dots,r-1,\qquad \alpha_r=e_{r-1}+e_r$$

and the fundamental weights are

$$\begin{aligned}
&\lambda_i=e_1+e_2+\dots+e_i,\qquad i=1\dots r-2,\\
&\lambda_{r-1}=\frac{1}{2}(e_1+e_2+\dots+e_{r-1}-e_r),\\
&\lambda_r=\frac{1}{2}(e_1+e_2+\dots+e_{r-1}+e_r).
\end{aligned}$$

The highest root is $\theta=\sum_{i=1}^r\alpha_i a_i=e_1+e_2$.

In the basis (e_i) the root (coroot) lattice of D_r is given by

$$Q=Q^\vee=\{\mathbf{n}\in\mathbb{Z}^r\mid|\mathbf{n}|\in2\mathbb{Z}\}.$$

The weight (coweight) lattice of D_r is

$$\oplus_{i=1}^r \mathbb{Z}\lambda_i = \bigcup_{\varepsilon=0,\frac{1}{2}} (\mathbb{Z}+\varepsilon)^r.$$

The inverse Cartan matrix is

$$\left(C^{D_r}\right)^{-1} = \|\lambda_i(\lambda_j^\vee)\|_{i,j}^r = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 2 & \cdots & 2 & \frac{2}{2} & \frac{2}{2} \\ 1 & 2 & 3 & 3 & \cdots & 3 & \frac{3}{2} & \frac{3}{2} \\ 1 & 2 & 3 & 4 & \cdots & 4 & \frac{4}{2} & \frac{4}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & 4 & \cdots & r-2 & \frac{r-2}{2} & \frac{r-2}{2} \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots & \frac{r-2}{2} & \frac{r}{4} & \frac{r-2}{4} \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots & \frac{r-2}{2} & \frac{r-2}{4} & \frac{r}{4} \end{pmatrix}.$$

The basis of the affine Cartan dual is $\{\lambda_0, \delta, e_1, \ldots, e_r\}$. The simple roots are $\widehat{\alpha}_0 = \delta - \theta$ and the roots α_i of D_r. The fundamental weights are

$$\widehat{\lambda}_0, \qquad \widehat{\lambda}_i = a_i \widehat{\lambda}_0 + \lambda_i, \quad i = 1, \ldots, r,$$

where λ_i denote the fundamental weights of D_r, and a_i are the Dynkin labels:

$$(a_0, \ldots, a_r) = (1, 1, 2, \ldots, 2, 1, 1).$$

D.3 E_6

The Cartan matrix of E_6 is

$$C^{E_6} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and the inverse is

$$\left(C^{E_6}\right)^{-1} = \begin{pmatrix} \frac{4}{3} & 1 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\ 1 & 2 & 2 & 3 & 2 & 1 \\ \frac{5}{3} & 2 & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} \\ 2 & 3 & 4 & 6 & 4 & 2 \\ \frac{4}{3} & 2 & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} \\ \frac{2}{3} & 1 & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} \end{pmatrix}$$

In the affine Cartan matrix the affine node "0" of $\widehat{E}_6$ connects to the node "2" of E_6. The Dynkin marks

$$(a_0, \ldots, a_6) = (1, 1, 2, 2, 3, 2, 1).$$

D.4 E_7

The Cartan matrix of E_7 is

$$C^{E_7} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and the inverse is

$$\left(C^{E_7}\right)^{-1} = \begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & \frac{7}{2} & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 3 & \frac{9}{2} & 6 & 9 & \frac{15}{2} & 5 & \frac{5}{2} \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & \frac{3}{2} & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2} \end{pmatrix}.$$

In the affine Cartan matrix the affine node "0" of $\widehat{E}_7$ connects to the node "1" of E_7. The Dynkin marks

$$(a_0, \ldots, a_7) = (1, 2, 2, 3, 4, 3, 2, 1).$$

D.5 E_8

The Cartan matrix of E_8 is

$$C^{E_8} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and the inverse is

$$\left(C^{E_8}\right)^{-1} = \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}.$$

In the affine Cartan matrix the node "0" of $\widehat{E}_8$ connects to the node "8" of E_8. The Dynkin marks

$$(a_0, \ldots, a_8) = (1, 3, 2, 4, 6, 5, 4, 3, 2).$$

E Affine Weyl group

For the class II theories the relevant reflection group turns out to be the affine Weyl group $W(\widehat{\mathfrak{g}})$. As a group, it is a semi-direct product of the finite Weyl group $W(\mathfrak{g})$ and the root lattice Q of $\mathfrak{g}$ (recall that for $\mathfrak{g}$ the root and the coroot lattices are identified):

$$W(\widehat{\mathfrak{g}}) = W(\mathfrak{g}) \ltimes \mathrm{Q}.$$

We can view $W(\widehat{\mathfrak{g}})$ as the group acting in $\widehat{\mathfrak{h}}$, preserving the non-degenerate scalar product $(\cdot,\cdot)$ which extends the Killing form on $\mathfrak{h}$ by the pairing between K and $\delta^\vee$ as follows:

$$\begin{aligned} &x = \tau\delta^\vee + \sigma K + \xi, \qquad \tau,\sigma \in \mathbb{C}, \quad \xi \in \mathfrak{h}, \\ &(x,x) = \langle \xi,\xi\rangle + 2\tau\sigma. \end{aligned} \tag{E.1}$$

The group $W(\widehat{\mathfrak{g}})$ is generated by simple reflections r_i, $i = 0,1,\dots,r$. The action of r_i on $\widehat{\mathfrak{h}}$ is given by

$$r_i\colon\ x \mapsto x - \alpha_i(x)\alpha_i^\vee, \qquad i = 0,\dots,r \tag{E.2}$$

for $x \in \widehat{\mathfrak{h}}$. Similarly, the action of r_i on $\widehat{\mathfrak{h}}^*$ is given by

$$r_i\colon\ p \mapsto p - p\big(\alpha_i^\vee\big)\alpha_i, \qquad i = 0,\dots,r. \tag{E.3}$$

Note that K is invariant under the reflections (E.2), while δ is invariant under the reflections (E.3), cf. (C.30).

On the hyperplane

$$H_\tau = \{x \mid \delta(x) = \tau\} \subset \widehat{\mathfrak{h}} \tag{E.4}$$

the group $W(\widehat{\mathfrak{g}})$ acts by orthogonal transformations, generated by the ordinary reflections, and by translations. In the decomposition (E.1), we have

$$r_0\big(\tau\delta^\vee + \sigma K + \xi\big) = \tau\delta^\vee + \sigma K + \xi - (\tau - \theta(\xi))\big(K - \theta^\vee\big),$$

where we introduced the highest root $\theta \in \mathfrak{h}^*$, and the highest coroot $\theta^\vee \in \mathfrak{h}$:

$$\theta = \delta - \alpha_0 = \sum_{i=1}^r a_i\alpha_i, \qquad \theta^\vee = K - \alpha_0^\vee = \sum_{i=1}^r a_i\alpha_i^\vee,$$

which obey $\theta(\theta^\vee) = \big\langle\theta^\vee,\theta^\vee\big\rangle = 2$, and also $\theta(\xi) = \big\langle\xi,\theta^\vee\big\rangle$ for any $\xi \in \mathfrak{h}$.

Now, to make the translational part of the $W(\widehat{\mathfrak{g}})$ action explicit, let us perform an additional reflection at θ:

$$r_\theta r_0(x) = r_0(x) - \theta\big(\xi - (\tau - \theta(\xi))\big(-\theta^\vee\big)\big)\theta^\vee = \tau\delta^\vee + (\sigma - \tau + \theta(\xi))K + \xi - \tau\theta^\vee.$$

Finally, the general element of $W(\widehat{\mathfrak{g}})$ can be represented as a pair (w,β), where $w \in W(\mathfrak{g})$ and $\beta \in \mathrm{Q}$, with the composition rule

$$(w_1,\beta_1)\cdot(w_2,\beta_2) = (w_1\cdot w_2, \beta_1 + \beta_2^{w_1}).$$

The element $\widehat{w} = (w,\beta)$ acts on $x \in H_\tau$ as follows:

$$\big(\tau\delta^\vee + \sigma K + \xi\big)^{\widehat{w}} = \tau\delta^\vee + \Big(\sigma - \langle\xi^w,\beta\rangle - \frac{\tau}{2}\langle\beta,\beta\rangle\Big)K + (\xi^w + \tau\beta). \tag{E.5}$$

The dual picture is the action of $W(\widehat{\mathfrak{g}})$ on the fixed level hyperplane $H_k^* = \{p \mid p(K) = k\} \subset \widehat{\mathfrak{h}}^*$. Write

$$p = k\lambda_0 + \mu\delta + \eta, \qquad \eta \in \mathfrak{h}^*, k, \mu \in \mathbb{C},$$

then

$$p(x) = k\sigma + \tau\mu + \eta(\xi).$$

Then the affine Weyl transformation w_β generated by $\beta \in \mathrm{Q}$ acts on p as follows:

$$w_\beta(p) = p + \left(-\langle \eta, \beta\rangle + \frac{k}{2}\langle\beta,\beta\rangle\right)\delta + k\beta. \tag{E.6}$$

We need the affine Weyl lattice action (E.6) to construct lattice theta-functions in Appendix I.

F Conjugacy classes and moduli of bundles

In this section we recall the relation of the moduli space of holomorphic G-bundles on elliptic curve and the space of conjugacy classes in $\widehat{\mathfrak{g}}$, see, e.g., [8].

Let H_τ be the hyperplane (E.4) and let B_τ be the quotient $H_\tau/\mathbb{C}K$, i.e., we forget about the σK part in the expansion

$$x = \tau\delta^\vee + \sigma K + \sum_{i=1}^{r} \xi_i \alpha_i^\vee.$$

We can identify the quotient with the Cartan subalgebra $\mathfrak{h}$ of the finite-dimensional Lie algebra $\mathfrak{g}$, $B_\tau \approx \mathfrak{h}$. In this way we arrive at the τ-dependent action of $W(\widehat{\mathfrak{g}})$ on $\mathfrak{h}$:

$$\xi \mapsto \xi^{\widehat{w}} = \xi^w + \tau\beta.$$

Note that this action does not preserve the coroot lattice $\mathrm{Q}^\vee \subset \mathfrak{h}$. However, it preserves the "doubled" coroot lattice $\mathrm{Q}^\vee + \tau\mathrm{Q}^\vee \subset \mathfrak{h}$.

Recall that *Bernstein–Schwarzman group* $W_\tau(\widehat{\mathfrak{g}})$ [12] is the semi-direct product of $W(\mathfrak{g})$ and the lattice $\mathrm{Q}^\vee + \tau\mathrm{Q}^\vee \subset \mathfrak{h}$. The quotient

$$\mathfrak{h}/W_\tau(\widehat{\mathfrak{g}}) \tag{F.1}$$

is identified with the coarse moduli space $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ of holomorphic principal semi-stable $\mathbf{G}$-bundles on the elliptic curve $\mathcal{E} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ [135, 164]. Let us denote by z the additive coordinate on $\mathcal{E}$:

$$z \sim z + m + n\tau, \qquad m, n \in \mathbb{Z}.$$

We can perform the quotient (F.1) in two steps: first, divide by $\mathrm{Q}^\vee$, and then, divide by the action of $W(\widehat{\mathfrak{g}})$. The first step is accomplished by the exponential map from $\mathfrak{h}$ to $\mathfrak{h}/\mathrm{Q}^\vee \approx \mathbf{T}$. In the second step we divide by the lattice $\tau\mathrm{Q}^\vee$, giving us $\mathcal{E}\otimes\mathrm{Q}^\vee$ (as an abelian group), and finally we divide by the Weyl group $W(\mathfrak{g})$:

$$\mathbf{T}/W(\widehat{\mathfrak{g}}) = \left(\mathbf{T}/\tau\mathrm{Q}^\vee\right)/W(\mathfrak{g}) = \left(\mathcal{E}\otimes\mathrm{Q}^\vee\right)/W(\mathfrak{g}).$$

It is instructive to recall "Dolbeault" realization of the moduli space of bundles, since it will be useful in our further analysis. A holomorphic structure in the G-bundle is a $(0,1)$-connection

$\bar\nabla_{\bar z} = \partial_{\bar z} + A_{\bar z}$ on a principal smooth G-bundle P, which is topologically trivial since we assume G simply connected, and therefore $A_{\bar z}$ is a $\mathfrak{g}$-valued $(0,1)$-form on $\mathcal{E}$. The gauge equivalent connections $\nabla_{\bar z}$ correspond to the isomorphic bundles:

$$A_{\bar z} \mapsto g^{-1} A_{\bar z} g + g^{-1}\partial_{\bar z} g, \qquad g \in C^\infty(\mathcal{E}, G).$$

Recall that the holomorphic structure on P is determined by the condition that the local holomorphic sections of P are annihilated by $\nabla_{\bar z}$, e.g., in an open neighborhood $U_\alpha \subset \mathcal{E}$, the holomorphic sections s_α obey

$$\nabla_{\bar z} s_\alpha = 0 \;\Leftrightarrow\; A_{\bar z}|_{U_\alpha} = -(\partial_{\bar z} s_\alpha) s_\alpha^{-1}.$$

One can find a gauge for $A_{\bar z}$, where it is a given by a constant $\mathfrak{h}$-valued $(0,1)$-form

$$\nabla_{\bar z} = \partial_{\bar z} + \frac{2\pi \mathrm{i}}{\tau - \bar\tau}\xi, \tag{F.2}$$

which still leaves a room for the residual gauge transformations, which preserve the fact that $\xi \in \mathfrak{h}$ and $\partial_z \xi = \partial_{\bar z}\xi = 0$. There are two kinds of transformations (the general such transformation is a composition of these two):

$$\xi \mapsto \xi^w = g_w^{-1}\xi g_w, \qquad g_w \in N(\mathbf{T})/\mathbf{T} = W(\mathfrak{g}) \tag{F.3}$$

and

$$\xi \mapsto \xi + \frac{\tau - \bar\tau}{2\pi\mathrm{i}}\partial_{\bar z}\log g = \xi - \beta_1 - \tau\beta_2, \tag{F.4}$$

where

$$g(z,\bar z) = \exp 2\pi\mathrm{i}\left(\frac{z-\bar z}{\tau-\bar\tau}\beta_1 + \frac{z\bar\tau - \bar z\tau}{\tau-\bar\tau}\beta_2\right)$$

with $\beta_1, \beta_2 \in \mathrm{Q}^\vee$, cf. (C.3). Thus the space of all the $(0,1)$-connections $\nabla_{\bar z}$ modulo the smooth gauge transformations is isomorphic to the quotient $(\mathcal{E}\otimes \mathrm{Q})/W(\mathfrak{g})$, as we claimed.

E. Loojienga's theorem [118] identifies the moduli space of $\mathbf{G}$-bundles on $\mathcal{E}$ with the weighted projective space:

$$\mathrm{Bun}_{\mathbf{G}}(\mathcal{E}) \approx \mathbb{WP}^{a_0,a_1,\ldots,a_r}. \tag{F.5}$$

There are several mathematical interpretations of this theorem [31, 55, 56, 59, 60]. We shall give yet another, more physical explanation of this result, using $\widehat{\mathfrak{g}}$ representation theory. This realization of (F.5) is closer related to our problem.

G Infinite matrices and their Weyl group

We shall also encounter the group $\widehat{\mathrm{GL}}_\infty$. It is the group whose Lie algebra is the central extension of the Lie algebra $\mathfrak{gl}(\infty)$ of infinite matrices which have only a finite number of non-vanishing elements away from the main diagonal:

$$A = \sum_{i,j\in\mathbb{Z}} a_{ij}E_{ij}, \qquad a_{ij} = 0 \quad \text{for } |i-j| \gg 0,$$

where E_{ij} is the matrix with all the entries vanishing except for the (i,j) entry. The central extension is given by the cocycle

$$[A\oplus\alpha K, B\oplus\beta K] = [A,B]\oplus\gamma(A,B)K, \qquad \gamma(A,B) = \mathrm{tr}\, A[J,B],$$

where

$$J=\frac{1}{2}\sum_{i\in\mathbb{Z}}\operatorname{sign}\left(i-\frac{1}{2}\right)E_{ii}.$$

The Cartan subalgebra $\widehat{\mathfrak{h}}_\infty\subset\widehat{\mathfrak{gl}(\infty)}$ consists of the diagonal matrices plus the span of the central generator:

$$\widehat{\mathfrak{h}}_\infty=\bigoplus_{i\in\mathbb{Z}}\mathbb{C}E_{ii}\oplus\mathbb{C}K.$$

The Weyl group $W(\widehat{\mathfrak{gl}(\infty)})$ is the group of finite permutations of the eigenvalues of the infinite diagonal matrix.

H The representation theory

Let us start over with the finite-dimensional simple Lie group $\mathbf{G}$. Recall that the weight $\mu\in\Lambda$ is *dominant*, $\mu\in\Lambda_+$ iff $\mu\bigl(\alpha_i^\vee\bigr)\geq 0$ for all $i=1,\dots,r$.

Recall that to every dominant weight $\mu\in\Lambda_+$ an irreducible highest weight finite-dimensional representation $\mathcal{V}_\mu$ of the group $\mathbf{G}$ is associated. In particular, the fundamental representations $R_i=\mathcal{V}_{\lambda_i}$ of the group $\mathbf{G}$ correspond to the fundamental weights λ_i. Moreover, the Grothendieck ring $\operatorname{Rep}(\mathbf{G})$ of finite-dimensional representations of $\mathbf{G}$ is generated by R_i.

Example H.1. For the A_r series, the group $\mathbf{G}=\mathrm{SL}(r+1,\mathbb{C})$, the fundamental representations $R_i=\bigwedge^i\mathbb{C}^{r+1}$, $i=1,\dots,r$, are the exterior powers of the defining $(r+1)$-dimensional representation. The center $Z=\mathbb{Z}_{r+1}$, the adjoint group $G^{\mathrm{ad}}=\mathrm{PGL}(r+1,\mathbb{C})=\mathrm{SL}(r+1,\mathbb{C})/\mathbb{Z}_{r+1}$.

Example H.2. For the D_r series, the group $\mathbf{G}=\mathrm{Spin}(2r,\mathbb{C})$, the fundamental representations $R_i=\bigwedge^i\mathbb{C}^{2r}$, $i=1,\dots,r-2$, i.e., antisymmetric tensors of the corresponding rank, while $R_{r-1}=\mathbb{S}_-$, $R_r=\mathbb{S}_+$ are the chiral spinors. The center $Z=\mathbb{Z}_4$ for r odd, and $Z=\mathbb{Z}_2\times\mathbb{Z}_2$ for r even.

H.1 Weights of the representation

The representations of $\mathbf{G}$ produce the $W(\mathfrak{g})$-invariant functions on $\mathbf{T}$. To every finite-dimensional representation V of $\mathbf{G}$, $\pi_V\colon\mathbf{G}\to\operatorname{End}(V)$, one assigns a function on $\mathbf{T}$, the character

$$\chi_V(t)=\operatorname{tr}_V\pi_V(t),\qquad t\in\mathbf{T}\subset\mathbf{G},$$

which is the trace of the element of $\mathbf{T}$ viewed as the element of $\mathbf{G}$ in the representation V. By definition of the Weyl group $W(\mathfrak{g})=N(\mathbf{T})/\mathbf{T}$ it acts on $\mathbf{T}$ by conjugation in $\mathbf{G}$, therefore the character χ_V is $W(\mathfrak{g})$-invariant. The representation V, viewed as a representation of $\mathbf{T}\subset\mathbf{G}$, splits as a direct sum of one-dimensional representations L_λ, i.e., the sum of weight subspaces L_λ for some $\lambda\in\Lambda_V\subset\operatorname{Hom}(\mathbf{T},\mathbb{C}^\times)=\Lambda$. Here Λ_V is the (finite, for finite-dimensional V) set of weights of the representation V. The character $\chi_V(t)$ can be written as

$$\chi_V(t)=\sum_{\lambda\in\Lambda_V}\chi_{V,\lambda}t^\lambda, \tag{H.1}$$

where $\chi_{V,\lambda}\in\mathbb{Z}_+$ is the multiplicity of the given weight λ in the decomposition of V:

$$V=\bigoplus_{\lambda\in\Lambda_V}\mathbb{C}^{\chi_{V,\lambda}}\otimes L_\lambda$$

and $t^\lambda \in \mathbb{C}^\times$ is the value of $\lambda \in \mathrm{Hom}(\mathbf{T}, \mathbb{C}^\times)$ on $t \in T$. For the highest weight representations $\mathcal{V}_\mu$ with the highest weight μ the set Λ_V contains μ, with multiplicity $\chi_{V,\mu} = 1$, while all the other elements λ of Λ_V obey

$$\mu - \lambda = \sum_{i=1}^{r} n_i \alpha_i, \qquad n_i \geq 0, \qquad \sum_i n_i > 0. \tag{H.2}$$

The Weyl group $W(\mathfrak{g})$ action on $t \in \mathbf{T}$ can be traded for the $W(\mathfrak{g})$-action on the set of weights Λ_V:

$$(t^w)^\lambda = t^{\lambda^w}.$$

In other words, the set $\Lambda_V \subset \Lambda$ is $W(\mathfrak{g})$ invariant, including the multiplicities

$$\chi_{V,\lambda} = \chi_{V,\lambda^w} \qquad \text{for any } w \in W(\mathfrak{g}).$$

The ring of $W(\mathfrak{g})$-invariant rational functions on $\mathbf{T}$ is generated, as a vector space, by the characters of all the irreducible representations of G, and polynomially by the characters $\chi_i = \chi_{R_i}$ of r fundamental representations of $\mathbf{G}$ (Chevalley):

$$\mathbb{C}[\mathbf{G}]^{\mathrm{Ad}(G)} = \mathbb{C}[\mathbf{T}]^{W(\mathfrak{g})} \equiv \mathbb{C}\big[z_1, \ldots, z_r, z_1^{-1}, \ldots, z_r^{-1}\big]^{W(\mathfrak{g})} = \mathbb{C}[\chi_1, \ldots, \chi_r].$$

H.2 The action of the center

Any element z of the center Z of $\mathbf{G}$ acts in the irreducible representation (V, π_V) of $\mathbf{G}$ by multiplication by a scalar character:

$$\pi_V(z) = \mu_V(z) \cdot \mathbf{1}_V, \qquad \mu_V \in \mathrm{Hom}(Z, \mathbb{C}^\times) \approx \Lambda/\mathrm{Q}.$$

For the highest weight module $V = \mathcal{V}_\lambda$ this character is easy to compute

$$\mu_{\mathcal{V}_\lambda}(\varpi_y) = \varpi_y^\lambda.$$

In particular, the character $\mathbf{m}_i \equiv \mu_{R_i}$ corresponds to the image of the fundamental weight λ_i in the coset Λ/Q:

$$\mathbf{m}_i(\varpi_y) = \zeta_y^{w_{yi}}.$$

H.3 Representation theory of affine Kac–Moody algebras

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ has the positive energy highest weight irreducible representations $\widehat{R}_i$, $i = 0, 1, \ldots, r$, which are integrable, i.e., lift to the representations of the corresponding Kac–Moody group $\widetilde{\mathrm{LG}}$. The highest weight of $\widehat{R}_i$ is $\widehat{\lambda}_i$. The representations $\widehat{R}_i$ are infinite-dimensional of finite type: the dimensions of the eigenspaces of $\delta^\vee$ are finite, so that the character

$$\widehat{\mathcal{X}}_i(\widehat{t}) = \mathrm{tr}_{\widehat{R}_i}\, \pi_{\widehat{R}_i}(\widehat{t}) = \sum_{\widehat{\nu}} \widehat{\chi}_{\widehat{R}_i, \widehat{\lambda}_i - \widehat{\nu}}\, \widehat{t}^{\widehat{\lambda}_i - \widehat{\nu}}$$

makes sense as a function in the domain $|\mathfrak{q}| < 1$, where

$$\mathfrak{q} = \widehat{t}^{-\delta} \in \mathbb{C}^\times.$$

H.4 On E. Loojienga's theorem

In view of what we explained above, this theorem has the following interpretation. The fundamental characters $\widehat{\chi}_i$ of $\widetilde{\mathrm{L}\mathbf{G}}$, with $i=0,1,\dots,r$ are the $W(\widehat{\mathfrak{g}})$-invariant functions on the maximal torus $\tilde{\mathbf{T}}$ of $\widetilde{\mathrm{L}\mathbf{G}}$. The $\tilde{\mathbf{T}}$ is a trivial fibre bundle over $\mathbb{C}^\times_{\delta^\vee}$

$$\tilde{\mathbf{T}}=\mathbb{C}^\times_{\delta^\vee}\times\widehat{\mathbf{T}} \tag{H.3}$$

with the fiber over the point $\mathfrak{q}=\mathrm{e}^{2\pi\mathrm{i}\tau}\in\mathbb{C}^\times_{\delta^\vee}$ being the $(r+1)$-dimensional torus

$$\widehat{\mathbf{T}}=\mathbb{C}^\times_K\times\mathbf{T}=H_\tau/\widehat{\mathrm{Q}}^\vee, \tag{H.4}$$

where $\mathbb{C}^\times_K$ is the center of $\widetilde{\mathrm{L}\mathbf{G}}$. Note that the central extension of $\mathrm{L}\mathbf{G}$ is topologically nontrivial. However, its restriction on any torus of $\mathrm{L}\mathbf{G}$ (which we can identify with the space of constant loops valued in $\mathbf{T}\subset\mathbf{G}$) is topologically trivial, hence (H.4). Now, since K generates the center of $\widetilde{\mathrm{L}\mathbf{G}}$, the c^K-dependence of $\widehat{\mathcal{X}}_i$ is a simple factor

$$\widehat{\mathcal{X}}_i\big(c^K\widehat{t}\big)=c^{a_i}\widehat{\mathcal{X}}_i(\widehat{t}).$$

On the other hand, as we explained above, forgetting the $\mathbb{C}^\times_K$-factor in the quotient

$$\widetilde{B}_\tau=\big(H_\tau/\widehat{\mathrm{Q}}^\vee\big)/W(\widehat{\mathfrak{g}})=\widehat{\mathbf{T}}/W(\widehat{\mathfrak{g}})$$

leads to the quotient $B_\tau/W(\widehat{\mathfrak{g}})=\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$. In other words, the space $\widetilde{B}_\tau$ is the total space of a $\mathbb{C}^\times_K$-bundle over $\mathrm{Bun}_G(\mathcal{E})$. The ring of holomorphic functions on $\widetilde{B}_\tau$ is identified with the ring of polynomials in the fundamental characters $\widehat{\chi}_i$:

$$\mathbb{C}\big[\widetilde{B}_\tau\big]=\mathbb{C}[\widehat{\chi}_0,\widehat{\chi}_1,\dots,\widehat{\chi}_r].$$

Dividing (or, better to say, grading) by $\mathbb{C}^\times_K$ makes $\mathrm{Bun}_{\mathbf{G}}(\mathcal{E})$ out of $\widetilde{B}_\tau$ on the left-hand side, and the space of holomorphic sections of the bundles $\mathcal{O}(k)$ over the weighted projective space $\mathbb{WP}^{a_0,a_1,a_2,\dots,a_r}$ on the right-hand side.

I Affine Weyl group, characters and theta-functions

The affine Weyl group of affine Lie algebra $\widehat{\mathfrak{g}}$ is the semi-direct product $W(\widehat{\mathfrak{g}})=W(\mathfrak{g})\ltimes\mathrm{Q}^\vee$. For each weight $\widehat{\lambda}\in\widehat{\mathfrak{h}}^\vee$ the affine transformation w_α by $\alpha\in\mathrm{Q}^\vee$ is a parabolic transformation which linearly shifts the $\mathfrak{h}$ component of $\widehat{\lambda}$ by α and quadratically adjusts the δ-component of $\widehat{\lambda}$ keeping the norm $\big(\widehat{\lambda},\widehat{\lambda}\big)$ invariant. Concretely, if $\widehat{\lambda}$ is of level k we represent it as $\widehat{\lambda}=k\lambda_0+\lambda$ with $\lambda\in\mathfrak{h}^\vee$, and then we have (cf. equation (E.6))

$$w_\alpha\big(\widehat{\lambda}\big)=\widehat{\lambda}+k\alpha-\tfrac{\delta}{2k}\big(\big(k\alpha+\widehat{\lambda}\big)^2-\widehat{\lambda}^2\big).$$

For each weight $\widehat{\lambda}$ we introduce theta-function $\Theta_{\widehat{\lambda}}$

$$\Theta_{\widehat{\lambda}}=\mathrm{e}^{-\frac{\lambda^2}{2k}\delta}\sum_{\alpha\in\mathrm{Q}^\vee}\mathrm{e}^{w_\alpha(\lambda)} \tag{I.1}$$

equivalently

$$\Theta_{\widehat{\lambda}}=\mathrm{e}^{k\Lambda_0}\sum_{\gamma\in\lambda+k\mathrm{Q}^\vee}\mathrm{e}^{\gamma-\frac{\delta\gamma^2}{2k}}, \tag{I.2}$$

where formally omitted the arguments in the exponentiated Cartan of $\widehat{\mathfrak{g}}$, i.e., $\tilde{T}$. The theta function $\Theta_{\widehat{\lambda}}$ can be evaluated on an element $\mathbf{g} = (c, \mathbf{t}, \mathfrak{q})$:

$$\Theta_\lambda(c, \mathbf{t}, \mathfrak{q}) = c^k \sum_{\gamma \in \widehat{\lambda} + k\mathrm{Q}^\vee} t^\gamma \mathfrak{q}^{\frac{\gamma^2}{2k}}.$$

Sometimes it is convenient not to include the prefactor in the definition of the theta-function (I.1) and for all fundamental weights we set

$$c_i(c, \mathbf{t}, \mathfrak{q}) := \mathfrak{q}^{-\frac{\lambda^2}{2k}} \Theta_{\widehat{\lambda}_i}(c, \mathbf{t}, \mathfrak{q}). \tag{I.3}$$

From (I.2), it is clear that $\Theta_{\widehat{\lambda}}$ depends only on the affine Weyl conjugacy class of $\widehat{\lambda}$.

The ring of affine Weyl group invariants is generated by the theta-functions $\Theta_{\widehat{\lambda}_0}, \dots, \Theta_{\widehat{\lambda}_r}$ for the fundamental weights $\widehat{\lambda}_0, \dots, \widehat{\lambda}_r$.

Another important function is the Weyl skewsymmetric function

$$R(t) = \prod_{\alpha \in \Delta^+} (1 - t^{-\alpha}) = \sum_{w \in W} (-1)^w t^{w(\rho) - \rho},$$

where ρ is the Weyl covector defined by $\rho(\alpha_i) = 1$ for all $i = 0, \dots, r$. The equality of the two representations above for Weyl denominator for affine algebras leads to the famous Macdonald identity. By definition, $R(h)$ is Weyl skewsymmetric.

J Characters

For the weight $\widehat{\lambda} \in \widehat{\Lambda}$ let us denote by $\widehat{\mathcal{R}}_{\widehat{\lambda}}$ the irreducible $\widehat{\mathfrak{g}}$-module with the highest weight $\widehat{\lambda}$. Let $\chi_{\widehat{\mathcal{R}}}(t)$ be the character evaluated on an element t of $\tilde{T}$ (H.3):

$$\chi_{\widehat{\mathcal{R}}}(t) = \sum_{\lambda \in \mathfrak{h}^*} t^\lambda \mathrm{mult}_{\widehat{\mathcal{R}}} \lambda.$$

The Weyl character formula literally holds for the affine Kac–Moody algebras

$$\chi_{\widehat{\mathcal{R}}}(t) = \frac{\sum_{w \in W} \epsilon(w) t^{w(\Lambda + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - t^{-\alpha})},$$

where Δ_+ is the set of positive roots in $\mathfrak{g}$, $\epsilon(w) = \pm 1$ is the parity of Weyl transformation $w \in W$.

If $\widehat{\mathfrak{g}}$ is simply laced affine ADE algebra and $\widehat{\mathcal{R}}$ is an irreducible $\mathfrak{g}$-module of level 1, then the character can be also computed in terms of theta-functions [93]

$$\chi_{\widehat{\mathcal{R}}}(t) = \phi(\mathfrak{q})^{-r} \sum_{\alpha \in \mathrm{Q}^\vee} t^{w_\alpha(\widehat{\lambda})}. \tag{J.1}$$

More generally,

$$\mathrm{ch}_{\widehat{\mathcal{R}}}(t) = \sum_{\lambda \in P_k \bmod k\mathrm{Q}^\vee} c_\lambda^\Lambda \Theta_\lambda(t), \tag{J.2}$$

where the string function $c_\lambda^\Lambda(\mathfrak{q})$ depends only on $\mathfrak{q} = t^{-\delta}$ and counts multiplicities of weights on the vertical rays $\widehat{\lambda} + \mathbb{Z}\delta$ in the $\mathfrak{g}$-module $L(\Lambda)$

$$c_\lambda^\Lambda(\mathfrak{q}) = \mathfrak{q}^{-\frac{\lambda^2}{2k}} \sum_{n \in \mathbb{Z}} \mathfrak{q}^n \mathrm{mult}_{\widehat{\mathcal{R}}}(\lambda - n\delta).$$

The formula (J.2) follows from the invariance of the multiplicities under affine Weyl group, with non-trivial information now contained in the finite set of the functions c^{Λ}_{λ}. If λ is of level $k=1$ and $\widehat{\mathfrak{g}}$ is simply laced affine ADE then there is only one non-zero inequivalent string function

$$c^{\Lambda}_{\lambda}(\mathfrak{q}) = \mathfrak{q}^{-\frac{\lambda^2}{2}}\phi(\mathfrak{q})^{-r}$$

reproducing formula (J.1).

In what follows, we use the notations

$$\mathbf{n} = (n_1,\dots,n_r)\in\mathbb{Z}^r, \qquad \tilde{\mathbf{n}} = (\tilde{n}_1,\dots,\tilde{n}_r)\in\left(\mathbb{Z}+\frac{1}{2}\right)^r,$$

$$|\mathbf{n}| = \sum_{i=1}^{r} n_i, \qquad |\tilde{\mathbf{n}}| = \sum_{i=1}^{r}\tilde{n}_i, \qquad \mathbf{n}^2 = \sum_{i=1}^{r} n_i^2, \qquad \tilde{\mathbf{n}}^2 = \sum_{i=1}^{r}\tilde{n}_i^2.$$

J.1 $\widehat{A}_r$ theta-functions

If we specialize (I.1) to fundamental weight $\widehat{\lambda}_j$, we get

$$\mathfrak{q}^{-\frac{j}{2}}\mathfrak{q}^{\frac{j^2}{2(r+1)}}\Theta_{\widehat{\lambda}_j} = \sum_{\alpha\in Q^\vee} \mathrm{e}^{w_\alpha(\widehat{\lambda}_j)}.$$

For each fundamental weight $\widehat{\lambda}_j$, $j=0,\dots,r$ we define the theta-function $\Theta_j\equiv\Theta_{\Lambda_j}$. We use the basis of multiplicative coordinates on $\exp(\widehat{\mathfrak{h}})\ni\mathbf{g}$ as follows: $c=\mathbf{g}^{\widehat{\lambda}_0}$, $\mathfrak{q}=\mathbf{g}^{-\delta}$, $\mathbf{t}=(t_1,\dots,t_{r+1})$ subject to $\prod_{i=1}^{r+1}t_i=1$ in our conventions for the basis in the root and weight spaces of $\widehat{A}_r$. We find

$$\Theta_j(\mathbf{g}) = c\mathfrak{q}^{-\frac{j^2}{2(r+1)}}\sum_{\mathbf{n}\in\Lambda_j}\mathbf{t}^{\mathbf{n}}\mathfrak{q}^{\frac{1}{2}\mathbf{n}^2}, \tag{J.3}$$

where

$$\Lambda_j = \{\mathbf{n}\in\mathbb{Z}^{r+1}\mid |\mathbf{n}| = j\}.$$

Consequently, from (J.1) and (I.1) the character of the j-th fundamental representation for $\widehat{A}_r$ is

$$\chi_j(c;\mathbf{t};\mathfrak{q}) = \phi(\mathfrak{q})^{-r}\mathfrak{q}^{\frac{-j}{2}+\frac{j^2}{2(r+1)}}\Theta_j(c;\mathbf{t};\mathfrak{q}). \tag{J.4}$$

J.2 $\widehat{D}_r$ theta-functions

Let $t\in\tilde{\mathbf{T}}$, $\mathbf{1}=(1,1,\dots,1,1)=2\lambda_r$, $t_i=t^{e_i}$, $\mathfrak{q}=t^{-\delta}$, $c=t^{\lambda_0}$. The level 1 theta-functions are associated to the fundamental weights λ_0, λ_1, λ_{r-1}, λ_r. According to our definition (I.2) we get

$$\Theta_0(c;\mathbf{t};\mathbf{q}) = c\sum_{\mathbf{n},|\mathbf{n}|\in2\mathbb{Z}}\mathbf{t}^{\mathbf{n}}\mathbf{q}^{\frac{1}{2}\mathbf{n}^2},$$

$$\Theta_1(c;\mathbf{t};\mathbf{q}) = c\sum_{\mathbf{n},|\mathbf{n}|-1\in2\mathbb{Z}}\mathbf{t}^{\mathbf{n}}\mathbf{q}^{\frac{1}{2}\mathbf{n}^2},$$

$$\Theta_{r-1}(c;\mathbf{t};\mathbf{q}) = c\sum_{\tilde{\mathbf{n}},|\tilde{\mathbf{n}}|-\frac{r}{2}+1\in2\mathbb{Z}}\mathbf{t}^{\tilde{\mathbf{n}}}\mathbf{q}^{\frac{1}{2}\tilde{\mathbf{n}}^2},$$

$$\Theta_r(c;\mathbf{t};\mathbf{q}) = c \sum_{\tilde{\mathbf{n}},|\tilde{\mathbf{n}}|-\frac{r}{2}\in 2\mathbb{Z}} \mathbf{t}^{\tilde{\mathbf{n}}}\mathbf{q}^{\frac{1}{2}\tilde{\mathbf{n}}^2}. \tag{J.5}$$

We can express $\widehat{D}_r$ theta-functions of level 1 in terms of Jacobi theta-functions as follows

$$\begin{aligned}
\Theta_0(c;\mathbf{t};\mathfrak{q}) &= \frac{c}{2}\left(\prod_{i=1}^r \theta_3(t_i;\mathfrak{q}) + \prod_{i=1}^r \theta_4(t_i;\mathfrak{q})\right),\\
\Theta_1(c;\mathbf{t};\mathfrak{q}) &= \frac{c}{2}\left(\prod_{i=1}^r \theta_3(t_i;\mathfrak{q}) - \prod_{i=1}^r \theta_4(t_i;\mathfrak{q})\right),\\
\Theta_r(c;\mathbf{t};\mathfrak{q}) &= \frac{c}{2}\left(\prod_{i=1}^r \theta_2(t_i;\mathfrak{q}) + \mathrm{i}^{-r}\prod_{i=1}^r \theta_1(t_i;\mathfrak{q})\right),\\
\Theta_{r-1}(c;\mathbf{t};\mathfrak{q}) &= \frac{c}{2}\left(\prod_{i=1}^r \theta_2(t_i;\mathfrak{q}) - \mathrm{i}^{-r}\prod_{i=1}^r \theta_1(t_i;\mathfrak{q})\right).
\end{aligned} \tag{J.6}$$

To construct D_r affine Weyl invariant functions at level 2 we use the embedding $\mathfrak{so}(2r) \subset \mathfrak{sl}(2r)$, i.e., $D_r \subset A_{2r-1}$ in which the maximal tori are mapped as

$$\begin{aligned}
&T_{\mathrm{SO}(2r)} \to T_{\mathrm{SL}(2r)},\\
&(t_1,\dots,t_r) \mapsto \left(t_1,\dots,t_r,t_r^{-1},\dots,t_1^{-1}\right).
\end{aligned} \tag{J.7}$$

It is easy to check that $\widehat{D}_r$ affine Weyl group action on $\widehat{\mathfrak{h}}_{D_r}$ at level 2 is a subgroup of $\widehat{A}_{2r-1}$ affine Weyl group action on $\widehat{\mathfrak{h}}_{A_{2r-1}}$ at level 1. Therefore, using (J.7), we can construct $\widehat{D}_r$ affine Weyl invariant functions at level 2 using $\widehat{A}_{2r-1}$ theta-functions at level 1. From (J.3) and (J.7), we get

$$\Xi_i(c;\mathbf{t};\mathfrak{q}) = c^2\mathfrak{q}^{-\frac{i^2}{4r}} \sum_{\mathbf{n},\mathbf{m}\in\mathbb{Z}^r,\,|\mathbf{m}|=i} \mathbf{t}^{\mathbf{2n}-\mathbf{m}}\mathfrak{q}^{\mathbf{n}\cdot(\mathbf{n}-\mathbf{m})+\frac{1}{2}\mathbf{m}^2} \tag{J.8}$$

with

$$\Xi_i = \Xi_{-i} = \Xi_{2r+i}.$$

The theta-functions Ξ_i for $i = 2,\dots,r-2$ compute the characters of the level 2 fundamental $\widehat{D}_r$ modules $\widehat{R}_i$ with the highest weight $\widehat{\lambda}_i = 2\lambda_0 + \bar{\lambda}_i$ and the character is

$$\chi_{\widehat{R}_i}(c;\mathbf{t};\mathfrak{q}) = c^2\frac{1}{\phi(\mathfrak{q})^{2r-1}}\mathfrak{q}^{-\frac{i}{2}}\mathfrak{q}^{\frac{i^2}{4r}}\Xi_i(\mathcal{Y}_0;\mathbf{t};\mathfrak{q}).$$

There are relations obtained immediately from the definitions (J.5), (J.6), (O.1)

$$\begin{aligned}
(\Theta_0+\Theta_1)^2 &= c^2\prod_{i=1}^r \theta_3(t_i;\mathfrak{q})^2 = c^2\sum_{j\in\mathbb{Z}}\mathfrak{q}^{\frac{j^2}{4r}}\Xi_j(\mathbf{t};\mathfrak{q}),\\
(\Theta_0-\Theta_1)^2 &= c^2\prod_{i=1}^r \theta_4(t_i;\mathfrak{q})^2 = c^2\sum_{j\in\mathbb{Z}}\mathfrak{q}^{\frac{j^2}{4r}}(-1)^j\Xi_j(\mathbf{t};\mathfrak{q}),\\
(-1)^r(\Theta_r-\Theta_{r-1})^2 &= c^2\prod_{i=1}^r \theta_1(t_i;\mathfrak{q})^2 = c^2\sum_{j\in\mathbb{Z}}\mathfrak{q}^{\frac{j^2}{4r}}(-1)^{j-r}\Xi_{j-r}(\mathbf{t};\mathfrak{q}),\\
(\Theta_r+\Theta_{r-1})^2 &= c^2\prod_{i=1}^r \theta_2(t_i;\mathfrak{q})^2 = c^2\sum_{j\in\mathbb{Z}}\mathfrak{q}^{\frac{j^2}{4r}}\Xi_{j-r}(\mathbf{t};\mathfrak{q}).
\end{aligned} \tag{J.9}$$

Notice

$$c^{-2}\phi(\mathfrak{q})^{2r}(\chi_r+\chi_{r-1})^2=\mathfrak{q}^{-\frac{r}{4}}(\Theta_r+\Theta_{r-1})^2$$
$$=\sum_{j\in\mathbb{Z}}q^{\frac{j^2}{4r}+\frac{j}{2}}\Xi_j(\mathbf{t};q)=c^{-2}\phi(\mathfrak{q})^{2r-1}\sum_{s\in\mathbb{Z}}\sum_{j'=0}^{2r-1}q^{rs(s+1)+(s+1)j'}\chi_{j'}$$

is similar the relation

$$S\otimes S=\bigoplus_{i\in 0\ldots 2r}\bigwedge^i V,$$

where $S=S^+\oplus S^-$ is the spin representation of D_r. As the fundamental invariants of $\widehat{D}_r$ we will take the set of theta-functions associated to the irreducible representations with the fundamental highest weight. For $i=2,\ldots,r-2$, we define the invariants of level 2, and for $s=1,\ldots,4$, we define invariants of level 1 as follows:

$$\begin{aligned}
\big({}_2\tilde{\mathfrak{X}}_i^{\widehat{D}}\big)&:=\Xi_i,\qquad i=2,\ldots,r-2,\\
\big({}_1\tilde{\mathfrak{X}}_s^{\widehat{D}}\big)&:=\begin{cases}\Theta_0+\Theta_1, & s=0,\\ \Theta_0-\Theta_1, & s=1,\\ \mathrm{i}^r(\Theta_r-\Theta_{r-1}), & s=r-1,\\ \Theta_r+\Theta_{r-1}, & s=r.\end{cases}
\end{aligned}\tag{J.10}$$

The linear relations (J.9) allow to express Ξ_0, Ξ_1, Ξ_{r-1}, Ξ_r as linear combinations of ${}_2\tilde{\mathfrak{X}}_2^{\widehat{D}},\ldots,$ ${}_2\tilde{\mathfrak{X}}_{r-2}^{\widehat{D}}$ and $\big({}_1\tilde{\mathfrak{X}}_1^{\widehat{D}}\big)^2,\ldots,\big({}_1\tilde{\mathfrak{X}}_4^{\widehat{D}}\big)^2$.

J.3 Infinite matrices

The fundamental representations $\mathcal{R}_i=$ of $\widehat{\mathrm{GL}}_\infty$ are labeled by a single integer $i\in\mathbb{Z}$. They are highest weight $\tilde{\lambda}_i$ representations, also infinite-dimensional of the finite type. Let us compute their characters (we shall remind the fermion construction of these representations in the next section). Let $\mathbf{t}=(t_i)_{i\in\mathbb{Z}}$ be the generic infinite diagonal matrix, with entries behaving as $\mathfrak{q}^i$ for large i, for some $|\mathfrak{q}|<1$. In other words $\mathbf{t}$ is close (in the topology of Hilbert space operators) to the evolution operator $\mathfrak{q}^{L_0}$ (with L_0 being the energy operator), and let c^K, for $c\in\mathbb{C}^\times$ represent the center of $\widehat{\mathrm{GL}}_\infty$. Let

$$\mathbf{g}=c^K\times\mathbf{t}.$$

The characters $\chi_i(\mathbf{g})$ can be arranged into the generating function

$$\sum_{i\in\mathbb{Z}}\zeta^i\chi_i(\mathbf{g})=c\prod_{n=1}^{\infty}(1+\zeta t_n)\big(1+\zeta^{-1}t_{1-n}^{-1}\big),\tag{J.11}$$

which is explicitly $W(\widehat{\mathfrak{gl}(\infty)})$-invariant, with the Weyl group acting on the eigenvalues t_i of $\mathbf{t}$ by the finite permutations,

$$(t_i)_{i\in\mathbb{Z}}\mapsto(t_{\sigma(i)})_{i\in\mathbb{Z}},\qquad\sigma(i)=i,\quad|i|\gg 0,$$

while the central element c^K transforming via

$$c\mapsto c\prod_{i=-\infty}^{0}\frac{t_{\sigma(i)}}{t_i}.$$

Example J.1. The transformation $t_0 \leftrightarrow t_1$ acts on c by

$$c \mapsto ct_1/t_0.$$

The character χ_i of $\mathcal{R}_i$ is given by the sum over the partitions. The formula is most simply obtained by expanding (J.11) as a sum over partitions

$$\chi_j(\mathbf{g}) = c \sum_{\lambda} \prod_{i=1}^{\infty} \frac{t_{\lambda_i - i+1+j}}{t_{-i+1}}. \tag{J.12}$$

In (J.12) the infinite product is to be understood in the following way

$$\prod_{i=1}^{\infty} \frac{t_{\lambda_i-i+1+j}}{t_{-i+1}} \equiv t^{[j]} \prod_{i=1}^{\ell(\lambda)} \frac{t_{\lambda_i-i+1+j}}{t_{-i+1+j}}.$$

The value of the highest weight $\tilde{\lambda}_i$ on $\mathbf{g}$ is equal to

$$\mathbf{g}^{\tilde{\lambda}_i} = ct^{[j]}. \tag{J.13}$$

K Free fermion representation

In describing the representations of $\widetilde{\mathrm{LG}}$ and $\widehat{\mathrm{GL}}_\infty$ it is convenient to use the free fermions.

One can realize the level 1 representations of $\mathfrak{g}$ using r flavors of chiral complex fermions in two dimensions:

$$L_1 = \sum_{a=1}^{r} \tilde{\psi}_a \bar{\partial} \psi^a.$$

The currents of $\widehat{\mathfrak{so}(2r)}$ are

$$J^{ab} = \psi^a \psi^b, \qquad J^a_b = :\tilde{\psi}_b \psi^a:, \qquad J_{ab} = \tilde{\psi}_a \tilde{\psi}_b. \tag{K.1}$$

The currents (K.1) do not preserve the $U(1)$ charge $J_0 = \oint :\tilde{\psi}_a \psi^a:$, however they preserve its parity. Therefore, one can construct the following four representations: take the fermions in the NS or R sector and project onto the even or odd J_0 subsectors. Recall that the NS sector fermions are anti-periodic on the circle $|z| = 1$,

$$\psi^a = \sum_{n \in \mathbb{Z}+\frac{1}{2}} \psi^a_n z^{-n} \left(\frac{\mathrm{d}z}{z}\right)^{\frac{1}{2}}, \qquad \tilde{\psi}_a = \sum_{n \in \mathbb{Z}+\frac{1}{2}} \tilde{\psi}_{a,n} z^{-n} \left(\frac{\mathrm{d}z}{z}\right)^{\frac{1}{2}}$$

and their modes obey the anti-commutation relations

$$\{\tilde{\psi}_{a,n}, \psi^b_m\} = \delta^b_a \delta_{n+m,0}.$$

The full space of states in the NS sector for the theory with r fermions is

$$\mathbb{H}^{\mathrm{NS}} = \mathbb{H}^{\mathrm{NS}}_{\mathrm{even}} \oplus \mathbb{H}^{\mathrm{NS}}_{\mathrm{odd}},$$

where

$$\mathbb{H}^{\mathrm{NS}}_{\mathrm{even/odd}} = \bigoplus_{\substack{\vec{M} \in \mathbb{Z}^r \\ \sum_i M_i \ \mathrm{even/odd}}} \otimes_{i=1}^{r} \mathcal{H}^{\mathrm{NS}}_{M_i}.$$

The character of the $\mathbb{H}^{\mathrm{NS}}_{\mathrm{even/odd}}$ representation is given by

$$\chi^{\mathrm{NS}}_{\mathrm{even/odd}} = \frac{1}{\phi^r(q)}\Theta_{r/r-1}.$$

In the Ramond sector the fermions are periodic on the circle $|z|=1$,

$$\psi^a = \sum_{n\in\mathbb{Z}} \psi^a_n z^{-n}\left(\frac{\mathrm{d}z}{z}\right)^{\frac{1}{2}}, \qquad \tilde{\psi}_a = \sum_{n\in\mathbb{Z}} \tilde{\psi}_{a,n} z^{-n}\left(\frac{\mathrm{d}z}{z}\right)^{\frac{1}{2}}$$

and their modes obey the anti-commutation relations

$$\{\tilde{\psi}_{a,n}, \psi^b_m\} = \delta^b_a \delta_{n+m,0}.$$

The characters of the Ramond representations are

$$\chi^R_{\mathrm{even(odd)}} = \frac{1}{\phi(q)}\Theta_{0/1}.$$

The remaining fundamental representations are constructed using $2r$ complex fermions (hence level two)

$$L_2 = \sum_{a=1}^{r} \big(\tilde{\psi}_a \bar{\partial}\psi^a + \tilde{\psi}^a \bar{\partial}\psi_a\big),$$

whose $\widehat{\mathfrak{g}_{\mathrm{q}}}$ symmetry is generated by

$$J^{ab} = \tilde{\psi}^a\psi^b - \tilde{\psi}^b\psi^a, \qquad J^a_b = :\tilde{\psi}_b\psi^a - \psi_b\tilde{\psi}^a:, \qquad J_{ab} = \tilde{\psi}_a\psi_b - \tilde{\psi}_b\psi_a,$$

which is actually a subalgebra of the level one $\widehat{\mathfrak{su}(2r)}$ symmetry. Accordingly, $\widehat{\mathfrak{g}_{\mathrm{q}}}$ commutes with the $\widehat{\mathfrak{u}(1)}$ Kac–Moody generated by

$$\mathcal{J} = :\tilde{\psi}_a\psi^a: + :\tilde{\psi}^a\psi_a:$$

we can construct the representations $\mathcal{V}_p$, $p=0,\dots,2r-1$ by taking the charge p subspace in the space of states of $2r$ fermions. Unlike the case of r complex or $2r$ real fermions, for the $2r$ complex fermions there is no difference between the NS or R boundary conditions (in fact, one can continuously interpolate between them using the spectral flow). The corresponding characters are given by Ξ_p.

The exceptional affine Kac–Moody algebras are realized using spin operators which map the NS sectors to R and vice versa. For example, to realize $\widehat{E}_8$, we take 8 complex fermions $\tilde{\psi}_i$, ψ_i.

The $\widehat{\mathrm{GL}}_\infty$ representations also have the free fermion realization.

The generators of $\widehat{\mathrm{GL}}_\infty$ are given by the expressions

$$\mathcal{O}_a = \sum_{i,j\in\mathbb{Z}} a_{i,j}\,:\psi_i\tilde{\psi}_j:.$$

L Examples of conformal extensions

We list here the examples of the groups $\mathbf{G}$, their conformal extensions $\mathrm{C}\mathbf{G}$, and the homomorphisms

$$D\colon\ \mathrm{C}\mathbf{G} \longrightarrow \mathrm{C} = (\mathbb{C}^\times)^{z_{\mathfrak{g}}}.$$

1. The A_r series. The conformal group $\mathrm{C}\mathbf{G} = \mathrm{GL}(r+1,\mathbb{C})$, i.e., the group of non-degenerate $(r+1)\times(r+1)$ matrices h. The homomorphism D is given by the determinant

$$D(h) = \det(h).$$

2. The D_r series. First we describe the conformal extension of the group $\mathrm{SO}(2r)$. This is a group of the conformal linear transformations of $\mathbb{C}^{2r}$, i.e., the group of $2r\times 2r$ matrices h, such that

$$hh^t = D(h)\cdot \mathbf{1}_{2r}, \qquad D(h)\in\mathbb{C}^\times$$

with the homomorphism $D(h_1h_2) = D(h_1)D(h_2)$, such that $D(h)^r = \det(h)$. In other words, these are the linear transformations which preserve the non-degenerate symmetric bilinear form up to a scalar multiple. The group $\mathrm{CSpin}(2r)$ for r even has two independent central elements c_+, c_-, so that c_+ acts trivially in S_- and c_- acts trivially in S_+ which act non-trivially in the spin representations S_+, while in the tensor representations $\wedge^i V$ both act as $c_\pm^i$, and their product is equal to $D(h)$. For r odd there is one central generator c which acts as c^i on $\wedge^i V$ and as $c^{\pm 1}$ on $S_\pm$.

3. The E_6, E_7 case. The E_6, E_7 groups can be characterized as the symmetries of certain polylinear forms in the 27 and the 56-dimensional vector spaces, respectively. Their conformal versions preserve these forms up to a scalar multiple. For example, E_6 is a stabilizer of the cubic form

$$Q(z,x,y) = \mathrm{Pf}(z) + z(x\wedge y),$$

where $x,y\in V\approx\mathbb{C}^6$, $z\in\Lambda^2 V^*\approx\mathbb{C}^{15}$.

L.1 The maximal torus T and the action of Z

The maximal torus $\mathbf{T}_{\mathrm{q}}$ can be coordinatized by $(\mathbb{C}^\times)^r$ via

$$(g_1,\dots,g_r)\in(\mathbb{C}^\times)^r \mapsto \prod_{i=1}^r g_i^{\alpha_i}\in\mathbf{T}$$

since in our conventions $\exp 2\pi\mathrm{i}\alpha_j = 1\in\mathbf{T}$.

1. The maximal torus of the A_r series is the group of diagonal matrices of size $r+1$

$$g = \mathrm{diag}\left(g_1, g_2g_1^{-1}, g_3g_2^{-1},\dots,g_rg_{r-1}^{-1}, g_r^{-1}\right) \tag{L.1}$$

with unit determinant. The formula (L.1) corresponds to the standard choice of the simple roots

$$\alpha_j = e_j - e_{j+1}, \qquad j=1,\dots,r,$$

where e_j represent the vectors of some orthonormal basis in $\mathbb{C}^{r+1}$. The center $Z = \mathbb{Z}/(r+1)\mathbb{Z}$ acts on $\mathbf{T}$, in the g_j coordinates, via

$$g_j\mapsto\omega^j g_j, \qquad \omega = \mathrm{e}^{\frac{2\pi\mathrm{i}}{r+1}}$$

so that the matrix g in (L.1) is multiplied by ω. The generator $\zeta\equiv\zeta_1$ of the center Z can be chosen as follows:

$$\zeta = \frac{1}{r+1}\sum_{j=1}^r j\alpha_j.$$

2. The maximal torus of the D_r series can be identified with the group of block-diagonal matrices of size $2r \times 2r$

$$g = \operatorname{diag}\left(R(g_1), R\left(g_2 g_1^{-1}\right), \ldots, R\left(g_{r-1} g_r g_{r-2}^{-1}\right), R\left(g_r g_{r-1}^{-1}\right)\right), \tag{L.2}$$

where for $t \in \mathbb{C}$:

$$R(t) = \frac{t}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} + \frac{t^{-1}}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}.$$

Of course, equation (L.2) defines only an element of the group $\mathrm{SO}(2r, \mathbb{C})$, in order to define the element of the spin cover $\mathrm{Spin}(2r, \mathbb{C})$ we have to specify the way g acts in the spin representations S_+, S_-. This is equivalent to the choice of the square root of g_{r-1}^2 and g_r^2, respectively, and the parametrization

$$g = \prod_{i=1}^{r} g_i^{\alpha_i} \tag{L.3}$$

corresponds to these square roots being g_{r-1} and g_r, respectively. In equation (L.3) the simple roots of the D_r algebra are

$$\alpha_i = e_i - e_{i+1}, \qquad i = 1, \ldots, r-1, \qquad \alpha_r = e_{r-1} + e_r.$$

The center acts on $\mathbf{T}$ as follows. If $r = 2s+1$, then the transformation generated by the 4-th root of unity ω, $\omega^4 = 1$, acts by

$$\begin{aligned} &g_{2m+1} \mapsto \omega^2 g_{2m+1}, \\ &g_{2m} \mapsto g_{2m}, \qquad m = 1, \ldots, s-1, \\ &g_{2s} \mapsto \omega^{-1} g_{2s}, \qquad g_{2s+1} \mapsto \omega g_{2s+1} \end{aligned}$$

and multiplies g by ω^2. The center is $Z = \mathbb{Z}/4\mathbb{Z}$ in this case. If $r = 2s$, then the transformations generated by ω_1, ω_2, such that $\omega_1^2 = \omega_2^2 = 1$, act on g_j via

$$\begin{aligned} &g_{2m+1} \mapsto \omega_1\omega_2 g_{2m+1}, \\ &g_{2m} \mapsto g_{2m}, \qquad m = 1, \ldots, s-1, \\ &g_{2s-1} \mapsto \omega_1 g_{2s-1}, \qquad g_{2s} \mapsto \omega_2 g_{2s} \end{aligned}$$

and multiply g by $\omega_1\omega_2$. The center is $Z = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ in this case.

3. In order to describe the action of the center $Z = \mathbb{Z}/3\mathbb{Z}$ on the maximal torus of the group E_6 let us choose the canonical basis of simple roots of E_6:

$$\begin{aligned} &\alpha_1 = e_1 - e_2, \qquad \alpha_3 = e_2 - e_3, \\ &\alpha_4 = e_3 - e_4, \qquad \alpha_5 = e_4 - e_5, \\ &\alpha_6 = e_5 - e_6, \qquad \alpha_2 = -e_1 - e_2 - e_3 + \eta e, \qquad e = \sum_{i=1}^{6} e_i, \end{aligned}$$

where $\eta = \frac{\sqrt{3}-1}{2\sqrt{3}}$. Then the element $\omega \in Z$, $\omega^3 = 1$ acts on the element of the maximal torus of E_6

$$g = \prod_{i=1}^{6} g_i^{\alpha_i} \in \mathbf{T}_{\mathrm{q}}$$

by

$$g \mapsto \prod_{i=1}^{6} \left(\omega^i g_i\right)^{\alpha_i} = \omega^{\alpha_1 - \alpha_2 + \alpha_4 - \alpha_5} g.$$

4. For the group E_7, the simple roots are

$$\alpha_1 = e_1 - e_2, \qquad \alpha_3 = e_2 - e_3, \qquad \alpha_4 = e_3 - e_4,$$
$$\alpha_5 = e_4 - e_5, \qquad \alpha_6 = e_5 - e_6, \qquad \alpha_7 = e_6 - e_7,$$
$$\alpha_2 = -e_1 - e_2 - e_3 + \eta e, \qquad e = \sum_{i=1}^{7} e_i,$$

where $\eta = \frac{3-\sqrt{2}}{7}$. The generator ω, $\omega^2 = 1$, of the center $Z = \mathbb{Z}/2\mathbb{Z}$ acts on the element of the maximal torus of E_7

$$g = \prod_{i=1}^{7} g_i^{\alpha_i} \in \mathbf{T}$$

by

$$g \mapsto \omega^{\alpha_5+\alpha_7+\alpha_2} g.$$

M The elliptic curve and elliptic functions

The combination

$$\mathfrak{q} = \prod_{i \in \mathrm{Vert}_\gamma} \mathfrak{q}_i^{a_i} = \mathrm{e}^{2\pi \mathrm{i} \tau} \tag{M.1}$$

of the multiplicative couplings $\mathfrak{q}_i$ plays a special role. It defines an elliptic curve

$$\mathcal{E} = \mathbb{C}^\times / \mathfrak{q}^{\mathbb{Z}}, \tag{M.2}$$

which underlies many constructions related to the class II and class II* theories.

The infinite product

$$\phi(\mathfrak{q}) = \prod_{n=1}^{\infty} (1 - \mathfrak{q}^n) \equiv (\mathfrak{q}, \mathfrak{q})_\infty$$

is the Euler product. Dedekind eta-function

$$\eta(\mathfrak{q}) = \frac{1}{\mathfrak{q}^{\frac{1}{24}} \phi(\mathfrak{q})}.$$

N Coordinates on $\mathcal{E}$

The elliptic curve $\mathcal{E}$ can be described in several coordinate systems. The multiplicative coordinate system

$$t \sim \mathfrak{q}^n t, \qquad t \in \mathbb{C}^\times, \qquad n \in \mathbb{Z}$$

is used in (M.2). The additive coordinate system

$$z \sim z + m + n\tau, \qquad z \in \mathbb{C},$$

where

$$\tau \in \mathbb{C}, \qquad \mathrm{Im}\, \tau > 0$$

is defined in (M.1), and $m, n \in \mathbb{Z}$, is related to the multiplicative one via:

$$t = \mathrm{e}^{2\pi \mathrm{i} z}.$$

O Elliptic functions

O.1 Jacobi theta-functions

Our conventions for the theta-functions, associated to the elliptic curve $\mathcal{E}=\mathbb{C}^\times/\mathfrak{q}^{\mathbb{Z}}$, are

$$\begin{aligned}
\theta_1(t;\mathfrak{q}) &= \mathrm{i}\sum_{n\in\mathbb{Z}+\frac12}(-1)^{n-\frac12}t^n\mathfrak{q}^{\frac12 n^2},\\
\theta_2(t;\mathfrak{q}) &= \sum_{n\in\mathbb{Z}+\frac12}t^n\mathfrak{q}^{\frac12 n^2},\\
\theta_3(t;\mathfrak{q}) &= \sum_{n\in\mathbb{Z}}t^n\mathfrak{q}^{\frac12 n^2},\\
\theta_4(t;\mathfrak{q}) &= \sum_{n\in\mathbb{Z}}(-1)^n t^n\mathfrak{q}^{\frac12 n^2}.
\end{aligned}\tag{O.1}$$

Note that in older literature on elliptic theta-functions $\mathfrak{q}$ is used to denote the nome" $q_{\text{nome}}=\mathrm{e}^{\pi\mathrm{i}\tau}=\mathfrak{q}^{\frac12}$. In order not to be confused we shall always specify the $\mathfrak{q}$ modulus in the formula for $\theta_\alpha(t;\mathfrak{q})$.

It is often convenient to use basic pseudo-elliptic θ-function

$$\theta(t;\mathfrak{q})=\sum_{b\in\mathbb{Z}}(-t)^b\mathfrak{q}^{\frac12 b(b-1)}=(1-t)+\mathfrak{q}\big(t^2-t^{-1}\big)+\cdots,$$

which is related in a simple manner to $\theta_1(t;\mathfrak{q})$:

$$\theta(t;\mathfrak{q})=-\mathrm{i}\mathfrak{q}^{-\frac18}t^{\frac12}\theta_1(t;\mathfrak{q})$$

and also to the $\widehat{A}_1$ Weyl denominator in the affine Kac–Weyl character formula

$$\theta(t;\mathfrak{q})=R(h)=\prod_{n\geq0}\big(1-t\mathfrak{q}^n\big)\big(1-\mathfrak{q}^{n+1}\big)\big(1-t^{-1}\mathfrak{q}^{n+1}\big).$$

The transformation rules for $\theta(t;\mathfrak{q})$ are

$$\theta(\mathfrak{q}t;\mathfrak{q})=\theta\big(t^{-1};\mathfrak{q}\big)=-t^{-1}\theta(t;\mathfrak{q})$$

and transformation properties of $\theta_1(t,\mathfrak{q})$ are

$$\theta_1(\mathfrak{q}t;\mathfrak{q})=-\mathfrak{q}^{-\frac12}t^{-1}\theta_1(t;\mathfrak{q}).$$

The relation of Jacobi theta functions to the Weierstraß functions is described below.

O.2 The ξ- and $\wp$-functions

Define the function

$$\xi(t;\mathfrak{q})=t\partial_t\log\theta_1(t;\mathfrak{q}),$$

which satisfies

$$\xi(\mathfrak{q}t;\mathfrak{q})=\xi(t;\mathfrak{q})-1,\qquad \xi\big(t^{-1};\mathfrak{q}\big)=-\xi(t;\mathfrak{q})$$

and has simple poles at $t=\mathfrak{q}^{\mathbb{Z}}$. Therefore, up to a shift by a linear function of $\log(t)$, $\xi(t;\mathfrak{q})$ is the Weierstraß ζ-function.

From the product representation of $\theta_1(t;q)$ we immediately get

$$\xi(t;\mathfrak{q}) = \frac{1}{2}\frac{t+1}{t-1} - \sum_{n\geq 1}\mathfrak{q}^n \sum_{d|n}\left(t^d - t^{-d}\right).$$

The expansion of $\xi(t;\mathfrak{q})$ near $t=1$ reads as

$$\xi(t,\mathfrak{q}) = \frac{1}{t-1} + \sum_{k\geq 0}\xi_k (t-1)^k$$

with $\xi_0 = 1/2$ and

$$\xi_1(\mathfrak{q}) = -2\sum_{n\geq 1,k\geq 1} k\mathfrak{q}^{nk} = 2\mathfrak{q}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{q}}\log\phi(\mathfrak{q}).$$

The function

$$\wp(t;\mathfrak{q}) = -t\partial_t\xi(t;\mathfrak{q}) + \wp_0, \qquad \wp_0 = 2\mathfrak{q}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{q}}\log\eta(\mathfrak{q})$$

is elliptic function, i.e., a meromorphic function on $\mathcal{E}$:

$$\wp(\mathfrak{q}t;\mathfrak{q}) = \wp(t;\mathfrak{q}),$$

which has a single second-order pole in $\mathcal{E}$, and on the covering space $\mathbb{C}^\times_{\langle t\rangle}$ it has a double pole at $t = \mathfrak{q}^{\mathbb{Z}}$.

The function $\wp(t;\mathfrak{q})$ is actually the Weierstraß $\wp$-function: the double-periodic version of the $\frac{1}{z^2}$ function in the plane $z\in\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$, with the constant $\wp_0$ chosen so that the constant term in the expansion of $\wp\big(t=\mathrm{e}^{2\pi\mathrm{i}z};\mathfrak{q}\big)$ in Laurent series in z at $z=0$ vanishes. This gives

$$\wp(t,\mathfrak{q}) = \frac{t}{(1-t)^2} + \frac{1}{12} + \sum_{k\geq 1} k\frac{\mathfrak{q}^k}{1-\mathfrak{q}^k}\left(t^k + t^{-k} - 2\right). \tag{O.2}$$

One more differentiation with respect to $\log t$ defines the Weierstraß elliptic function with the pole of the third order

$$\wp'(t,\mathfrak{q}) := t\partial_t\wp(t,\mathfrak{q})$$

and the expansion near $t=1$:

$$\wp'(t,\mathfrak{q}) = t\frac{1+t}{(1-t)^3} + \sum_{k\geq 0} k^2\frac{\mathfrak{q}^k}{1-\mathfrak{q}^k}\left(t^k - t^{-k}\right).$$

Define the functions $X(t,\mathfrak{q})$ and $Y(t,\mathfrak{q})$ by

$$X(t,\mathfrak{q}) = \wp(t,\mathfrak{q}), \qquad Y(t,\mathfrak{q}) = \wp'(t,\mathfrak{q})$$

and define the weight 4 modular form $g_2(\mathfrak{q})$ and weight 6 modular form $g_3(\mathfrak{q})$

$$g_2(\mathfrak{q}) := \frac{1}{12} + 20\sum_{k=1}^{\infty} k^3\frac{\mathfrak{q}^k}{1-\mathfrak{q}^k},$$

$$g_3(\mathfrak{q}) := -\frac{1}{216} + \frac{7}{3}\sum_{k=1}^{\infty} k^5\frac{\mathfrak{q}^k}{1-\mathfrak{q}^k}.$$

The Weierstraß functions $X(t,\mathfrak{q})$ and $Y(t,\mathfrak{q})$ satisfy the cubic equation

$$Y(t,\mathfrak{q})^2 = 4X(t,\mathfrak{q})^3 - g_2(\mathfrak{q})X(t,\mathfrak{q}) - g_3(\mathfrak{q}) = 4\prod_{\alpha=1}^{3}(X(t,\mathfrak{q}) - e_\alpha), \tag{O.3}$$

where

$$e_1 = \wp(-1;\mathfrak{q}), \qquad e_2 = \wp\big(-\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}\big), \qquad e_3 = \wp\big(\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}\big). \tag{O.4}$$

The periods of the holomorphic differential $\mathrm{d}X/Y$ on $\mathcal{E}$ are equal to

$$\frac{1}{2\pi\mathrm{i}}\oint_A \frac{\mathrm{d}X}{Y} = 1, \qquad \frac{1}{2\pi\mathrm{i}}\oint_B \frac{\mathrm{d}X}{Y} = \tau.$$

O.3 ϕ_i and $\tilde{\phi}_{\tilde{j}}$ -functions

We shall also need to deal with meromorphic functions on $\mathcal{E}$ with the higher-order poles at $t=1$:[8] on $\mathcal{E}$ with the only pole of order $\le (r+1)$ at $t=1$. Explicitly,

$$\phi_i(t;\mathfrak{q}) = \frac{(-\mathfrak{q}/t)^i\theta\big(-(-t)^{r+1}\mathfrak{q}^{-i};\mathfrak{q}^{r+1}\big)}{\theta(t;\mathfrak{q})^{r+1}}. \tag{O.5}$$

The $r+1$ meromorphic functions $\phi_i(t;\mathfrak{q})$, $i=0\dots r$ with the pole of order $r+1$ at $t=1$ form a basis of $H^0(\mathcal{E},\mathcal{O}((r+1)p_0)) \approx \mathbb{C}^{r+1}$. On the other hand, if we represent elliptic curve $\mathcal{E}$ in the Weierstraß form (O.3), $Y^2 = 4X^3 - g_2X - g_3$, then the basis in the space of meromorphic elliptic functions on $\mathcal{E}$ with the pole at $t=1$ of less then $r+2$ order can also be chosen as the monomials

$$\tilde{\phi}_{\tilde{j}}(t,\mathfrak{q}) = \begin{cases} X^{\tilde{j}/2} & \text{if } 2|\tilde{j}, \\ X^{(\tilde{j}-3)/2}Y & \text{if } 2|\big(\tilde{j}+1\big), \end{cases} \qquad \tilde{j}\in\{0,2,\dots,r+1\},$$

since $X(t;\mathfrak{q}) = \wp(t;\mathfrak{q})$ is the Weierstraß elliptic function with the pole of second order at $t=1$ and $Y(t;\mathfrak{q}) = t\partial_t\wp(t;\mathfrak{q})$ has the pole of third order at $t=1$. There is a linear relation between the Weierstraß monomials $\tilde{\phi}_{\tilde{j}}(t;\mathfrak{q})$ and the ratio (O.5) of theta-functions $\phi_i(t;\mathfrak{q})$ which we shall denote as

$$\phi_i(t;\mathfrak{q}) = \sum_{\tilde{j}} M_{i\tilde{j}}(\mathfrak{q})\tilde{\phi}_{\tilde{j}}(t;\mathfrak{q}), \tag{O.6}$$

where $M_{j\tilde{j}}(\mathfrak{q})$ is a certain $\mathfrak{q}$-modular $(r+1)\times(r+1)$ transformation matrix, see example in Appendix O.4.

O.4 Expansion of Weierstraß functions in terms of theta-functions

The matrix $M(\mathfrak{q})$ for the change of basis from meromorphic elliptic functions (O.5) to Weierstraß functions for $r=1$ is

$$\begin{pmatrix} 1 \\ X(t,\mathfrak{q}) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} \phi_0(t,\mathfrak{q}) \\ \phi_1(t,\mathfrak{q}) \end{pmatrix}$$

[8] It is easy to verify that $\phi_i(t\mathfrak{q},\mathfrak{q}) = \phi_i(t,\mathfrak{q})$.

with

$$
\begin{aligned}
M_{11} &= \theta_3(1,\mathfrak{q}),\\
M_{12} &= \theta_2(1,\mathfrak{q}),\\
M_{21} &= M_{11}\Big(-\frac{1}{6}m_2(2;\mathfrak{q}) + \frac{1}{2}m_2(4;\mathfrak{q})^{(1)} - 4m_2(4;\mathfrak{q})^{(2)}\Big),\\
M_{22} &= M_{12}\Big(-\frac{1}{6}m_2(2;\mathfrak{q}) - \frac{1}{2}m_2(4;\mathfrak{q})^{(1)} + 4m_2(4;\mathfrak{q})^{(2)}\Big),
\end{aligned}
$$

where $m_k(N;\mathfrak{q})^{(i)}$ denotes i-th modular form for modular group $\Gamma_0(N)$ of weight k in a certain basis. Concretely

$$
\begin{aligned}
m_2(2,\mathfrak{q}) &= \Theta_{D_4}(\mathfrak{q}) = \frac{1}{2}\big(\theta_3(\mathfrak{q})^4 + \theta_4(\mathfrak{q})^4\big),\\
m_2(4,\mathfrak{q})^{(1)} &= m_2\big(2,\mathfrak{q}^2\big),\\
m_2(4,\mathfrak{q})^{(2)} &= \frac{1}{16}\theta_2(\mathfrak{q})^4 = \frac{\eta\big(\mathfrak{q}^4\big)^8}{\eta\big(\mathfrak{q}^2\big)^4}.
\end{aligned}
$$

O.5 Even ϕ functions for D series

From (J.8), we find

$$
\mathcal{Y}_0(x)^2\prod_{i=1}^{r}\frac{\theta(t/t_i(x);\mathfrak{q})}{\theta(t;\mathfrak{q})}\frac{\theta\big(t/t_i(x)^{-1};\mathfrak{q}\big)}{\theta(t;\mathfrak{q})} = \sum_{i=0}^{2r-1}(-1)^i\mathfrak{q}^{\frac{i}{2}+\frac{i^2}{4r}}\Xi_i(\mathcal{Y}_0;\mathfrak{t}(x);\mathfrak{q})\phi_i^{(2r)}(t;\mathfrak{q}), \tag{O.7}
$$

where $\phi_i^{(2r)}(t;\mathfrak{q})$ are the meromorphic functions, with the pole at $t=1$ of order not greater than $2r$, associated to the holomorphic sections $H^0(\mathcal{E},\mathcal{O}(2rp_0))$

$$
\phi_i^{(2r)}(t;\mathfrak{q}) = \frac{t^{-i}\theta\big(-t^{2r}\mathfrak{q}^{-i};\mathfrak{q}^{2r}\big)}{\theta(t;\mathfrak{q})^{2r}}.
$$

Using $\Xi_{2r-j}=\Xi_j$, we get

$$
\mathcal{Y}_0^2 s(t,x) = \sum_{i=0}^{r}\Xi_i(\mathcal{Y}_0;\mathfrak{t}(x);\mathfrak{q})M_{ij}(\mathfrak{q})X^j(t;\mathfrak{q}) \tag{O.8}
$$

with

$$
\begin{aligned}
\tilde\phi_0(t;\mathfrak{q}) &= \phi_0^{(2r)}(t;\mathfrak{q}),\\
\tilde\phi_i(t;\mathfrak{q}) &= (-1)^i\mathfrak{q}^{\frac{i}{2}+\frac{i^2}{4r}}\big(\phi_i^{(2r)}(t;\mathfrak{q}) + \mathfrak{q}^{2r-2i}\phi_{2r-i}^{(2r)}(t;\mathfrak{q})\big), \qquad i=1,\dots,r-1,\\
\tilde\phi_r(t;\mathfrak{q}) &= (-1)^r\mathfrak{q}^{\frac{3}{4}r}\phi_r^{(2r)}(t;\mathfrak{q}).
\end{aligned}
$$

The $r+1$ functions $\tilde\phi_i(t;\mathfrak{q})$ for $j=0,\dots,r$ form the basis in the space $H^0_{\text{even}}(\mathcal{E},\mathcal{O}(2rp_0))$ of meromorphic functions on elliptic curve symmetric under the reflection $t\to t^{-1}$ and with a pole of order no greater then $2r$ at the origin. Another basis in $H^0_{\text{even}}(\mathcal{E},\mathcal{O}(2rp_0))$ is given by the powers of Weierstraß $\wp$-function

$$
X(t;\mathfrak{q})^j, \qquad j=0,\dots,r.
$$

Let $M_{ij}(\mathfrak{q})$ be the transformation matrix between these bases in $H^0_{\text{even}}(\mathcal{E},\mathcal{O}(2rp_0))$

$$
\tilde\phi_i(t;\mathfrak{q}) = \sum_{j=0}^{r}M_{ij}(\mathfrak{q})X^j(t;\mathfrak{q}),
$$

then (O.8) can be written as (10.51).

P Affine E spectral curves

P.1 The E_r spectral curves from del Pezzo

P.1.1 The E_6 spectral curve from del Pezzo

```
E6Pezzo[X0_, X1_, X2_, X3_] := -X1 X3^2 + 4 X2^3 - g2 X2 X1^2 -
 g3 X1^3 +
 X0 (p[0] X1^2 + p[1] X2 X1 + p[6] X2 X3) +
 X0^2 (p[2] X1 + p[3] X2 + p[5] X3) + X0^3 p[4];
E6Xi = CoefficientList[E6Pezzo[z, 1, X + vx z, Y + vy z ], z];
vys = First[Solve[E6Xi[[2]] == 0, vy]];
E6PezzoCurveU = -4 Y^4 Resultant[ E6Xi[[3]] /. vys, E6Xi[[4]] /. vys,
 vx];
E6PezzoCurve =
 PolynomialRemainder[E6PezzoCurveU, E6Pezzo[0, 1, X, Y], Y] //
 Expand
```

P.1.2 The E_7 spectral curve from del Pezzo

```
E7Pezzo[X0_, X1_, X2_, X3_] := -X3^2 + 4 X1 X2^3 - g2 X2 X1^3 -
 g3 X1^4 +
 X0 (p[0] X1^3 + p[7] X1^2 X2 ) +
 X0^2 ( p[1] X1^2 + p[2] X1 X2 + p[6] X2^2 ) +
 X0^3 ( p[3] X1 + p[5] X2) + p[4] X0^4 ;
E7Xi = CoefficientList[
 E7Pezzo[z, 1, X + vx z, Y + vy z + 1/2 z^2 wy], z];
vys = First[Solve[E7Xi[[2]] == 0, vy]];
wys = First[Solve[E7Xi[[3]] == 0, wy]] /. vys;
E7PezzoCurveU = -1024 Y^18 Resultant[(E7Xi[[4]] /. wys /. vys),
 E7Xi[[5]] /. wys /. vys, vx];
E7PezzoCurve =
 PolynomialRemainder[E7PezzoCurveU, E7Pezzo[0, 1, X, Y], Y] //
 Expand
```

P.2 The map from del Pezzo to theta-function coordinates

In this section, we record the modular matrix $M_{i,\{j_1,\dots,j_r\}}(\mathfrak{q})$ discussed in Section 10.21 for E_6 bundles on $\mathcal{E}(\mathfrak{q})$. The result is presented in terms of a certain basis (m_h) in the space of modular forms of $\Gamma(6)$ of weight 1 with the expansion at the cusp $\mathfrak{q}=0$ starting as

$$m_h = \mathfrak{q}^h\left(1+\sum_{n=1}^{\infty}\mu_{h,n}\mathfrak{q}^n\right).$$

(If the argument of m_h is not explicitly spelled out it is always assumed to be $\mathfrak{q}$, i.e., in this section we set $m_h \equiv m_h(\mathfrak{q})$). Concretely, the basis of weight 1 modular forms of $\Gamma(6)$ is

$$m_0 = \left[1^{-3},2^6,3^1,6^{-2}\right]+3\left[1^1,2^{-2},3^{-3},6^6\right] = 1+6\mathfrak{q}+6\mathfrak{q}^3+\cdots,$$

$$m_{\frac{1}{6}} = \left[1^{-2},2^3,3^2,6^{-1}\right] = \mathfrak{q}^{\frac{1}{6}}\left(1+2\mathfrak{q}+2\mathfrak{q}^2+2\mathfrak{q}^3+\cdots\right),$$

$$m_{\frac{1}{3}} = \left[1^{-1},3^3\right] = \mathfrak{q}^{\frac{1}{3}}\left(1+\mathfrak{q}+2\mathfrak{q}^2+2\mathfrak{q}^4+\cdots\right),$$

$$m_{\frac12} = \big[1^{-1},2^2,3^{-1},6^2\big] = \mathfrak{q}^{\frac12}\big(1+\mathfrak{q}+2\mathfrak{q}^3+\mathfrak{q}^4+\cdots\big),$$
$$m_{\frac23} = \big[2^{-1},6^3\big] = \mathfrak{q}^{\frac23}\big(1+\mathfrak{q}^2+2\mathfrak{q}^4+\cdots\big),$$
$$m_1 = \big[1^1,2^{-2},3^{-3},6^6\big] = \mathfrak{q}\big(1-\mathfrak{q}+\mathfrak{q}^2+\mathfrak{q}^3-\mathfrak{q}^5+2\mathfrak{q}^6+\cdots\big),$$

where we use the standard notation for the eta-products

$$\big[1^{p_1},2^{p_2},3^{p_3},\dots,L^{p_L}\big] = \prod_{j=1}^{L}\eta\big(\mathfrak{q}^j\big)^{p_j},$$

where $\eta(\mathfrak{q})$ is the Dedekind modular function

$$\eta(\mathfrak{q}) = \mathfrak{q}^{\frac{1}{24}}\prod_{j=1}^{\infty}\big(1-\mathfrak{q}^j\big).$$

In particular, notice that the Weierstraß parameters $g_2(\mathfrak{q})$, $g_3(\mathfrak{q})$ are expressed as

$$g_2(\mathfrak{q}) = m_0\big(\tfrac{1}{12}m_0^3+18m_{\frac13}^3\big) = \tfrac{1}{12}+20\mathfrak{q}+180\mathfrak{q}^2+560\mathfrak{q}^3+1460\mathfrak{q}^4+\cdots,$$
$$g_3(\mathfrak{q}) = -\frac{1}{216}m_0^6+\tfrac{15}{6}m_0^3m_{\frac13}^3+27m_{\frac13}^6 = -\tfrac{1}{216}+\tfrac{7}{3}\mathfrak{q}+77\mathfrak{q}^2+\tfrac{1708}{3}\mathfrak{q}^3+\tfrac{7399}{3}\mathfrak{q}^4+\cdots.$$

We remark that $m_0(\mathfrak{q})$ and $m_{\frac13}(\mathfrak{q})$ are modular forms for $\Gamma(3)$ and they are equal to the theta-constants of the $\widehat{A}_2$ lattice associated respectively to the fundamental weight Λ_0 and Λ_1

$$m_0 = \sum_{\alpha\in Q(A_2)}\mathfrak{q}^{\frac12(\alpha,\alpha)}, \qquad m_{\frac13} = \sum_{\alpha\in\Lambda_1+Q(A_2)}\mathfrak{q}^{\frac12(\alpha,\alpha)}.$$

With all notations in place we finally present the explicitly computed[9] components of the modular matrix $M_{i,\{j_1,\dots\}}(\mathfrak{q})$ (the fixed argument $\mathfrak{q}$ is assumed for all $M_{i,\{j_1,\dots\}}$ and m_h) in the following table of relations

$$M_{0,\{0\}} = m_0^3+54m_{\frac13}^3,$$
$$M_{0,\{1\}} = \frac{1}{12}m_0^2\big(m_0^3-270m_{\frac13}^3\big),$$
$$M_{0,\{6\}} = 0,$$
$$M_{1,\{0\}} = 27m_0m_{\frac13}^2,$$
$$M_{1,\{1\}} = -\frac{3}{4}m_{\frac13}^2\big(5m_0^3+108m_{\frac13}^3\big),$$
$$M_{1,\{6\}} = -m_{\frac13}^2\big(3m_{\frac13}-m_0\big)^2\big(m_0^2+3m_{\frac13}m_0+9m_{\frac13}^2\big)^2,$$
$$M_{2,\{2\}} = -4m_0m_{\frac12}(m_0-12m_1)^3(m_0-4m_1)(m_0-3m_1)\big(m_0^2+12m_1m_0-72m_1^2\big),$$
$$M_{2,\{3\}} = -\frac{1}{3}m_{\frac12}(m_0-12m_1)^3(m_0-4m_1)(m_0-3m_1)\big(m_0^2+12m_1m_0-72m_1^2\big)$$
$$\times\big(m_0^3-108m_1m_0^2+864m_1^2m_0-1728m_1^3\big),$$
$$M_{2,\{5\}} = 0,$$

[9] The authors used computer algebra MATHEMATICA to find these expressions for modular forms by matching the first coefficients in the $\mathfrak{q}$-expansion.

$$
\begin{aligned}
M_{2,\{0,0\}} &= 72m_0m_{\frac{1}{2}}(m_0-3m_1),\\
M_{2,\{0,1\}} &= -12m_{\frac{1}{2}}(m_0-3m_1)\big(m_0^3+52m_1m_0^2-384m_1^2m_0+576m_1^3\big),\\
M_{2,\{1,1\}} &= -\frac{1}{2}m_{\frac{1}{2}}(m_0-3m_1)\big(m_0^5-200m_1m_0^4-96m_1^2m_0^3+20736m_1^3m_0^2-110592m_1^4m_0\\
&\quad +165888m_1^5\big),\\
M_{2,\{6,6\}} &= -\frac{1}{144}m_{\frac{1}{2}}(m_0-12m_1)^3(m_0-4m_1)(m_0-3m_1)\big(m_0^7-780m_1m_0^6\\
&\quad +40680m_1^2m_0^5-648000m_1^3m_0^4+4914432m_1^4m_0^3-19906560m_1^5m_0^2\\
&\quad +41803776m_1^6m_0-35831808m_1^7\big),\\
M_{3,\{3\}} &= 3m_0^2m_{\frac{1}{6}}m_{\frac{2}{3}}(m_0-12m_1)^3(m_0-4m_1)(m_0-3m_1)\big(m_0^2+12m_1m_0-72m_1^2\big),\\
M_{3,\{5\}} &= -4m_{\frac{1}{6}}m_{\frac{2}{3}}(m_0-12m_1)^5(m_0-4m_1)(m_0-3m_1)^2\big(m_0^2+12m_1m_0-72m_1^2\big),\\
M_{3,\{0,0\}} &= 216m_{\frac{1}{6}}m_{\frac{2}{3}}(m_0-3m_1),\\
M_{3,\{0,1\}} &= -12m_{\frac{1}{6}}m_{\frac{2}{3}}(m_0-3m_1)\big(7m_0^2+48m_1m_0-288m_1^2\big),\\
M_{3,\{0,6\}} &= -16m_{\frac{1}{6}}m_{\frac{2}{3}}(m_0-12m_1)^2(m_0-3m_1)^2\big(m_0^2+12m_1m_0-72m_1^2\big),\\
M_{3,\{1,1\}} &= \frac{1}{2}m_0m_{\frac{1}{6}}m_{\frac{2}{3}}(m_0-3m_1)\big(13m_0^3+360m_1m_0^2-2592m_1^2m_0+3456m_1^3\big),\\
M_{3,\{1,6\}} &= \frac{2}{3}m_0m_{\frac{1}{6}}m_{\frac{2}{3}}(m_0-12m_1)^2(m_0-3m_1)^2\big(m_0^3+216m_1m_0^2-1728m_1^2m_0\\
&\quad +3456m_1^3\big),\\
M_{4,\{4\}} &= 16m_{\frac{1}{3}}^3\big(3m_{\frac{1}{3}}-m_0\big)^3\big(m_0^2+3m_{\frac{1}{3}}m_0+9m_{\frac{1}{3}}^2\big)^3\big(36m_{\frac{1}{3}}^3-m_0^3\big),\\
M_{4,\{0,2\}} &= 24m_{\frac{1}{3}}^3\big(3m_{\frac{1}{3}}-m_0\big)\big(m_0^2+3m_{\frac{1}{3}}m_0+9m_{\frac{1}{3}}^2\big)\big(11m_0^3+108m_{\frac{1}{3}}^3\big),\\
M_{4,\{0,3\}} &= 2m_0^2\big(m_0-3m_{\frac{1}{3}}\big)m_{\frac{1}{3}}^3\big(m_0^2+3m_{\frac{1}{3}}m_0+9m_{\frac{1}{3}}^2\big)\big(13m_0^3+864m_{\frac{1}{3}}^3\big),\\
M_{4,\{1,2\}} &= 2m_0^2\big(m_0-3m_{\frac{1}{3}}\big)m_{\frac{1}{3}}^3\big(m_0^2+3m_{\frac{1}{3}}m_0+9m_{\frac{1}{3}}^2\big)\big(13m_0^3+864m_{\frac{1}{3}}^3\big),\\
M_{4,\{1,3\}} &= -\frac{1}{6}m_0\big(m_0-3m_{\frac{1}{3}}\big)m_{\frac{1}{3}}^3\big(m_0+6m_{\frac{1}{3}}\big)\big(m_0^2+3m_{\frac{1}{3}}m_0+9m_{\frac{1}{3}}^2\big)\\
&\quad \times\big(m_0^2-6m_{\frac{1}{3}}m_0+36m_{\frac{1}{3}}^2\big)\big(11m_0^3+108m_{\frac{1}{3}}^3\big),\\
M_{4,\{5,6\}} &= \frac{2}{3}m_0^5\big(m_0-3m_{\frac{1}{3}}\big)^3m_{\frac{1}{3}}^3\big(m_0^2+3m_{\frac{1}{3}}m_0+9m_{\frac{1}{3}}^2\big)^3,\\
M_{4,\{0,0,0\}} &= 720m_{\frac{1}{3}}^3,\\
M_{4,\{0,0,1\}} &= -540m_0^2m_{\frac{1}{3}}^3,\\
M_{4,\{0,1,1\}} &= 15m_0m_{\frac{1}{3}}^3\big(7m_0^3+54m_{\frac{1}{3}}^3\big),\\
M_{4,\{0,6,6\}} &= \frac{1}{24}m_0\big(m_0-3m_{\frac{1}{3}}\big)m_{\frac{1}{3}}^3\big(m_0^2+3m_{\frac{1}{3}}m_0+9m_{\frac{1}{3}}^2\big)\big(133m_0^6+5292m_{\frac{1}{3}}^3m_0^3\\
&\quad -338256m_{\frac{1}{3}}^6\big),\\
M_{4,\{1,1,1\}} &= -\frac{5}{12}m_{\frac{1}{3}}^3\big(17m_0^6+54m_{\frac{1}{3}}^3m_0^3+5832m_{\frac{1}{3}}^6\big),\\
M_{4,\{1,6,6\}} &= -\frac{1}{288}m_{\frac{1}{3}}^3\big(3m_{\frac{1}{3}}-m_0\big)\big(m_0^2+3m_{\frac{1}{3}}m_0+9m_{\frac{1}{3}}^2\big)\big(61m_0^9-36936m_{\frac{1}{3}}^3m_0^6\\
&\quad +1014768m_{\frac{1}{3}}^6m_0^3+6298560m_{\frac{1}{3}}^9\big),\\
M_{5,\{2\}} &= -12m_{\frac{1}{6}}m_{\frac{2}{3}}(m_0-12m_1)^3(m_0-4m_1)(m_0-3m_1)\big(m_0^2+12m_1m_0-72m_1^2\big),
\end{aligned}
$$

$$M_{5,\{6,6\}} = \frac{1}{48} m_0 m_{\frac{1}{6}} m_{\frac{2}{3}} (m_0 - 12m_1)^3 (m_0 - 4m_1)(m_0 - 3m_1)\big(7m_0^5 - 780m_1 m_0^4 + 11160 m_1^2 m_0^3 - 74304 m_1^3 m_0^2 + 228096 m_1^4 m_0 - 248832 m_1^5\big),$$

$$M_{6,\{0\}} = 27 m_0 m_{\frac{1}{3}}^2,$$

$$M_{6,\{1\}} = -\frac{3}{4} m_{\frac{1}{3}}^2 \big(5m_0^3 + 108 m_{\frac{1}{3}}^3\big),$$

$$M_{6,\{6\}} = m_{\frac{1}{3}}^2 \big(3m_{\frac{1}{3}} - m_0\big)^2 \big(m_0^2 + 3m_{\frac{1}{3}} m_0 + 9m_{\frac{1}{3}}^2\big)^2.$$

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