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Cohomology of $\mathfrak{sl}_3$ and $\mathfrak{gl}_3$ with Coefficients in Simple Modules and Weyl Modules in Positive Characteristics

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Abstract. We calculate the cohomology of $\mathfrak{sl}_3(k)$ over an algebraically closed field k of characteristic $p > 3$ with coefficients in simple modules and Weyl modules. We also give descriptions of the corresponding cohomology of $\mathfrak{gl}_3(k)$.

Key words: Lie algebra; simple module; cohomology

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1 Introduction

There are many remarkable results in the cohomology theory of modular Lie algebras. Any simple module over the restricted Lie algebra with nontrivial cohomology is restricted, see [9, Theorem 2]. Modules over a Lie algebra with nonzero cohomology are called *peculiar*. For any finite-dimensional Lie algebra, the number of non-isomorphic peculiar indecomposable modules is finite, see [10, Theorem 1].

Now, let $\mathfrak{g} = \mathfrak{sl}_3(k)$. The cohomology with coefficients in simple $\mathfrak{g}$ -modules is completely described only in small characteristics $p = 2, 3$, see [18, Theorem 1], [19, Theorem 1]. In the case where $p > 3$, the cohomology of simple $\mathfrak{g}$-modules are known in the following cases: for $H^1(\mathfrak{g}, M)$, see [21, p. 301], for $H^2(\mathfrak{g}, M)$, see [11, Theorem 1.1]. In small degrees, the cohomology has the following interpretations: $H^1(\mathfrak{g}, \mathfrak{g})$ is identified with the space of outer differentiation and $H^2(\mathfrak{g}, \mathfrak{g})$ is identified with the space of local deformations of the Lie algebra $\mathfrak{g}$. It is known that these spaces are trivial, see [23, p. 124], [6, p. 125], [2, Lemma 2.2.1b], [3, p. 32]. The cohomology of $\mathfrak{g}$ with coefficients in the trivial module is also known, see [3, p. 42]. In other cases, the interpretation of the cohomology with coefficients in simple modules remains open. In this paper, we give a complete description of the cohomology of $\mathfrak{sl}_3(k)$ with coefficients in the simple modules for $p > 3$.

1.1 Notation

Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ over an algebraically closed field k of characteristic $p > 3$ and M a simple $\mathfrak{g}$-module. Let $L(r, s)$ denote a simple $\mathfrak{g}$-module with the highest weight $r\omega_1 + s\omega_2$, where ω_1, ω_2 are fundamental weights.

Let $G = \mathrm{SL}_3(k)$; it is an algebraic group, its Lie algebra is $\mathfrak{sl}_3(k)$. We will consider cohomology of $\mathfrak{sl}_3(k)$ as G-modules. Let V be a G-module and M be a simple G-module. We define a *composition coefficient* $[V : M]$ for M from the formula

$$\mathrm{ch}(V) = \sum_{M \text{ is simple}} [V : M]\,\mathrm{ch}(M),$$

where $\mathrm{ch}(V)$ is the formal character of the G-module V. If $[V : M] \neq 0$, then we say that M is a *composition factor of* V.

For a vector space L over k, we denote by $L^{(1)}$ the vector space over k that coincides with L as an additive group and with the scalar multiplication given by

$$a \cdot v = \sqrt[p]{a}v \qquad \text{for all} \quad a \in k, \quad v \in L,$$

where the left hand side is the new multiplication and the right hand side the old one. If L is a G-module, then $L^{(1)}$ is also a G-module using the given action of any $g \in G$ on the additive group $L^{(1)} = L$. The new G-module $L^{(1)}$ is called the *Frobenius twist* of L. We define *higher Frobenius twists* inductively: $L^{(d+1)} = \left(L^{(d)}\right)^{(1)}$. To each weight μ of the space L there corresponds the weight $p^d\mu$ of the space $L^{(d)}$.

A weight $r\omega_1 + s\omega_2$ is *restricted* if $0 \leq r, s \leq p-1$. The composition factors of $H^n(\mathfrak{g}, M)$ are Frobenius twists of some simple G-modules with restricted highest weights.

A G-module L is *rational* if the corresponding representation is a homomorphism from G to $\mathrm{GL}(L)$. Suppose V is the Frobenius twist of some rational G-module. Then, there is a unique $d > 0$ and rational G-module L such that $L^{(d)} = V$. Denote this module by $V^{(-d)}$.

We will usually denote $H^n(\mathfrak{g}, k)$ by $H^n(\mathfrak{g})$, and use the following short notation:

$$mV := V \oplus \cdots \oplus V \qquad (m \text{ summands}),$$

where V is a G-module.

1.2 Main result

In this paper, k is always algebraically closed field k of characteristic $p > 3$.

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ and M be a simple $\mathfrak{g}$-module. Then, the following isomorphisms of G-modules hold:*

(a) $H^n(\mathfrak{g}) \cong k$ *for* $n = 0, 3, 5, 8$;

(b) $H^n(\mathfrak{g}, L(p-2, 1)) \cong \begin{cases} L(1,0)^{(1)} & \text{if } n = 1, 7, \\ 2L(1,0)^{(1)} & \text{if } n = 4; \end{cases}$

(c) $H^n(\mathfrak{g}, L(1, p-2)) \cong \begin{cases} L(0,1)^{(1)} & \text{if } n = 1, 7, \\ 2L(0,1)^{(1)} & \text{if } n = 4; \end{cases}$

(d) $H^n(\mathfrak{g}, L(p-3, 0)) \cong L(1,0)^{(1)}$ *for* $n = 2, 3, 5, 6$;

(e) $H^n(\mathfrak{g}, L(0, p-3)) \cong L(0,1)^{(1)}$ *for* $n = 2, 3, 5, 6$;

(f) $H^n(\mathfrak{g}, L(p-2, p-2)) \cong \begin{cases} k & \text{if } n = 1, 7, \\ L(1,1)^{(1)} & \text{if } n = 3, 5, \\ 2L(1,1)^{(1)} \oplus 2k & \text{if } n = 4. \end{cases}$

Otherwise, $H^n(\mathfrak{g}, M) = 0$.

This theorem completes the description of the cohomology of $\mathfrak{sl}_3(k)$ with coefficients in simple modules over an algebraically closed fields of positive characteristics.

1.3 Some applications of the main result

Using Theorem 1.1, one can easily describe the cohomology of $\mathfrak{gl}_3(k)$ with coefficients in simple modules. Let M be an $\mathfrak{sl}_3(k)$-module. Since $\mathfrak{gl}_3(k) \cong \mathfrak{sl}_3(k) \oplus I$, where I is the subspace spanned by the identity 3×3 matrix, a $\mathfrak{gl}_3(k)$-module structure on M can be determined by setting

$$(x, a)m = xm + \mu(a)m \qquad \text{for any} \quad (x, a) \in \mathfrak{gl}_3(k), \quad x \in \mathfrak{sl}_3(k), \quad \text{and} \quad a \in I, \tag{1.1}$$

where μ is a linear form on I. We denote the obtained $\mathfrak{gl}_3(k)$-module also by M. Using Theorem 1.1 and the isomorphism (see [18, p. 737])

$$H^n(\mathfrak{gl}_3(k), M) \cong H^n(\mathfrak{sl}_3(k), M) \oplus H^{n-1}(\mathfrak{sl}_3(k), M)$$

for $\mu = 0$, we obtain for the cohomology of simple $\mathfrak{gl}_3(k)$-modules the following

Corollary 1.2. *Let $\mathfrak{g} = \mathfrak{gl}_3(k)$, let M be a simple $\mathfrak{g}$-module defined by the formula (1.1). If $\mu = 0$, then the following isomorphisms of G-modules hold:*

(a) $H^n(\mathfrak{g}) \cong k$ *for* $n = 0, 1, 3, 4, 5, 6, 8, 9$;

(b) $H^n(\mathfrak{g}, L(p-2, 1)) \cong \begin{cases} L(1,0)^{(1)} & \text{if } n = 1, 2, 7, 8, \\ 2L(1,0)^{(1)} & \text{if } n = 4, 5; \end{cases}$

(c) $H^n(\mathfrak{g}, L(1, p-2)) \cong \begin{cases} L(0,1)^{(1)} & \text{if } n = 1, 2, 7, 8, \\ 2L(0,1)^{(1)} & \text{if } n = 4, 5; \end{cases}$

(d) $H^n(\mathfrak{g}, L(p-3, 0)) \cong \begin{cases} L(1,0)^{(1)} & \text{if } n = 2, 4, 5, 7, \\ 2L(1,0)^{(1)} & \text{if } n = 3, 6; \end{cases}$

(e) $H^n(\mathfrak{g}, L(0, p-3)) \cong \begin{cases} L(0,1)^{(1)} & \text{if } n = 2, 4, 5, 7, \\ 2L(0,1)^{(1)} & \text{if } n = 3, 6; \end{cases}$

(f) $H^n(\mathfrak{g}, L(p-2, p-2)) \cong \begin{cases} k & \text{if } n = 1, 2, 7, 8, \\ L(1,1)^{(1)} & \text{if } n = 3, 6, \\ 3L(1,1)^{(1)} \oplus 2k & \text{if } n = 4, 5. \end{cases}$

Otherwise, $H^n(\mathfrak{g}, M) = 0$.

The results of Corollary 1.2 can be applied to describe the cohomology of the general Lie algebra of Cartan type $W_3(\mathbf{m})$ (for the definition of $W_n(\mathbf{m})$, see [18, Section 1.3]). For example, using in [25, Theorem 0.2] and the statement (a) of Corollary 1.2, one can easily describe the cohomology of the restricted Lie algebra of Cartan type $W_3(\mathbf{1})$ with coefficients in the divided power algebra.

Let $V(\lambda)$ be the Weyl module with highest weight $\lambda = r\omega_1 + s\omega_2$ (for the definition, see Section 2.2) and $H^0(\lambda) = V(-w_0(\lambda))^*$, where w_0 is the longest element of the Weyl group W of the Lie algebra $\mathfrak{g}$. As an G-module, $H^0(\lambda)$ is isomorphic to the induced G-module $\mathrm{Ind}_B^G(k_\lambda)$, where B is the Borel subgroup of G, corresponding to the negative roots, and k_λ is a one-dimensional B-module. A module V over G is *G-acyclic*, if $H^n(G, V) = 0$ for all $n > 0$. The restricted weights λ and μ are *linked* if there is $w \in W$ such that

$$\lambda + \rho \equiv w(\mu + \rho) \mod pX(T),$$

where ρ is the half-sum of positive roots and $X(T)$ is the additive character group of the maximal torus T of G. We say that two G-modules with highest weights are linked if their highest weights are linked. As is well-known, $H^0(\lambda)$ and $V(\lambda)$ are G-acyclic, see [8, Corollary 3.4]. But, from the proof of Theorem 1.1, we will see that the $\mathfrak{g}$-modules $H^0(\lambda)$ and $V(\lambda)$, linked with simple peculiar modules, are peculiar for $\mathfrak{g}$. For the cohomology of these modules, the following result occurs:

Corollary 1.3. *Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ and $V = H^0(\lambda)$. Then, the following isomorphisms of G-modules hold:*

(a) $H^n(\mathfrak{g}, H^0(0,0)) \cong k$ *for* $n = 0, 3, 5, 8$;

(b) $H^n(\mathfrak{g}, H^0(p-2,1)) \cong \begin{cases} L(1,0)^{(1)} & \text{if } n = 1,2,3,5,6,7, \\ 2L(1,0)^{(1)} & \text{if } n = 4; \end{cases}$

(c) $H^n(\mathfrak{g}, H^0(1,p-2)) \cong \begin{cases} L(0,1)^{(1)} & \text{if } n = 1,2,3,5,6,7, \\ 2L(0,1)^{(1)} & \text{if } n = 4; \end{cases}$

(d) $H^n(\mathfrak{g}, H^0(p-3,0)) \cong L(1,0)^{(1)}$ *for* $n = 2, 3, 5, 6$;

(e) $H^n(\mathfrak{g}, H^0(0,p-3)) \cong L(0,1)^{(1)}$ *for* $n = 2, 3, 5, 6$;

(f) $H^n(\mathfrak{g}, H^0(p-2,p-2)) \cong \begin{cases} L(1,1)^{(1)} & \text{if } n = 3, \\ 2L(1,1)^{(1)} \oplus k & \text{if } n = 4, \\ L(1,1)^{(1)} \oplus k & \text{if } n = 5, \\ k & \text{if } n = 7, 8. \end{cases}$

Otherwise, $H^n(\mathfrak{g}, V) = 0$.

Corollary 1.4. *Let* $\mathfrak{g} = \mathfrak{sl}_3(k)$ *and* $V = V(\lambda)$. *Then, the following isomorphisms of* G*-modules hold:*

(a) $H^n(\mathfrak{g}, V(0,0)) \cong k$ *for* $n = 0, 3, 5, 8$;

(b) $H^n(\mathfrak{g}, V(p-2,1)) \cong \begin{cases} L(1,0)^{(1)} & \text{if } n = 1,2,3,5,6,7, \\ 2L(1,0)^{(1)} & \text{if } n = 4; \end{cases}$

(c) $H^n(\mathfrak{g}, V(1,p-2)) \cong \begin{cases} L(0,1)^{(1)} & \text{if } n = 1,2,3,5,6,7, \\ 2L(0,1)^{(1)} & \text{if } n = 4; \end{cases}$

(d) $H^n(\mathfrak{g}, V(p-3,0)) \cong L(1,0)^{(1)}$ *for* $n = 2, 3, 5, 6$;

(e) $H^n(\mathfrak{g}, V(0,p-3)) \cong L(0,1)^{(1)}$ *for* $n = 2, 3, 5, 6$;

(f) $H^n(\mathfrak{g}, V(p-2,p-2)) \cong \begin{cases} k & \text{if } n = 0, 1, \\ L(1,1)^{(1)} \oplus k & \text{if } n = 3, \\ 2L(1,1)^{(1)} \oplus k & \text{if } n = 4, \\ L(1,1)^{(1)} & \text{if } n = 5. \end{cases}$

Otherwise, $H^n(\mathfrak{g}, V) = 0$.

2 Preliminary facts

2.1 Properties of cohomology

In this section, we give some properties of the cohomology for the Lie algebra $\mathfrak{g}$ that are used to prove the main results. Cohomology

$$H^\bullet(\mathfrak{g}, M) = \bigoplus_{n \geq 0} H^n(\mathfrak{g}, M)$$

can be computed using a complex $\left(\bigwedge^\bullet \mathfrak{g}^* \bigotimes M, d\right)$, see [22, Section I.9.17]. Therefore, we can identify the space of cochains $C^n(\mathfrak{g}, M)$ with the space $\bigwedge^n \mathfrak{g}^* \bigotimes M$ and regard the space

$C^n(\mathfrak{g}, M)$ as the G-module. We decompose the space of cochains $C^n(\mathfrak{g}, M)$ into a direct sum of the eigenspaces with respect to the maximal torus T of the group G:

$$C^n(\mathfrak{g}, M) = \bigoplus_{\mu \in X(T)} C^n_\mu(\mathfrak{g}, M),$$

where $X(T)$ is the additive character group of the torus T. Then,

$$H^n(\mathfrak{g}, M) = \bigoplus_{\mu \in X(T)} H^n_\mu(\mathfrak{g}, M).$$

Denote by $\prod(V)$ the set of weights of the subspace V of the G-module $C^n(\mathfrak{g}, M)$. Since

$$\prod(H^n(\mathfrak{g}, M)) \subseteq pX(T) \bigcap \prod \Big(\bigwedge{}^n \mathfrak{g}^* \bigotimes M \Big),$$

we will consider elements of the subspace $\overline{C}^n(\mathfrak{g}, M)$ of the space $C^n(\mathfrak{g}, M)$ with weights from the set

$$pX(T) \bigcap \prod \Big(\bigwedge{}^n \mathfrak{g}^* \bigotimes M \Big).$$

The corresponding subspaces of cocycles and cohomology are denoted by $\overline{Z}^n(\mathfrak{g}, M)$ and $\overline{H}^n(\mathfrak{g}, M)$, respectively. Note that

$$H^n(\mathfrak{g}, M) = \overline{H}^n(\mathfrak{g}, M).$$

By the definition of $H^n(\mathfrak{g}, M)$,

$$\dim H^n(\mathfrak{g}, M) = \dim Z^n(\mathfrak{g}, M) - \dim B^n(\mathfrak{g}, M),$$

and by the definition of $B^n(\mathfrak{g}, M)$,

$$\dim B^n(\mathfrak{g}, M) = \dim C^{n-1}(\mathfrak{g}, M) - \dim Z^{n-1}(\mathfrak{g}, M).$$

Then, we get

$$\dim H^n(\mathfrak{g}, M) = \dim \overline{Z}^n(\mathfrak{g}, M) + \dim \overline{Z}^{n-1}(\mathfrak{g}, M) - \dim \overline{C}^{n-1}(\mathfrak{g}, M). \tag{2.1}$$

Since $\mathrm{Tr}(\mathrm{ad}_x) = 0$ for all $x \in \mathfrak{g}$, then, according to the main theorem in [15, p. 639], we get the following isomorphism:

$$H^n(\mathfrak{g}, M^*) \cong \big(H^{\dim \mathfrak{g} - n}(\mathfrak{g}, M)\big)^*. \tag{2.2}$$

The weight subspaces are invariant under the coboundary operator. Therefore, the formula (2.1) also holds for weight subspaces:

$$\dim H^n_\mu(\mathfrak{g}, M) = \dim \overline{Z}^n_\mu(\mathfrak{g}, M) + \dim \overline{Z}^{n-1}_\mu(\mathfrak{g}, M) - \dim \overline{C}^{n-1}_\mu(\mathfrak{g}, M). \tag{2.3}$$

2.2 Peculiar modules

In this section, we describe simple peculiar $\mathfrak{sl}_3(k)$-modules. Let $\{e_1, e_2, e_3, h_1, h_2, f_1, f_2, f_3\}$ be the Chevalley basis of $\mathfrak{g}$ with the nonzero brackets

$$\begin{aligned} &[e_i, f_i] = h_i, \qquad [h_i, e_i] = 2e_i, \qquad [h_i, f_i] = -2f_i, \qquad i = 1, 2, 3, \\ &[h_1, e_2] = -e_2, \qquad [h_1, e_3] = e_3, \qquad [h_2, e_1] = -e_1, \qquad [h_2, e_3] = e_3, \end{aligned}$$

$$[h_1, f_2] = f_2, \qquad [h_1, f_3] = -f_3, \qquad [h_2, f_1] = f_1, \qquad [h_2, f_3] = -f_3,$$
$$[e_1, e_2] = e_3, \qquad [e_3, f_1] = -e_2, \qquad [e_3, f_2] = e_1,$$
$$[f_1, f_2] = -f_3, \qquad [e_1, f_3] = -f_2, \qquad [e_2, f_3] = f_1,$$

where $h_3 = h_1 + h_2$. It is known (see [11, p. 145]) that there are six simple peculiar $\mathfrak{g}$-modules:

$$L(0,0), \quad L(p-2,1), \quad L(1,p-2), \quad L(p-3,0), \quad L(0,p-3), \quad L(p-2,p-2).$$

A *linear span* of a set $\{v_1, \dots, v_m\}$ of vectors of a vector space V over k is the smallest linear subspace of V that contains the set $\{v_1, \dots, v_m\}$. Let $\langle v_1, \dots, v_m \rangle_k$ denote the linear span of the set $\{v_1, \dots, v_m\}$ of vectors of the vector space V over k. For a detailed description of the peculiar simple modules, consider the restricted Verma module

$$W(r,s) := \left\langle v_{i,j,t} := \frac{f_3^t f_2^j f_1^i}{t!j!i!} u_{r,s} \,\middle|\, 0 \le i,j,t \le p-1 \right\rangle_k$$

with the following action of $\mathfrak{sl}_3(k)$:

$$e_1 v_{i,j,t} = -(j+1) v_{i,j+1,t-1} + (r-i+1) v_{i-1,j,t},$$
$$e_2 v_{i,j,t} = (s+i-j-t+1) v_{i,j-1,t} + (i+1) v_{i+1,j,t-1},$$
$$e_3 v_{i,j,t} = (r+s-i-j-t+1) v_{i,j,t-1} + (r-i+1) v_{i-1,j-1,t},$$
$$h_1 v_{i,j,t} = (r-2i+j-t) v_{i,j,t}, \qquad h_2 v_{i,j,t} = (s+i-2j-t) v_{i,j,t},$$
$$f_1 v_{i,j,t} = -(t+1) v_{i,j-1,t+1} + (i+1) v_{i+1,j,t},$$
$$f_2 v_{i,j,t} = (j+1) v_{i,j+1,t}, \qquad f_3 v_{i,j,t} = (t+1) v_{i,j,t+1}.$$

The restricted Verma module $W(r,s)$ has a submodule $I(r,s)$ generated by the vectors $v_{r+1,0,0}$ and $v_{0,s+1,0}$. The quotient $V(r,s) = W(r,s)/I(r,s)$ is also restricted; let us call it the *Weyl module*. In the modular case, the term "Weyl module" was first used in [26, p. 321], see also [13, p. 59]. For groups of Lie type over a field of positive characteristic, the term "Weyl module" has also been used for a long time (see, for example, [5, p. 213], [17, p. 262], [20, p. 291]). Obviously, for the Weyl module $V(r,s)$, the following relations hold:

$$v_{r+1,0,0} = 0, \qquad v_{0,s+1,0} = 0.$$

The submodule structure of $V(r,s)$ is well-known, see [4, p. 484], [24, pp. 151 and 157]. The results of these papers say that the quotient of $V(r,s)$ by the maximal submodule is the restricted simple module isomorphic to $L(r,s)$. In particular, for the peculiar simple $\mathfrak{g}$-modules we get

$$L(0,0) = V(0,0), \qquad L(p-3,0) = V(p-3,0), \qquad L(0,p-3) = V(0,p-3),$$
$$L(p-2,1) = V(p-2,1)/L(p-3,0), \qquad L(1,p-2) = V(1,p-2)/L(0,p-3),$$
$$L(p-2,p-2) = V(p-2,p-2)/L(0,0).$$

To describe simple modules, we will use the basis vectors of the corresponding restricted Verma modules. The maximal submodules of these Weyl modules are generated by the highest weight vectors

$$w_{p-3,0} = v_{1,1,0} - 2v_{0,0,1} \qquad \text{for} \quad V(p-2,1),$$
$$w_{0,p-3} = v_{1,1,0} + v_{0,0,1} \qquad \text{for} \quad V(1,p-2),$$
$$w_{0,0} = \sum_{i=0}^{p-2} (p-2-i)i! v_{p-2-i,p-2-i,i} \qquad \text{for} \quad V(p-2,p-2),$$

respectively. Then, for simple non-trivial peculiar modules, we obtain the following descriptions in terms of the basis vectors of the restricted Verma module:

$$
\begin{aligned}
&L(p-3,0) = \langle v_{i,j,t} \mid 0 \le i \le p-3,\ 0 \le j \le i,\ 0 \le t \le p-3-i\rangle_k,\\
&L(0,p-3) = \langle v_{i,j,t} \mid 0 \le i \le j,\ 0 \le j \le p-3,\ 0 \le t \le p-3-j\rangle_k,\\
&L(p-2,1) = \langle v_{i,j,t} \mid 0 \le i \le p-2,\ 0 \le j \le i+1,\ 0 \le t \le p-1-i;\ w_{p-3,0} = 0\rangle_k,\\
&L(1,p-2) = \langle v_{i,j,t} \mid 0 \le i \le j+1,\ 0 \le j \le p-2,\ 0 \le t \le p-1-j;\ w_{0,p-3} = 0\rangle_k,\\
&L(p-2,p-2) = \langle v_{i,j,t} \mid 0 \le i \le p-2,\ 0 \le j \le p-2,\ 0 \le t \le p-2-i-j;\ w_{0,0} = 0\rangle_k.
\end{aligned}
$$

3 Proof of Theorem 1.1

As noted above, there are only six peculiar simple modules. Let us prove the theorem for each peculiar simple module separately.

(a) Since $p > 3$, then the Killing form on $\mathfrak{g}$ is non-degenerate. Then, for the trivial one-dimensional module $M = L(0,0)$, the result obtained earlier for zero characteristic remains true in our case as well. So, we consider only non-trivial peculiar simple modules.

(b) Let $M = L(p-2,1)$.

Lemma 3.1. *Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ and $M = L(p-2,1)$. Then, $H^n(\mathfrak{g}, M) = 0$, except for the following cases:*

(i) $H^1(\mathfrak{g}, L(p-2,1)) \cong H^7(\mathfrak{g}, L(p-2,1)) \cong L(1,0)^{(1)}$,

(ii) $H^4(\mathfrak{g}, L(p-2,1)) \cong 2L(1,0)^{(1)}$.

Proof. Obviously, $H^0(\mathfrak{g}, L(p-2,1)) = 0$. It is also known that $H^1(\mathfrak{g}, L(p-2,1)) \cong L(1,0)^{(1)}$, see [21, p. 301], and $H^2(\mathfrak{g}, L(p-2,1)) = 0$, see [11, Theorem 1.1].

Now we show that $H^3(\mathfrak{g}, L(p-2,1)) = 0$. We get

$$\prod\left(\overline{C}^{\bullet}(\mathfrak{g}, L(p-2,1))\right) = \{p\omega_1, p(-\omega_1+\omega_2), -p\omega_2\}.$$

The subspace $\overline{C}^2(\mathfrak{g}, L(p-2,1))$ is 21-dimensional and its set of weights consists of three elements $p\omega_1$, $p(-\omega_1+\omega_2)$, $-p\omega_2$. We have

$$
\begin{aligned}
\dim \overline{C}^2_{p\omega_1}(\mathfrak{g}, L(p-2,1)) &= \dim \overline{C}^2_{p(-\omega_1+\omega_2)}(\mathfrak{g}, L(p-2,1))\\
&= \dim \overline{C}^2_{-p\omega_2}(\mathfrak{g}, L(p-2,1)) = 7.
\end{aligned}
$$

The subspace $\overline{C}^2_{p\omega_1}(\mathfrak{g}, L(p-2,1))$ is spanned by the 2-cochains

$$
\begin{aligned}
&\psi_1^2 = h_1^* \wedge f_1^* \otimes v_{0,0,0}, && \psi_2^2 = h_2^* \wedge f_1^* \otimes v_{0,0,0}, && \psi_3^2 = e_2^* \wedge f_3^* \otimes v_{0,0,0},\\
&\psi_4^2 = h_1^* \wedge f_3^* \otimes v_{0,1,0}, && \psi_5^2 = h_2^* \wedge f_3^* \otimes v_{0,1,0}, &&\\
&\psi_6^2 = f_1^* \wedge f_2^* \otimes v_{0,1,0}, && \psi_7^2 = f_1^* \wedge f_3^* \otimes v_{0,0,1}. &&
\end{aligned}
$$

Let $\sum_{i=1}^{7} a_i\psi_i^2 \in Z^2(\mathfrak{g}, L(p-2,1))$, where $a_i \in k$ for all i. Then, by the cocycle condition,

$$a_1 = a_2 = a_4 = a_5 = 0, \qquad a_3 = a_6 = a_7.$$

This means that $\dim Z^2_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 1$.

The subspace $\overline{C}^3_{p\omega_1}(\mathfrak{g}, L(p-2,1))$ is spanned by the 3-cochains

$$\psi_1^3 = h_1^* \wedge h_2^* \wedge f_1^* \otimes v_{0,0,0}, \qquad \psi_2^3 = h_1^* \wedge e_2^* \wedge f_3^* \otimes v_{0,0,0},$$

$$\psi_3^3 = h_2^* \wedge e_2^* \wedge f_3^* \otimes v_{0,0,0}, \qquad \psi_4^3 = e_2^* \wedge f_1^* \wedge f_1^* \otimes v_{0,0,0},$$
$$\psi_5^3 = e_3^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,0}, \qquad \psi_6^3 = e_2^* \wedge f_1^* \wedge f_3^* \otimes v_{1,0,0},$$
$$\psi_7^3 = e_1^* \wedge f_1^* \wedge f_3^* \otimes v_{0,1,0}, \qquad \psi_8^3 = h_1^* \wedge h_2^* \wedge f_3^* \otimes v_{0,1,0},$$
$$\psi_9^3 = e_2^* \wedge f_2^* \wedge f_3^* \otimes v_{0,1,0}, \qquad \psi_{10}^3 = h_2^* \wedge f_1^* \wedge f_2^* \otimes v_{0,1,0},$$
$$\psi_{11}^3 = h_1^* \wedge f_1^* \wedge f_2^* \otimes v_{0,1,0}, \qquad \psi_{12}^3 = h_1^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,1},$$
$$\psi_{13}^3 = h_2^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,1}, \qquad \psi_{14}^3 = f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,1,1}.$$

So, $\dim \overline{C}^3_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 14$. Suppose $\sum_{i=1}^{14} b_i\psi_i^3 \in Z^3(\mathfrak{g}, L(p-2,1))$, where $b_i \in k$ for all i. Then, using the cocycle condition, we get

$$b_1 = b_8 = 0, \qquad b_2 + b_4 - b_5 - 2b_6 - b_7 = 0, \qquad -b_2 - b_3 + b_5 + b_7 + b_9 - 2b_{14} = 0,$$
$$b_{10} - b_3 = 0, \qquad b_{11} - b_2 = 0, \qquad b_{12} - b_2 = 0, \qquad b_{13} - b_3 = 0.$$

Consider these equalities as a system of equations for b_i, where $i = 1, \dots, 14$. The rank of the matrix of this system is equal to 8. Therefore, $\dim Z^3_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 14 - 8 = 6$. Then, by (2.3),

$$\begin{aligned}\dim H^3_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = {} & \dim Z^3_{p\omega_1}(\mathfrak{g}, L(p-2,1)) + \dim Z^2_{p\omega_1}(\mathfrak{g}, L(p-2,1)) \\ & - \dim C^2_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 6 + 1 - 7 = 0.\end{aligned}$$

Thus, all 3-cocycles with dominant highest weight $p\omega_1$ are coboundaries. Therefore,

$$H^3(\mathfrak{g}, L(p-2,1)) = 0.$$

Now we will calculate $H^4(\mathfrak{g}, L(p-2,1))$. The subspace $\overline{C}^4_{p\omega_1}(\mathfrak{g}, L(p-2,1))$ is 18-dimensional and is spanned by the 4-cochains

$$\psi_1^4 = h_1^* \wedge h_2^* \wedge e_2^* \wedge f_3^* \otimes v_{0,0,0}, \qquad \psi_2^4 = h_1^* \wedge e_2^* \wedge f_1^* \wedge f_2^* \otimes v_{0,0,0},$$
$$\psi_3^4 = h_2^* \wedge e_2^* \wedge f_1^* \wedge f_2^* \otimes v_{0,0,0}, \qquad \psi_4^4 = h_1^* \wedge e_3^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,0},$$
$$\psi_5^4 = h_2^* \wedge e_3^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,0}, \qquad \psi_6^4 = e_1^* \wedge e_2^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,0},$$
$$\psi_7^4 = h_1^* \wedge e_2^* \wedge f_1^* \wedge f_3^* \otimes v_{1,0,0}, \qquad \psi_8^4 = h_2^* \wedge e_2^* \wedge f_1^* \wedge f_3^* \otimes v_{1,0,0},$$
$$\psi_9^4 = h_1^* \wedge h_2^* \wedge f_1^* \wedge f_2^* \otimes v_{1,0,0}, \qquad \psi_{10}^4 = h_1^* \wedge e_1^* \wedge f_1^* \wedge f_3^* \otimes v_{0,1,0},$$
$$\psi_{11}^4 = h_2^* \wedge e_1^* \wedge f_1^* \wedge f_3^* \otimes v_{0,1,0}, \qquad \psi_{12}^4 = h_1^* \wedge e_2^* \wedge f_2^* \wedge f_3^* \otimes v_{0,1,0},$$
$$\psi_{13}^4 = h_2^* \wedge e_2^* \wedge f_2^* \wedge f_3^* \otimes v_{0,1,0}, \qquad \psi_{14}^4 = e_3^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,1,0},$$
$$\psi_{15}^4 = h_1^* \wedge h_2^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,1}, \qquad \psi_{16}^4 = e_2^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,0,1},$$
$$\psi_{17}^4 = h_1^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,1,1}, \qquad \psi_{18}^4 = h_2^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,1,1}.$$

Suppose $\sum_{i=1}^{18} c_i\psi_i^4 \in Z^4(\mathfrak{g}, L(p-2,1))$, where $c_i \in k$ for all i. Then, using the cocycle condition, we get

$$c_1 = c_9 = c_{15}, \qquad c_1 - c_3 + c_5 + 2c_8 + c_{11} = 0, \qquad c_2 + c_3 + c_5 - c_6 - c_{14} - c_{16} = 0,$$
$$c_2 - 2c_7 + c_{12} - c_{15} - 2c_{17} = 0, \qquad c_3 - 2c_8 + c_{13} - 2c_{18} = 0,$$
$$c_4 - c_2 + 2c_7 + c_{10} = 0, \qquad c_4 - c_9 + c_{10} + c_{12} - 2c_{17} = 0,$$
$$c_5 + c_9 + c_{11} + c_{13} - 2c_{18} = 0, \qquad c_6 - c_{11} - c_{12} + c_{14} + c_{16} = 0.$$

The rank of the matrix of this system is equal to 8. Therefore,

$$\dim Z^4_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 18 - 8 = 10.$$

Then, by (2.3),

$$\begin{aligned}\dim H^4_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = {} & \dim Z^4_{p\omega_1}(\mathfrak{g}, L(p-2,1)) + \dim Z^3_{p\omega_1}(\mathfrak{g}, L(p-2,1)) \\ & - \dim C^3_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 10+6-14 = 2.\end{aligned}$$

Thus, $H^4(\mathfrak{g}, L(p-2,1))$ is generated by the two cohomological classes with weight $p\omega_1$. So, $H^4(\mathfrak{g}, L(p-2,1)) \cong 2L(1,0)^{(1)}$.

Similar calculations give us

$$\begin{aligned}&\dim C^5_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 14, && \dim Z^5_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 8,\\ &\dim C^6_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 7, && \dim Z^6_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 6,\\ &\dim C^7_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 2, && \dim Z^7_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 2.\end{aligned}$$

Then, using (2.3), we get

$$\begin{aligned}&\dim H^5_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 0, && \dim H^6_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 0,\\ &\dim H^7_{p\omega_1}(\mathfrak{g}, L(p-2,1)) = 1.\end{aligned}$$

This completes the proof of the lemma. ∎

(c) Let $M = L(1,p-2)$. Obviously, M is dual to $L(p-2,1)$. Then, using (2.2) and Lemma 3.1, we get the following

Lemma 3.2. *Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ and $M = L(1,p-2)$. Then, $H^n(\mathfrak{g}, M) = 0$, except for the following cases:*

(i) $H^1(\mathfrak{g}, L(1,p-2)) \cong H^7(\mathfrak{g}, L(1,p-2)) \cong L(0,1)^{(1)}$,

(ii) $H^4(\mathfrak{g}, L(1,p-2)) \cong 2L(0,1)^{(1)}$.

(d) Let $M = L(p-3,0)$.

Lemma 3.3. *Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ and $M = L(p-3,0)$. Then, $H^n(\mathfrak{g}, M) = 0$, except for the following cases:*

(i) $H^2(\mathfrak{g}, L(p-3,0)) \cong H^6(\mathfrak{g}, L(p-3,0)) \cong L(1,0)^{(1)}$,

(ii) $H^3(\mathfrak{g}, L(p-3,0)) \cong H^5(\mathfrak{g}, L(p-3,0)) \cong L(1,0)^{(1)}$.

Proof. It is easy to see that

$$\prod\left(\overline{C}^{\bullet}(\mathfrak{g}, L(p-3,0))\right) = \{p\omega_1, p(-\omega_1+\omega_2), -p\omega_2\}.$$

Then, it is obvious that

$$\prod\left(\overline{C}^{i}(\mathfrak{g}, L(p-3,0))\right) \bigcap \prod\left(\overline{C}^{*}(\mathfrak{g}, L(p-3,0))\right) = \varnothing \qquad \text{for} \quad i = 1,2.$$

Therefore,

$$H^0(\mathfrak{g}, L(p-3,0)) = 0 \qquad \text{and} \qquad H^1(\mathfrak{g}, L(p-3,0)) = 0.$$

Further, we get

$$\prod\left(\overline{C}^{2}(\mathfrak{g}, L(p-3,0))\right) = \{p\omega_1, p(-\omega_1+\omega_2), -p\omega_2\}.$$

Any composition factor of $H^2(\mathfrak{g}, L(p-3,0))$, as a G-module, is uniquely determined by its highest weight. The highest weight of a simple G-module is dominant, see [17, p. 260]. Recall that the weight $\lambda = r\omega_1 + s\omega_2$ is dominant if $r \geq 0$ and $s \geq 0$. Then, $H^2(\mathfrak{g}, L(p-3,0))$ can be generated only by the classes of cocycles with dominant weight $p\omega_1$. Therefore, it is sufficient to determine the multiplicity of $p\omega_1$. Consider the subspace of 2-cochains with dominant weight $p\omega_1$. The subspace $\overline{C}^2_{p\omega_1}(\mathfrak{g}, L(p-3,0))$ is one-dimensional and is spanned by the 2-cochain $\psi^2 = f_1^* \wedge f_3^* \otimes v_{0,0,0}$. It is easy to see that ψ^2 is a 2-cocycle. Since $\overline{C}^1_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 0$, it follows that ψ^2 cannot be a coboundary. Therefore, $H^2(\mathfrak{g}, L(p-3,0))$, as a G-module, is generated by the class $\left[\psi^2\right]$ of 2-cocycles with weight $p\omega_1$ and is isomorphic to $L(1,0)^{(1)}$.

The set of weights of the subspace $\prod\left(\overline{C}^3(\mathfrak{g}, L(p-3,0))\right)$ is also equal to $\{p\omega_1, p(-\omega_1+\omega_2), -p\omega_2\}$. The subspace $\overline{C}^3_{p\omega_1}(\mathfrak{g}, L(p-3,0))$ is two-dimensional and is spanned by the 3-cochains

$$\psi_1^3 = h_1^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,0}, \qquad \psi_2^3 = h_2^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,0}.$$

If $a_1\psi_1^3 + a_2\psi_2^3$ is a 3-cocycle, then it follows from the cocycle condition that $a_2 = 0$. Since

$$\dim \overline{C}^2_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = \dim \overline{Z}^2_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 1,$$

by (2.3), we see that $\dim H^3_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 1+1-1 = 1$. Therefore, $H^3(\mathfrak{g}, L(p-3,0))$, as a G-module, is generated by the class $\left[\psi_1^3\right]$ of 3-cocycles with weight $p\omega_1$ and is isomorphic to $L(1,0)^{(1)}$.

Now, we will prove that $H^4(\mathfrak{g}, L(p-3,0)) = 0$. The weight subspace $\overline{C}^4_{p\omega_1}(\mathfrak{g}, L(p-3,0))$ is two-dimensional and is spanned by the 4-cochains

$$\psi_1^4 = h_1^* \wedge h_2^* \wedge f_1^* \wedge f_3^* \otimes v_{0,0,0}, \qquad \psi_2^4 = e_2^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,0,0}.$$

If $b_1\psi_1^4 + b_2\psi_2^4$ is a 4-cocycle, then it follows from the cocycle condition that $b_1 = 0$. Since

$$\dim \overline{C}^3_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 2 \qquad \text{and} \qquad \dim \overline{Z}^3_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 1,$$

by (2.3), it follows that

$$\dim H^4_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 1+1-2 = 0.$$

Therefore,

$$\dim H^4(\mathfrak{g}, L(p-3,0)) = \dim H^4_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 0.$$

The weight subspace $\overline{C}^5_{p\omega_1}(\mathfrak{g}, L(p-3,0))$ is two-dimensional and is spanned by the 5-cochains

$$\psi_1^5 = h_1^* \wedge e_2^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,0,0}, \qquad \psi_2^5 = h_2^* \wedge e_2^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,0,0}.$$

It follows from the cocycle condition that $c_1\psi_1^5 + c_2\psi_2^5$ is a 5-cocycle for any $c_1, c_2 \in k$. So, $\dim \overline{Z}^5_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 2$. Since

$$\dim \overline{C}^4_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 2 \qquad \text{and} \qquad \dim \overline{Z}^4_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 1,$$

then by (2.3), we see that

$$\dim H^5_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 2+1-2 = 1.$$

Therefore, $H^5(\mathfrak{g}, L(p-3,0))$, as a G-module, is generated by the class $\left[\psi_1^5\right]$ of 5-cocycles with weight $p\omega_1$ and is isomorphic to $L(1,0)^{(1)}$.

The weight subspace $\overline{C}^6_{p\omega_1}(\mathfrak{g}, L(p-3,0))$ is one-dimensional and is spanned by the 6-cochain

$$\psi^6 = h_1^* \wedge h_2^* \wedge e_2^* \wedge f_1^* \wedge f_2^* \wedge f_3^* \otimes v_{0,0,0}.$$

It follows from the cocycle condition that ψ^6 is a 6-cocycle. So, $\dim \overline{Z}^6_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 1$. Since

$$\dim \overline{C}^5_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 2 \qquad \text{and} \qquad \dim \overline{Z}^5_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 2,$$

then by (2.3),

$$\dim H^6_{p\omega_1}(\mathfrak{g}, L(p-3,0)) = 1.$$

Therefore, $H^6(\mathfrak{g}, L(p-3,0))$, as a G-module, is generated by the class $[\psi^6]$ of 6-cocycles with weight $p\omega_1$ and is isomorphic to $L(1,0)^{(1)}$.

Finally, the subspaces $\overline{C}^7(\mathfrak{g}, L(p-3,0))$ and $\overline{C}^8(\mathfrak{g}, L(p-3,0))$ are trivial, therefore,

$$H^7(\mathfrak{g}, L(p-3,0)) = 0 \qquad \text{and} \qquad H^8(\mathfrak{g}, L(p-3,0)) = 0.$$

(e) Let $M = L(0,p-3)$. Obviously, M is dual to $L(p-3,0)$. Then, using (2.2) and Lemma 3.3, we get the following

Lemma 3.4. *Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ and $M = L(0,p-3)$. Then, $H^n(\mathfrak{g}, M) = 0$, except for the following cases:*

(i) $H^2(\mathfrak{g}, L(0,p-3)) \cong H^6(\mathfrak{g}, L(0,p-3)) \cong L(0,1)^{(1)}$,

(ii) $H^3(\mathfrak{g}, L(0,p-3)) \cong H^5(\mathfrak{g}, L(0,p-3)) \cong L(0,1)^{(1)}$.

(f) Finally, let $M = L(p-2,p-2)$. In this case, we will use some properties of the connection between ordinary and restricted cohomologies. The restricted cohomolology of a restricted Lie algebra with coefficients in a restricted module was introduced by Hochschild in [16, p. 561]. The restricted n-cohomology of $\mathfrak{g}$ with coefficients in a restricted $\mathfrak{g}$-module V is denoted by $H^n_{\text{res}}(\mathfrak{g}, V)$.

For $M = L(p-2,p-2)$, there is the following short exact sequence of $\mathfrak{g}$-modules:

$$0 \longrightarrow M \longrightarrow H^0(p-2,p-2) \longrightarrow k \longrightarrow 0. \tag{3.1}$$

If the cohomology of $H^0(p-2,p-2)$ is known, then using the long exact cohomology sequence

$$\cdots \longrightarrow H^n(\mathfrak{g}, M) \longrightarrow H^n\big(\mathfrak{g}, H^0(p-2,p-2)\big) \longrightarrow H^n(\mathfrak{g}) \longrightarrow \cdots, \tag{3.2}$$

corresponding to the short exact sequence (3.1), we can obtain information about $H^n(\mathfrak{g}, M)$. We calculate $H^n\big(\mathfrak{g}, H^0(p-2,p-2)\big)$ in two steps.

First, we calculate the restricted cohomology $H^n_{\text{res}}\big(\mathfrak{g}, H^0(p-2,p-2)\big)$, using the equivalence of the cohomologies $H^n_{\text{res}}\big(\mathfrak{g}, H^0(p-2,p-2)\big)$ and $H^n\big(G_1, H^0(p-2,p-2)\big)$, where G_1 is the first Frobenius kernel for G, see [22, Section I.9.6], and Andersen–Jantzen formula on cohomology of G_1 with coefficients in $H^0(\lambda)$, see [1]. Let $p > 3$, and $\lambda = w \cdot 0 + p\nu$. Then, see [1, p. 501],

$$H^i\big(G_1, H^0(\lambda)\big)^{(-1)} \cong \begin{cases} \operatorname{Ind}_B^G \left(S^{(i-l(w))/2}(\mathfrak{u}^*) \otimes k_\nu\right) & \text{if } i - l(w) \text{ is even}, \\ 0 & \text{if } i - l(w) \text{ is odd}, \end{cases} \tag{3.3}$$

where $\mathfrak{u}$ is the maximal nilpotent subalgebra of $\mathfrak{g}$, corresponding to the negative roots. The Lie algebra $\mathfrak{u}$ is the Lie algebra of the unipotent radical U of B.

Then, to pass to the usual cohomology $H^n\big(\mathfrak{g}, H^0(p-2,p-2)\big)$, we use the Friedlander–Parshall–Farnsteiner spectral sequence, see [14, Section 5] and [12, Theorem 4.1]. In [12, Theorem 4.1], choosing the zero ideal as an ideal of a given Lie algebra, we obtain the following spectral sequence for the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module V:

$$\bigoplus_{i+j=n} \mathrm{Hom}_k\left(\bigwedge{}^i(\mathfrak{g}), H^j_{\mathrm{res}}(\mathfrak{g},V)\right) \Longrightarrow H^n(\mathfrak{g},V).$$

In particular, the following lemma can be directly obtained from the last spectral sequence:

Lemma 3.5. *Let $\mathfrak{g}=\mathfrak{sl}_3(k)$ $p>3$ and V a $\mathfrak{g}$-module. Then*

(i) if $H^i_{\mathrm{res}}(\mathfrak{g},V)=0$ for all $i\le n$, then $H^i(\mathfrak{g},V)=0$ for all $i\le n$,

(ii) if $H^i(\mathfrak{g},V)=0$ for all $i\le n-2$, then

$$H^{n-1}(\mathfrak{g},V)\cong H^{n-1}_{\mathrm{res}}(\mathfrak{g},V) \tag{3.4}$$

and the following sequence is exact:

$$\begin{aligned}0 &\longrightarrow H^n_{\mathrm{res}}(\mathfrak{g},V)\longrightarrow H^n(\mathfrak{g},V)\longrightarrow \mathrm{Hom}_k\big(\mathfrak{g}, H^{n-1}_{\mathrm{res}}(\mathfrak{g},V)\big)\longrightarrow H^{n+1}_{\mathrm{res}}(\mathfrak{g},V)\\ &\longrightarrow H^{n+1}(\mathfrak{g},V).\end{aligned} \tag{3.5}$$

We start by calculating the cohomology $H^n\big(G_1, H^0(p-2,p-2)\big)$ with $n\le \dim\mathfrak{g}$.

Lemma 3.6. *Let G_1 be the first Frobenius kernel of G, and $V=H^0(p-2,p-2)$ the G_1-module. Then,*

(i) $H^i(G_1,V)=0$ for $i=0,1,2,4,6,8$,

(ii) $H^3(G_1,V)\cong L(1,1)^{(1)}$,

(iii) $H^5(G_1,V)\cong L(3,0)^{(1)}\oplus L(0,3)^{(1)}\oplus L(2,2)^{(1)}$.

Proof. (i) Since

$$\lambda=(p-2)(\omega_1+\omega_2)=s_1s_2s_1\cdot 0+p(\omega_1+\omega_2),$$

we get

$$w=s_1s_2s_1,\qquad l(w)=3,\qquad \text{and}\qquad \nu=\omega_1+\omega_2.$$

Then, by (3.3), $H^i(G_1,V)=0$ for $i=0,1,2,4,6,8$. The statement (i) is proved.

(ii) We have

$$S^{(3-l(w))/2}(\mathfrak{u}^*)\otimes k_\nu=S^0(\mathfrak{u}^*)\otimes k_\nu\cong k_\nu=k_{\omega_1+\omega_2}$$

and

$$H^0\big(S^{(3-l(w))/2}(\mathfrak{u}^*)\otimes k_\nu\big)\cong H^0(k_{\omega_1+\omega_2})=\mathrm{Ind}_B^G(k_{\omega_1+\omega_2})\cong L(1,1).$$

Then, by (3.3),

$$H^3\big(G_1,H^0(p-2,p-2)\big)^{(-1)}\cong L(1,1).$$

(iii) We have

$$S^{(5-l(w))/2}(\mathfrak{u})^*\otimes k_\nu\cong(k_{\alpha_1}\oplus k_{\alpha_2}\oplus k_{\alpha_1+\alpha_2})\otimes k_{\omega_1+\omega_2}\cong k_{3\omega_1}\oplus k_{3\omega_2}\oplus k_{2\omega_1+2\omega_2}$$

and

$$H^0\big(S^{(5-l(w))/2}(\mathfrak{u}^*)\otimes k_\nu\big)=\mathrm{Ind}_B^G(k_{3\omega_1}\oplus k_{3\omega_2}\oplus k_{2\omega_1+2\omega_2})\cong L(3,0)\oplus L(0,3)\oplus L(2,2).$$

Then, by (3.3),

$$H^5\big(G_1,H^0(p-2,p-2)\big)^{(-1)}\cong L(3,0)\oplus L(0,3)\oplus L(2,2).$$

Now, we calculate the cohomology $H^n\big(\mathfrak{g}, H^0(p-2,p-2)\big)$.

Lemma 3.7. *Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ and $V = H^0(p-2,p-2)$. Then,*

(i) $H^i(\mathfrak{g}, V) = 0$ *for* $i = 0, 1, 2$,

(ii) $H^3(\mathfrak{g}, V) \cong L(1,1)^{(1)}$,

(iii) $H^4(\mathfrak{g}, V) \cong 2L(1,1)^{(1)} \oplus k$.

Proof. (i) Follows from the statements (i) of Lemmas 3.5 and 3.6. (ii) Follows from the statements (ii) of Lemma 3.6 and formula (3.4). (iii) We get

$$\begin{aligned}\mathrm{Hom}_k\left(\mathfrak{g}, H^3_{\mathrm{res}}\big(\mathfrak{g}, H^0(p-2,p-2)\big)^{(-1)}\right) &\cong \mathfrak{g}^* \otimes H^3\big(G_1, H^0(p-2,p-2)\big)^{(-1)}\\ &\cong L(1,1)\otimes L(1,1)\\ &\cong L(3,0)\oplus L(0,3)\oplus L(2,2)\oplus 2L(1,1)\oplus k.\end{aligned}$$

Then, by (3.5) and the statement (iii) of Lemma 3.6,

$$H^4\big(\mathfrak{g}, H^0(p-2,p-2)\big)^{(-1)} \cong 2L(1,1)\oplus k.$$

For $H^n(\mathfrak{g}, M)$, where $M = L(p-2,p-2)$, we obtain the following result:

Lemma 3.8. *Let $\mathfrak{g} = \mathfrak{sl}_3(k)$ and $M = L(p-2,p-2)$. Then, $H^n(\mathfrak{g}, M) = 0$, except for the following cases:*

(i) $H^1(\mathfrak{g}, M) \cong H^7(\mathfrak{g}, M) \cong k$,

(ii) $H^3(\mathfrak{g}, M) \cong H^5(\mathfrak{g}, M) \cong L(1,1)^{(1)}$,

(iii) $H^4(\mathfrak{g}, M) \cong 2L(1,1)^{(1)} \oplus 2k$.

Proof. Obviously, $H^0\big(\mathfrak{g}, H^0(p-2,p-2)\big) = H^8\big(\mathfrak{g}, H^0(p-2,p-2)\big) = 0$.

(i) The initial terms of the exact sequence (3.2) give us the following exact sequence:

$$0 \longrightarrow H^0(\mathfrak{g}) \longrightarrow H^1(\mathfrak{g}, L(p-2,p-2)) \longrightarrow H^1\big(\mathfrak{g}, H^0(p-2,p-2)\big).$$

By Lemma 3.7,

$$H^1(\mathfrak{g}, L(p-2,p-2)) \cong H^0(\mathfrak{g}) \cong k.$$

Since $L(p-2,p-2)$ is a self-dual module, then by (2.2), we get

$$H^7(\mathfrak{g}, L(p-2,p-2)) \cong H^1(\mathfrak{g}, L(p-2,p-2))^* \cong k.$$

Let us prove that $H^2(\mathfrak{g}, L(p-2,p-2)) = 0$. Obviously, $H^1(\mathfrak{g}) = 0$, and by Lemma 3.7, $H^2\big(\mathfrak{g}, H^0(p-2,p-2)\big) = 0$. Then, it follows from the exactness of the sequence (3.2) that $H^2(\mathfrak{g}, L(p-2,p-2)) = 0$. By (2.2),

$$H^6(\mathfrak{g}, L(p-2,p-2)) \cong H^2(\mathfrak{g}, L(p-2,p-2))^* = 0.$$

(ii) Since $H^2(\mathfrak{g}) = 0$, it follows from the exactness of the sequence (3.2) that the sequence

$$0 \longrightarrow H^3(\mathfrak{g}, L(p-2,p-2)) \longrightarrow H^3\big(\mathfrak{g}, H^0(p-2,p-2)\big) \longrightarrow H^3(\mathfrak{g}). \tag{3.6}$$

is exact. It is known that $H^3(\mathfrak{g}) \cong k$, see [7, p. 113]. Moreover, by Lemma 3.7,

$$H^3\big(\mathfrak{g}, H^0(p-2,p-2)\big) \cong L(1,1)^{(1)}.$$

Then, the exactness of the sequence (3.6) implies that

$$H^3(\mathfrak{g}, L(p-2,p-2)) \cong H^3\big(\mathfrak{g}, H^0(p-2,p-2)\big) \cong L(1,1)^{(1)},$$

since there is no G-homomorphism between the modules $L(1,1)^{(1)}$ and k. By (2.2),

$$H^5(\mathfrak{g}, L(p-2,p-2)) \cong H^3(\mathfrak{g}, L(p-2,p-2))^* \cong L(1,1)^{(1)}.$$

(iii) In the previous statement, we proved that

$$H^3(\mathfrak{g}, L(p-2,p-2)) \cong H^3\big(\mathfrak{g}, H^0(p-2,p-2)\big).$$

Then, since the sequence (3.2) is exact, the following sequence is exact:

$$0 \longrightarrow H^3(\mathfrak{g}) \longrightarrow H^4(\mathfrak{g}, L(p-2,p-2)) \longrightarrow H^4\big(\mathfrak{g}, H^0(p-2,p-2)\big) \longrightarrow H^4(\mathfrak{g}).$$

Since $H^3(\mathfrak{g}) \cong k$ and $H^4(\mathfrak{g}) = 0$, then the sequence

$$0 \longrightarrow k \longrightarrow H^4(\mathfrak{g}, L(p-2,p-2)) \longrightarrow H^4\big(\mathfrak{g}, H^0(p-2,p-2)\big) \longrightarrow 0$$

is exact. By the statement (iii) of Lemma 3.7,

$$H^4\big(\mathfrak{g}, H^0(p-2,p-2)\big) \cong 2L(1,1)^{(1)} \oplus k.$$

There is no G-homomorphism between the modules $L(1,1)^{(1)}$ and k, so the last exact sequence is split. Then, we get an isomorphism of G-modules of the statement (iii). ■

Theorem 1.1 follows from Lemmas 3.1–3.4 and 3.8.

4 Cohomology with coefficients in Weyl modules

In this section, we prove Corollaries 1.3 and 1.4. Let us start with Corollary 1.3. We use the following *linkage principle* (see [17, p. 264]): *Let V be indecomposable G-module, having $L(\lambda)$ and $L(\mu)$ as composition factors. Then λ and μ are linked.*

Obviously, $H^0(\lambda)$ is peculiar, if it contains a peculiar composition factor. According to the linkage principle, any composition factor of $H^0(\lambda)$ is linked to $L(\lambda)$. By Theorem 1.1, there are only six peculiar simple modules. Therefore, $H^0(\lambda)$ is peculiar only in the following cases, which appear in Theorem 1.1:

$$\lambda = 0, \quad (p-2)\omega_1 + \omega_2, \quad \omega_1 + (p-2)\omega_2, \quad (p-3)\omega_1, \quad (p-3)\omega_2, \quad (p-2)(\omega_1+\omega_2).$$

Let us consider each of these cases separately.

(a) Obviously, $H^0(0,0) \cong k$. Then, the needed statement follows from the statement (a) of Theorem 1.1.

Further, we will proceed as in the proof of Lemma 3.8.

(b) There is the short exact sequence

$$0 \longrightarrow L(p-2,1) \longrightarrow H^0(p-2,1) \longrightarrow L(p-3,0) \longrightarrow 0.$$

Consider the corresponding long cohomological exact sequence

$$\cdots \longrightarrow H^n(\mathfrak{g}, L(p-2,1)) \longrightarrow H^n\big(\mathfrak{g}, H^0(p-2,1)\big) \longrightarrow H^n(\mathfrak{g}, L(p-3,0)) \longrightarrow \cdots.$$

According to Lemmas 3.1, 3.3, the last long cohomological exact sequence splits into the following exact sequences:

$$\begin{aligned}
&0 \longrightarrow H^0\big(\mathfrak{g}, H^0(p-2,1)\big) \longrightarrow 0,\\
&0 \longrightarrow L(1,0)^{(1)} \longrightarrow H^1\big(\mathfrak{g}, H^0(p-2,1)\big) \longrightarrow 0,\\
&0 \longrightarrow H^2\big(\mathfrak{g}, H^2(p-2,1)\big) \longrightarrow L(1,0)^{(1)} \longrightarrow 0,\\
&0 \longrightarrow H^3\big(\mathfrak{g}, H^0(p-2,1)\big) \longrightarrow H^3(\mathfrak{g}, L(p-3,0)) \longrightarrow 2L(1,0)^{(1)}\\
&\quad\longrightarrow H^4\big(\mathfrak{g}, H^0(p-2,1)\big) \longrightarrow 0,\\
&0 \longrightarrow H^5\big(\mathfrak{g}, H^0(p-2,1)\big) \longrightarrow L(1,0)^{(1)} \longrightarrow 0,\\
&0 \longrightarrow H^6(\mathfrak{g}, H^0(p-2,1)) \longrightarrow L(1,0)^{(1)} \longrightarrow H^7(\mathfrak{g}, L(p-2,1))\\
&\quad\longrightarrow H^7\big(\mathfrak{g}, H^0(p-2,1)\big) \longrightarrow 0,\\
&0 \longrightarrow H^8\big(\mathfrak{g}, H^0(p-2,1)\big) \longrightarrow 0.
\end{aligned}$$

The first and last exact sequences yield $H^n\big(\mathfrak{g}, H^0(p-2,1)\big) = 0$ for $n = 0, 8$. The second, third, and fifth exact sequences yield isomorphisms

$$H^1\big(\mathfrak{g}, H^0(p-2,1)\big) \cong L(1,0)^{(1)}, \qquad H^2\big(\mathfrak{g}, H^0(p-2,1)\big) \cong L(1,0)^{(1)},$$

and

$$H^5\big(\mathfrak{g}, H^0(p-2,1)\big) \cong L(1,0)^{(1)},$$

respectively. Consider the fourth exact sequence. Composition factors of $H^3\big(\mathfrak{g}, H^0(p-2,1)\big)$ can only be H^3 with coefficients in either $L(p-2,1)$ or the socle of $H^0(p-2,1)/L(p-2,1)$. According to Lemma 3.1, $H^3(\mathfrak{g}, L(p-2,1)) = 0$. The socle of $H^0(p-2,1)/L(p-2,1)$ is isomorphic to the simple module $L(p-3,0)$. According to Lemma 3.3,

$$H^3(\mathfrak{g}, L(p-3,0)) \cong L(1,0)^{(1)}.$$

Therefore,

$$H^3\big(\mathfrak{g}, H^0(p-2,1)\big) \cong L(1,0)^{(1)}.$$

Then, the fourth exact sequence yields an isomorphism

$$H^4\big(\mathfrak{g}, H^0(p-2,1)\big) \cong 2L(1,0)^{(1)}.$$

According to Lemma 3.1, in the sixth exact sequence, the map

$$H^7(\mathfrak{g}, L(p-2,1)) \longrightarrow H^7\big(\mathfrak{g}, H^0(p-2,1)\big)$$

is an epimorphism. Consequently, there are the isomorphisms

$$H^7(\mathfrak{g}, L(p-2,1)) \cong H^7\big(\mathfrak{g}, H^0(p-2,1)\big) \cong L(1,0)^{(1)}.$$

Then the sixth exact sequence yields an isomorphism

$$H^6\big(\mathfrak{g}, H^0(p-2,1)\big) \cong L(1,0)^{(1)}.$$

(c) The proof is similar to the previous statement.

(d) Since $H^0(p-3,0) \cong L(p-3,0)$, the statement follows from the statement (d) of Theorem 1.1.

(e) Since $H^0(0, p-3) \cong L(0, p-3)$, the statement follows from the statement (e) of Theorem 1.1.

(f) A part of this statement is proved in Lemma 3.7. We will prove only the rest of the statement. Using the statement (a) of Theorem 1.1 and Lemma 3.8, and the long cohomological exact sequence (3.2), we obtain the following exact sequences:

$$\begin{aligned}
&0 \longrightarrow L(1,1)^{(1)} \longrightarrow H^5\big(\mathfrak{g}, H^0(p-2,p-2)\big) \longrightarrow k \longrightarrow 0,\\
&0 \longrightarrow H^6\big(\mathfrak{g}, H^0(p-2,p-2)\big) \longrightarrow 0,\\
&0 \longrightarrow k \longrightarrow H^7\big(\mathfrak{g}, H^0(p-2,p-2)\big) \longrightarrow 0,\\
&0 \longrightarrow H^8\big(\mathfrak{g}, H^0(p-2,p-2)\big) \longrightarrow k \longrightarrow 0.
\end{aligned}$$

These exact sequences yield the isomorphisms

$$\begin{aligned}
&H^5\big(\mathfrak{g}, H^0(p-2,p-2)\big) \cong L(1,1)^{(1)} \oplus k,\\
&H^6\big(\mathfrak{g}, H^0(p-2,p-2)\big) = 0,\\
&H^7\big(\mathfrak{g}, H^0(p-2,p-2)\big) \cong k,\\
&H^8\big(\mathfrak{g}, H^0(p-2,p-2)\big) \cong k.
\end{aligned}$$

Using (2.2) and Corollary 1.3 for $H^n(\mathfrak{g}, V(\lambda))$, we get Corollary 1.4.

Corollaries 1.3 and 1.4 show that $H^n\big(\mathfrak{g}, H^0(\lambda)\big) \cong H^n(\mathfrak{g}, V(\lambda))$, except for the case where

$$\lambda = (p-2)(\omega_1 + \omega_2).$$

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