Symmetry, Integrability and Geometry: Methods and Applications SIGMA **18** (2022), 031, 19 pages

Spinors in Five-Dimensional Contact Geometry

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Received January 31, 2022, in final form April 13, 2022; Published online April 16, 2022
https://doi.org/10.3842/SIGMA.2022.031

Abstract. We use classical (Penrose) two-component spinors to set up the differential geometry of two parabolic contact structures in five dimensions, namely G_2 contact geometry and Legendrean contact geometry. The key players in these two geometries are invariantly defined directional derivatives defined only in the contact directions. We explain how to define them and their usage in constructing basic invariants such as the harmonic curvature, the obstruction to being locally flat from the parabolic viewpoint. As an application, we calculate the invariant torsion of the G_2 contact structure on the configuration space of a flying saucer (always a five-dimensional contact manifold).

Key words: spinors; contact geometry; parabolic geometry

2020 Mathematics Subject Classification: 53B05; 53D10; 58J10

Dedicated to Roger Penrose
on the occasion of his 90th birthday

1 Introduction

Two-component spinors are widely used in four-dimensional Lorentzian geometry. The seminal books 'Spinors and space-time' [12, 13] are devoted to such usage. At a very basic level, two-component spinors arise via the 2–1 covering of Lie groups

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SO}^{\uparrow}(3,1),$$

where $\mathrm{SO}^{\uparrow}(3,1)$ is the identity-connected component of the Lorentz group. Similarly, in three dimensions, the 2–1 covering

$$\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}^{\uparrow}(2,1)$$

is responsible for the utility of two-component spinors in three dimensions (as in Section 5).

At a very basic level, the two-component spinors in this article arise via inclusions

$$\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{R}),$$

where $\mathrm{Sp}(4,\mathbb{R})$ is the subgroup of $\mathrm{SL}(4,\mathbb{R})$ consisting of matrices that preserve a fixed non-degenerate skew form on $\mathbb{R}^4$. There are two such inclusions:

- by writing $\mathbb{R}^4 = \bigodot^3 \mathbb{R}^2$, where $\bigodot$ denotes symmetric tensor product,
- by writing $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$,

This paper is a contribution to the Special Issue on Twistors from Geometry to Physics in honour of Roger Penrose. The full collection is available at https://www.emis.de/journals/SIGMA/Penrose.html

giving rise to G_2 contact geometry (as in Section 7) and Legendrean contact geometry (as in Section 8), respectively. These two geometries are defined on a five-dimensional contact manifold M as extra structure on the contact distribution $H \subset TM$. A contact form α is a 1-form so that $H = \ker \alpha$. It gives rise to a nondegenerate skew form, the Levi form, namely $d\alpha|_H$. The extra spin structures in Sections 7 and 8 are required to be compatible with the Levi form.

Another important theme in [12, 13] is conformal geometry. It is concerned with what happens if the (Lorentzian) metric is replaced by a smooth positive multiple of itself. The resulting formulæ, for example (5.1), fit very well with spinors. For the contact geometries in this article, the corresponding freedom is in choosing a contact form α. The resulting formulæ, for example (7.6), (8.7), (8.8), also fit very well with spinors. These 'conformal' structures are examples of *parabolic geometries* [4]. In particular, *invariant differential operators* play a key part in parabolic constructions.

Ideally, one would like to approach the natural differential geometric calculus on these various geometries via invariant differential operators. More specifically, for a chosen 'scale' (a metric in the conformal class or a choice of contact form) one expects a canonical (partial) connection on all the natural irreducible vector bundles. This expectation follows from the Čap-Slovák theory of *Weyl structures and scales* in parabolic geometry (see [3] or [4, Section 5.1]). In conformal geometry, it is just the Levi-Civita connection: see Sections 2 and 3 for details. For parabolic contact structures, one expects partial connections defined only in the contact directions. We shall see in Sections 7 and 8 that a suitable collection of invariant operators is, indeed, sufficient for these purposes. For the five-dimensional Legendrean contact structures in Section 8, these operators can easily be found (within the Rumin complex, explained in Section 6). The two key invariant operators (7.7) in G_2 contact geometry remain somewhat mysterious: it is shown in Section 7 that these operators are, indeed, invariant but we have not been able to find a suitable shortcut to their construction. Nevertheless, we are able to construct, in Section 9, the general G_2 contact geometry from a suitable Legendrean contact structure and, thereby, calculate its basic spinor invariant (a certain septic).

Through this article we shall use Penrose's abstract index notation and other conventions from [12]. Also, to ease the notational burden, we shall not carefully distinguish between bundles and sections thereof, for example writing $\omega_a \in \wedge^1$ instead of $\omega_a \in \Gamma(\wedge^1)$ or even $\omega_a \in \Gamma(M, \wedge^1)$, for a 1-form ω_a. Especially as this article is concerned with *local* differential geometry, this should cause no confusion.

2 The Levi-Civita connection

On a general smooth manifold, the exterior derivative and the Lie derivative are defined independently of local coördinates (and there is little else with this property [9]). Both of these operations can be defined in terms of an arbitrary torsion-free affine connection. For the exterior derivative on 1-forms, we have (following the conventions of [12, equation (4.3.14)])

$$\omega_b \longmapsto \nabla_{[a}\omega_{b]}.$$

For the Lie derivative on covariant 2-tensors, we have

$$\phi_{bc} \longmapsto (\mathcal{L}_X\phi)_{bc} \equiv X^a\nabla_a\phi_{bc} + (\nabla_b X^a)\phi_{ac} + (\nabla_c X^a)\phi_{ba}, \tag{2.1}$$

for a vector field X^a and, irrespective of the usual interpretation of $\mathcal{L}_X$ in terms of the flow of X^a, it easy to check that this expression does not depend on the choice of ∇_a. It is convenient to regard the right hand side of (2.1), let's say for a symmetric tensor ϕ_{ab}, as an invariantly defined differential *pairing*

$$TM \times \bigodot^2\wedge^1 \longrightarrow \bigodot^2\wedge^1.$$

In particular, if g_{ab} is a semi-Riemannian metric, that is to say a nondegenerate symmetric form, then the Lie derivative of g_{ab}, with a convenient factor of $\frac{1}{2}$ thrown in, can be regarded as a canonically defined linear differential operator

$$TM \ni X^a \longmapsto \tfrac{1}{2}\mathcal{L}_X g_{bc} \in \bigodot^2\wedge^1. \tag{2.2}$$

Of course, the tensor g_{ab} also defines an isomorphism $TM \cong \wedge^1$ by $X^a \mapsto g_{ab}X^b$, and so we have obtained an invariantly defined linear differential operator

$$\wedge^1 \cong TM \longrightarrow \bigodot^2\wedge^1. \tag{2.3}$$

In combination with the exterior derivative $\mathrm{d}\colon \wedge^1 \to \wedge^2$, we have obtained

$$\wedge^1 \longrightarrow \bigodot^2\wedge^1 \oplus \wedge^2 = \wedge^1 \otimes \wedge^1 \tag{2.4}$$

and we claim that this is the Levi-Civita connection defined by g_{ab}. This is easy to check: if we use the Levi-Civita connection in (2.1), then

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}_X g_{ab} + \nabla_{[a}X_{b]} &= \tfrac{1}{2}\big(X^c\nabla_c g_{ab} + (\nabla_a X^c)g_{cb} + (\nabla_b X^c)g_{ac}\big) + \nabla_{[a}X_{b]}\\ &= \nabla_{(a}X_{b)} + \nabla_{[a}X_{b]} = \nabla_a X_b,\end{aligned}$$

as required. But we can play the moves (2.2), (2.3), and (2.4) to *define* the Levi-Civita connection. An advantage of this viewpoint is that it is easily modified to define other connections (and partial connections), as we shall soon see.

3 Conformal differential geometry

In semi-Riemannian conformal differential geometry, instead of a metric g_{ab}, we are given only a *conformal class* of metrics. A convenient way of expressing this is to say that we are given η_{ab}, a nondegenerate section of $\bigodot^2\wedge^1 \otimes L^2$ for some line bundle L. Thus, if σ is a non-vanishing section of L, then $g_{ab} \equiv \sigma^{-2}\eta_{ab}$ is a genuine metric. We shall refer to σ as a *scale* and, if we choose a different section $\widehat{\sigma} = \Omega^{-1}\sigma$ for some nowhere vanishing function Ω, then we encounter a *rescaled* metric $\widehat{g}_{ab} \equiv \widehat{\sigma}^{-2}\eta_{ab} = \Omega^2 g_{ab}$. For a given conformal structure η_{ab}, let us define η^{ab}, a section of $\bigodot^2 TM \otimes L^{-2}$, by $\eta_{ac}\eta^{bc} = \delta_a{}^b$, where $\delta_a{}^b$ is the invariant (Kronecker delta) pairing $TM \otimes \wedge^1 \to \wedge^0$.

On an n-dimensional oriented conformal manifold, we may attempt to normalise a section $\epsilon_{ab\ldots d}$ of the bundle $\wedge^n$, compatible with the orientation, by insisting that

$$\epsilon_{ab\ldots d}\epsilon_{ef\ldots h}\eta^{ae}\eta^{bf}\cdots\eta^{dh} \equiv n!, \tag{3.1}$$

just as we may do on an n-dimensional oriented Riemannian manifold to normalise and define the volume form. The problem with (3.1) is that the left hand side takes values in the line bundle L^{-2n}. To remedy this problem, we may insist that $\epsilon_{ab\ldots d}$ take values in $\wedge^n \otimes L^n$. The normalisation (3.1) now makes good sense and $\epsilon_{ab\ldots d}$ is uniquely determined. Equivalently, we may view $\epsilon_{ab\ldots d}$ as providing a canonical identification

$$L^{-n} = \wedge^n \qquad \text{given by} \quad \phi \mapsto \phi\epsilon_{ab\ldots d}.$$

It it usual to write $L \equiv \wedge^0[1]$ and refer to sections of $L^w \equiv \wedge^0[w]$ as *conformal densities of weight* w. In the presence of a scale σ, defining a metric $g_{ab} = \sigma^{-2}\eta_{ab}$, the normalisation (3.1) reads

$$(\sigma^n\epsilon_{ab\ldots d})(\sigma^n\epsilon_{ef\ldots h})g^{ae}g^{bf}\cdots g^{dh} = n!$$

so $\sigma^n\epsilon_{ab\ldots d}$ is the usual volume form for the metric g_{ab}. This is consistent with the scale σ trivialising all the density bundles $\wedge^0[w]$. To summarise, in the presence of a scale σ, a conformal density of weight w can be regarded as an ordinary function but if we change to a new scale $\widehat{\sigma}=\Omega^{-1}\sigma$, then the same density is represented by a new function $\widehat{f}=\Omega^w f$.

Once we have decided that $\wedge^0[1]=L=(\wedge^n)^{-1/n}$, the Lie derivative $\mathcal{L}_X\colon\wedge^1\to\wedge^1$ induces invariantly defined differential pairings $TM\times B\to B$ for all conformally weighted tensor bundles B and, in particular,

$$TM\times\bigodot^2\wedge^1[2]\to\bigodot^2\wedge^1[2].$$

Recall that a conformal structure is a nondegenerate section η_{ab} of this bundle $\bigodot^2\wedge^1[2]$ and hence we obtain a conformally invariant first-order linear differential operator

$$TM\ni X^a\longmapsto\mathcal{L}_X\eta_{ab}\in\bigodot^2\wedge^1[2].\tag{3.2}$$

But recall that $\epsilon_{ab\ldots d}$ is the canonical section of $\wedge^n[n]=\wedge^0$ corresponding to the constant function 1. It follows that $\mathcal{L}_X\epsilon_{ab\ldots d}=0$ for any vector field X^a and therefore, from (3.1), that $\eta^{ab}\mathcal{L}_X\eta_{ab}=0$. In addition, we may use the conformal metric η_{ab} to lower indices, at the expense of a conformal weight so that $TM=\wedge^1[2]$. The Lie derivative in (3.2) thus yields a conformal invariant differential operator

$$\wedge^1[2]\longrightarrow\bigodot^2_\circ\wedge^1[2],$$

where $\circ$ denotes the trace-free part (a manifestly conformally invariant notion). If $\phi_{b\ldots d}$ is an $(n-1)$-form then

$$\epsilon_{ab\ldots d}\eta^{be}\cdots\eta^{dg}\phi_{e\ldots g}$$

is a 1-form of conformal weight $n-2(n-1)=2-n$. In other words, the bundles $\wedge^{n-1}$ and $\wedge^1[2-n]$ are canonically isomorphic. Similarly, we have $\wedge^n=\wedge^0[-n]$ and so the exterior derivative $\mathrm{d}\colon\wedge^{n-1}\to\wedge^n$ may be viewed as an invariant differential operator

$$\wedge^1[2-n]\to\wedge^0[-n].$$

Together with the exterior derivative $\mathrm{d}\colon\wedge^1\to\wedge^2$, we now have three conformally invariant linear differential operators defined on variously weighted 1-forms, namely

$$\wedge^1[2]\to\bigodot^2_\circ\wedge^1[2],\qquad\wedge^1[2-n]\to\wedge^0[-n],\qquad\wedge^1\to\wedge^2.\tag{3.3}$$

As in Section 2, we are now in a position to define a preferred connection in the presence of a scale $\sigma\in\Gamma(\wedge^0[1])$. Specifically, we may drop all the weights in (3.3) and combine them to obtain

$$\wedge^1\xrightarrow{\nabla}\wedge^1\otimes\wedge^1=\bigodot^2_\circ\wedge^1\oplus\wedge^0\oplus\wedge^2.$$

The three bundles on the right constitute the orthogonal decomposition of $\wedge^1\otimes\wedge^1$ into irreducibles with respect to the metric $g_{ab}=\sigma^{-2}\eta_{ab}$. Of course, it is easily verified that we have found the Levi-Civita connection for g_{ab} but there are two advantages of this particular route. One is that we are able to read off the change in the Levi-Civita connection under conformal rescaling $\sigma\mapsto\widehat{\sigma}=\Omega^{-1}\sigma$. The other is that this route may be employed in other *parabolic geometries*. Regarding the first advantage, another way of pinning down the connection defined by (3.3) and a scale $\sigma\in\Gamma\big(\wedge^0[1]\big)$, is to require that $\nabla_a\sigma=0$. This is consistent with the connections induced on the line bundles $\wedge^0[w]$ in the sense that, for $\phi \in \Gamma(\wedge^0[w])$, we have $\nabla\phi = \sigma^w \mathrm{d}(\sigma^{-w}\phi)$. In particular, for a new scale $\widehat{\sigma} = \Omega^{-1}\sigma$ and $\phi \in \Gamma(\wedge^0[w])$, we have

$$\widehat{\nabla}\phi = \widehat{\sigma}^w \mathrm{d}(\widehat{\sigma}^{-w}\phi) = \Omega^{-w}\sigma^w \mathrm{d}(\Omega^w\sigma^{-w}\phi) = \sigma^w \mathrm{d}(\sigma^{-w}\phi) + w\Omega^{-1}(\mathrm{d}\Omega)\phi = \nabla\phi + w\Upsilon\phi,$$

where $\Upsilon \equiv \Omega^{-1}\mathrm{d}\Omega$. It follows that, regarding the change in the Levi-Civita connection on a 1-form ω_b, we have

$$\widehat{\nabla}_a\omega_b = \nabla_a\omega_b - \Upsilon_a\omega_b - \Upsilon_b\omega_a + \Upsilon^c\omega_c g_{ab}$$

since then, if ω_b has conformal weight w, we deduce that

$$\widehat{\nabla}_a\omega_b = \nabla_a\omega_b + (w-1)\Upsilon_a\omega_b - \Upsilon_b\omega_a + \Upsilon^c\omega_c g_{ab},$$

which is exactly so that the three operators (3.3), namely

$$\phi_b \mapsto \nabla_{(a}\phi_{b)} - \tfrac{1}{n}(\nabla^c\phi_c)g_{ab}, \qquad \phi_b \mapsto \nabla^b\phi_b, \qquad \phi_b \mapsto \nabla_{[a}\phi_{b]},$$

are invariantly defined.

Regarding the second advantage, it is convenient to adopt a universal notation for the natural irreducible vector bundles available on a parabolic geometry, as detailed in [1, 4]. Let's see how this works in 5-dimensional conformal geometry. The general irreducible spin bundle is

$$\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\Rightarrow\!\overset{c}{\bullet}, \qquad \text{where } a \in \mathbb{R} \text{ and } b, c \in \mathbb{Z}_{\geq 0}$$

and for tensor bundles we restrict c to be even. (Roughly speaking, this indicates a (complex) irreducible representation of $\mathbb{R}_{>0}\times\mathrm{Spin}(5)$ with the real number a over the crossed node recording the action of $\mathbb{R}_{>0}$.) The de Rham complex is

$$\overset{0}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \to \overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \to \overset{-3}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\Rightarrow\!\overset{2}{\bullet} \to \overset{-4}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\Rightarrow\!\overset{2}{\bullet} \to \overset{-5}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \to \overset{-5}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \tag{3.4}$$

and the three conformally invariant first-order linear differential operators on weighted 1-forms are

$$\overset{0}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \to \overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\Rightarrow\!\overset{0}{\bullet}, \qquad \overset{-5}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \to \overset{-5}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\Rightarrow\!\overset{0}{\bullet}, \qquad \overset{-2}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \to \overset{-3}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\Rightarrow\!\overset{2}{\bullet}.$$

Notice that $\wedge^0[w] = \overset{w}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\Rightarrow\!\overset{0}{\bullet}$ and that the identification $\wedge^5 = \wedge^0[-5]$ is built into the notation. There are many other advantages of this seemingly arcane notation: its utility in 5-dimensional spin geometry is the subject of the following section.

4 Five-dimensional conformal spin geometry

In this section we discuss five-dimensional conformal geometry by means of spinors (that is to say four-spinors in this dimension (also known as *twistors*)). For simplicity, we shall suppose that the conformal metric has split signature. There is some elementary linear algebra behind this discussion as follows. Suppose that $\mathbb{T}$ is a four-dimensional real vector space equipped with a nondegenerate skew form $\epsilon_{\alpha\beta}$. Let us write $\epsilon^{\alpha\beta}$ for its inverse so that $\epsilon_{\alpha\beta}\epsilon^{\alpha\gamma} = \delta_\alpha{}^\gamma$. The vector space $\wedge^2\mathbb{T}$ naturally splits:

$$\wedge^2\mathbb{T} = \{P_{\alpha\beta} \,|\, \epsilon^{\alpha\beta}P_{\alpha\beta} = 0\} \oplus \{P_{\alpha\beta} = \lambda\epsilon_{\alpha\beta}\} \equiv \wedge^2_\perp\mathbb{T} \oplus \mathbb{R}. \tag{4.1}$$

Otherwise said, if we let $\mathrm{Sp}(4,\mathbb{R})$ denote the linear automorphisms of $\mathbb{T} \cong \mathbb{R}^4$ preserving the symplectic form $\epsilon_{\alpha\beta}$, then (4.1) is the decomposition of $\wedge^2\mathbb{T}$ into $\mathrm{Sp}(4,\mathbb{R})$-irreducibles. The 5-dimensional vector space $\wedge^2_\perp\mathbb{T}$ is acquires a split signature metric

$$\|P_{\alpha\beta}\|^2 \equiv P_{\alpha\beta}P_{\gamma\eta}\epsilon^{\alpha\gamma}\epsilon^{\beta\eta}.$$

Otherwise said, we have constructed the isomorphism $\mathrm{Sp}(4,\mathbb{R}) \cong \mathrm{Spin}(3,2)$. According to the Plücker relations, the null vectors in $\wedge^2_\perp \mathbb{T}$ are the decomposable tensors.

With the conventions of [1], the bundle version of this discussion yields the splitting

$$\wedge^2\big(\overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}\big) = \overset{0}{\times}\!-\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \oplus \overset{1}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} = \wedge^1[2] \oplus \wedge^0[1].$$

More precisely, we may view a split signature conformal spin structure on a five-dimensional split signature conformal manifold as a rank 4 bundle, denoted by $\overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}$ and equipped with

- a nondegenerate section of $\wedge^2\big(\overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}\big)$ with values in $\wedge^0[-1]$,
- an identification $\wedge^2_\perp\big(\overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}\big) = \wedge^1[2]$.

A scale $\sigma \in \Gamma\big(\wedge^0[1]\big)$ induces a nondegenerate section of $\wedge^2\big(\overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}\big)$ (and, according to the discussion in Section 3, a compatible nondegenerate (split signature) symmetric form on $\wedge^1$). The first summand in the splitting

$$\operatorname{End}\big(\overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}\big) = \overset{-1}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{2}{\bullet} \oplus \overset{-1}{\times}\!-\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \oplus \overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \tag{4.2}$$

captures the trace-free endomorphisms of $\overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}$ that preserve its conformal skew form $\big($and, simultaneously, the second summand in

$$\operatorname{End}\big(\wedge^1\big) = \operatorname{End}\big(\overset{-2}{\times}\!-\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet}\big) = \overset{-2}{\times}\!-\!\overset{2}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \oplus \overset{-1}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{2}{\bullet} \oplus \overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{0}{\bullet}$$

plays the same rôle with respect to the conformal metric on $\wedge^1\big)$. From this viewpoint, a scale $\sigma \in \Gamma\big(\wedge^0[1]\big)$ gives rise to a connection on the spin bundle $\overset{0}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}$ as follows. We firstly insist that this connection preserve its conformal skew form and also that the induced connection on $\wedge^0[1]$ annihilate σ. According to (4.2), the freedom in choosing such a connection lies in

$$\wedge^1 \otimes \overset{-1}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{2}{\bullet} = \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \otimes \overset{-1}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{2}{\bullet}. \tag{4.3}$$

On the other hand, the induced operator $\nabla\colon \wedge^1 \to \wedge^2$ differs from the exterior derivative by a homomorphism $\wedge^1 \to \wedge^2$ (it is the *torsion* of the induced affine connection) and, according to (3.4), lies in

$$\operatorname{Hom}\big(\wedge^1, \wedge^2\big) = \overset{0}{\times}\!-\!\overset{1}{\bullet}\!\Rightarrow\!\overset{0}{\bullet} \otimes \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!\Rightarrow\!\overset{2}{\bullet}. \tag{4.4}$$

Comparing the bundles (4.3) and (4.4) and noting that they are canonically isomorphic suggests that the homomorphism $\wedge^1 \to \wedge^2$ may be precisely eliminated by the allowed freedom. It is easy to check that this is, indeed, the case.

5 Three-dimensional conformal spin geometry

A split signature three-dimensional conformal spin structure may be viewed as a rank two 'spin bundle' S equipped with an identification $\bigodot^2 S = \wedge^1[2]$ (cf. [15]). With the Dynkin diagram notation from [1],

$$S = \overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}, \qquad \bigodot^2 \overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times} = \overset{2}{\bullet}\!\Leftarrow\!\overset{0}{\times}, \qquad \wedge^1 = \overset{2}{\bullet}\!\Leftarrow\!\overset{-2}{\times}, \qquad \wedge^0[1] = \overset{0}{\bullet}\!\Leftarrow\!\overset{1}{\times},$$

and the de Rham complex is

$$\wedge^0 = \overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\times} \to \overset{2}{\bullet}\!\Leftarrow\!\overset{-2}{\times} \to \overset{2}{\bullet}\!\Leftarrow\!\overset{-3}{\times} \to \overset{0}{\bullet}\!\Leftarrow\!\overset{-3}{\times} = \wedge^3.$$

The conformal structure can now be characterised by decreeing that the simple spinors in $\overset{2}{\bullet}\!\Leftarrow\!\overset{0}{\times}$ are the null vectors in $\wedge^1[2] = \overset{2}{\bullet}\!\Leftarrow\!\overset{0}{\times}$.

The first summand in

$$\mathrm{End}\left(\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}\right) = \overset{2}{\bullet}\!\Leftarrow\!\overset{-1}{\times} \oplus \overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\times}$$

captures the trace-free endomorphisms of $\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}$ and it follows that the freedom in choosing a connection on this spin bundle annihilating a scale, i.e., a nowhere vanishing section of $\wedge^2\big(\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}\big) = \overset{0}{\bullet}\!\Leftarrow\!\overset{1}{\times}$, lies in

$$\wedge^1 \otimes \overset{2}{\bullet}\!\Leftarrow\!\overset{-1}{\times} = \overset{2}{\bullet}\!\Leftarrow\!\overset{-2}{\times} \otimes \overset{2}{\bullet}\!\Leftarrow\!\overset{-1}{\times}.$$

On the other hand the torsion of the induced affine connection lies in the canonically isomorphic bundle

$$\mathrm{Hom}\left(\wedge^1, \wedge^2\right) = \overset{2}{\bullet}\!\Leftarrow\!\overset{0}{\times} \otimes \overset{2}{\bullet}\!\Leftarrow\!\overset{-3}{\times}.$$

It is easy to check that the freedom in choice of connection on $\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}$ may be used exactly to eliminate this torsion. More precisely, we have the following:

Proposition 5.1. *Given a scale, i.e., a nowhere vanishing $\sigma \in \Gamma\big(\overset{0}{\bullet}\!\Leftarrow\!\overset{1}{\times}\big)$, there is a unique connection on $\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}$ so that*

- *the induced connection on $\wedge^2\big(\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}\big) = \overset{0}{\bullet}\!\Leftarrow\!\overset{1}{\times}$ annihilates σ,*
- *the torsion of the induced connection on $\wedge^1 = \overset{2}{\bullet}\!\Leftarrow\!\overset{-2}{\times}$ vanishes.*

It is straightforward to figure out how this preferred connection changes under change of scale. To do this, let us adapt the classical two-spinor notation of [12] to write

$$\nabla\colon\ \overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times} \longrightarrow \overset{2}{\bullet}\!\Leftarrow\!\overset{-2}{\times} \otimes \overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times} \quad \text{as } \phi_C \longmapsto \nabla_{AB}\phi_C.$$

Proposition 5.2. *Let us change scale $\sigma \in \Gamma\big(\wedge^0[1]\big)$ by $\widehat{\sigma} = \Omega^{-1}\sigma$. Then, for $\phi_C \in \Gamma\big(\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}\big)$,*

$$\widehat{\nabla}_{AB}\phi_C = \nabla_{AB}\phi_C + \Upsilon_{AB}\phi_C - \Upsilon_{C(A}\phi_{B)}, \qquad \textit{where } \Upsilon_{AB} \equiv \Omega^{-1}\nabla_{AB}\Omega. \tag{5.1}$$

Proof. Given ∇_{AB} we use (5.1) to define $\widehat{\nabla}_{AB}$ and then verify that it has the characterising properties from Proposition 5.1 for the scale $\widehat{\sigma}$. Firstly, if $\sigma_{CD} \in \Gamma\big(\wedge^2\big(\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times}\big)\big)$, then

$$\widehat{\nabla}_{AB}\sigma_{CD} = \nabla_{AB}\sigma_{CD} + 2\Upsilon_{AB}\sigma_{CD} - \Upsilon_{C(A}\sigma_{B)D} + \Upsilon_{D(A}\sigma_{B)C} = \nabla_{AB}\sigma_{CD} + \Upsilon_{AB}\sigma_{CD}. \tag{5.2}$$

Therefore $\widehat{\nabla}_{AB}\widehat{\sigma}_{CD} = \nabla_{AB}\big(\Omega^{-1}\sigma_{CD}\big) + \Upsilon_{AB}\big(\Omega^{-1}\sigma_{CD}\big) = \Omega^{-1}\nabla_{AB}\sigma_{CD}$. Thus, if $\nabla_{AB}\sigma_{CD} = 0$, then $\widehat{\nabla}_{AB}\widehat{\sigma}_{CD} = 0$, which accounts for the first condition in Proposition 5.1. Now equation (5.2) may be abbreviated as $\widehat{\nabla}_{AB}\sigma = \nabla_{AB}\sigma + \Upsilon_{AB}\sigma$ for $\sigma \in \Gamma\big(\overset{0}{\bullet}\!\Leftarrow\!\overset{1}{\times}\big)$ and it follows that

$$\widehat{\nabla}_{AB}\rho = \nabla_{AB}\rho + w\Upsilon_{AB}\rho, \qquad \text{for } \rho \in \Gamma\big(\overset{0}{\bullet}\!\Leftarrow\!\overset{w}{\times}\big).$$

Combining this with (5.1), it follows that $\widehat{\nabla}_{AB}\phi_C = \nabla_{AB}\phi_C - \Upsilon_{C(A}\phi_{B)}$, for $\phi_C \in \Gamma\big(\overset{1}{\bullet}\!\Leftarrow\!\overset{-1}{\times}\big)$ and hence that

$$\widehat{\nabla}_{AB}\omega_{CD} = \nabla_{AB}\omega_{CD} - \Upsilon_{C(A}\omega_{B)D} - \Upsilon_{D(A}\omega_{B)C},$$

for $\omega_{CD} \in \Gamma\big(\overset{2}{\bullet}\!\Leftarrow\!\overset{-2}{\times}\big) = \Gamma\big(\wedge^1\big)$. It follows that

$$\widehat{\nabla}_{AB}\omega_{CD} - \widehat{\nabla}_{CD}\omega_{AB} = \nabla_{AB}\omega_{CD} - \nabla_{CD}\omega_{AB},$$

which accounts for the second condition in Proposition 5.1. ∎

6 The Rumin complex

Contact geometry is the geometry of a *maximally non-integrable* corank one subbundle $H \subset TM$, where M is of dimension $2n+1$. Maximal non-integrability is to say that locally H is given as the kernel of a *contact form* α such that $\alpha \wedge (\mathrm{d}\alpha)^n$ is non-vanishing.

Alternatively, writing $\wedge^1_H$ for the bundle of one forms $\wedge^1$ restricted naturally to H, and L for the annihilator of H, we have a short exact sequence

$$0 \to L \to \wedge^1 \to \wedge^1_H \to 0$$

and, therefore, short exact sequences

$$0 \to \wedge^{k-1}_H \otimes L \to \wedge^k \to \wedge^k_H \to 0 \tag{6.1}$$

for $k = 1, \dots, 2n$. If we now consider the exterior derivative

$$\begin{array}{ccc} \wedge^1_H & & \wedge^2_H \\ \uparrow & & \uparrow \\ \wedge^1 & \xrightarrow{\mathrm{d}} & \wedge^2 \\ \uparrow & & \uparrow \\ L & & \wedge^1_H \otimes L, \end{array}$$

then, by the Leibniz rule, the composition $L \to \wedge^1 \xrightarrow{\mathrm{d}} \wedge^2 \to \wedge^2_H$ is actually a vector bundle homomorphism known as the *Levi form*. The maximal non-integrability condition is equivalent to the Levi form being injective with image consisting of nondegenerate forms.

If one writes out the de Rham sequence along with the short exact sequences (6.1), one can obtain by diagram chasing, the Rumin complex [14]. The Rumin complex is a replacement for the de Rham complex on any contact manifold in that it computes the de Rham cohomology, but it is in some sense more efficient in that derivatives are only taken in contact directions. We are concerned with the case $\dim M = 5$ in which case the diagram to chase is

$$\begin{array}{ccccccccccc} & & \wedge^1_H & & \wedge^2_H & & \wedge^3_H & & \wedge^4_H & & \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \wedge^0 & \xrightarrow{\mathrm{d}} & \wedge^1 & \xrightarrow{\mathrm{d}} & \wedge^2 & \xrightarrow{\mathrm{d}} & \wedge^3 & \xrightarrow{\mathrm{d}} & \wedge^4 & \xrightarrow{\mathrm{d}} & \wedge^5 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ & & L & & \wedge^1_H \otimes L & & \wedge^2_H \otimes L & & \wedge^3_H \otimes L, & & \end{array}$$

and, writing $\wedge^2_H = \wedge^2_{H\perp} \oplus L$, where $\wedge^2_{H\perp}$ comprises 2-forms on H that are trace-free with respect to the Levi form, one obtains the invariantly defined complex

$$\wedge^0 \xrightarrow{\mathrm{d}_\perp} \wedge^1_H \xrightarrow{\mathrm{d}_\perp} \wedge^2_{H\perp} \xrightarrow{\mathrm{d}_\perp^{(2)}} \wedge^2_{H\perp} \otimes L \xrightarrow{\mathrm{d}_\perp} \wedge^3_H \otimes L \xrightarrow{\mathrm{d}_\perp} \wedge^5.$$

A difference to the de Rham complex is that one obtains $\mathrm{d}_\perp^{(2)} \colon \wedge^2_{H\perp} \to \wedge^2_{H\perp} \otimes L$, which is a second-order differential operator.

7 Spinors in G_2 contact geometry

A G_2 contact geometry is an additional structure on the contact distribution of a five-dimensional contact manifold. As observed in the previous section, a contact geometry is naturally equipped with its Levi form $L \to \wedge^2_H$ and the contact distribution H thereby inherits a nondegenerate skew form defined up to scale. This is just what is needed to talk about *Legendrean varieties* [10] in the projective bundle $\mathbb{P}(H) \to M$. A G_2 *contact* structure on M is a field of Legendrean twisted cubics in $\mathbb{P}(H)$. Precisely, this means that, for all $m \in M$, there is a twisted cubic $C_m \subset \mathbb{P}(H_m)$, varying smoothly with $m \in M$, such that the 2-planes in H_m covering the tangent lines to the cubic are null for the Levi form. Equivalently, such a G_2 contact structure may be viewed as a rank two 'spin bundle' S equipped with a 'Levi-compatible' identification $\bigodot^3 S = \wedge^1_H[2]$, where $\wedge^5 = \wedge^0[-3]$. Levi-compatibility means that the Levi form

$$L[4] \to \wedge^2_H[4] = \wedge^2\big(\bigodot^3 S\big) = \big(\bigodot^4 S \otimes \wedge^2 S\big) \oplus \big(\wedge^2 S\big)^3$$

has its range in the second summand and thus provides an identification $L[4] = \big(\wedge^2 S\big)^3$. In these circumstances, notice that

$$\wedge^4_H[8] = \wedge^4\big(\bigodot^3 S\big) = \big(\wedge^2 S\big)^6 \quad \Rightarrow \quad \wedge^0[9] = \wedge^5[12] = \wedge^4_H[8] \otimes L[4] = \big(\wedge^2 S\big)^9$$

and, therefore, we find canonical identifications $\wedge^2 S = \wedge^0[1]$ and $L = \wedge^0[-1]$. In any case, the G_2 contact structure can now be characterised by decreeing that the simple spinors in $H = \bigodot^3 S[-1]$ constitute the cone over the twisted cubic (and there is a clear analogy with conformal spin geometry in dimension three). More detail can be found in [7, Section 5] and the flat model is presented in [6, Section 4]. We should also point out that the geometry of the rank four bundle H follows that of Bryant's 'H_3-structures' on the tangent bundle in four dimensions [2].

The reason for the name 'G_2 contact structure' is that this geometric data defines a parabolic geometry of type (G_2, P) where G_2 is the simply-connected exceptional Lie group of split type G_2 and P is a particular parabolic subgroup such that G_2/P is a contact manifold: see, for example, [4, Section 4.2.8] (and, in particular, it is explained in [6] that this particular realisation of the Lie algebra of G_2 goes back to Engel [8]). With the Dynkin diagram notation from [1], this motivates our writing $\bigodot^k S[w] = \overset{k}{\bullet}\!\Lleftarrow\!\overset{w}{\times}$ (just another way of organising the irreducible representations of $\mathrm{GL}_+(2,\mathbb{R})$) so that

$$S = \overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}, \qquad \bigodot^3 \overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times} = \overset{3}{\bullet}\!\Lleftarrow\!\overset{0}{\times}, \qquad \wedge^1_H = \overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times}, \qquad H = \overset{3}{\bullet}\!\Lleftarrow\!\overset{-1}{\times}, \qquad \wedge^0[1] = \overset{0}{\bullet}\!\Lleftarrow\!\overset{1}{\times},$$

and the Rumin complex is

$$\overset{0}{\bullet}\!\Lleftarrow\!\overset{0}{\times} \xrightarrow{\mathrm{d}_\perp} \overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times} \xrightarrow{\mathrm{d}_\perp} \overset{4}{\bullet}\!\Lleftarrow\!\overset{-3}{\times} \xrightarrow{\mathrm{d}_\perp^{(2)}} \overset{4}{\bullet}\!\Lleftarrow\!\overset{-4}{\times} \xrightarrow{\mathrm{d}_\perp} \overset{3}{\bullet}\!\Lleftarrow\!\overset{-4}{\times} \to \overset{0}{\bullet}\!\Lleftarrow\!\overset{-3}{\times}$$

(consistent with the basic *BGG complex* from [1]). Notice that

$$\wedge^2_H = \wedge^2\big(\overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times}\big) = \overset{4}{\bullet}\!\Lleftarrow\!\overset{-3}{\times} \oplus \overset{0}{\bullet}\!\Lleftarrow\!\overset{-1}{\times}$$

so that the Levi form $L = \overset{0}{\bullet}\!\Lleftarrow\!\overset{-1}{\times} \hookrightarrow \wedge^2\big(\wedge^1_H\big)$ is built into the notation.

Regarding calculus on a contact manifold, it is natural to consider *partial connections*, rather than connections, on vector bundles in which directional derivatives are defined, in the first instance, only in the contact directions. More precisely, a *partial connection* on a vector bundle E is a linear differential operator

$$\nabla_H \colon\ E \to \wedge^1_H \otimes E$$

satisfying a *partial Leibniz rule* $\nabla_H(fs) = f\nabla_H s + \mathrm{d}_\perp f \otimes s$. (In fact, a partial connection can be uniquely promoted [5, Proposition 3.5] to a full connection but we shall not need this trick.) In analogy with three-dimensional spin geometry, we may construct a preferred partial connection on $\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$ in the presence of a 'scale' $\sigma \in \Gamma\big(\overset{0}{\bullet}\!\Lleftarrow\!\overset{1}{\times}\big)$.

The construction of this preferred partial connection follows the same route save for a minor yet crucial distinction. For any contact manifold of dimension ≥ 5, a partial connection on $\wedge^1_H$ gives rise to a linear differential operator $\nabla_\perp\colon \wedge^1_H \to \wedge^2_{H\perp}$ defined as the composition

$$\wedge^1_H \xrightarrow{\nabla_H} \wedge^1_H \otimes \wedge^1_H \xrightarrow{\wedge} \wedge^2_H \to \wedge^2_{H\perp}$$

with the same symbol as the invariantly defined Rumin operator. It follows that the difference

$$\nabla_\perp - \mathrm{d}_\perp\colon\ \wedge^1_H \to \wedge^2_{H\perp}$$

is actually a homomorphism of vector bundles. By definition, this is the *partial torsion* of a partial connection $\nabla_H\colon \wedge^1_H \to \wedge^1_H \otimes \wedge^1_H$.

In the case of a G_2 contact structure, bearing in mind that $\wedge^2\big(\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}\big) = \overset{0}{\bullet}\!\Lleftarrow\!\overset{1}{\times}$, a partial connection on the spin bundle $\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$ induces partial connections on all spin bundles $\overset{k}{\bullet}\!\Lleftarrow\!\overset{w}{\times}$ and, in particular, on $\wedge^1_H = \overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times}$. Thus, we may ask about its partial torsion, which lies in

$$\operatorname{Hom}\big(\wedge^1_H, \wedge^2_{H\perp}\big) = \operatorname{Hom}\big(\overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times}, \overset{4}{\bullet}\!\Lleftarrow\!\overset{-3}{\times}\big) = \overset{7}{\bullet}\!\Lleftarrow\!\overset{-4}{\times} \oplus \overset{5}{\bullet}\!\Lleftarrow\!\overset{-3}{\times} \oplus \overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times} \oplus \overset{1}{\bullet}\!\Lleftarrow\!\overset{-1}{\times}. \tag{7.1}$$

This decomposition is crucial in characterising preferred spin connections as follows.

Proposition 7.1. *Given a scale, i.e., a nowhere vanishing $\sigma \in \Gamma\big(\overset{0}{\bullet}\!\Lleftarrow\!\overset{1}{\times}\big)$, there is a unique partial connection on $\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$ so that*

- *the induced partial connection on $\wedge^2\big(\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}\big) = \overset{0}{\bullet}\!\Lleftarrow\!\overset{1}{\times}$ annihilates σ,*
- *the partial torsion of the induced partial connection on $\wedge^1_H = \overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times}$ lies in $\overset{7}{\bullet}\!\Lleftarrow\!\overset{-4}{\times}$.*

Proof. The first summand in

$$\operatorname{End}\big(\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}\big) = \overset{2}{\bullet}\!\Lleftarrow\!\overset{-1}{\times} \oplus \overset{0}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$$

captures the trace-free endomorphisms of $\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$ and it follows that the freedom in choosing a partial connection on this spin bundle annihilating σ lies in

$$\wedge^1_H \otimes \overset{2}{\bullet}\!\Lleftarrow\!\overset{-1}{\times} = \overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times} \otimes \overset{2}{\bullet}\!\Lleftarrow\!\overset{-1}{\times} = \overset{5}{\bullet}\!\Lleftarrow\!\overset{-3}{\times} \oplus \overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times} \oplus \overset{1}{\bullet}\!\Lleftarrow\!\overset{-1}{\times}. \tag{7.2}$$

Comparison with (7.1) certainly suggests that all but the piece in $\overset{7}{\bullet}\!\Lleftarrow\!\overset{-4}{\times}$ can be uniquely eliminated. We may verify this using spinors. With the familiar conventions of [12], let us write $\epsilon_{AB} \in \Gamma\big(\wedge^2\big(\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}\big)\big)$ rather than $\sigma \in \Gamma\big(\overset{0}{\bullet}\!\Lleftarrow\!\overset{1}{\times}\big)$. Then, choosing a partial connection

$$\phi_D \xrightarrow{\nabla_H} \nabla_{ABC}\phi_D \qquad \text{on} \qquad \overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times},$$

the general partial connection on $\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$ with $\nabla_{ABC}\epsilon_{DE} = 0$ has the form

$$\nabla_{ABC}\phi_D + \Gamma_{ABCD}{}^E\phi_E,$$

where $\Gamma_{ABCDE} = \Gamma_{(ABC)(DE)}$ (i.e., lying in $\overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times} \otimes \overset{2}{\bullet}\!\Lleftarrow\!\overset{-1}{\times}$, as in (7.2)). By the Leibniz rule, the induced operator $\overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times} \to \overset{4}{\bullet}\!\Lleftarrow\!\overset{-3}{\times}$ is

$$\omega_{DEF} \longmapsto \nabla_{(AB}{}^E\omega_{CD)E} + 2\Gamma_{(AB}{}^E{}_C{}^G\omega_{D)EG} - \Gamma_{E(AB}{}^{EG}\omega_{CD)G}. \tag{7.3}$$

Therefore, according to the decomposition (7.2), we should now write

$$\Gamma_{ABC}{}^{DE} = \lambda_{ABC}{}^{DE} + \mu_{(AB}{}^{(D}\delta_{C)}{}^{E)} + \nu_{(A}\delta_B{}^D\delta_{C)}{}^E, \tag{7.4}$$

where λ_{ABCDE} and μ_{ABC} are symmetric spinors and compute

$$2\Gamma_{(AB}{}^{E}{}_{C}{}^{G}\omega_{D)EG} - \Gamma_{E(AB}{}^{EG}\omega_{CD)G}$$

for each term on the right hand side of (7.4). Clearly, this entails computing

$$\Gamma_{(AB}{}^{(E}{}_{C)}{}^{G)} \qquad \text{and} \qquad \Gamma_{EAB}{}^{EG}. \tag{7.5}$$

Firstly, if $\Gamma_{ABC}{}^{DE} = \lambda_{ABC}{}^{DE}$, where $\lambda_{ABCDE} = \lambda_{(ABCDE)}$, then the second term in (7.5) vanishes so for (7.3) we end up with

$$\omega_{DEF} \longmapsto \nabla_{(AB}{}^{E}\omega_{CD)E} + 2\lambda_{(ABC}{}^{EG}\omega_{D)EG},$$

which is perfect for eliminating the $\overset{5}{\bullet}\!\Lleftarrow\!\overset{-3}{\times}$-component of partial torsion.

Secondly, if $\Gamma_{ABC}{}^{DE} = \mu_{(AB}{}^{(D}\delta_{C)}{}^{E)}$, where $\mu_{ABC} = \mu_{(ABC)}$, then straightforward spinor computations show that

$$\Gamma_{(AB}{}^{(E}{}_{C)}{}^{G)} = \tfrac{1}{6}\mu_{(AB}{}^{(E}\delta_{C)}{}^{G)} \qquad \text{and} \qquad \Gamma_{EAB}{}^{EG} = \tfrac{5}{6}\mu_{AB}{}^{G}$$

so for (7.3) we end up with

$$\begin{aligned}\omega_{DEF} &\mapsto \nabla_{(AB}{}^{E}\omega_{CD)E} + \tfrac{1}{3}\mu_{(AB}{}^{E}\omega_{CD)E} - \tfrac{5}{6}\mu_{(AB}{}^{G}\omega_{CD)G}\\ &= \nabla_{(AB}{}^{E}\omega_{CD)E} - \tfrac{1}{2}\mu_{(AB}{}^{E}\omega_{CD)E},\end{aligned}$$

which is perfect for eliminating the $\overset{3}{\bullet}\!\Lleftarrow\!\overset{-2}{\times}$-component of partial torsion.

Thirdly, if $\Gamma_{ABC}{}^{DE} = \nu_{(A}\delta_{B}{}^{D}\delta_{C)}{}^{E}$, then straightforward spinor calculations yield

$$\Gamma_{(AB}{}^{(E}{}_{C)}{}^{G)} = -\tfrac{1}{3}\nu_{(A}\delta_{B}{}^{E}\delta_{C)}{}^{G} \qquad \text{and} \qquad \Gamma_{EAB}{}^{EG} = \tfrac{4}{3}\nu_{(A}\delta_{B)}{}^{G}$$

so for (7.3) we end up with

$$\omega_{DEF} \mapsto \nabla_{(AB}{}^{E}\omega_{CD)E} - \tfrac{2}{3}\nu_{(A}\omega_{BCD)} - \tfrac{4}{3}\nu_{(A}\omega_{BCD)} = \nabla_{(AB}{}^{E}\omega_{CD)E} - 2\nu_{(A}\omega_{BCD)},$$

which is perfect for eliminating the $\overset{1}{\bullet}\!\Lleftarrow\!\overset{-1}{\times}$-component of partial torsion. ∎

Several remarks are in order. Firstly, the preferred connection of Proposition 7.1 is constructed by eliminating all but the $\overset{7}{\bullet}\!\Lleftarrow\!\overset{-4}{\times}$-component of the partial torsion of the induced partial connection on $\wedge^1_H$, decomposed according to (7.1). In fact, it is clear from the proof that the component lying in $\overset{7}{\bullet}\!\Lleftarrow\!\overset{-4}{\times}$ is the same for any choice of partial connection on $\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$ and is, therefore, an invariant of the structure. It is called the *torsion* of our G_2 contact structure. In the general theory of parabolic geometry [4], this is the only component of *harmonic curvature* and is therefore the only obstruction to *local flatness*, i.e., to being locally isomorphic to the flat model G_2/P. Secondly, we should point out that the spinor identities established by direct calculation in our proof can be avoided by judicious use of Lie algebra cohomology (as in done in [4]). Thirdly, we note that a *scale*, a nowhere vanishing section σ of $\overset{0}{\bullet}\!\Lleftarrow\!\overset{1}{\times}$, has a nice geometric interpretation. Since $\overset{0}{\bullet}\!\Lleftarrow\!\overset{-1}{\times} = L \hookrightarrow \wedge^1$ is the bundle of contact forms, we can interpret σ^{-1} as a choice of contact form. In other words, the preferred partial connection on the spin bundle $S = \overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$ is obtained in the presence of a contact form.

The transformation law for preferred partial connections in G_2 contact geometry is obtained by analogy with Proposition 5.2. Its proof will therefore be omitted.

Proposition 7.2. *Let us change scale $\sigma \in \Gamma\big(\wedge^0[1]\big)$ by $\widehat{\sigma} = \Omega^{-1}\sigma$. Then, for $\phi_D \in \Gamma\big(\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}\big)$,*

$$\widehat{\nabla}_{ABC}\phi_D = \nabla_{ABC}\phi_D + \Upsilon_{ABC}\phi_D - \Upsilon_{D(AB}\phi_{C)}, \qquad \textit{where} \quad \Upsilon_{ABC} \equiv \Omega^{-1}\nabla_{ABC}\Omega. \tag{7.6}$$

As an immediate consequence of (7.6), if $\phi_D \in \Gamma(\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times})$, then $\widehat{\nabla}_{(ABC}\phi_{D)} = \nabla_{(ABC}\phi_{D)}$. Furthermore, if $\phi_D \in \Gamma(\overset{1}{\bullet}\!\Lleftarrow\!\overset{-4/3}{\times})$, then

$$\widehat{\nabla}_{ABC}\phi_D = \nabla_{ABC}\phi_D - \tfrac{1}{3}\Upsilon_{ABC}\phi_D - \Upsilon_{D(AB}\phi_{C)},$$

whence $\widehat{\nabla}_{AB}{}^C\phi_C = \nabla_{AB}{}^C\phi_C$. We have found two *invariant operators*

$$\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times} \longrightarrow \overset{4}{\bullet}\!\Lleftarrow\!\overset{-3}{\times} \qquad \text{and} \qquad \overset{1}{\bullet}\!\Lleftarrow\!\overset{-4/3}{\times} \longrightarrow \overset{2}{\bullet}\!\Lleftarrow\!\overset{-7/3}{\times}, \tag{7.7}$$

defined only in terms of the G_2 contact structure itself. Conversely, it is easy to see that the existence of these two operators is sufficient to define the preferred partial connection on $\overset{1}{\bullet}\!\Lleftarrow\!\overset{0}{\times}$ associated with a scale and to capture the transformation law (7.6). Sadly, we have not been able to manufacture either of the invariant operators (7.7) directly.

In three-dimensional conformal spin geometry, the transformation law (5.1) leads to a pair of basic first-order invariant differential operators

$$\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\times} \longrightarrow \overset{3}{\bullet}\!\Leftarrow\!\overset{-2}{\times} \qquad \text{and} \qquad \overset{1}{\bullet}\!\Leftarrow\!\overset{-3/2}{\times} \longrightarrow \overset{1}{\bullet}\!\Leftarrow\!\overset{-5/2}{\times},$$

given by $\phi_C \mapsto \nabla_{(AB}\phi_{C)}$ and $\phi_C \mapsto \nabla_{AB}\phi^B$. This suggests that we should refer to the operators (7.7) as the 'twistor' or 'tractor' operator and 'Dirac' operator, respectively, on a G_2 contact manifold. Sure enough, this tractor operator is overdetermined and, in the flat case, has a 7-dimensional kernel corresponding to the embedding $G_2 \hookrightarrow \mathrm{SO}^{\uparrow}(4,3)$. The prolongation of this tractor operator (leading to the *standard tractor* bundle) is detailed in [11].

8 Legendrean contact geometry in five dimensions

A Legendrean contact geometry in five dimensions is a 5-dimensional contact manifold equipped with a splitting of the contact distribution as two rank 2 subbundles

$$H = E \oplus F$$

each of which is null with respect to the Levi form (so that the Levi form reduces to a perfect pairing $E \otimes F \to L^*$). The flat model naturally arises in [6, Section 2] as the moduli space of flying saucers in 'attacking mode' and [6, Proposition 2.5] gives the 15-dimensional symmetry algebra.

In the spirit of previous sections our aim will be, in the presence of a scale (equivalently, a choice of contact form), to construct preferred partial connections on all the natural irreducible bundles on such a geometry. As before, these connections can be obtained by means of the canonical differential operators present on this type of geometry. In fact, we shall only need to examine the Rumin complex to find sufficiently many canonical differential operators for these purposes.

A Legendrean contact geometry is a type of parabolic geometry [4, Section 4.2.3], specifically $\times\!\!-\!\!\bullet\!\!-\!\!\times$ in the notation of [1]. Then

$$H = \begin{array}{c} E \\ \oplus \\ F \end{array} = \begin{array}{c} \overset{1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-1}{\times} \\ \oplus \\ \overset{-1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\times}. \end{array}$$

The general irreducible bundle has the form $\overset{u}{\times}\!-\!\overset{k}{\bullet}\!-\!\overset{v}{\times}$ for $k \in \mathbb{Z}_{\geq 0}$ and $u, v, \in \mathbb{R}$. It is convenient to take $S \equiv \overset{-1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-1}{\times}$ as our basic 'spin' bundle. If we also let $\wedge^0[u,v] \equiv \overset{u}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times}$, then $L = \wedge^0[-1,-1]$ and the general irreducible bundle is $\overset{u}{\times}\!-\!\overset{k}{\bullet}\!-\!\overset{v}{\times} = \bigodot^k S[u+k, v+k]$. Notice that

$$E \otimes F = \overset{1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-1}{\times} \otimes \overset{-1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\times} = \overset{0}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\times} \oplus \overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}$$

and the Levi form is built into the notation as projection onto the second summand. (Without the Dynkin diagram notation, it follows from the perfect pairing $E \otimes F \to L^*$ and the canonical identification $E^* = E \otimes \det E^*$ that

$$E \otimes (\det E^*)^{1/2} = \big(E \otimes (\det E^*)^{1/2}\big)^* = F \otimes (\det F^*)^{1/2}$$

and we may take $S \equiv E \otimes (\det E^*)^{1/2} \otimes L^{1/2} \equiv F \otimes (\det F^*)^{1/2} \otimes L^{1/2}$.)

In order to avoid confusion, by default we shall write a section of $\overset{u}{\times}\!\!-\!\!\overset{k}{\bullet}\!\!-\!\!\overset{v}{\times}$ with *lower* spinor indices

$$\phi_{\underbrace{AB\ldots C}_{k}} = \phi_{(AB\ldots C)} \in \Gamma\big(\overset{u}{\times}\!\!-\!\!\overset{k}{\bullet}\!\!-\!\!\overset{v}{\times}\big)$$

with no special terminology to record the bundle of which it is a section (in other words, we shall forgo any systematic notion of 'weight'). Of course, we may use the tautological identification $S = S^* \otimes \det S = S^* \otimes L$ to replace lower spinor indices by upper spinor indices (with an appropriate change in 'weight' if we were to assign one) so there is no loss in using lower indices by default. As an example of these conventions in action, we may write the first operator $\mathrm{d}_\perp : \wedge^0 \to \wedge^1_H$ in the Rumin complex as

$$f \longmapsto \mathrm{d}_\perp f \equiv \begin{bmatrix} \nabla_A f \\ \bar\nabla_A f \end{bmatrix} \in \begin{array}{c} \overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times} \\ \oplus \\ \overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times} \end{array} = \begin{array}{c} E^* \\ \oplus \\ F^* \end{array} = \wedge^1_H, \tag{8.1}$$

where ∇_A, respectively $\bar\nabla_A$, is the directional derivative in the E, respectively F, direction, both of which are manifestly invariantly defined. (Although the notion of Legendrean contact geometry pertains in any odd dimension, it is only in five dimensions that one has the convenience of spinors and, in particular, that the bundles E and F agree save for a line bundle factor.) Of course, the directional derivatives $\nabla_A f$ and $\bar\nabla_A f$ end up as sections of different bundles even though each of them has a single spinor index.

Soon (as with all parabolic geometries [3]), we shall find it convenient to work in a particular *scale*, i.e., with a nowhere vanishing section $\sigma \in \Gamma\big(\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times}\big)$. As with all parabolic contact structures [4, Section 4.2], we may interpret the section σ^{-1} of $\overset{-1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{-1}{\times} = L \hookrightarrow \wedge^1$ as a choice of contact form. For 5-dimensional Legendrean contact geometry, however, we may also interpret a scale as a choice of skew spinor ϵ_{AB} by dint of

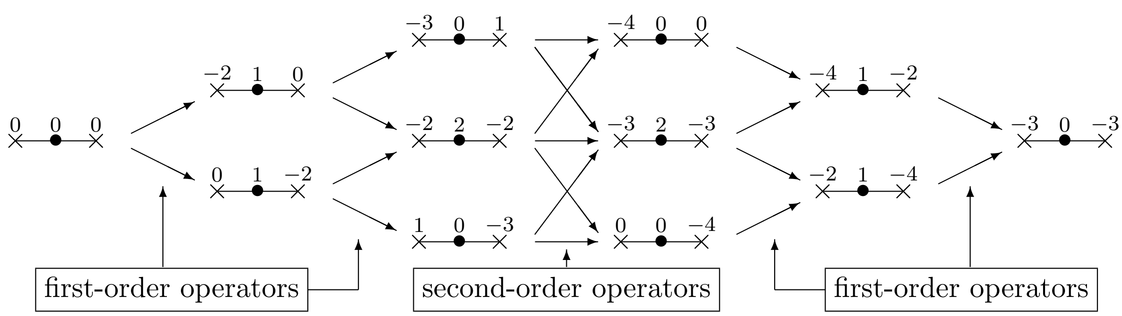
$$\sigma^{-1} \in \Gamma\big(\overset{-1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{-1}{\times}\big) = \Gamma\big(\wedge^2\big(\overset{-1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-1}{\times}\big)\big) = \Gamma\big(\wedge^2 S\big) \ni \epsilon_{AB}, \tag{8.2}$$

which we may use to raise and lower spinor indices with the familiar conventions of [12]. In particular, for $\phi_A \in \Gamma\big(\overset{-1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-1}{\times}\big)$ and $\psi_A \in \Gamma\big(\overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times}\big)$, we may write

$$\sigma^{-1}\phi_{[A}\psi_{B]} = \tfrac{1}{2}\epsilon_{AB}\psi^C\phi_C \qquad \text{to define } \psi^C\phi_C \in \Gamma\big(\wedge^0\big),$$

independent of choice of σ and identifying $\big(\overset{-1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-1}{\times}\big)^* = \overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times}$, as expected.

Continuing from (8.1), we may decompose the entire Rumin complex into its constituent parts via the usual spin-bundle decompositions to obtain an array of invariantly defined linear differential operators



(8.3)

In this diagram we have omitted arrows that correspond to homomorphisms. For example one can check that the part of the Rumin complex $\overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times} \to \overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{-3}{\times}$ is actually a homomorphism, and its vanishing is equivalent to the integrability of F. As already noted, the operator

$$\nabla_A\colon\ \overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\times} \to \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times}$$

is just the directional derivative on functions in the E direction: $f \mapsto \mathrm{d}f|_E$. More generally, the partial connections we aim to construct naturally split into a part that differentiates along E (which we shall denote by ∇_A) and a part that differentiates along F (denoted by $\bar{\nabla}_A$). In particular, amongst the invariant operators in (8.3) we find $\nabla_A\colon \overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{-4}{\times} \to \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-4}{\times}$ and, therefore, by insisting on the Leibniz rule, invariantly defined derivatives in the E direction

$$\nabla_A\colon\ \overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times} \to \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times} \otimes \overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times}, \qquad \text{for all } v \in \mathbb{R}.$$

Now suppose we are given a nowhere vanishing *scale* $\sigma \in \Gamma\big(\overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}\big)$. Then we can define

$$\nabla_A\colon\ \overset{u}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times} \to \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times} \otimes \overset{u}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times}, \qquad \text{for all } u, v \in \mathbb{R} \tag{8.4}$$

by $\nabla_A(f\sigma^u) \equiv (\nabla_A f)\sigma^u$ for smooth sections f of $\overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v-u}{\times}$. We may compute how this operator changes under a change of scale $\widehat{\sigma} = \Omega^{-1}\sigma$, for some nowhere vanishing smooth function Ω. Firstly, note that $\nabla_A(\Omega^{-u}f) = \Omega^{-u}(\nabla_A f - u\Upsilon_A f)$, where $\Upsilon_A \equiv \Omega^{-1}\nabla_A\Omega$. Hence,

$$\begin{aligned}\widehat{\nabla}_A(f\widehat{\sigma}^u) &\equiv (\nabla_A f)\widehat{\sigma}^u = \Omega^{-u}(\nabla_A f)\sigma^u \\ &= \nabla_A(\Omega^{-u}f)\sigma^u + u\Upsilon_A f\widehat{\sigma}^u = \nabla_A(\Omega^{-u}f\sigma^u) + u\Upsilon_A f\widehat{\sigma}^u\end{aligned}$$

and, writing $s = f\widehat{\sigma}^u \in \Gamma\big(\overset{u}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times}\big)$, we obtain

$$\widehat{\nabla}_A s = \nabla_A s + u\Upsilon_A s.$$

In particular, this transformation law records the invariance of ∇_A when $u = 0$. Similarly, starting with $\bar{\nabla}_A\colon \overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\times} \to \overset{-4}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-2}{\times}$ from (8.3), in the presence of a scale σ, we obtain

$$\bar{\nabla}_A\colon\ \overset{u}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times} \to \overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-2}{\times} \otimes \overset{u}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times}, \qquad \text{for all } u, v \in \mathbb{R}$$

and, under change of scale $\widehat{\sigma} = \Omega^{-1}\sigma$, we find that

$$\widehat{\bar{\nabla}}_A s = \bar{\nabla}_A s + v\bar{\Upsilon}_A s,$$

where $\bar{\Upsilon}_A \equiv \Omega^{-1}\bar{\nabla}_A\Omega$.

Referring back to the Rumin complex (8.3) we also have canonical differential operators (in the E direction)

$$\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-2}{\times} \to \overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{-2}{\times} \qquad \text{and} \qquad \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times} \to \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}, \tag{8.5}$$

which may be combined, via the Leibniz, rule with (8.4) to obtain, in the presence of a scale, a first-order differential operator

$$\nabla_A\colon\ \overset{u}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{v}{\times} \to \overset{u-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{v}{\times} \oplus \overset{u-1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v+1}{\times} = \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times} \otimes \overset{u}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{v}{\times}.$$

To express the transformation of this operator under change of scale, we may split it as

$$\phi_B \mapsto \nabla_A\phi_B = \nabla_{(A}\phi_{B)} + \nabla_{[A}\phi_{B]},$$

and recall that operator (8.4) on densities $s \in \Gamma\big(\overset{u}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{v}{\times}\big)$ transforms as

$$\widehat{\nabla}_A s = \nabla_A + u\Upsilon_A s, \qquad \text{where} \quad \Upsilon_A \equiv \Omega^{-1}\nabla_A\Omega. \tag{8.6}$$

Proposition 8.1. *Suppose we change scale* $\sigma \in \Gamma(\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times})$ *by* $\widehat{\sigma} = \Omega^{-1}\sigma$. *Then, for* ϕ_B *a section of* $\overset{u}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{v}{\times}$ *we have*

$$\widehat{\nabla}_A\phi_B = \nabla_A\phi_B + (u+1)\Upsilon_A\phi_B - \Upsilon_B\phi_A. \tag{8.7}$$

Proof. It suffices to note that this transformation law is consistent with the invariance of the operators (8.5), which may be written as

$$\phi_B \longmapsto \nabla_{(A}\phi_{B)} \qquad \text{and} \qquad \phi_B \longmapsto \nabla_{[A}\phi_{B]}$$

and also with (8.6) on densities. ∎

Similarly, from the canonical operators

$$\bar{\nabla}\colon \overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times} \to \overset{-2}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{-2}{\times} \qquad \text{and} \qquad \bar{\nabla}\colon \overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times} \to \overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{-3}{\times},$$

from (8.3) we may construct, in the presence of a scale,

$$\Gamma(\overset{u}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{v}{\times}) \ni \phi_B \longmapsto \bar{\nabla}_A\phi_B$$

differentiating in the F direction and transforming by

$$\widehat{\bar{\nabla}}_A\phi_B = \bar{\nabla}_A\phi_B + (v+1)\bar{\Upsilon}_A\phi_B - \bar{\Upsilon}_B\phi_A, \qquad \text{where} \quad \bar{\Upsilon}_A \equiv \Omega^{-1}\bar{\nabla}_A\Omega. \tag{8.8}$$

Finally, we may combine ∇_A and $\bar{\nabla}_A$ to define, in the presence of a scale $\sigma \in \Gamma(\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times})$, partial connections

$$\Gamma(\overset{u}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{v}{\times}) \ni \phi_B \to \begin{bmatrix} \nabla_A\phi_B \\ \bar{\nabla}_A\phi_B \end{bmatrix} \in \begin{matrix} \overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times} \\ \oplus \\ \overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times} \end{matrix} \otimes \overset{u}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{v}{\times} = \wedge^1_H \otimes \overset{u}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{v}{\times}$$

and, indeed by the Leibniz rule, on all weighted spinor bundles. These partial connections are generated by $\wedge^1_H \to \wedge^1_H \otimes \wedge^1_H$ and this basic one is characterised as follows.

Proposition 8.2. *Let* σ *be a nowhere vanishing section of* $\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times}$, *equivalently a choice of contact form. Then, there is a unique partial connection* $\nabla_H\colon \wedge^1_H \to \wedge^1_H \otimes \wedge^1_H$ *so that the induced partial connection on* $\wedge^4_H = \overset{-2}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{-2}{\times}$ *annihilates* σ^{-2} *and so that* ∇_H *has minimal partial torsion in the sense that the induced operator* $\wedge^1_H \to \wedge^2_{H\perp}$ *agrees with the Rumin operator* $\mathrm{d}_\perp$ *modulo the homomorphisms that are the obstructions to integrability.*

Proof. Recall that we constructed this partial connection from the Rumin complex (8.3) modulo the obstructions to integrability. The only ingredients in this argument not immediately visible in $\mathrm{d}_\perp\colon \wedge^1_H \to \wedge^2_{H\perp}$, where the two operators

$$\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{-4}{\times} \to \overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-4}{\times} \qquad \text{and} \qquad \overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\times} \to \overset{-4}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times}$$

coming from further along the Rumin complex. However, it is straightforward to check that, for example,

$$\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{-4}{\times} = \wedge^5 \otimes \big(\overset{-3}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times}\big)^* \longrightarrow \wedge^5 \otimes \big(\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times}\big)^* = \overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-4}{\times}$$

is the adjoint of $\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times} \to \overset{-3}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times}$ so these two hidden ingredients are also secretly carried by $\mathrm{d}_\perp\colon \wedge^1_H \to \wedge^2_{H\perp}$. ∎

In fact, this proposition also follows from the general theory [4], or by a more explicit spinor calculation [11]. The prolongation of $\overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times} \ni \phi_B \longmapsto \big(\nabla_{(A}\phi_{B)}, \bar{\nabla}_{(A}\phi_{B)}\big)$ gives an especially convenient tractor bundle and its (partial) connection (cf. [4, 11]).

A more familiar way [12] of saying that ∇_A annihilates the scale $\sigma \in \Gamma\big(\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times}\big)$ and hence $\sigma^{-1} \in \Gamma\big(\overset{-1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{-1}{\times}\big)$, is to say that $\nabla_A \epsilon_{BC} = 0$ for the corresponding skew form ϵ_{BC} under (8.2). As a final consistency check, we may verify that this constraint is invariant under (8.7), as follows. For ϕ_B a section of $\overset{-1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-1}{\times}$, the transformation (8.7) reads

$$\widehat{\nabla}_A \phi_B = \nabla_A \phi_B - \Upsilon_B \phi_A$$

(cf. conformal spin geometry in four dimensions [12, equation (5.6.15)]) and, therefore, if ϕ_{BC} is a section of $\overset{-1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-1}{\times} \otimes \overset{-1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-1}{\times}$, then the Leibniz rule implies that

$$\widehat{\nabla}_A \phi_{BC} = \nabla_A \phi_{BC} - \Upsilon_B \phi_{AC} - \Upsilon_C \phi_{BA}.$$

Hence, if ϕ_{BC} is skew, then $\Upsilon_A \phi_{BC} = \Upsilon_B \phi_{AC} + \Upsilon_C \phi_{BA}$ and it follows that

$$\widehat{\nabla}_A \phi_{BC} = \nabla_A \phi_{BC} - \Upsilon_A \phi_{BC}.$$

Finally, when $\widehat{\sigma} = \Omega^{-1}\sigma$, we find that $\widehat{\epsilon}_{AB} = \Omega \epsilon_{AB}$ (cf. [12, equation (5.6.2)]) and, hence, that

$$\widehat{\nabla}_A \widehat{\epsilon}_{BC} = \widehat{\nabla}_A(\Omega \epsilon_{BC}) = \Omega\big(\widehat{\nabla}_A \epsilon_{BC} + \Upsilon_A \epsilon_{BC}\big) = \Omega \nabla_A \epsilon_{BC},$$

as required.

9 Flying saucers via spinors

Distilling the construction in [7] down to its key ingredients, we will explain how to construct a G_2 contact structure starting from a Legendrean contact structure plus some additional data, a choice of appropriately weighted sections of the Legendrean subbundles. We then will calculate the torsion of the resulting G_2 contact structure in terms of the input data and the preferred partial connection in the previous section.

Using the Levi form we may identify $F^* = E \otimes L = E[-1,-1]$ and so can write

$$\wedge^1_H \ni \omega_a = \begin{bmatrix} \omega_A \\ \bar{\omega}^A \end{bmatrix} \in \begin{array}{c} E^* \\ \oplus \\ E[-1,-1] \end{array}.$$

In what follows we will fix a scale $\sigma \in \overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times}$, or equivalently a contact form. The expression for the torsion will turn out to be independent of this choice, but to write it down we will need the distinguished connection from the previous section. The distinguished partial connection annihilating the scale can be written on E^* as

$$E^* \ni \omega_A \mapsto \begin{bmatrix} \nabla_A \omega_B \\ \bar{\nabla}^A \omega_B \end{bmatrix} \in \begin{array}{c} E^* \\ \oplus \\ E[-1,-1] \end{array} \otimes E^*.$$

Recall from Section 8 that a choice of scale gives rise to a skew spinor ϵ_{AB}, covariant constant under the distinguished partial connection, together with its inverse ϵ^{AB} (such that $\epsilon_{AB}\epsilon^{AC} = \delta_B{}^C$), which we may use to raise and lower indices as per [12]. Covariant constancy implies that we may equate $\nabla_A \sigma^B = \nabla_A\big(\sigma_C \epsilon^{BC}\big)$ and $(\nabla_A \sigma_C)\epsilon^{BC}$, as in the usual spinor calculus.

We decompose $\wedge^2_{H\perp} = \wedge^2 E^* \oplus (E^* \otimes E)_\circ[-1,-1] \oplus \wedge^2 E[-2,-2]$, and Proposition 8.2 then means we can write the Rumin operator as

$$d_\perp \begin{bmatrix} \sigma_A \\ \tau^A \end{bmatrix} = \begin{bmatrix} -\nabla^A \sigma_A + \Pi_A \tau^A \\ \left(\nabla_A \tau^B - \bar{\nabla}^B \sigma_A\right)_\circ \\ \bar{\nabla}_A \tau^A + \Sigma^A \sigma_A \end{bmatrix},$$

where Σ^A and Π_A are the obstructions to integrability and $\left(\Xi_A{}^B\right)_\circ = \Xi_A{}^B - \frac{1}{2}\delta_A{}^B \Xi_C{}^C$ for $\Xi_A{}^B \in E \otimes E^*$.

Now suppose we are given sections $o_A \in E^*[2,-1]$ and $\iota^A \in E[-2,1]$ such that $o_A \iota^A = 1$, classically known as a *spin-frame* [12, pp. 110–115]. Then we may define a G_2 contact geometry as follows. These sections determine an isomorphism

$$\wedge^0[0,3] \oplus \wedge^0[1,2] \oplus \wedge^0[2,1] \oplus \wedge^0[3,0] \cong E^*[2,2] \oplus E[1,1]$$

given by

$$(x,y,z,w) \longmapsto \left(x o_A - \tfrac{1}{\sqrt{3}} y \iota_A, w \iota^A - \tfrac{1}{\sqrt{3}} z o^A\right). \tag{9.1}$$

If we set $S = \wedge^0[0,1] \oplus \wedge^0[1,0]$, we have an isomorphism

$$\textstyle\bigodot^3 S \cong E^*[2,2] \oplus E[1,1] = \wedge^1_H[2,2]$$

as required in the definition of a G_2 contact structure. The peculiar factors here are chosen so that the isomorphism also satisfies the additional Levi-compatibility condition.

Starting by considering the general connection on S that annihilates the scale, we can calculate the torsion of the G_2 contact structure by calculating the obstruction to the differential operator $\Lambda^1_H \to \Lambda^2_{H\perp}$ induced by the defining isomorphism being equal to the Rumin operator.

We can write any partial connection on S as

$$\nabla_a \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \nabla_a y + \kappa_a y + \lambda_a z \\ \nabla_a z + \mu_a y + \nu_a z \end{bmatrix}$$

for appropriately weighted sections κ_a, λ_a, μ_a, ν_a, where we will take ∇_a to be the partial connection distinguished by the contact form. The connection on S will be compatible with the contact form when $\kappa_a = -\nu_a$. Given this, the induced partial connection on $\bigodot^3 S \cong \wedge^0[0,3] \oplus \wedge^0[1,2] \oplus \wedge^0[2,1] \oplus \wedge^0[3,0]$ is

$$\nabla_a \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \nabla_a x \\ \nabla_a y \\ \nabla_a z \\ \nabla_a w \end{bmatrix} + \begin{bmatrix} 3\kappa_a & \lambda_a & 0 & 0 \\ 3\mu_a & \kappa_a & 2\lambda_a & 0 \\ 0 & 2\mu_a & -\kappa_a & 3\lambda_a \\ 0 & 0 & \mu_a & -3\kappa_a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \wedge^1_H \otimes \textstyle\bigodot^3 S.$$

Rewriting the right hand side using the defining isomorphism (9.1) yields the differential operator $\bigodot^3 S \to \wedge^1_H \otimes (E^*[2,2] \oplus E[1,1])$ given by

$$\nabla_a \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} (\nabla_a x + 3x\kappa_a + y\lambda_a) o_A - \frac{1}{\sqrt{3}}(\nabla_a y + 3x\mu_a + y\kappa_a + 2z\lambda_a)\iota_A \\ -\frac{1}{\sqrt{3}}(\nabla_a z + 2y\mu_a - z\kappa_a + 3w\lambda_a) o^A + (\nabla_a w + z\mu_a - 3w\kappa_a)\iota^A \end{bmatrix}.$$

To calculate the induced operator $\wedge^1_H \to \wedge^2_{H\perp}$ we may use the canonical identification $\wedge^1_H = E^* \oplus E[-1,-1]$ and then project the right hand side above onto the direct sum

$$\wedge^2 E^*[2,2] \oplus (E^* \otimes E)_\circ[1,1] \oplus \wedge^2 E.$$

We write $\kappa_a = (\kappa_A, \bar\kappa^A) \in E^* \oplus E[-1,1]$ and so on to denote the projections. The induced operator (pulled back via the isomorphism (9.1)) is therefore

$$\bigodot^3 S[-2,-2] \to \wedge^2 E^* \oplus (E^*\otimes E)_\circ[-1,-1] \oplus \wedge^2 E[-2,-2]$$

given by

$$\begin{bmatrix} x\\ y\\ z\\ w\end{bmatrix} \mapsto \begin{bmatrix} (\nabla_A x + 3x\kappa_A + y\lambda_A)o^A - \frac{1}{\sqrt3}(\nabla_A y + 3x\mu_A + y\kappa_A + 2z\lambda_A)\iota^A \\ \Big(-\frac{1}{\sqrt3}(\nabla_A z + 2y\mu_A - z\kappa_A + 3w\lambda_A)o^B + (\nabla_A w + z\mu_A - 3w\kappa_A)\iota^B \\ -(\bar\nabla^B x + 3x\bar\kappa^B + y\bar\lambda^B)o_A + \frac{1}{\sqrt3}(\bar\nabla^B y + 3x\bar\mu^B + y\bar\kappa^B + 2z\bar\lambda^B)\iota_A\Big)_\circ \\ \frac{1}{\sqrt3}(\bar\nabla^A z + 2y\bar\mu^A - z\bar\kappa^A + 3w\bar\lambda^A)o_A - (\bar\nabla^A w + z\bar\mu^A - 3w\bar\kappa^A)\iota_A \end{bmatrix}.$$

This is the operator that we should compare to the Rumin operator (pulled back via the isomorphism (9.1)), which can be written

$$\mathrm{d}_\perp \begin{bmatrix} x\\ y\\ z\\ w\end{bmatrix} = \begin{bmatrix} -\nabla^A(xo_A) + \frac{1}{\sqrt3}\nabla^A(y\iota_A) + \Pi_A\big(w\iota^A - \frac{1}{\sqrt3}zo^A\big) \\ \big(\nabla_A\big(w\iota^B - \frac{1}{\sqrt3}zo^B\big) - \bar\nabla^B\big(xo_A - \frac{1}{\sqrt3}y\iota_A\big)\big)_\circ \\ \bar\nabla_A(w\iota^A) - \frac{1}{\sqrt3}\bar\nabla_A(zo^A) + \Sigma^A\big(xo_A - \frac{1}{\sqrt3}y\iota_A\big) \end{bmatrix}.$$

Insisting that these differential operators are equal (and hence that the torsion vanishes) we obtain a system of twelve spinor equations

$$\begin{aligned}
&\nabla^A o_A = 3\kappa^A o_A - \sqrt3\mu^A\iota_A, \qquad \nabla^A\iota_A = -\sqrt3\lambda^A o_A + \kappa^A\iota_A,\\
&\Pi_A o^A = -2\lambda^A\iota_A \qquad\qquad\qquad \Pi_A\iota^A = 0,\\
&\big(\bar\nabla^B o_A\big)_\circ = \big(-\sqrt3\bar\mu^B\iota_A + 3\bar\kappa^B o_A\big)_\circ,\\
&\big(\bar\nabla^B \iota_A\big)_\circ = \big(-2\mu_A o^B - \sqrt3\bar\lambda^B o_A + \bar\kappa^B\iota_A\big)_\circ,\\
&\big(\nabla_A o^B\big)_\circ = \big(-\kappa_A o^B - \sqrt3\mu_A\iota^B - 2\bar\lambda^B\iota_A\big)_\circ,\\
&\big(\nabla_A \iota^B\big)_\circ = \big(-\sqrt3\lambda_A o^B - 3\kappa_A\iota^B\big)_\circ,\\
&\bar\nabla_A o^A = -\bar\kappa_A o^A - \sqrt3\bar\mu_A\iota^A, \qquad \bar\nabla_A\iota^A = -\sqrt3\bar\lambda_A o^A - 3\bar\kappa_A\iota^A,\\
&\Sigma^A\iota_A = 2\bar\mu_A o^A, \qquad \Sigma^A o_A = 0.
\end{aligned}$$

Contracting the middle four equations above with combinations of o_A, ι^B and their counterparts with indices raised and lowered, respectively, produces (together with the other eight) a system of twenty independent linear equations over $\mathbb{R}$ in twelve unknowns $\kappa^A o_A, \mu^A\iota_A, \dots$ (owing to the trace-free condition, the middle four equations above yield three independent equations each). This system is consistent if and only if the following eight obstructions vanish:

$$\begin{aligned}
\psi_0 &= \Pi_A\iota^A,\\
\psi_1 &= \Pi_A o^A - \tfrac{2}{\sqrt3}\iota^A\big(\nabla_A\iota^B\big)\iota_B,\\
\psi_2 &= \iota_A\big(\bar\nabla^A o_B\big)o^B + o_A\big(\bar\nabla^A o_B\big)\iota^B - \bar\nabla_A o^A + 2o_A\big(\bar\nabla^A\iota_B\big)o^B,\\
\psi_3 &= \iota_A\big(\bar\nabla^A\iota_B\big)o^B + o_A\big(\bar\nabla^A\iota_B\big)\iota^B - \tfrac13\bar\nabla_A\iota^A + \tfrac23\iota_A\big(\bar\nabla^A o_B\big)\iota^B + \tfrac{2}{\sqrt3}o^A\big(\nabla_A o^B\big)o_B,\\
\psi_4 &= o^A\big(\nabla_A o^B\big)\iota_B + \iota^A\big(\nabla_A o^B\big)o_B - \tfrac13\nabla^A o_A + \tfrac23 o^A\big(\nabla_A\iota^B\big)o_B + \tfrac{2}{\sqrt3}\iota_A\big(\bar\nabla^A\iota_B\big)\iota^B,\\
\psi_5 &= o^A\big(\nabla_A\iota^B\big)\iota_B + \iota^A\big(\nabla_A\iota^B\big)o_B - \nabla^A\iota_A + 2\iota^A\big(\nabla_A o^B\big)\iota_B,\\
\psi_6 &= \Sigma^A\iota_A + \tfrac{2}{\sqrt3}o_A\big(\bar\nabla^A o_B\big)o^B,\\
\psi_7 &= \Sigma^A o_A.
\end{aligned}$$

The above eight functions vanish with the invariant torsion of the G_2 contact structure. One can check using the formulæ (8.7) and (8.8) that these expressions are invariant under change of scale, as they should be.

This construction and resulting formulæ apply in some generality (locally all G_2 contact structures arise this way [11]). In particular, they generalise [7, equation (32)] in case that the spin-frame o_A, ι^A arises from flying saucer data [7, equation (24)] (precisely, with the notation from [7], this means that $\iota^A = \pi^!\psi$ and $o_A = \Theta^{-1}\pi^!\phi$).

Acknowledgements

We would like to thank all staff at SIGMA in Kyiv for their extraordinary courage, continuing their work despite the shocking Russian invasion and unconscionable aggression.

We would also like to thank the referees for their careful reading of our manuscript and for their valuable suggestions and corrections.

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