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Reduction of the 2D Toda Hierarchy and Linear Hodge Integrals

Si-Qi LIU, Zhe WANG and Youjin ZHANG

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China
E-mail: *liusq@tsinghua.edu.cn*, *zhe-wang17@mails.tsinghua.edu.cn*, *youjin@tsinghua.edu.cn*

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Abstract. We construct a certain reduction of the 2D Toda hierarchy and obtain a tau-symmetric Hamiltonian integrable hierarchy. This reduced integrable hierarchy controls the linear Hodge integrals in the way that one part of its flows yields the intermediate long wave hierarchy, and the remaining flows coincide with a certain limit of the flows of the fractional Volterra hierarchy which controls the special cubic Hodge integrals.

Key words: integrable hierarchy; limit fractional Volterra hierarchy; intermediate long wave hierarchy

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1 Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves of genus g with n marked points, and $\mathbb{L}_i$ be the i-th tautological line bundle of $\overline{\mathcal{M}}_{g,n}$ whose first Chern class is denoted by ψ_i for $i = 1, \dots, n$. Let $\mathbb{E}_{g,n}$ be the Hodge bundle of $\overline{\mathcal{M}}_{g,n}$ and $\gamma_j \in H^j(\overline{\mathcal{M}}_{g,n})$ be the degree j component of its Chern character. In [6], the following generating function of Hodge integrals is studied:

$$\mathcal{H}(\mathbf{t};\mathbf{s};\varepsilon) = \sum_{\substack{g,n,m\geq 0\\ k_1,\dots,k_n\geq 0\\ l_1,\dots,l_m\geq 1}} \varepsilon^{2g-2}\frac{t_{k_1}\cdots t_{k_n}}{n!}\frac{s_{l_1}\cdots s_{l_m}}{m!}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\gamma_{2l_1-1}\cdots\gamma_{2l_m-1}. \tag{1.1}$$

Note that only odd components of the Chern character are considered due to the vanishing of even components γ_{2j} by Mumford's relation [15]. It is proved that the evolutions of the two-point function

$$w = \varepsilon^2\frac{\partial^2}{\partial t_0^2}\mathcal{H}(\mathbf{t};\mathbf{s};\varepsilon)$$

along the time variables t_n form a tau-symmetric Hamiltonian integrable hierarchy which is called the Hodge hierarchy.

When the parameters s_k are taken to be equal to some special values, the Hodge hierarchy degenerates to some well-known integrable hierarchies. For example, by taking $s_k = 0$, we recover the Korteweg–de Vries (KdV) hierarchy. When $s_k = (2k-2)!s^{2k-1}$, (1.1) reduces to a generating function of linear Hodge integrals and the corresponding integrable hierarchy is proved to be the intermediate long wave (ILW) hierarchy in [3]. For arbitrary given non-zero numbers p, q, r satisfying the local Calabi–Yau condition

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 0,$$

we obtain the generating function of the special cubic Hodge integrals

$$\mathcal{H}(\mathbf{t};p,q,r;\varepsilon)=\sum_{g\geq 0}\varepsilon^{2g-2}\sum_{n\geq 0}\frac{t_{k_1}\cdots t_{k_n}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\mathcal{C}_g(-p)\mathcal{C}_g(-q)\mathcal{C}_g(-r) \tag{1.2}$$

from (1.1) by setting

$$s_j=-(2j-2)!\big(p^{2j-1}+q^{2j-1}+r^{2j-1}\big),$$

where $\mathcal{C}_g(z)$ is the Chern polynomial of the Hodge bundle $\mathbb{E}_{g,n}$. In [11], it is proved that the corresponding integrable hierarchy is the fractional Volterra (FV) hierarchy which is constructed in [12]. This fact is called the Hodge-FVH correspondence.

For a fixed parameter $p\neq 0$, we see from the local Calabi–Yau condition $pq+qr+rp=0$ that when q tends to zero so does $r=-pq/(p+q)$. Hence by taking such a limit, the generating function (1.2) becomes a generating function of linear Hodge integrals. A natural question is whether we can also take the limit of the FV hierarchy to obtain the ILW hierarchy? The answer to this question is not straightforward due to the fact that the construction of the FV hierarchy involves a complicated infinite linear combination of the time variables [11]. It turns out that one family of the flows of the FV hierarchy does not admit a limit when $q,r\to 0$ (see Section 3 for details) and the other family does have a limit, and this limit can be viewed as infinite linear combinations of the flows of the ILW hierarchy.

To illustrate the limit procedure more explicitly, let us consider the following equation which is one of the flows of the FV hierarchy:

$$\frac{\partial u}{\partial t}=\frac{\big(\Lambda^{-1/r}-1\big)\big(1-\Lambda^{-1/q}\big)}{\varepsilon\big(\Lambda^{1/p}-1\big)}\mathrm{e}^u,$$

where $\Lambda=\exp(\varepsilon\partial_x)$ is the shift operator. After the rescaling $\varepsilon\mapsto q\varepsilon$, $t\mapsto q^2t$, we can take the limit $q\to 0$ of the above equation, and obtain the following equation:

$$\frac{\partial u}{\partial t}=p\frac{\big(1-\Lambda^{-1}\big)(\Lambda-1)}{\varepsilon^2\partial_x}\mathrm{e}^u=p\mathrm{e}^u u_x+p\frac{\varepsilon^2}{12}\mathrm{e}^u\big(u_{xxx}+3u_xu_{xx}+u_x^3\big)+O\big(\varepsilon^4\big). \tag{1.3}$$

On the other hand, from the Lax equations of the ILW hierarchy [4] it follows that the first nontrivial flow of the ILW hierarchy reads

$$\frac{\partial w}{\partial s_1}=ww_x+\varepsilon\frac{\tau\partial_x^2}{2}\frac{\Lambda+1}{\Lambda-1}w-\tau w_x,$$

and all other flows have the form

$$\frac{\partial w}{\partial s_n}=\frac{w^n}{n!}w_x+\varepsilon^2\tau\big(a_nw^{n-1}w_{xxx}+b_nw^{n-2}w_{xx}w_x+c_nw^{n-3}w_x^3\big)+O\big(\varepsilon^4\big),$$
$$a_n=\frac{(n-1)!}{12},\qquad b_n=\frac{(n-2)!}{6},\qquad c_n=\frac{(n-3)!}{24},\qquad n\geq 1,$$

here τ is a parameter of the ILW hierarchy and we assume $n!=0$ for $n<0$. Let us define a flow by an infinite linear combination of the flows of the ILW hierarchy:

$$\frac{\partial}{\partial s}:=\sum_{n\geq 1}\frac{\partial}{\partial s_n},$$

then we obtain the following expression for this flow

$$\frac{\partial w}{\partial s}=\mathrm{e}^w w_x+\frac{\varepsilon^2}{24}\mathrm{e}^w\big(2w_{xxx}+4w_xw_{xx}+w_x^3\big)+O\big(\varepsilon^4\big),$$

here we take $\tau = 1$ for simplicity. It is straightforward to verify that after a Miura type transformation

$$u = w + \frac{\varepsilon^2}{24} w_{xx} + O(\varepsilon^4),$$

we arrive at

$$p\frac{\partial u}{\partial s} = p\mathrm{e}^u u_x + p\frac{\varepsilon^2}{12}\mathrm{e}^u\left(u_{xxx} + 3u_x u_{xx} + u_x^3\right) + O(\varepsilon^4),$$

which coincides with the flow (1.3). Therefore we see that the limit of the FV hierarchy is related to the ILW hierarchy by infinite linear combinations of flows.

Since both the ILW hierarchy and the limit of the FV hierarchy correspond to the linear Hodge integrals, we expect that these two integrable hierarchies are compatible with each other and that there exists an integrable hierarchy that contains these two hierarchies. It turns out that such an integrable hierarchy indeed exists and it is given by a certain reduction of the 2D Toda hierarchy.

The 2D Toda hierarchy [19, 20] is one of the central objects of study in the theory of integrable systems and its various reductions play important roles in the field of mathematical physics. For example, the 1D Toda hierarchy (and its extension), the equivariant Toda hierarchy and the Ablowitz–Ladik hierarchy control the Gromov–Witten theory of $\mathbb{P}^1$ [5, 7], the equivariant Gromov–Witten theory of $\mathbb{P}^1$ [16] and the Gromov–Witten theory of local $\mathbb{P}^1$ [2] respectively. For more applications of the 2D Toda hierarchy, one may refer to [17] and the references therein. Recall that the 2D Toda hierarchy can be described in terms of the operators [20]

$$L = \Lambda + \sum_{n\geq 0} u_n \Lambda^{-n}, \qquad \overline{L} = \bar{u}_{-1}\Lambda^{-1} + \sum_{n\geq 0} \bar{u}_n \Lambda^n$$

by the following Lax equations:

$$\frac{\partial L}{\partial t_{1,n}} = \left[(L^n)_+, L\right], \qquad \frac{\partial L}{\partial t_{2,n}} = -\left[\left(\overline{L}^n\right)_-, L\right],$$
$$\frac{\partial \overline{L}}{\partial t_{1,n}} = \left[(L^n)_+, \overline{L}\right], \qquad \frac{\partial \overline{L}}{\partial t_{2,n}} = -\left[\left(\overline{L}^n\right)_-, \overline{L}\right].$$

Let us consider the following reduction of the 2D Toda hierarchy

$$\log L = p\overline{L} - \log \overline{L} = pK,$$

where the precise definition of the logarithm of L and $\overline{L}$ will be given in Section 2,

$$K = \frac{1}{p}\varepsilon\partial_x + \mathrm{e}^u\Lambda^{-1}, \tag{1.4}$$

and p is a parameter. Then we define the reduction of the flows by

$$\varepsilon\frac{\partial K}{\partial t_{1,n}} = [(L^n)_+, K], \qquad \varepsilon\frac{\partial K}{\partial t_{2,n}} = -\left[\left(\overline{L}^n\right)_-, K\right], \qquad n \geq 1.$$

This is a well-defined reduction of the 2D Toda hierarchy, and due to the reason that the construction of this reduction is inspired by taking the limit of the FV hierarchy, we call this integrable hierarchy the limit fractional Volterra (LFV) hierarcy. We summarize its main properties in the following theorem.

Theorem 1.1. *The LFV hierarchy is a tau-symmetric Hamiltonian integrable hierarchy with hydrodynamic limit. Moreover, the flows $\frac{\partial}{\partial t_{1,n}}$ of the LFV hierarchy are certain limits of the flows of the FV hierarchy and the flows $\frac{\partial}{\partial t_{2,n}}$ is equivalent to the ILW hierarchy under a certain Miura-type transformation.*

The paper is organized as follows. In Section 2, we give the definition of the LFV hierarchy and prove that it is a tau-symmetric Hamiltonian integrable hierarchy. In Section 3, we explain how the construction of the LFV hierarchy is inspired by taking a certain limit of the FV hierarchy, and we also obtain a limit of the Hodge-FVH correspondence. In Section 4, we relate the LFV hierarchy to the ILW hierarchy. Finally in Section 5, we give some concluding remarks about the relation between our work and the Gromov–Witten/Hurwitz theory.

2 The LFV hierarchy and its properties

Throughout this paper, we work with the ring of differential polynomials $\mathcal{R}(u)$. It consists of formal power series in ε with coefficients being elements in the polynomial ring $C^\infty(u) \otimes \mathbb{C}\big[u^{(k)}\colon k \geq 1\big]$. Let us define a derivation ∂_x and an automorphism Λ on $\mathcal{R}(u)$ by

$$\partial_x = \sum_{k\geq 0} u^{(k+1)} \frac{\partial}{\partial u^{(k)}}, \qquad \Lambda = \exp(\varepsilon\partial_x).$$

If we view $u = u(x)$ as a function of the spatial variable x, then it is easy to see that $u^{(k)} = \partial_x^k u(x)$ and $\Lambda u(x) = u(x+\varepsilon)$. For this reason, the operator Λ is called the shift operator. Note that the ring $\mathcal{R}(u)$ is graded with respect to the differential degree $\deg_x$ given by $\deg_x u^{(k)} = k$.

Let us consider the following two difference operators given by

$$L = \Lambda + a_0 + a_1\Lambda^{-1} + \cdots , \tag{2.1}$$

$$\overline{L} = \mathrm{e}^u\Lambda^{-1} + b_0 + b_1\Lambda + \cdots , \tag{2.2}$$

where $a_i, b_i \in \mathcal{R}(u)$ for $i \geq 0$. In [20], the dressing operators P, Q for these difference operators are defined by

$$L = P\Lambda P^{-1}, \qquad P = 1 + \sum_{k\geq 1} p_k\Lambda^{-k}, \tag{2.3}$$

$$\overline{L} = Q\Lambda^{-1}Q^{-1}, \qquad Q = \sum_{k\geq 0} q_k\Lambda^{k}. \tag{2.4}$$

The coefficients p_k and q_k of the dressing operators P and Q are not in the ring $\mathcal{R}(u)$ but only exist in a certain extension of $\mathcal{R}(u)$ (for details, see [20]). Note that the choice of P and Q is unique up to the right multiplication by difference operators with constant coefficients. Following [5], we define the logarithm of the operators L, $\overline{L}$ as follows:

$$\log L := P(\varepsilon\partial_x)P^{-1} = \varepsilon\partial_x - \varepsilon P_x P^{-1},$$

$$\log \overline{L} := Q(-\varepsilon\partial_x)Q^{-1} = -\varepsilon\partial_x + \varepsilon Q_x Q^{-1},$$

where $P_x = \sum_{k\geq 1}(\partial_x p_k)\Lambda^{-k}$ and $Q_x = \sum_{k\geq 0}(\partial_x q_k)\Lambda^{k}$. The ambiguities of choices of P and Q are canceled in the operators P_xP^{-1} and Q_xQ^{-1}, and they are difference operators with coefficients belonging to $\mathcal{R}(u)$ [5]. Before giving the definition of the LFV hierarchy, let us make some preparations by proving the following lemmas.

Lemma 2.1. *There exist unique differential polynomials a_k such that*

$$\lim_{\varepsilon\to 0} a_k = \frac{p^{k+1}}{(k+1)!}\mathrm{e}^{(k+1)u} \tag{2.5}$$

and the difference operator L defined in (2.1) satisfies the relation

$$\frac{1}{p}\log L = K, \tag{2.6}$$

here p is a formal parameter and K is the differential-difference operator defined by (1.4).

Proof. Let us first find $a_i \in \mathcal{R}(u)$ such that the operator L defined in (2.1) satisfies the identities

$$\operatorname{res}\left[L^n, p\mathrm{e}^u\Lambda^{-1}\right] = \varepsilon\partial_x \operatorname{res} L^n, \qquad n \geq 1, \tag{2.7}$$

here and henceforth, for any difference operator $D = \sum_k f_k\Lambda^k$ with $f_k \in \mathcal{R}(u)$, we define its residue by $\operatorname{res} D = f_0$. By taking $n = 1$ in the equation (2.7), we arrive at

$$p(\Lambda - 1)\mathrm{e}^u = \varepsilon\partial_x a_0.$$

Then the above equation for a_0 has a unique solution by taking the integral constant to be zero, i.e.,

$$a_0 = p\frac{\Lambda - 1}{\varepsilon\partial_x}\mathrm{e}^u. \tag{2.8}$$

For general $n \geq 1$, one can show by induction that if we represent $L^n = \sum_k f_k^n\Lambda^k$, then the differential polynomials f_k^n can be viewed as functions in a_i and we have

$$f_k^n = f_k^n(a_0, \dots, a_{n-1-k}), \qquad k \leq n-1,$$
$$f_0^n = \left(1 + \Lambda + \dots + \Lambda^{n-1}\right)a_{n-1} + g_n(a_0, \dots, a_{n-2}).$$

Therefore we see that the differential polynomials can be found recursively by taking $n = 2, 3, \dots$ in the equation (2.7). More explicitly, if we have found differential polynomials $a_0, \dots, a_{n-1}$, to determine a_n, we consider the equation

$$\operatorname{res}\left[L^{n+1}, p\mathrm{e}^u\Lambda^{-1}\right] = \varepsilon\partial_x \operatorname{res} L^{n+1}$$

and obtain that

$$p\left(1 - \Lambda^{-1}\right)\left(f_1^{n+1}(a_0, \dots, a_{n-1})\Lambda\mathrm{e}^u\right) = \left(1 + \Lambda + \dots + \Lambda^n\right)\varepsilon\partial_x a_n + \varepsilon\partial_x g_{n+1}(a_0, \dots, a_{n-1}).$$

Hence we can solve a_n uniquely by taking the integral constant to be zero.

Next we show that the difference operator L determined by the equations (2.7) satisfies the relation (2.6). Indeed, by using the results given in [5], we see that the equations (2.7) imply that

$$\varepsilon P_x P^{-1} = -p\mathrm{e}^u\Lambda^{-1},$$

and therefore the relation $\frac{1}{p}\log L = K$ holds true.

Finally we show that the differential polynomials a_i determined above satisfy the relation (2.5). From the relation (2.6) it follows that $[L, pK] = 0$, which is equivalent to the following recursion relation:

$$\frac{1}{p}\varepsilon\partial_x a_{k+1} = a_k\Lambda^{-k}\mathrm{e}^u - \mathrm{e}^u\Lambda^{-1}a_k, \qquad k \geq 0. \tag{2.9}$$

Then (2.5) can be verified using this recursion relation and the initial condition (2.8). The lemma is proved. ■

Lemma 2.2. *There exist unique differential polynomials b_k such that*

$$\lim_{\varepsilon\to 0} b_k = \frac{\mathrm{e}^{-ku}}{p^{k+1}}\beta_k$$

and the difference operator $\overline{L}$ defined in (2.2) satisfies the relations

$$\overline{L} - \frac{1}{p}\log \overline{L} = K,$$

where K is the differential-difference operator given by (1.4) and β_k are polynomials in u satisfying the recursion relations

$$\beta_{k+1} = \beta_k - \int_0^u k\beta_k \,\mathrm{d}u, \qquad k \geq 0,$$

with the initial condition $\beta_0 = u$.

Proof. The lemma can be proved by using a similar method that is used in the proof of Lemma 2.1, so we omit the details here. For later use, we write down the following recursion relations satisfied by b_k:

$$\frac{1}{p}\varepsilon\partial_x b_k = b_{k+1}\Lambda^{k+1}\mathrm{e}^u - \mathrm{e}^u\Lambda^{-1}b_{k+1}, \qquad k \geq -1, \tag{2.10}$$

with $b_{-1} = \mathrm{e}^u$. ■

Definition 2.3. The limit fractional Volterra (LFV) hierarchy consists of the flows

$$\varepsilon\frac{\partial K}{\partial t_{1,n}} = [(L^n)_+, K], \qquad \varepsilon\frac{\partial K}{\partial t_{2,n}} = -\big[\big(\overline{L}^n\big)_-, K\big], \qquad n \geq 1, \tag{2.11}$$

where the operator K is given by (1.4), and the opertors L and $\overline{L}$ are determined by Lemma 2.1 and Lemma 2.2 respectively. Here and in what follows, for a difference operator $\sum_k f_k\Lambda^k$, we define its positive part and negative part by

$$\left(\sum_k f_k\Lambda^k\right)_+ = \sum_{k\geq 0} f_k\Lambda^k, \qquad \left(\sum_k f_k\Lambda^k\right)_- = \sum_{k<0} f_k\Lambda^k.$$

It follows from Lemmas 2.1 and 2.2 that the operators L and $\overline{L}$ commute with K, therefore we have the following identities:

$$[(L^n)_+, K] = -[(L^n)_-, K], \qquad \big[\big(\overline{L}^n\big)_+, K\big] = -\big[\big(\overline{L}^n\big)_-, K\big].$$

Since $[(L^n)_+, K]$ is a difference operator of the form $\sum_{j\geq -1} f_j\Lambda^j$ and $[(L^n)_-, K]$ is of the form $\sum_{j\leq -1} g_j\Lambda^j$, it follows that the first set of equations given in (2.11) yields a hierarchy of well-defined equations of u. Similarly the second set of equations given in (2.11) is also well-defined. Moreover, it is easy to see that the flows of the LFV hierarchy are given by differential polynomials with hydrodynamic limits.

Example 2.4. The simplest flows of (2.11) read

$$\frac{\partial u}{\partial t_{1,1}} = p\frac{\big(1-\Lambda^{-1}\big)(\Lambda - 1)}{\varepsilon^2\partial_x}\mathrm{e}^u,$$
$$\frac{\partial u}{\partial t_{2,1}} = \frac{1}{p}u_x, \qquad \frac{\partial u}{\partial t_{2,2}} = \frac{u_x}{p^2}\varepsilon\partial_x\frac{\Lambda+1}{\Lambda-1}u + \frac{1}{p^2}\varepsilon\partial_x^2\frac{\Lambda+1}{\Lambda-1}u.$$

Let us proceed to present some basic properties of the LFV hierarchy. We will show that it is a tau-symmetric Hamiltonian integrable hierarchy. We start with proving that the flows defined in (2.11) mutually commute. The following lemma is standard in the theory of integrable hierarchies.

Lemma 2.5. *The following equations are satisfied by the operators L and $\overline{L}$:*

$$\varepsilon\frac{\partial L}{\partial t_{1,n}} = [(L^n)_+, L], \qquad \varepsilon\frac{\partial L}{\partial t_{2,n}} = -\big[\big(\overline{L}^n\big)_-, L\big],$$
$$\varepsilon\frac{\partial \overline{L}}{\partial t_{1,n}} = \big[(L^n)_+, \overline{L}\,\big], \qquad \varepsilon\frac{\partial \overline{L}}{\partial t_{2,n}} = -[\big(\overline{L}^n\big)_-, \overline{L}\,].$$

From Lemma 2.5, it is straightforward to derive the commutation relations of the flows (2.11).

Theorem 2.6. *The flows* (2.11) *of the LFV hierarchy mutually commute, i.e.,*

$$\left[\frac{\partial}{\partial t_{1,n}}, \frac{\partial}{\partial t_{1,m}}\right] = \left[\frac{\partial}{\partial t_{2,n}}, \frac{\partial}{\partial t_{2,m}}\right] = \left[\frac{\partial}{\partial t_{1,n}}, \frac{\partial}{\partial t_{2,m}}\right] = 0, \qquad n, m \geq 1.$$

To derive the Hamiltonian formalism of the LFV hierarchy, we need to compute the variational derivatives of the local functionals of the form

$$\int \operatorname{res} L^n, \qquad \int \operatorname{res} \overline{L}^n.$$

Let us give a general description on how such a variational derivative can be computed (see [5]).

Consider the 1-form

$$\sum_{k\geq 0} f_k \mathrm{d}u^{(k)}, \qquad f_k \in \mathcal{R}(u),$$

where d is the natural exterior differential operator on $\mathcal{R}(u)$. For two 1-forms $\sum_{k\geq 0} f_k \mathrm{d}u^{(k)}$ and $\sum_{k\geq 0} g_k \mathrm{d}u^{(k)}$, we denote $\sum_{k\geq 0} f_k \mathrm{d}u^{(k)} \sim \sum_{k\geq 0} g_k \mathrm{d}u^{(k)}$ if there exists another 1-form $\sum_{k\geq 0} h_k \mathrm{d}u^{(k)}$ such that:

$$\sum_{k\geq 0} f_k \mathrm{d}u^{(k)} - \sum_{k\geq 0} g_k \mathrm{d}u^{(k)} = \partial_x\left(\sum_{k\geq 0} h_k \mathrm{d}u^{(k)}\right),$$

here the derivation ∂_x acts on the 1-form by $\partial_x \mathrm{d}u^{(k)} = \mathrm{d}u^{(k+1)}$. Now for a local functional $\int h$, $h \in \mathcal{R}(u)$, we can compute its variational derivative as follows:

$$\mathrm{d}h = \sum_{k\geq 0} \frac{\partial h}{\partial u^{(k)}} \mathrm{d}u^{(k)} \sim \left(\frac{\delta}{\delta u}\int h\right)\mathrm{d}u. \tag{2.12}$$

Lemma 2.7. *Consider the local functionals defined by*

$$H_n = \frac{1}{np}\int \operatorname{res} L^n, \qquad n \geq 1.$$

Then their variational derivatives have the expressions

$$\frac{\delta H_n}{\delta u} = \frac{1}{p}\frac{\varepsilon\partial_x}{\Lambda - 1}\operatorname{res} L^n, \qquad n \geq 1. \tag{2.13}$$

Proof. We need the following identity whose proof can be found in [5]:

$$\operatorname{res} L^{n-1}\mathrm{d}L \sim -\operatorname{res} L^{n}\mathrm{d}\big(\varepsilon P_x P^{-1}\big), \tag{2.14}$$

where the operator P is defined in (2.3). From this identity and the relation (2.6) it follows that

$$\mathrm{d}\left(\frac{1}{np}\operatorname{res} L^n\right) \sim \frac{1}{p}\operatorname{res} L^{n-1}\mathrm{d}L \sim -\frac{1}{p}\operatorname{res} L^n\mathrm{d}\big(\varepsilon P_x P^{-1}\big) = \operatorname{res} L^n\big(\mathrm{e}^u\mathrm{d}u\Lambda^{-1}\big).$$

Here and henceforth, by abusing the notations we denote

$$\mathrm{d}\left(\sum_n f_k\Lambda^k\right) = \sum_k (\mathrm{d}f_k)\Lambda^k$$

for a difference operator $\sum_k f_k\Lambda^k$.

Let us represent L^n in the form $L^n = \sum_k a_{k,n}\Lambda^k$, we then arrive at the following relations:

$$\mathrm{d}\left(\frac{1}{np}\operatorname{res} L^n\right) \sim a_{1,n}\Lambda\big(\mathrm{e}^u\mathrm{d}u\big) \sim \big(\mathrm{e}^u\Lambda^{-1}a_{1,n}\big)\mathrm{d}u.$$

Hence by using the formula (2.12) we obtain

$$\frac{\delta H_n}{\delta u} = \mathrm{e}^u\Lambda^{-1}a_{1,n}, \qquad n \geq 1.$$

Finally by using the identity $\operatorname{res}[L^n, p\mathrm{e}^u\Lambda^{-1}] = \operatorname{res}\varepsilon\partial_x L^n$ obtained from Lemma 2.1, we prove the desired identity (2.13). The lemma is proved. ∎

Lemma 2.8. *The variational derivatives of the local functionals*

$$\overline{H}_n = \int\left(\operatorname{res}\frac{\overline{L}^{n+1}}{n+1} - \operatorname{res}\frac{\overline{L}^n}{np}\right), \qquad n \geq 1$$

can be represented as

$$\frac{\delta\overline{H}_n}{\delta u} = \frac{1}{p}\frac{\varepsilon\partial_x}{\Lambda - 1}\operatorname{res}\overline{L}^n, \qquad n \geq 1. \tag{2.15}$$

Proof. Similar to the identity(2.14) we have

$$\operatorname{res}\overline{L}^{n-1}\mathrm{d}\overline{L} \sim \operatorname{res}\overline{L}^n\mathrm{d}\big(\varepsilon Q_x Q^{-1}\big).$$

By using Lemma 2.2 we obtain the relations

$$\begin{aligned}\operatorname{res}\overline{L}^n\mathrm{d}K &= \operatorname{res}\overline{L}^n\mathrm{d}\overline{L} - \frac{1}{p}\operatorname{res}\overline{L}^n\mathrm{d}\big(\varepsilon Q_x Q^{-1}\big)\\ &\sim \operatorname{res}\overline{L}^n\mathrm{d}\overline{L} - \frac{1}{p}\operatorname{res}\overline{L}^{n-1}\mathrm{d}\overline{L}\\ &\sim \operatorname{res}\mathrm{d}\left(\frac{\overline{L}^{n+1}}{n+1} - \frac{\overline{L}^n}{np}\right).\end{aligned}$$

We then arrive at (2.15) by applying the calculation similar to the one we do in the proof of Lemma 2.7. The lemma is proved. ∎

Theorem 2.9. *The flows (2.11) can be represented as Hamiltonian systems as follows:*

$$\varepsilon\frac{\partial u}{\partial t_{1,n}} = \{u(x), H_n\}, \qquad \varepsilon\frac{\partial u}{\partial t_{2,n}} = \big\{u(x), \overline{H}_n\big\}, \qquad n \geq 1,$$

where the Poisson bracket $\{-,-\}$ is defined by the Hamiltonian operator

$$\mathscr{P} = p\frac{\big(1-\Lambda^{-1}\big)(\Lambda-1)}{\varepsilon^2\partial_x}.$$

Proof. From the definition (2.11) it is straightforward to see that

$$\varepsilon\frac{\partial u}{\partial t_{1,n}} = \big(1-\Lambda^{-1}\big)\operatorname{res} L^n = \varepsilon\mathscr{P}\frac{\delta H_n}{\delta u} = \varepsilon\{u(x), H_n\}.$$

As for the flows $\frac{\partial u}{\partial t_{2,n}}$, let us denote $\overline{L}^n = \sum_k b_{k,n}\Lambda^k$, then by combining the definition (2.11) and the definitions of L and $\overline{L}$, we arrive at

$$\varepsilon \mathrm{e}^u\frac{\partial u}{\partial t_{2,n}} = \frac{1}{p}\varepsilon\partial_x b_{-1,n} = \mathrm{e}^u\big(1-\Lambda^{-1}\big)\operatorname{res}\overline{L}^n,$$

which implies the Hamiltonian formalism of the flows $\frac{\partial u}{\partial t_{2,n}}$ due to Lemma 2.8. Finally it is obvious that $\mathscr{P}$ is indeed a Hamiltonian operator and thus the theorem is proved. ∎

Finally we are going to consider the tau-structure of the LFV hierarchy. The constructions and proofs are standard and very similar to those presented in [12].

Lemma 2.10. *We define the following functions for $k, l \geq 1$:*

$$\Omega_{1,k;1,l} = \sum_{n=1}^{l}\frac{\Lambda^n-1}{\Lambda-1}\big(\operatorname{res}\big(\Lambda^{-n}L^l\big)\operatorname{res}\big(L^k\Lambda^n\big)\big), \tag{2.16}$$

$$\Omega_{2,k;1,l} = \Omega_{1,l;2,k} = \sum_{n=1}^{l}\frac{\Lambda^n-1}{\Lambda-1}\big(\operatorname{res}\big(\Lambda^{-n}L^l\big)\operatorname{res}\big(\overline{L}^k\Lambda^n\big)\big), \tag{2.17}$$

$$\Omega_{2,k;2,l} = \sum_{n=1}^{l}\frac{\Lambda^n-1}{\Lambda-1}\big(\operatorname{res}\big(\Lambda^{-n}\overline{L}^l\big)\operatorname{res}\big(\overline{L}^k\Lambda^n\big)\big). \tag{2.18}$$

Then the following relations hold true:

$$\varepsilon\frac{\partial}{\partial t_{1,l}}\operatorname{res} L^k = \varepsilon\frac{\partial}{\partial t_{1,k}}\operatorname{res} L^l = (\Lambda-1)\Omega_{1,k;1,l},$$
$$\varepsilon\frac{\partial}{\partial t_{1,l}}\operatorname{res} \overline{L}^k = \varepsilon\frac{\partial}{\partial t_{2,k}}\operatorname{res} L^l = (\Lambda-1)\Omega_{2,k;1,l},$$
$$\varepsilon\frac{\partial}{\partial t_{2,l}}\operatorname{res} \overline{L}^k = \varepsilon\frac{\partial}{\partial t_{2,k}}\overline{\operatorname{res}} L^l = (\Lambda-1)\Omega_{2,k;2,l}.$$

Lemma 2.11. *The functions defined in (2.16)–(2.18) satisfy the following identities:*

$$\Omega_{i,k;j,l} = \Omega_{j,l;i,k}, \qquad \frac{\partial\Omega_{j,l;m,n}}{\partial t_{i,k}} = \frac{\partial\Omega_{i,k;m,n}}{\partial t_{j,l}}, \qquad i,j,m = 1,2, \quad k,l,n \geq 1.$$

Proof. Let us start by proving the first set of identities. It follows from Lemma 2.10 that

$$(\Lambda-1)\Omega_{i,k;j,l}=(\Lambda-1)\Omega_{j,l;i,k}.$$

Therefore there exist constants $c_{i,k;j,l}$ such that

$$\Omega_{i,k;j,l}-\Omega_{j,l;i,k}=c_{i,k;j,l}.$$

To verify that the constants $c_{j,k;j,l}$ vanish, we may compute the limits of the differential polynomials $\Omega_{i,k;j,l}$ by setting $u=0$ and $u^{(k)}=0$. It is easy to see from Lemmas 2.1 and 2.2 that the only non-trivial case is $i=j=1$. By using Lemma 2.1, it follows from a straightforward computation that

$$\Omega_{1,k;1,l}|_{u=u^{(i)}=0}=p^{k+l}k^kl^l\sum_{n=1}^{l}\left(\frac{k}{l}\right)^n\frac{n}{(l-n)!(k+n)!}.$$

The summation on the right hand side of the above identity is evaluated using Gosper's algorithm [8].[1] Let us denote

$$y_n=\left(\frac{k}{l}\right)^n\frac{n}{(l-n)!(k+n)!},\qquad 1\le n\le l,$$

and

$$z_n=-\left(\frac{k}{l}\right)^n\frac{l}{(k+l)(l-n)!(k+n-1)!},\qquad 1\le n\le l.$$

Then it is straightforward to verify that

$$z_{n+1}-z_n=y_n,\qquad 1\le n\le l-1,\qquad z_l=-y_l.$$

Hence we conclude that

$$\sum_{n=1}^{l}\left(\frac{k}{l}\right)^n\frac{n}{(l-n)!(k+n)!}=-z_1=\frac{1}{(k+l)(k-1)!(l-1)!}.$$

So we see that $\Omega_{1,k;1,l}|_{u=u^{(i)}=0}$ is symmetric with respect to the indices k and l and therefore we see that $\Omega_{1,k;1,l}=\Omega_{1,l;1,k}$.

On the other hand, it follows from Lemma 2.10 that

$$(\Lambda-1)\frac{\partial\Omega_{j,l;m,n}}{\partial t_{i,k}}=(\Lambda-1)\frac{\partial\Omega_{i,k;m,n}}{\partial t_{j,l}}. \tag{2.19}$$

Since the differential polynomials $\frac{\partial\Omega_{j,l;m,n}}{\partial t_{i,k}}$ have differential degrees greater than 1, from (2.19) we arrive at the validity of the second set of identities of the lemma. The lemma is proved. ■

Theorem 2.12. *For any solution $u(x;\boldsymbol{t})$ of the LFV hierarchy, there exists a tau-function $\tau(x;\boldsymbol{t})$ such that:*

$$(\Lambda-1)\left(1-\Lambda^{-1}\right)\log\tau=u, \tag{2.20}$$

$$\varepsilon(\Lambda-1)\frac{\partial\log\tau}{\partial t_{1,k}}=\operatorname{res}L^k, \tag{2.21}$$

$$\varepsilon(\Lambda-1)\frac{\partial\log\tau}{\partial t_{2,k}}=\operatorname{res}\overline{L}^k, \tag{2.22}$$

$$\varepsilon^2\frac{\partial^2\log\tau}{\partial t_{i,k}\partial t_{j,l}}=\Omega_{i,k;j,l}. \tag{2.23}$$

[1] Here we use a Mathematica package provided on the website https://www2.math.upenn.edu/~wilf/progs.html.

Proof. The compatibility of the equations (2.20)–(2.23) is given by the definition of LFV hierarchy (2.11), Lemmas 2.10 and 2.11. Hence such a tau-function exists and we prove the theorem. ∎

3 The LFV hierarchy as a limit of the FV hierarchy

Let us explain how the LFV hierarchy can be viewed as a certain limit of the FV hierarchy given in [12]. Let p, q, r be any given non-zero complex numbers satisfying the condition $pq + qr + rp = 0$. We introduce the shift operators

$$\Lambda_1 = \Lambda^{1/q}, \qquad \Lambda_2 = \Lambda^{1/p}, \qquad \Lambda_3 = \Lambda_1\Lambda_2 = \Lambda^{-1/r}, \qquad \Lambda = \mathrm{e}^{\varepsilon\partial_x},$$

and consider the following Lax operator

$$L^{\mathrm{FV}} = \Lambda_2 + \mathrm{e}^v\Lambda_1^{-1}$$

with $v = v(x,\varepsilon)$. The fractional powers

$$\mathcal{A} = \left(L^{\mathrm{FV}}\right)^{-p/r} = \Lambda_3 + \sum_{k\geq 0} f_k\Lambda_3^{-k}, \qquad \mathcal{B} = \left(L^{\mathrm{FV}}\right)^{-q/r} = g_{-1}\Lambda_3^{-1} + \sum_{k\geq 0} g_k\Lambda_3^{k}$$

are well-defined, and their coefficients f_k, g_k belong to the ring $\mathcal{R}(v)$. It follows from the relations

$$\left[\mathcal{A}, L^{\mathrm{FV}}\right] = \left[\mathcal{B}, L^{\mathrm{FV}}\right] = 0$$

that the coefficients f_n and g_n satisfy the following recursion relations:

$$(\Lambda_2 - 1)f_{k+1} = f_k\Lambda_3^{-k}\mathrm{e}^v - \mathrm{e}^v\Lambda_1^{-1}f_k, \qquad k\geq 0, \qquad f_0 = \frac{\Lambda_3 - 1}{\Lambda_2 - 1}\mathrm{e}^v, \tag{3.1}$$

$$(\Lambda_2 - 1)g_k = g_{k+1}\Lambda_3^{k+1}\mathrm{e}^v - \mathrm{e}^v\Lambda_1^{-1}g_{k+1}, \qquad k\geq -1, \qquad g_{-1} = \mathrm{e}^{\frac{1-\Lambda_3^{-1}}{1-\Lambda_1^{-1}}v}. \tag{3.2}$$

The fractional Volterra hierarchy is defined by the following Lax equations:

$$\varepsilon\frac{\partial L^{\mathrm{FV}}}{\partial T_{1,n}} = \left[\mathcal{A}_+^n, L^{\mathrm{FV}}\right], \qquad \varepsilon\frac{\partial L^{\mathrm{FV}}}{\partial T_{2,n}} = -\left[\mathcal{B}_-^n, L^{\mathrm{FV}}\right]. \tag{3.3}$$

Here for an operator $\mathcal{C}$ of the form $\sum c_k\Lambda_3^k$, we denote

$$\mathcal{C}_+ = \sum_{k\geq 0} c_k\Lambda_3^k, \qquad \mathcal{C}_- = \sum_{k<0} c_k\Lambda_3^k.$$

In order to take a certain limit of the FV hierarchy, we first introduce a new dispersion parameter $\tilde\varepsilon = \varepsilon/q$ and the associated shift operator $\tilde\Lambda = \mathrm{e}^{\tilde\varepsilon\partial_x}$. We have the relations

$$\Lambda_1 = \tilde\Lambda, \qquad \Lambda_3 = \tilde\Lambda^{\frac{p+q}{p}},$$

and we can rewrite the recursion relations (3.1) and (3.2) as follows:

$$\sum_{i\geq 1}\frac{\tilde\varepsilon^i}{i!}\frac{q^{i-1}}{p^i}\partial_x^i\tilde f_{k+1} = \tilde f_k\tilde\Lambda^{-\frac{k(p+q)}{p}}\mathrm{e}^v - \mathrm{e}^v\tilde\Lambda^{-1}\tilde f_k, \qquad k\geq 0, \tag{3.4}$$

$$\sum_{i\geq 1}\frac{\tilde\varepsilon^i}{i!}\frac{q^{i-1}}{p^i}\partial_x^i\tilde g_k = \tilde g_{k+1}\tilde\Lambda^{\frac{(k+1)(p+q)}{p}}\mathrm{e}^v - \mathrm{e}^v\tilde\Lambda^{-1}\tilde g_{k+1}, \qquad k\geq -1, \tag{3.5}$$

here

$$\tilde{f}_k = q^{k+1} f_k, \qquad \tilde{g}_k = \frac{g_k}{q^{k+1}}.$$

Then it is easy to see that the recursion relations (3.4) and (3.5) become the relations (2.9) and (2.10) after taking the limit $q \to 0$.

Let us look at the relations (3.1) and (3.4) more carefully. The coefficients f_i of the operator $\mathcal{A}$ can be uniquely determined from (3.1) by requiring that

$$\lim_{\varepsilon \to 0} f_k = \frac{p^{k+1} \mathrm{e}^{kv}}{(k+1)! q^{k+1}} \prod_{i=1}^{k+1} \left(1 + \frac{(i-k)q}{p}\right).$$

Therefore if we assume the limit

$$u(x, \tilde{\varepsilon}) = \lim_{q \to 0} v(x, q\tilde{\varepsilon})$$

exists, then it is easy to see from (3.4) that the limits

$$a_k(x, \varepsilon) = \lim_{q \to 0} q^{k+1} f_k(x, q\tilde{\varepsilon})|_{\tilde{\varepsilon} \mapsto \varepsilon} = \lim_{q \to 0} \tilde{f}_k(x, q\tilde{\varepsilon})|_{\tilde{\varepsilon} \mapsto \varepsilon}, \qquad k \geq 0 \tag{3.6}$$

satisfy the relations (2.8) and (2.9), and are exactly the coefficients of the operator L described in Lemma 2.1.

By using the above-mentioned observation, we can relate the flows $\frac{\partial u}{\partial t_{1,n}}$ of the LFV hierarchy to the flows $\frac{\partial v}{\partial T_{1,n}}$ of the FV hierarchy as follows. From the definition (2.11) and (3.3), we can write these flows as follows:

$$\varepsilon \frac{\partial u}{\partial t_{1,n}} = \left(1 - \Lambda^{-1}\right) \operatorname{res} L^n, \qquad \varepsilon \frac{\partial v}{\partial T_{1,n}} = \left(1 - \Lambda_1^{-1}\right) \operatorname{res} \mathcal{A}^n.$$

By using the relations (3.6), one can prove that

$$\operatorname{res} L^n = \lim_{q \to 0} \left(\operatorname{res} q^n \mathcal{A}^n|_{\varepsilon \mapsto q\varepsilon}\right).$$

Thus we arrive at the relation

$$\frac{\partial u}{\partial t_{1,n}} = \lim_{q \to 0} \left(q^{n+1} \frac{\partial v}{\partial T_{1,n}} \bigg|_{\varepsilon \mapsto q\varepsilon} \right).$$

Moreover, the Hamiltonian operator of the FV hierarchy is given by

$$\mathscr{P}^{\mathrm{FV}} = \frac{\left(1 - \Lambda_1^{-1}\right)(\Lambda_3 - 1)}{\Lambda_2 - 1},$$

and it is easy to verify that the Hamiltonian operator of the LFV hierarchy can be obtained by taking the following limit:

$$\mathscr{P} = \lim_{q \to 0} \left(q \mathscr{P}^{\mathrm{FV}}|_{\varepsilon \to q\varepsilon} \right).$$

We note that the flows $\frac{\partial}{\partial t^{2,n}}$ and $\frac{\partial}{\partial T^{2,n}}$ are not related by means of taking the limit $q \to 0$ even though after taking such a limit the recursion relations (3.5) coincide with the relations (2.10). Indeed, let us write down the flows

$$\frac{\partial u}{\partial T_{2,1}} = \frac{1}{p} u_x,$$
$$\frac{\partial v}{\partial T_{2,1}} = \frac{\Lambda_2 - \Lambda_1^{-1}}{\varepsilon} \exp\left(\frac{1 - \Lambda_2^{-1}}{1 - \Lambda_1^{-1}} v \right) = \frac{p+q}{pq} \mathrm{e}^v v_x + O(\varepsilon).$$

We conclude that we cannot obtain the flow $\frac{\partial u}{\partial t_{2,1}}$ by taking the limit of the flow $\frac{\partial v}{\partial T_{2,1}}$.

The fact that the limits of the flows $\frac{\partial v}{\partial T_{2,n}}$ do not exist can also be obtained in view of the Hodge-FVH correspondence [11]. It is proved in [11] that the following function gives a solution of the FVH hierarchy:

$$v(x;\mathbf{T};\varepsilon) = \left(\Lambda_1^{1/2} - \Lambda_1^{-1/2}\right)\left(\Lambda_3^{1/2} - \Lambda_3^{-1/2}\right)\mathcal{H}\left(\mathbf{t}(x;\mathbf{T});p,q,r;\frac{\sqrt{p+q}}{pq}\varepsilon\right),$$

where the variables t_i and $T_{\alpha,k}$ are related by

$$t_i(x;\mathbf{T}) = x\delta_{i,0} + \delta_{i,1} - 1 + \frac{1}{pq}\sum_{k>0}(kp)^{i+1}\binom{k(p+q)/q}{k}T_{1,k} + (kq)^{i+1}\binom{k(p+q)/p}{k}T_{2,k},$$

and the generating function $\mathcal{H}\left(\mathbf{t}(x;\mathbf{T});p,q,r;\varepsilon\right)$ of the special cubic Hodge integral is defined in (1.2). Therefore it follows from the relations between t_i and $T_{\alpha,k}$ that there does not exist a way to rescale $T_{2,k}$ by multiplying a suitable power of q such that after taking the limit $q \to 0$ of t_i the variables $T_{2,k}$ are still preserved.

However, by setting $T_{2,k} = 0$ and by performing the change of variables $T_{1,k} \mapsto q^{k+1}t_{1,k}$ and $\varepsilon \mapsto q\varepsilon$, we can obtain the following corollary which can be viewed as a limit of the Hodge-FVH correspondence.

Corollary 3.1. *Let us denote by $\mathcal{H}(\boldsymbol{t};p;\varepsilon)$ the following generating function of the linear Hodge integrals:*

$$\mathcal{H}(\boldsymbol{t};p;\varepsilon) = \sum_{g\geq 0}\varepsilon^{2g-2}\sum_{n\geq 0}\frac{t_{k_1}\cdots t_{k_n}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\mathcal{C}_g(-p)$$

and denote by $u(x;t_{1,k};\varepsilon)$ the function

$$u(x;t_{1,k};\varepsilon) = \left(\Lambda^{1/2} - \Lambda^{-1/2}\right)^2\mathcal{H}(\boldsymbol{t}(x;t_{1,k});p;\varepsilon),$$

where the relations between the variables t_i and $t_{1,k}$ are given by

$$t_i = \sum_{k\geq 0}\frac{k^{i+1+k}}{k!}p^{k+i}t_{1,k} - 1 + x\delta_{i,0} + \delta_{i,1}.$$

Then the function $u(x;t_{1,k};\varepsilon)$ satisfies the equations which form a part of the LFV hierarchy:

$$\varepsilon\frac{\partial K}{\partial t_{1,n}} = [(L^n)_+, K],$$

where the differential-difference operator K and the difference operator L are given by

$$K = \frac{1}{p}\varepsilon\partial_x + \mathrm{e}^u\Lambda^{-1}, \qquad \log L = pK.$$

4 Relation between the ILW hierarchy and the LFV hierarchy

Due to the discussion given in the last section, we expect that the LFV hierarchy should control a certain generating function of the linear Hodge integrals over the moduli space of stable curves. It is proved in [3] that the integrable hierarchy corresponds to the linear Hodge integral is the intermediate-long wave (ILW) hierarchy. Therefore it is natural to establish relations between the LFV hierarchy and the ILW hierarchy.

We start by reviewing the Lax pair formalism of the ILW hierarchy given in [4]. Compared to the original convention used in [4], we do some rescalings for the convenience of our presentation given below. Consider the following operators $\mathcal{K}$ and $\mathcal{L}$ defined by

$$\mathcal{L} - \frac{1}{p}\log\mathcal{L} = \mathcal{K}, \qquad \mathcal{K} = \Lambda + w - \frac{1}{p}\varepsilon\partial_x, \tag{4.1}$$

where the operator $\mathcal{L}$ can be written as

$$\mathcal{L} = \Lambda + w + c_1\Lambda^{-1} + c_2\Lambda^{-2} + \cdots, \qquad c_k \in \mathcal{R}(w).$$

Similar as before, we denote by $\mathcal{R}(w)$ the ring of differential polynomials of w. The logarithm of the operator $\mathcal{L}$ is defined, similar to $\log L$, via the dressing operator as follows:

$$\mathcal{L} = \mathcal{P}\Lambda\mathcal{P}^{-1}, \qquad \log\mathcal{L} = \varepsilon\partial_x - \varepsilon\mathcal{P}_x\mathcal{P}^{-1}.$$

Using the operators defined above, the ILW hierarchy can be represented as

$$\varepsilon\frac{\partial\mathcal{K}}{\partial s_{1,n}} = [\mathcal{L}_+^n, \mathcal{K}], \qquad n \geq 1. \tag{4.2}$$

The differential polynomials c_k are uniquely determined by the recursion relations

$$c_k\big(1 - \Lambda^{-k}\big)w + (\Lambda - 1)c_{k+1} = \frac{1}{p}\varepsilon\partial_x c_k, \qquad k \geq 0, \qquad c_0 = w, \tag{4.3}$$

and by the condition that the dispersionless limits of c_k are given by $\lim_{\varepsilon\to 0} c_k = \gamma_k$, where γ_k are polynomials given by

$$\gamma_{k+1} = \frac{1}{p}\gamma_k - \int_0^w \gamma_k\,\mathrm{d}w, \qquad k \geq 0, \qquad \gamma_0 = w.$$

The following theorem gives a direct relation between the ILW hierarchy and the LFV hierarchy.

Theorem 4.1. *The operator $\overline{L}$ defined in Lemma 2.2 and the operator $\mathcal{L}$ defined in (4.1) are related via the following relation:*

$$\Lambda^{1/2}\operatorname{res}\mathcal{L}^k = \operatorname{res}\overline{L}^k, \qquad k \geq 1, \tag{4.4}$$

where the functions $u(x)$ and $w(x)$ in the above identity are identified by the Miura-type transformation

$$u(x) = p\frac{\Lambda^{1/2} - \Lambda^{-1/2}}{\varepsilon\partial_x}w(x). \tag{4.5}$$

Proof. We start with considering the dressing operator Q of the operator $\overline{L}$ described in Lemma 2.2. From the definition of Q we see that its residue q_0 satisfies the following equation:

$$q_0 = \mathrm{e}^u\Lambda^{-1}q_0. \tag{4.6}$$

Taking derivatives of both sides of the above equation with respect to x, we arrive at

$$\frac{\partial_x q_0}{q_0} = \frac{\partial_x}{1 - \Lambda^{-1}}u. \tag{4.7}$$

Now let us perform a gauge transformation on $\overline{L}$ by the adjoint action of q_0 to obtain

$$\overline{\mathscr{L}} = q_0^{-1}\overline{L}q_0 = \mathscr{Q}\Lambda^{-1}\mathscr{Q}^{-1}, \qquad \mathscr{Q} = q_0^{-1}Q.$$

By using the relation

$$\overline{L} - \frac{1}{p}\log\overline{L} = K$$

together with (4.6) and (4.7), we can check the validity of the following relation:

$$\overline{\mathscr{L}} - \frac{1}{p}\log\overline{\mathscr{L}} = \Lambda^{-1} + \frac{1}{p}\frac{\varepsilon\partial_x}{1-\Lambda^{-1}}u + \frac{1}{p}\varepsilon\partial_x, \tag{4.8}$$

here the logarithm of $\overline{\mathscr{L}}$ is defined by $\mathscr{Q}$ as follows:

$$\log\overline{\mathscr{L}} = -\varepsilon\partial_x + \varepsilon\mathscr{Q}_x\mathscr{Q}^{-1}.$$

Denote $\overline{\mathscr{L}} = \Lambda^{-1} + \sum_{k\geq 0} f_k\Lambda^k$, then it follows from the identity

$$\left[\overline{\mathscr{L}}, \Lambda^{-1} + \frac{1}{p}\frac{\varepsilon\partial_x}{1-\Lambda^{-1}}u + \frac{1}{p}\varepsilon\partial_x\right] = 0$$

that we can rewrite (4.8) as the following recursion relations for the coefficients f_k:

$$f_k\big(\Lambda^k - 1\big)\Lambda^{1/2}w + \big(1-\Lambda^{-1}\big)f_{k+1} = \frac{1}{p}\varepsilon\partial_x f_k, \qquad k\geq 0, \qquad f_0 = \Lambda^{1/2}w. \tag{4.9}$$

Here we have already identified $u(x)$ and $w(x)$ via the relation (4.5). The dispersionless limits of f_k can be computed by using Lemma 2.2 and we obtain that $\lim_{\varepsilon\to 0} f_k = \delta_k$, where δ_k are polynomials given by

$$\delta_{k+1} = \frac{1}{p}\delta_k - \int_0^w k\delta_k\,\mathrm{d}w, \qquad k\geq 0, \qquad \delta_0 = w.$$

Recall that for a difference operator $\sum_k g_k\Lambda^k$, its adjoint is defined to be $\big(\sum_k g_k\Lambda^k\big)^* = \sum_k\big(\Lambda^{-k}g_k\big)\Lambda^{-k}$. Therefore if we denote $\overline{\mathscr{L}}^* = \Lambda + \sum_{k\geq 0} f_k^*\Lambda^{-k}$, we easily obtain from (4.9) the following recursion relations:

$$f_k^*\big(1-\Lambda^{-k}\big)\Lambda^{1/2}w + (\Lambda - 1)f_{k+1} = \frac{1}{p}\varepsilon\partial_x f_k^*, \qquad k\geq 0, \qquad f_0^* = \Lambda^{1/2}w. \tag{4.10}$$

Finally by comparing (4.10) with (4.3), we arrive at

$$\Lambda^{1/2}\mathcal{L}\Lambda^{-1/2} = \overline{\mathscr{L}}^*,$$

which implies (4.4). The theorem is proved. ■

The following corollary is straightforward and gives the exact correspondence between the flows of the LFV hierarchy and the ILW hierarchy:

Corollary 4.2. *The flows* $\frac{\partial w}{\partial s_{1,n}}$ *of the ILW hierarchy defined in* (4.2) *coincide with the flows* $\frac{\partial u}{\partial t_{2,n}}$ *of the LFV hierarchy given in* (2.11) *after identifying* $w(x)$ *and* $u(x)$ *by the Miura-type transformation* (4.5).

Proof. From the definition (4.2) of the ILW flows it follows that

$$\varepsilon\frac{\partial w}{\partial s_{1,n}} = \frac{1}{p}\varepsilon\partial_x \operatorname{res}\mathcal{L}^n.$$

By using the Miura-type transformation (4.5) we arrive at

$$\varepsilon\frac{\partial u}{\partial s_{1,n}} = \left(\Lambda^{1/2}-\Lambda^{-1/2}\right)\operatorname{res}\mathcal{L}^n.$$

On the other hand, from the definition (2.11) of the LFV flows we know that

$$\varepsilon\frac{\partial u}{\partial t_{2,n}} = \left(1-\Lambda^{-1}\right)\operatorname{res}\overline{L}^k,$$

hence we complete the proof of the corollary by using the identity (4.4). ■

From Corollary 4.2 we conclude that after the Miura-type transformation (4.5), the flows $\frac{\partial u}{\partial t_{1,n}}$ of the LFV hierarchy are transformed to symmetries of the ILW hierarchy, which has a Lax pair description given by the following constructions. The proofs of what follows are similar to the ones for the LFV hierarchy and we omit the details.

Lemma 4.3. *There exists a difference operator*

$$\overline{\mathcal{L}} = d_{-1}\Lambda^{-1} + \sum_{k\geq 0} d_k\Lambda^k = \mathcal{Q}\Lambda^{-1}\mathcal{Q}^{-1}, \qquad d_k\in\mathcal{R}(w),$$

such that the following relation holds true

$$\frac{1}{p}\log\overline{\mathcal{L}} = \mathcal{K}, \tag{4.11}$$

here $\mathcal{K}$ is defined in (4.1) and the logarithm of $\overline{\mathcal{L}}$ is defined by

$$\log\overline{\mathcal{L}} = -\varepsilon\partial_x + \varepsilon\mathcal{Q}_x\mathcal{Q}^{-1}.$$

Definition 4.4. We define the following symmetries for the ILW hierarchy:

$$\varepsilon\frac{\partial\mathcal{K}}{\partial s_{2,n}} = -\left[\overline{\mathcal{L}}^n_-,\mathcal{K}\right], \tag{4.12}$$

where $\overline{\mathcal{L}}$ is introduced in Lemma 4.3.

Example 4.5. The first flow defined in (4.12) reads

$$\varepsilon\frac{\partial w}{\partial s_{2,1}} = (\Lambda-1)\exp\left(p\frac{1-\Lambda^{-1}}{\varepsilon\partial_x}w\right).$$

In a similar way as we prove Theorem 4.1, we can prove the following theorem.

Theorem 4.6. *The operator L defined in Lemma 2.1 and the operator $\overline{\mathcal{L}}$ defined in (4.11) satisfy the following identity:*

$$\Lambda^{1/2}\operatorname{res}\overline{\mathcal{L}}^k = \operatorname{res}L^k, \qquad k\geq 1,$$

where the functions $u(x)$ and $w(x)$ in the above identity are related by (4.5).

Corollary 4.7. *The flows $\frac{\partial w}{\partial s_{2,n}}$ defined in (4.12) coincide with the flows $\frac{\partial u}{\partial t_{1,n}}$ of the LFV hierarchy (2.11) after identifying $w(x)$ and $u(x)$ by (4.5). In particular, we have*

$$\left[\frac{\partial}{\partial s_{1,n}},\frac{\partial}{\partial s_{2,m}}\right] = \left[\frac{\partial}{\partial s_{2,n}},\frac{\partial}{\partial s_{2,m}}\right] = 0, \qquad n,m\geq 1.$$

5 Concluding remarks

In this paper, we consider the limiting procedure from the special cubic Hodge integrals to the linear Hodge integrals in view of the theory of integrable hierarchies. By taking a certain limit of the FV hierarchy, we obtain an integrable hierarchy which, together with the ILW hierarchy, forms a reduction of the 2D Toda hierarchy. We call the resulting hierarchy the LFV hierarchy, which is a Hamiltonian tau-symmetric integrable hierarchy with hydrodynamic limit.

This limiting procedure is quite natural in geometric setting, for example in the Gromov–Witten theory or the Hurwitz theory. Our result can be viewed as an integrable hierarchy theoretical interpretation of the Bouchard–Mariño conjecture [1] which is proved in [14]. In [1], it is conjectured that the generating function of linear Hodge integrals can be computed in the scheme of Eynard and Orantin topological recursion associated with the spectral curve

$$C = \{x = y\mathrm{e}^{-y}\},$$

which is related to the symbol of the constraint (1.4). Their conjecture is based on a limiting procedure of the Mariño–Vafa formula which relates the open amplitude of the A-model topological string on $\mathbb{C}^3$ with a framed brane on one leg of the toric diagram to the cubic Hodge integrals [9, 10, 13]. The Mariño–Vafa formula can be used to derive a special case of Hodge-FVH correspondence [18], and the results of the present paper explain the above limiting procedure in view of the theory of integrable hierarchies. We thank the anonymous referee for pointing out this relation to us.

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