Modular Ordinary Differential Equations on $\mathrm{SL}(2,\mathbb{Z})$ of Third Order and Applications

Zhijie CHEN [a], *Chang-Shou LIN* [b] and *Yifan YANG* [c]

[a] *Department of Mathematical Sciences, Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China*
E-mail: *zjchen2016@tsinghua.edu.cn*

[b] *Center for Advanced Study in Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan*
E-mail: *cslin@math.ntu.edu.tw*

[c] *Department of Mathematics, National Taiwan University and National Center for Theoretical Sciences, Taipei 10617, Taiwan*
E-mail: *yangyifan@ntu.edu.tw*

Received June 24, 2021, in final form February 13, 2022; Published online February 22, 2022
https://doi.org/10.3842/SIGMA.2022.013

Abstract. In this paper, we study third-order modular ordinary differential equations (MODE for short) of the following form $y''' + Q_2(z)y' + Q_3(z)y = 0$, $z \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, where $Q_2(z)$ and $Q_3(z) - \frac{1}{2}Q_2'(z)$ are meromorphic modular forms on $\mathrm{SL}(2,\mathbb{Z})$ of weight 4 and 6, respectively. We show that any quasimodular form of depth 2 on $\mathrm{SL}(2,\mathbb{Z})$ leads to such a MODE. Conversely, we introduce the so-called Bol representation $\hat{\rho}\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(3,\mathbb{C})$ for this MODE and give the necessary and sufficient condition for the irreducibility (resp. reducibility) of the representation. We show that the irreducibility yields the quasimodularity of some solution of this MODE, while the reducibility yields the modularity of all solutions and leads to solutions of certain $\mathrm{SU}(3)$ Toda systems. Note that the $\mathrm{SU}(N+1)$ Toda systems are the classical Plücker infinitesimal formulas for holomorphic maps from a Riemann surface to $\mathbb{CP}^N$.

Key words: modular differential equations; quasimodular forms; Toda system

2020 Mathematics Subject Classification: 11F11; 34M03

1 Introduction

Let $Ly = 0$ be a Fuchsian ordinary differential equation of third order defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$:

$$Ly := y''' + Q_2(z)y' + Q_3(z)y = 0, \qquad z \in \mathbb{H}, \tag{1.1}$$

where $' := \frac{\mathrm{d}}{\mathrm{d}z}$. Near a regular point z_0 of $Ly = 0$, a local solution $y(z)$ can be obtained by giving the initial values $y^{(k)}(z_0)$, $k = 0, 1, 2$, and then $y(z)$ could be globally defined through analytic continuation. However, globally $y(z)$ might be multi-valued. If $Ly = 0$ is defined on $\mathbb{C}$, then the monodromy representation from $\pi_1(\mathbb{C} \setminus \{\text{singular points}\})$ to $\mathrm{SL}(3,\mathbb{C})$ is introduced to characterize the multi-valueness of solutions. If the potentials $Q_2(z)$ and $Q_3(z)$ are elliptic functions with periods 1 and τ ($\operatorname{Im}\tau > 0$) and any solution of $Ly = 0$ is single-valued and meromorphic, then the monodromy representation reduces to a homomorphism from $\pi_1(E_\tau)$ to $\mathrm{SL}(3,\mathbb{C})$, where $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is the elliptic curve. The well-known examples are the integral Lamé equations and its generalizations; see, e.g., [4, 5, 6] for some recent developments of this subject. Note that $\pi_1(E_\tau)$ is abelian. In this paper, we consider the case that $Ly = 0$ is defined on $\mathbb{H}$ and instead of the monodromy representation defined on $\pi_1(\mathbb{H}\setminus\{\text{singular points}\})$, we study a new representation defined on a discrete non-abelian group Γ that is related to the modular property of Γ acting on $\mathbb{H}$. It is called the Bol representation in this paper as in [25] where the Bol representation was first introduced for second order differential equations.

Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ that is commensurable with $\mathrm{SL}(2,\mathbb{Z})$. Equation (1.1) is called a *modular ordinary differential equation* (MODE for short) on Γ if $Q_2(z)$ and $Q_3(z)-\frac{1}{2}Q_2'(z)$ are meromorphic modular forms on Γ of weight 4 and 6, respectively. Modular ordinary differential equations (of general order) appear prominently in the study of rational conformal field theories (see, e.g., [1, 10, 12, 14, 19, 21, 27, 34]). They provide a practical tool for classifying rational conformal field theories. As an object in the theory of modular forms, modular differential equations have also been studied by mathematicians. See, for example, [9, 11, 13, 17, 18, 29].

Given a MODE (1.1), it is natural to ask whether there are solutions satisfying some modular property. The main goal of this paper is to study when MODE (1.1) has solutions that lead to modular forms or quasimodular forms. The approach is to calculate the Bol representation, which will be explained below.

First we recall some basic notions from the ODE aspect. Equation (1.1) is called *Fuchsian* if the order of any pole of $Q_j(z)$ is at most j, $j=2,3$. At the cusp ∞, we let $q_N=\mathrm{e}^{2\pi\mathrm{i}z/N}$, where N is the width of ∞ in Γ. Then $\frac{\mathrm{d}}{\mathrm{d}z}=\frac{2\pi\mathrm{i}}{N}q_N\frac{\mathrm{d}}{\mathrm{d}q_N}$ and so (1.1) becomes

$$\left(q_N\frac{\mathrm{d}}{\mathrm{d}q_N}\right)^3 y+\left(\frac{N}{2\pi\mathrm{i}}\right)^2 Q_2(z)q_N\frac{\mathrm{d}}{\mathrm{d}q_N}y+\left(\frac{N}{2\pi\mathrm{i}}\right)^3 Q_3(z)y=0. \tag{1.2}$$

From here we see that

(1.1) *is Fuchsian at* ∞ *if and only if* $Q_2(z)$ *and* $Q_3(z)$ *are holomorphic at* ∞,

and similar conclusions hold for other cusps of Γ. By (1.2), the indicial equation at the cusp ∞ is given by

$$\kappa^3+\left(\frac{N}{2\pi\mathrm{i}}\right)^2 Q_2(\infty)\kappa+\left(\frac{N}{2\pi\mathrm{i}}\right)^3 Q_3(\infty)=0,$$

the roots of which are called the local exponents of (1.1) at ∞, denoted by $\kappa_\infty^{(1)}$, $\kappa_\infty^{(2)}$ and $\kappa_\infty^{(3)}$, satisfying $\sum_j\kappa_\infty^{(j)}=0$. In this paper, we always assume that the exponent differences $\kappa_\infty^{(j)}-\kappa_\infty^{(1)}$ are *integers* for $j=2,3$, Then $\sum_j\kappa_\infty^{(j)}=0$ implies $\kappa_\infty^{(j)}\in\frac{1}{3}\mathbb{Z}$ for all j and so we may assume $\kappa_\infty^{(1)}\le\kappa_\infty^{(2)}\le\kappa_\infty^{(3)}$. Similar assumptions are made for other cusps.

On the other hand, let $z_0\in\mathbb{H}$ be a singular point of (1.1) and write

$$Q_j(z)=A_j(z-z_0)^{-j}+O\big((z-z_0)^{-j+1}\big)\qquad \textit{at } z_0,$$

then the indicial equation at z_0 is given by

$$\kappa(\kappa-1)(\kappa-2)+A_2\kappa+A_3=0,$$

the roots of which are the local exponents of (1.1) at z_0, denoted by $\kappa_{z_0}^{(1)}$, $\kappa_{z_0}^{(2)}$ and $\kappa_{z_0}^{(3)}$, satisfying $\sum_j\kappa_{z_0}^{(j)}=3$. In this paper, we always assume that the exponent differences $\kappa_{z_0}^{(j)}-\kappa_{z_0}^{(1)}$ are *integers* for $j=2,3$. Then $\sum_j\kappa_{z_0}^{(j)}=3$ implies $\kappa_{z_0}^{(j)}\in\frac{1}{3}\mathbb{Z}$ for all j and so we may assume $\kappa_{z_0}^{(1)}\le\kappa_{z_0}^{(2)}\le\kappa_{z_0}^{(3)}$. Since the exponent differences are integers, (1.1) might have solutions with logarithmic singularities at z_0. See Appendix A for all possibilities of the solution structure of (1.1) at z_0. The singularity z_0 is called *apparent* if (1.1) has no solutions with logarithmic singularities at z_0. In this case, the three local exponents must be distinct, i.e., $\kappa_{z_0}^{(1)} < \kappa_{z_0}^{(2)} < \kappa_{z_0}^{(3)}$; see, e.g., Appendix A. In this paper, we always assume that L is *apparent at any singularity* $z_0 \in \mathbb{H}$. More precisely, we assume that the MODE (1.1) satisfies

(H1) The MODE (1.1) is Fuchsian on $\mathbb{H} \cup \{\text{cusps}\}$;

(H2) At any singular point $z_0 \in \mathbb{H}$, $\kappa_{z_0}^{(1)} < \kappa_{z_0}^{(2)} < \kappa_{z_0}^{(3)}$ satisfy $\kappa_{z_0}^{(1)} \in \frac{1}{3}\mathbb{Z}_{\le 0}$ and $\kappa_{z_0}^{(j)} - \kappa_{z_0}^{(1)} \in \mathbb{Z}$ for $j = 2, 3$. Furthermore, z_0 is apparent.

(H3) At any cusp s of Γ, $\kappa_s^{(1)} \le \kappa_s^{(2)} \le \kappa_s^{(3)}$ satisfies $\kappa_s^{(1)} \in \frac{1}{3}\mathbb{Z}_{\le 0}$ and $\kappa_s^{(j)} - \kappa_s^{(1)} \in \mathbb{Z}$ for $j = 2, 3$.

The motivation of all these assumptions will be clear from Theorem 1.1 below.

Note $\ln q_N = \frac{2\pi \mathrm{i}}{N} z$. Under our assumption (H3), (1.1) might have solutions containing $(\ln q_N)^2 = -\frac{4\pi^2}{N^2} z^2$ terms; see Remark A.9. In this case, we call the cusp ∞ to be *completely not apparent* or *maximally unipotent*, because under the Bol representation $\hat{\rho}$ that will be introduced below, the corresponding matrix $\hat{\rho}(T) \in \mathrm{SL}(3,\mathbb{C})$ of $T = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ has eigenvalues $\{1, 1, 1\}$ but $\operatorname{rank}(\hat{\rho}(T) - I_3) = 2$, i.e., $\hat{\rho}(T)$ is maximally unipotent. Here I_3 denotes the 3×3 identity matrix.

One class of the MODEs can be derived from quasimodular forms of depth 2. The notion of quasimodular forms was first introduced by Kaneko and Zagier [20]. See Section 2 for a brief overview of basic properties of quasimodular forms. In particular, given a holomorphic function $\phi(z)$ satisfying

$$(\phi\big|_2\gamma)(z) := (cz+d)^{-2}\phi(\gamma z) = \phi(z) + \frac{\alpha c}{cz+d}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for some nonzero complex number α, any quasimodular form $f(z)$ of weight k and depth 2 with character χ can be expressed as

$$f(z) = f_0(z) + f_1(z)\phi(z) + f_2(z)\phi(z)^2,$$

where $f_j(z)$ is a modular form on Γ of weight $k - 2j$ with character χ and $f_2 \ne 0$. Consider

$$\begin{pmatrix} h_1(z) \\ h_2(z) \\ h_3(z) \end{pmatrix} := \begin{pmatrix} z^2 f(z) + \alpha z(f_1(z) + 2f_2(z)\phi(z)) + \alpha^2 f_2(z) \\ 2zf(z) + \alpha(f_1(z) + 2f_2(z)\phi(z)) \\ f(z) \end{pmatrix} \tag{1.3}$$

and define

$$W_f(z) := \det \begin{pmatrix} h_1 & h_1' & h_1'' \\ h_2 & h_2' & h_2'' \\ h_3 & h_3' & h_3'' \end{pmatrix} \tag{1.4}$$

to be the Wronskian associated to f. Then $W_f(z)$ is a modular form on Γ of weight $3k$ with character χ^3; see Lemma 2.1 for a proof. This $W_f(z)$ was first introduced by Pellarin [28] for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $\phi(z) = E_2(z)$.

Now we define $g_j(z) := \frac{h_j(z)}{\sqrt[3]{W_f(z)}}$, then

$$\det \begin{pmatrix} g_1 & g_1' & g_1'' \\ g_2 & g_2' & g_2'' \\ g_3 & g_3' & g_3'' \end{pmatrix} = 1,$$

and a further differentiation leads to

$$\det \begin{pmatrix} g_1 & g_1' & g_1''' \\ g_2 & g_2' & g_2''' \\ g_3 & g_3' & g_3''' \end{pmatrix} = 0, \tag{1.5}$$

so $g_3(z)$ is a solution of (1.1) with

$$Q_2(z) := \frac{g_1'''g_2 - g_1g_2'''}{g_1g_2' - g_1'g_2}, \qquad Q_3(z) := \frac{g_1'g_2''' - g_2'g_1'''}{g_1g_2' - g_1'g_2}. \tag{1.6}$$

It is easy to see that $Q_2(z)$ and $Q_3(z)$ are single-valued, and g_1, g_2 are also solutions of (1.1). Our first result reads as follows.

Theorem 1.1. *Let $Q_2(z)$ and $Q_3(z)$ be given by* (1.6)*. Then*

(1) (1.1) *is a MODE, i.e., $Q_2(z)$ and $Q_3(z) - \frac{1}{2}Q_2'(z)$ are meromorphic modular forms on Γ* (*with trivial character*) *of weight 4 and 6, respectively.*

(2) *$(H1)$–$(H3)$ hold for* (1.1)*.*

Furthermore, for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, we have that

(3) *At the elliptic point* i, $\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\} \equiv \{0,0,1\} \bmod 2$.

(4) *At the elliptic point* $\rho = \frac{-1+\sqrt{3}i}{2}$, $\kappa_\rho^{(j)} \in \mathbb{Z}$ *for all j and* $\{\kappa_\rho^{(1)}, \kappa_\rho^{(2)}, \kappa_\rho^{(3)}\} \equiv \{0,1,2\} \bmod 3$.

We emphasize that (1) and (2) in Theorem 1.1 hold for any Γ, not only for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. They will lay the ground for our future study of general MODEs on other congruence subgroups.

As an example, in Section 6, we will work out the MODE in the case $f(z)$ is an extremal quasimodular form on $\mathrm{SL}(2,\mathbb{Z})$, introduced first in [18]; see Theorem 6.2.

Now we introduce the notion of the Bol representation of Γ associated to the MODE (1.1), which was first introduced in [25] for second order MODEs. It is well known that any (local) solution $y(z)$ of (1.1) can be extended to a multi-valued function in $\mathbb{H}$ through analytic continuation. Fix a point $z_0 \in \mathbb{H}$ that is not a singular point of (1.1) and let U be a simply-connected neighborhood of z_0 that contains no singularities of (1.1). For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, choose a path σ from z_0 to γz_0 and consider the analytic continuation of $y(z)$, $z \in U$, along the path. Then $y(\gamma z)$ is well-defined in U. Define

$$(y|_{-2}\gamma)(z) := (cz+d)^2 y(\gamma z), \qquad z \in U,$$

then by a direct computation or by using Bol's identity [3], we see that $(y|_{-2}\gamma)(z)$ is also a solution of (1.1). Thus, given a fundamental system of solutions $Y(z) = (y_1(z), y_2(z), y_3(z))^t$, there is $\hat{\gamma} \in \mathrm{SL}(3,\mathbb{C})$ such that

$$(Y\big|_{-2}\gamma)(z) = \hat{\gamma}Y(z),$$

where the fact $\det\hat{\gamma} = 1$ follows from that the Wronskians of Y and $(Y\big|_{-2}\gamma)$ are the same. Obviously, this matrix $\hat{\gamma}$ depends on the choice of the path σ. However, under the above assumptions, all local monodromy matrices are εI_3 with $\varepsilon^3 = 1$, so different choices of σ will only possibly change $\hat{\gamma}$ to $\mathrm{e}^{\pm\frac{2\pi \mathrm{i}}{3}}\hat{\gamma}$. From here, we see that there is a well-defined homomorphism $\rho\colon \Gamma \to \mathrm{PSL}(3,\mathbb{C})$ such that

$$(Y\big|_{-2}\gamma)(z) = \mathrm{e}^{\frac{2\pi \mathrm{i}k}{3}}\rho(\gamma)Y(z), \qquad k \in \{0, \pm 1\},$$

where $y_j(\gamma z)$ are always understood to take analytic continuation along the same path for $j = 1,2,3$. This homomorphism ρ will be called the *Bol representation* as in [25]. For the convenience of computations, it is better to lift ρ to a homomorphism $\hat{\rho}\colon \Gamma \to \mathrm{GL}(3,\mathbb{C})$ as follows. Suppose that we can find a multi-valued meromorphic function $F(z)$ such that: (i) The analytic continuation of $\hat{y}(z) := F(z)y(z)$, where $y(z)$ is any solution of (1.1), gives rise to a single-valued holomorphic function on $\mathbb{H}$, and (ii) $F(z)^3$ is a modular form on Γ of weight $3k$

with some character, where $k \in \mathbb{N}$. Such $F(z)$ can be constructed explicitly when Γ is a triangle group. Then by letting $\hat{Y}(z) := F(z)Y(z)$, there is $\hat{\rho}(\gamma) \in \mathrm{GL}(3, \mathbb{C})$ such that

$$\big(\hat{Y}\big|_\ell \gamma\big)(z) = \hat{\rho}(\gamma)\hat{Y}(z), \qquad \text{where } \ell = k-2.$$

This homomorphism $\hat{\rho}\colon \Gamma \to \mathrm{GL}(3, \mathbb{C})$, as a lift of ρ, will also be called the *Bol representation* since there is no confusion arising. Naturally we consider the following problem:

Question. *Can we characterize, in terms of local exponents, the MODEs* (1.1) *whose Bol representations are irreducible?*

One purpose of this paper is to answer this question for the case $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. For $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, the above $F(z)$ can be taken to be

$$F(z) := \Delta(z)^{-\kappa_\infty^{(1)}} E_4(z)^{-\kappa_\rho^{(1)}} E_6(z)^{-\kappa_{\mathrm{i}}^{(1)}} \prod_{j=1}^{m} F_j(z)^{-\kappa_{z_j}^{(1)}}, \tag{1.7}$$

where

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \qquad E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}, \qquad q = \mathrm{e}^{2\pi \mathrm{i} z}, \tag{1.8}$$

are the Eisenstein series of weight 4 and 6, respectively,

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots,$$

$\mathrm{i} = \sqrt{-1}$ and $\rho = \big(-1 + \sqrt{3}\mathrm{i}\big)/2$ are the elliptic points of $\mathrm{SL}(2, \mathbb{Z})$, $\{z_1, \dots, z_m\} \sqcup \{\mathrm{i}, \rho, \infty\}$ denotes the set of singular points of the MODE (1.1) mod $\mathrm{SL}(2, \mathbb{Z})$, $t_j := E_4(z_j)^3/E_6(z_j)^2$ and $F_j(z) := E_4(z)^3 - t_j E_6(z)^2$. Then $F(z)^3$ is a modular form of weight $3(\ell+2)$, where the integer $\ell = k-2$ is given by

$$\ell := -2 - 12\kappa_\infty^{(1)} - 4\kappa_\rho^{(1)} - 6\kappa_{\mathrm{i}}^{(1)} - 12\sum_{j=1}^{m} \kappa_{z_j}^{(1)}.$$

In other words, besides the assumptions (H1)–(H3), we need to assume further that $\kappa_\rho^{(1)} \in \mathbb{Z}$ such that $\ell \in \mathbb{Z}$. Consequently, we will see from Lemma 3.2 that the Bol representation $\hat{\rho}$ is indeed a group homomorphism from $\mathrm{SL}(2, \mathbb{Z})$ to $\mathrm{SL}(3, \mathbb{C})$.

Remark 1.2. The choice of $F(z)$ is not unique since we can multiply $F(z)$ by a holomorphic modular form to obtain a new one. Different choices of $F(z)$'s may give different weights k (and so ℓ) but keeping $\hat{\rho}(\gamma)$ invariant. For example, when the MODE (1.1) comes from a quasimodular form $f(z)$ of depth 2 on Γ as shown in Theorem 1.1, then one choice is to take $\sqrt[3]{W_f(z)}$ as $F(z)$, i.e., $\hat{Y} = (h_1, h_2, h_3)^t$ defined in (1.3). Note that for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, $\sqrt[3]{W_f(z)}$ might be different from the $F(z)$ given by (1.7). To obtain that $\sqrt[3]{W_f(z)}$ equals to the $F(z)$ given by (1.7), we need to assume that f_0, f_1, f_2 have no common zeros.

Note from $\sum_j \kappa_{\mathrm{i}}^{(j)} = 3$ that we have either $\big\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\big\} \equiv \{0,0,1\}$ mod 2 or $\big\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\big\} \equiv \{1,1,1\}$ mod 2. Our second result of this paper is

Theorem 1.3. *Let* $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ *and suppose that the MODE* (1.1) *satisfies* (*H*1)–(*H*3) *and* $\kappa_\rho^{(1)} \in \mathbb{Z}$. *Then the Bol representation* $\hat{\rho}$ *is irreducible if and only if* $\big\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\big\} \equiv \{0,0,1\}$ mod 2.

Note that all irreducible representations of $\mathrm{SL}(2,\mathbb{Z})$ of rank up to 5 have been classified by Tuba and Wenzl [31].[1] One may use their results, the work of Westbury [32], and Lemma 3.10 below to give another proof of Theorem 1.3 different from that given in Section 3. See Remark 3.12.

As an application of Theorem 1.3, we can show that the converse statement of Theorem 1.1 holds. More precisely, we have

Theorem 1.4. *Let $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ and suppose that the MODE (1.1) satisfies $(H1)$–$(H3)$ and $\kappa_\rho^{(1)}\in\mathbb{Z}$. Let $y_+(z)$ be the solution of (1.1) of the form $y_+(z)=q^{\kappa_\infty^{(3)}}\sum_{j=0}^\infty c_jq^j$, $c_0=1$, and $F(z)$ be defined by (1.7).*

1. *If $\{3\kappa_{\mathrm{i}}^{(1)},3\kappa_{\mathrm{i}}^{(2)},3\kappa_{\mathrm{i}}^{(3)}\}\equiv\{0,0,1\}\bmod 2$, then $\hat{y}_+(z):=F(z)y_+(z)$ is a quasimodular form of weight $\ell+2$ and depth 2.*
2. *If $\{3\kappa_{\mathrm{i}}^{(1)},3\kappa_{\mathrm{i}}^{(2)},3\kappa_{\mathrm{i}}^{(3)}\}\equiv\{1,1,1\}\bmod 2$, then the Bol representation $\hat\rho$ is trivial, i.e., $\hat\rho(\gamma)=I_3$ for all $\gamma\in\mathrm{SL}(2,\mathbb{Z})$. In particular, $12|\ell$ and $\hat{y}(z):=F(z)y(z)$ is a modular form of weight ℓ for any solution $y(z)$ of (1.1).*

Together with Theorems 1.4 and 1.1, we can obtain

Corollary 1.5. *Let $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ and suppose $(H1)$–$(H3)$ and $\kappa_\rho^{(1)}\in\mathbb{Z}$ hold for the MODE (1.1). Then (3) and (4) in Theorem 1.1 are equivalent.*

In the reducible case, Theorem 1.4(2) can be applied to construct solutions of the $\mathrm{SU}(3)$ Toda system. See Section 4 for the precise statement. The Toda system is an important integrable system in mathematical physics. In algebraic geometry, the $\mathrm{SU}(N+1)$ Toda system is exactly the classical infinitesimal Plücker formula associated with holomorphic maps from Riemann surfaces to $\mathbb{CP}^N$; see, e.g., [7, 23, 24] and references therein for the recent development of the Toda system.

The rest of this paper is organized as follows. In Section 2, we give the proof of Theorem 1.1, namely we will prove that every quasimodular form of depth 2 leads to a MODE (1.1) satisfying the conditions (H1)–(H3). We focus on the case $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ from Section 3. Theorems 1.3–1.4 and Corollary 1.5 will be proved in Section 3. In Section 4, we discuss the reducible case and prove the converse statement of Theorem 1.4(2). We also give an application to the $\mathrm{SU}(3)$ Toda system. In Section 5, we discuss the criterion on the existence of the MODE (1.1) which is Fuchsian and apparent throughout $\mathbb{H}$ with prescribed local exponents at singularities and at cusps. In Section 6, as examples of MODEs, we will work out the explicit expressions of $Q_j(z)$'s for an extremal quasimodular form $f(z)$. Finally in Appendix A, we recall the theory of the solution structure of third order ODEs at a regular singular point.

2 Quasimodular forms of depth 2 and its associated 3rd order MODE

The main purpose of this section is to prove Theorem 1.1. Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ that is commensurable with $\mathrm{SL}(2,\mathbb{Z})$ and $\chi\colon\Gamma\to\mathbb{C}^\times$ be a character of Γ of finite order. A holomorphic function $f(z)$ defined on the upper half plane $\mathbb{H}$ is a modular form of weight k with character χ if the following conditions hold:

1. $(f|_k\gamma)(z):=(cz+d)^{-k}f(\gamma z)=\chi(\gamma)f(z)$ for any $\gamma\in\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$;
2. f is holomorphic at any cusp s of Γ.

[1] We thank the referee for providing the reference.

For example, the Eisenstein series $E_4(z)$ and $E_6(z)$ in (1.8) are modular forms of weight 4 and 6 on $\mathrm{SL}(2,\mathbb{Z})$, respectively. We let $\mathfrak{M}_k(\Gamma,\chi)$ denote the space of modular forms of weight k with character χ on Γ. For example, for $\Gamma=\mathrm{SL}(2,\mathbb{Z})$, ∞ is the only cusp and we assume that the character is trivial, i.e., $\chi\equiv 1$. Then condition (1) implies $f(z+1)=f(z)$, which implies that $f(z)$ can be viewed as a function of $q=\mathrm{e}^{2\pi\mathrm{i}z}$, and condition (2) just means that f is holomorphic at $q=0$.

The notion of quasimodular forms was introduced by Kaneko and Zagier [20]. Originally, they are defined as the holomorphic parts of nearly holomorphic modular forms. For our purpose, it suffices to know that a holomorphic function $f(z)$ is a quasimodular form of weight k and depth r with character χ on Γ if and only if $f(z)$ can be expressed as

$$f(z)=\sum_{j=0}^{r} f_j(z)\phi(z)^j,$$

where $f_j(z)\in\mathfrak{M}_{k-2j}(\Gamma,\chi)$ with $f_r\not\equiv 0$ and $\phi(z)$ is a holomorphic function satisfying that

$$\left(\phi\big|_2\gamma\right)(z):=(cz+d)^{-2}\phi(\gamma z)=\phi(z)+\frac{\alpha c}{cz+d} \tag{2.1}$$

for all $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$ for some nonzero complex number α and $\phi(z)$ is holomorphic at cusps of Γ. This $\phi(z)$ is called a quasimodular form of weight 2 and depth 1 on Γ. For example, if Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$, we can always let

$$\phi(z)=E_2(z):=\frac{1}{2\pi\mathrm{i}}\frac{\Delta'(z)}{\Delta(z)}=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$

and so $\alpha=\frac{6}{\pi\mathrm{i}}$. We let $\widetilde{\mathfrak{M}}_k^{\le r}(\Gamma,\chi)$ denote the space of quasimodular forms of weight k and depth $\le r$ with character χ. One basic property is that the quasi-modularity is invariant under the differentiation, namely if $f(z)\in\widetilde{\mathfrak{M}}_k^{\le r}(\Gamma,\chi)$, then $f'(z)\in\widetilde{\mathfrak{M}}_{k+2}^{\le r+1}(\Gamma,\chi)$; see [33, Proposition 20]. We refer the reader to [8, 20, 33] for the general theory of quasimodular forms.

Now we consider the Wronskian $W_f(z)$ (see [28]) associated to

$$f(z)=f_0(z)+f_1(z)\phi(z)+f_2(z)\phi(z)^2\in\widetilde{\mathfrak{M}}_k^{\le 2}(\Gamma,\chi),$$

where $f_j(z)\in\mathfrak{M}_{k-2j}(\Gamma,\chi)$ with $f_2\ne 0$. Then

$$\begin{aligned}\left(f\big|_k\gamma\right)(z)&:=(cz+d)^{-k}f(\gamma z)\\ &=\chi(\gamma)\sum_{j=0}^{2}f_j(z)\left(\phi(z)+\frac{\alpha c}{cz+d}\right)^j,\qquad \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma.\end{aligned} \tag{2.2}$$

As in [28], we set

$$\begin{aligned}P_f(t)&:=\sum_{j=0}^{2}f_j(z)\left(\phi(z)+\alpha t\right)^j,\\ Q_f(t)&:=t^2P_f(1/t)=f(z)t^2+\alpha(f_1(z)+2f_2(z)\phi(z))t+\alpha^2 f_2(z),\\ F_f(z)&:=\begin{pmatrix} Q_f(t)\\ \frac{\partial}{\partial t}Q_f(t)\\ \frac{1}{2}\frac{\partial^2}{\partial t^2}Q_f(t)\end{pmatrix}_{|t=z}\end{aligned}$$

$$= \begin{pmatrix} z^2 f(z) + \alpha z(f_1(z) + 2f_2(z)\phi(z)) + \alpha^2 f_2(z) \\ 2zf(z) + \alpha(f_1(z) + 2f_2(z)\phi(z)) \\ f(z) \end{pmatrix} =: \begin{pmatrix} h_1(z) \\ h_2(z) \\ h_3(z) \end{pmatrix}. \tag{2.3}$$

and define $W_f(z)$ as in (1.4) to be the Wronskian associated to f.

Lemma 2.1. *$W_f(z)$ is a modular form on Γ of weight $3k$ with character χ^3.*

Proof. For $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, this result is proved in [28] and can be also derived from Mason [26, Lemma 3.1], and the approaches in [26, 28] can be easily applied to general discrete subgroups Γ. Here we provide an elementary proof for general Γ for completeness. By (2.2) and $ad - bc = 1$ we have

$$\begin{aligned} f(\gamma z) &= \chi(\gamma)(cz+d)^k \left[f + \frac{\alpha c}{cz+d}(f_1 + 2f_2\phi) + \frac{\alpha^2 c^2}{(cz+d)^2} f_2 \right] \\ &= \chi(\gamma)(cz+d)^{k-2}\big(c^2 h_1 + cdh_2 + d^2 h_3\big), \\ h_2(\gamma z) &= 2\frac{az+b}{cz+d} f(\gamma z) + \alpha(f_1(\gamma z) + 2f_2(\gamma z)\phi(\gamma z)) \\ &= \chi(\gamma)(cz+d)^{k-2}\Big[2\frac{az+b}{cz+d}((cz+d)^2 f + \alpha c(cz+d)(f_1 + 2f_2\phi) \\ &\quad + \alpha^2 c^2 f_2) + \alpha(f_1 + 2f_2\phi) + \frac{2\alpha^2 c}{cz+d} f_2 \Big] \\ &= \chi(\gamma)(cz+d)^{k-2}[2ach_1 + (ad+bc)h_2 + 2bdh_3], \\ h_1(\gamma z) &= \frac{(az+b)^2}{(cz+d)^2} f(\gamma z) + \alpha\frac{az+b}{cz+d}(f_1(\gamma z) + 2f_2(\gamma z)\phi(\gamma z)) + \alpha^2 f_2(\gamma z) \\ &= \chi(\gamma)(cz+d)^{k-2}\Big[\frac{(az+b)^2}{(cz+d)^2}((cz+d)^2 f + \alpha c(cz+d)(f_1 + 2f_2\phi) \\ &\quad + \alpha^2 c^2 f_2) + \frac{az+b}{cz+d}\Big[\alpha(f_1 + 2f_2\phi) + \frac{2\alpha^2 c}{cz+d} f_2\Big] + \frac{\alpha^2}{(cz+d)^2} f_2 \Big] \\ &= \chi(\gamma)(cz+d)^{k-2}\big[a^2 h_1 + abh_2 + b^2 h_3\big]. \end{aligned}$$

Thus

$$F_f(\gamma z) = \chi(\gamma)(cz+d)^{k-2} A F_f(z), \qquad \text{where} \quad A := \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

Together with the fact $\det A = (ad-bc)^3 = 1$, it is easy to see that

$$W_f(\gamma z) = \det(F_f(\gamma z), F_f'(\gamma z), F_f''(\gamma z)) = \chi(\gamma)^3 (cz+d)^{3k} W_f(z).$$

In particular, for the case $\gamma = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$, where N is the width of the cusp ∞, the transformation law shows that $W_f(z)$ is a polynomial only in f_0, f_1, f_2, ϕ, and their derivatives. Thus, by the computation above and the fact that the ring of quasimodular forms is invariant under differentiation, $W_f(z)$ is a quasimodular form that is actually modular. In other words, $W_f(z) \in \mathfrak{M}_{3k}(\Gamma, \chi^3)$. This completes the proof. ∎

Now we define $g_j(z) := h_j(z)/\sqrt[3]{W_f(z)}$. Then (1.5) holds, so $g_3(z)$ is a solution of

$$Ly := y''' + Q_2(z)y' + Q_3(z)y = 0, \tag{2.4}$$

where

$$Q_2(z) := \frac{g_1'''g_2 - g_1 g_2'''}{g_1 g_2' - g_1' g_2}, \qquad Q_3(z) := \frac{g_1' g_2''' - g_2' g_1'''}{g_1 g_2' - g_1' g_2}. \tag{2.5}$$

Note that

(i) $g_1(z)$ and $g_2(z)$ are also solutions of (2.4) and g_1, g_2, g_3 are linearly independent.

(ii) Although g_j might be multi-valued, g_j'/g_j is single-valued. Thus $Q_j(z)$ is single-valued and meromorphic on $\mathbb{H}$ for $j = 1, 2$. We will see from Theorem 2.2 below that any pole of $Q_j(z)$ comes from zeros of $W_f(z)$.

For $z_0 \in \mathbb{H}$, we denote $\kappa_{z_0}^{(1)} \le \kappa_{z_0}^{(2)} \le \kappa_{z_0}^{(3)}$ to be the local exponents of (2.4) at z_0.

Theorem 2.2. *Under the above notations, $Q_2(z)$ and $Q_3(z) - \frac{1}{2}Q_2'(z)$ are meromorphic modular forms on Γ (with trivial character) of weight 4 and 6, respectively. Furthermore, all singular points of (2.4) on $\mathbb{H}$ comes from the zeros of $W_f(z)$, (H1)–(H3) hold, and every cusp of Γ is completely not apparent for (2.4).*

Proof. To prove the modularity of Q_j, we consider

$$\tilde{y}(z) := \big(y\big|_{-2}\gamma\big)(z) = (cz+d)^2 y(\gamma z), \qquad \gamma \in \Gamma.$$

Then

$$\tilde{y}'(z) = y'(\gamma z) + 2c(cz+d)y(\gamma z), \qquad \tilde{y}'''(z) = (cz+d)^{-4} y'''(\gamma z),$$

so

$$\begin{aligned} L\tilde{y} = (cz+d)^{-4}\big\{ & y'''(\gamma z) + (cz+d)^4 Q_2(z) y'(\gamma z) \\ & + \big[(cz+d)^6 Q_3(z) + 2c(cz+d)^5 Q_2(z)\big] y(\gamma z)\big\}. \end{aligned} \tag{2.6}$$

Recalling $g_3(z) = \frac{f(z)}{\sqrt[3]{W_f(z)}}$, we have

$$\begin{aligned} \big(g_3\big|_{-2}\gamma\big)(z) &= \left(\frac{f}{\sqrt[3]{W_f}}\Big|_{-2}\gamma\right)(z) = \frac{(cz+d)^2 f(\gamma z)}{\sqrt[3]{W_f(\gamma z)}} = \frac{\varepsilon}{\sqrt[3]{W_f(z)}}\big(c^2 h_1 + cd h_2 + d^2 h_3\big) \\ &= \varepsilon\big(c^2 g_1(z) + cd g_2(z) + d^2 g_3(z)\big), \end{aligned}$$

where $\varepsilon^3 = 1$. Thus $\big(g_3\big|_{-2}\gamma\big)(z)$ is also a solution of (2.4). From this and (2.6), we have

$$\begin{aligned} Q_2(\gamma z) &= (cz+d)^4 Q_2(z), \\ Q_3(\gamma z) &= (cz+d)^6 Q_3(z) + 2c(cz+d)^5 Q_2(z), \end{aligned}$$

so $\big(Q_2\big|_4\gamma\big) = Q_2$ and $\big((Q_3 - \frac{1}{2}Q_2')\big|_6\gamma\big) = Q_3 - \frac{1}{2}Q_2'$. This proves the modularity of Q_2 and Q_3.

To prove (H1)–(H3), we let z_0 be any pole of $Q_j(z)$ for some $j = 1, 2$. Clearly $g_j(z) = (z - z_0)^{\alpha_j}(c_j + O(z - z_0)^j)$ near z_0 for some $\alpha_j \in \frac{1}{3}\mathbb{Z}$ and $c_j \neq 0$. By replacing g_2 by $g_2 - \frac{c_2}{c_1}g_1$ if necessary, we may assume $\alpha_1 \neq \alpha_2$. Then we easily deduce from (2.5) that

$$\begin{aligned} Q_2(z) &= \frac{\alpha_1(\alpha_1-1)(\alpha_1-2) - \alpha_2(\alpha_2-1)(\alpha_2-2)}{(\alpha_2-\alpha_1)(z-z_0)^2} + O\big((z-z_0)^{-1}\big), \\ Q_3(z) &= \frac{\alpha_1\alpha_2[(\alpha_2-1)(\alpha_2-2) - (\alpha_1-1)(\alpha_1-2)]}{(\alpha_2-\alpha_1)(z-z_0)^3} + O\big((z-z_0)^{-2}\big), \end{aligned}$$

so z_0 is a regular singular point of (2.4). Thus, (2.4) is Fuchsian on $\mathbb{H}$.

Let z_0 be any singular point. It follows from $g_j(z) = \frac{h_j(z)}{\sqrt[3]{W_f(z)}}$ that

$$g_j(z) = (z-z_0)^{\frac{-\mathrm{ord}_{z_0} W_f}{3}} \sum_{l\geq 0} d_j (z-z_0)^j, \tag{2.7}$$

where $\mathrm{ord}_{z_0} W_f$ denotes the zero order of $W_f(z)$ at z_0. Since (g_1, g_2, g_3) is a fundamental system of solutions of (2.4) and g_j's have no logarithmic singularities at z_0, we conclude from (2.7) and Remark A.8 that (1) the local exponents $\kappa_{z_0}^{(j)} \in \frac{1}{3}\mathbb{Z}$ and are all distinct; (2) the exponent differences are all nonzero integers, namely $m_{z_0}^{(j)} := \kappa_{z_0}^{(j+1)} - \kappa_{z_0}^{(j)} - 1$ are nonnegative integers for $j = 1, 2$; (3) z_0 is an apparent singularity of (2.4).

Since

$$Q_j(z) = A_j (z-z_0)^{-j} + O\big((z-z_0)^{-j+1}\big), \qquad j = 2, 3, \tag{2.8}$$

then the indicial equation of (2.4) at z_0 is

$$\kappa(\kappa-1)(\kappa-2) + A_2\kappa + A_3 = 0, \tag{2.9}$$

which implies $\sum_{j=1}^{3} \kappa_{z_0}^{(j)} = 3$ and so $\kappa_{z_0}^{(1)} = -\frac{2m_{z_0}^{(1)} + m_{z_0}^{(2)}}{3} \in \frac{1}{3}\mathbb{Z}_{\leq 0}$. This proves (H2). Remark that if z_0 is not a zero of $W_f(z)$, i.e., $\mathrm{ord}_{z_0} W_f = 0$, then it follows from (2.7) that $\kappa_{z_0}^{(1)} \in \mathbb{Z}_{\geq 0}$ and so $\kappa_{z_0}^{(1)} = m_{z_0}^{(1)} = m_{z_0}^{(2)} = 0$, i.e., the local exponents at z_0 are $\{0, 1, 2\}$. This already implies $A_2 = A_3 = 0$. Together with the fact that z_0 is apparent, we easily deduce from the Frobenius method that both $Q_2(z)$ and $Q_3(z)$ are holomorphic at z_0, a contradiction with that z_0 is a singular point. Thus z_0 is a zero of $W_f(z)$. This proves that all singular points of (2.4) on $\mathbb{H}$ come from the zeros of $W_f(z)$.

Let N be the width of the cusp ∞ and $q_N = \mathrm{e}^{2\pi \mathrm{i} z/N}$. Since modular forms $f_j(z)$, $W_f(z)$ are holomorphic in terms of q_N and $z = \frac{N}{2\pi \mathrm{i}} \ln q_N$, we see from (2.3) that

$$g_j(z) = \frac{h_j(z)}{\sqrt[3]{W_f(z)}} = \sum_{k=0}^{3-j} (\ln q_N)^k q_N^{-\frac{\mathrm{ord}_\infty W_f}{3}} \sum_{t=0}^{\infty} c_{j,k,t} q_N^t, \qquad j = 1, 2, 3,$$

so ∞ is also a regular singular point of (2.4), i.e., (2.4) is Fuchsian at ∞ and so $Q_j(z)$ is holomorphic at ∞ for $j = 2, 3$. Since $(\ln q_N)^2$ appears in the expression of $g_1(z)$, we see from Remark A.9 that ∞ is completely not apparent and the local exponents $\kappa_\infty^{(1)} \leq \kappa_\infty^{(2)} \leq \kappa_\infty^{(3)}$ satisfy $\kappa_\infty^{(j)} \in \frac{1}{3}\mathbb{Z}$ and

$$m_\infty^{(1)} := \kappa_\infty^{(2)} - \kappa_\infty^{(1)} \in \mathbb{Z}_{\geq 0}, \qquad m_\infty^{(2)} := \kappa_\infty^{(3)} - \kappa_\infty^{(2)} \in \mathbb{Z}_{\geq 0}.$$

Note that the indicial equation at ∞ is

$$\kappa^3 + \left(\frac{N}{2\pi \mathrm{i}}\right)^2 Q_2(\infty)\kappa + \left(\frac{N}{2\pi \mathrm{i}}\right)^3 Q_3(\infty) = 0,$$

which implies $\sum \kappa_\infty^{(j)} = 0$ and so $\kappa_\infty^{(1)} = -\frac{2m_\infty^{(1)} + m_\infty^{(2)}}{3}$. We now consider other cusps.

Assume that s is another cusp of Γ different from ∞. Let $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ be a matrix such that $\sigma\infty = s$. Regarding $f(z)$ as a quasimodular form on $\Gamma' = \ker\chi \cap \mathrm{SL}(2, \mathbb{Z})$, we can express $f(z)$ as $f(z) = \widetilde{f}_0(z) + \widetilde{f}_1(z) E_2(z) + \widetilde{f}_2(z) E_2(z)^2$ for some $\widetilde{f}_j(z) \in \mathfrak{M}_{k-2j}(\Gamma')$. We check that

$$\left(g_3|_{-2}\sigma\right)(z) = \frac{(cz+d)^2 f(\sigma z)}{\sqrt[3]{W_f(\sigma z)}} = \epsilon \frac{(cz+d)^2 (f|_k\sigma)(z)}{\sqrt[3]{(W_f|_{3k}\sigma)(z)}}$$

$$= \epsilon \frac{(cz+d)^2 p_1(z) + \alpha c(cz+d)p_2(z) + \alpha^2 c^2 p_3(z)}{\sqrt[3]{(W_f|_{3k}\sigma)(z)}},$$

where

$$p_1(z) = \big(\widetilde{f}_0|_k\sigma\big)(z) + \big(\widetilde{f}_1|_{k-2}\sigma\big)(z)E_2(z) + \big(\widetilde{f}_2|_{k-4}\sigma\big)(z),$$
$$p_2(z) = \big(\widetilde{f}_1|_{k-2}\sigma\big)(z) + 2\big(\widetilde{f}_2|_{k-4}\sigma\big)(z)E_2(z),$$
$$p_3(z) = \big(\widetilde{f}_2|_{k-4}\sigma\big)(z),$$

$\alpha = 6/\pi\mathrm{i}$ and ϵ is a third root of unity. Except for $cz+d$, every term in the expression has a q_M-expansion, where M is the width of the cusp σ and $q_M = \mathrm{e}^{2\pi\mathrm{i}z/M}$. Since $s \neq \infty$, we have $c \neq 0$. This shows that there is a local solution at the cusp s having a factor z^2. According to the solution structure discussed in the appendix, the point s must be completely not apparent and we have $\kappa_s^{(2)} - \kappa_s^{(1)}, \kappa_s^{(3)} - \kappa_s^{(1)} \in \mathbb{Z}$. By the same reasoning as in the case of the cusp ∞, the sum $\kappa_s^{(1)} + \kappa_s^{(2)} + \kappa_s^{(3)}$ is equal to 0 and hence $\kappa_s^{(j)} \in \frac{1}{3}\mathbb{Z}$ for all j. This proves (H1), (H3), and that every cusp is completely not apparent. ■

For a MODE, the local exponents are invariant under $z_0 \to \gamma z_0$ for any $\gamma \in \Gamma$.

Proposition 2.3. *Let z_0 be a singular point of (2.4). Then the local exponents of (2.4) at γz_0 are the same for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.*

Proof. Let

$$Q_j(z) = \tilde{A}_j(z-\gamma z_0)^{-j} + O\big((z-\gamma z_0)^{-j+1}\big), \qquad j = 2, 3.$$

Recalling (2.8) and (2.9). we only need to prove $\big(\tilde{A}_2, \tilde{A}_3\big) = (A_2, A_3)$.

Since

$$\gamma z - \gamma z_0 = \frac{z - z_0}{(cz+d)(cz_0+d)} = \frac{z-z_0}{(cz_0+d)^2}(1 + O(z-z_0)) \qquad \text{as } z \to z_0,$$

we have for $z \to z_0$ that

$$\begin{aligned} Q_2(z) &= (cz+d)^{-4} Q_2(\gamma z) \\ &= (cz_0+d)^{-4}(1+O(z-z_0))\big[\tilde{A}_2(\gamma z - \gamma z_0)^{-2} + O\big((\gamma z - \gamma z_0)^{-1}\big)\big] \\ &= \tilde{A}_2(z-z_0)^{-2} + O\big((z-z_0)^{-1}\big), \end{aligned}$$

so $\tilde{A}_2 = A_2$. Since $Q_3 - \frac{1}{2}Q_2'$ is a modular form of weight 6, a similar argument implies $\tilde{A}_3 + \tilde{A}_2 = A_3 + A_2$ and so $\tilde{A}_3 = A_3$. ■

Now we consider $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and discuss the local exponents of (2.4) at the elliptic points i and ρ, namely to complete the proof of Theorem 1.1. For this purpose, we note that the remark below could be used to simplify some computations.

Remark 2.4. If f_0, f_1, f_2 have a common zero $z_0 \in \mathbb{H} \cup \{\infty\}$, we take $M(z)$ to be a holomorphic modular form such that it has only one simple zero at z_0. Then $f_j(z)/M(z)$ are holomorphic modular forms and $f(z)/M(z)$ is a quasimodular form. Since $W_f(z) = M(z)^3 W_{f/M}(z)$ and so

$$g_j(z) = \frac{h_j(z)}{\sqrt[3]{W_f(z)}} = \frac{h_j(z)/M(z)}{\sqrt[3]{W_{f/M}(z)}},$$

namely all $g_j(z)$'s are invariant by replacing $f(z)$ by $f(z)/M(z)$, so the differential equation (2.4) derived from $f(z)$ and that from $f(z)/M(z)$ are the same (Note that $g_j(z)$ are solutions of (2.4) but neither $f(z)$ nor $f(z)/M(z)$ are solutions of (2.4), so we do not mean that $f(z)/M(z)$ satisfies the same differential equation as $f(z)$). Therefore, without loss of generality, we assume throughout the section that f_0, f_1, f_2 have no common zeros on $\mathbb{H} \cup \{\infty\}$.

By applying this remark, we have

Lemma 2.5. *Let $\Gamma_\infty = \pm\langle T\rangle$ be the stabilizer subgroup of ∞ in Γ. Then there are at most two right cosets $\Gamma_\infty\gamma$ in $\Gamma_\infty\backslash\Gamma$ (the set of right cosets of Γ_∞ in Γ) such that $f(\gamma z_0) = 0$.*

Proof. Suppose that there are three distinct right cosets $\Gamma_\infty\gamma$ in $\Gamma_\infty\backslash\Gamma$ such that $f(\gamma z_0) = 0$. Without loss of generality, we assume that one of them is the coset of I, i.e., $f(z_0) = 0$. Now we have

$$(f|_k\gamma)(z) = \chi(\gamma)\left(f(z) + \frac{\alpha c}{cz+d}(f_1(z) + 2f_2(z)\phi(z)) + \left(\frac{\alpha c}{cz+d}\right)^2 f_2(z)\right)$$

for $\gamma \in \Gamma$. If $f(\gamma_1 z_0) = f(\gamma_2 z_0) = 0$ for γ_1 and γ_2 in two different cosets in $\Gamma_\infty\backslash\Gamma$, then we have $f_1(z_0) + 2f_2(z_0)\phi(z_0) = f_2(z_0) = 0$. However, this implies that $f_0(z_0) \neq 0$ since $f_j(z)$ are assumed to have no common zeros on $\mathbb{H}$, and hence $f(z_0) \neq 0$, a contradiction. ∎

Now we let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Given a character χ, we have $\chi(T) = \mathrm{e}^{2\pi i m/24}$ for some integer $m \in [0, 23]$. Recall the Dedekind eta function

$$\eta(z) = \mathrm{e}^{2\pi \mathrm{i} z/24}\prod_{n=1}^{\infty}\left(1 - \mathrm{e}^{2n\pi \mathrm{i} z}\right).$$

Since the MODE associated to f is the same as that associated to f/η^m, by considering f/η^m if necessary, we can always assume $\chi(T) = 1$ and so $\chi(S) = \chi(R) = 1$, i.e., we can always assume that the character χ is trivial for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.

Proof of Theorem 1.1. The conclusions (1) and (2) are proved in Theorem 2.2. It suffices to consider $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and prove (3) and (4).

Let $z_0 \in \{\mathrm{i}, \rho\}$. By Lemma 2.5, we have $f(\gamma z_0) \neq 0$ for some $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Then it follows from Proposition 2.3 that

$$\kappa_{z_0}^{(1)} = \kappa_{\gamma z_0}^{(1)} = \mathrm{ord}_{\gamma z_0}\frac{f(z)}{\sqrt[3]{W_f(z)}} = -\frac{1}{3}\,\mathrm{ord}_{z_0} W_f(z).$$

Since $W_f(z)$ is a modular form of weight $3k$ on $\mathrm{SL}(2,\mathbb{Z})$, it follows from the valence formula for modular forms (see, e.g., [30]) that

$$\frac{\mathrm{ord}_{\mathrm{i}} W_f}{2} + \frac{\mathrm{ord}_\rho W_f}{3} \equiv \frac{k}{4} \mod 1.$$

This implies $\kappa_\rho^{(1)} = -\frac{\mathrm{ord}_\rho W_f}{3} \in \mathbb{Z}_{\leq 0}$ and $3\kappa_{\mathrm{i}}^{(1)} = -\mathrm{ord}_{\mathrm{i}} W_f \equiv k/2 \mod 2$.

Recall that we may assume that f_0, f_1, and f_2 have no common zeros. When $k \equiv 0 \mod 4$, we have

$$m_{\mathrm{i}}^{(2)} \equiv 3\kappa_{\mathrm{i}}^{(1)} \equiv \frac{k}{2} \equiv 0 \mod 2,$$

and we are done. When $k \equiv 2 \mod 4$, the weights of f_0 and f_2 are congruent to 2 modulo 4 and their expansions in $w = (z-\mathrm{i})/(z+\mathrm{i})$ are of the form

$$f_0(z) = (1-w)^k \sum_{n=0}^{\infty} a_{2n+1} w^{2n+1}, \qquad f_2(z) = (1-w)^{k-4}\sum_{n=0}^{\infty} c_{2n+1} w^{2n+1},$$

while the expansion of $f_1(z)$ is of the form

$$f_1(z) = (1-w)^{k-2}\sum_{n=0}^{\infty} b_{2n}w^{2n}.$$

(See Proposition 5.1 and Remark 5.2 below.)

Let $h_j(z)$, $j=1,2,3$, be given by (2.3). Then the local exponent of $ah_1(z)+bh_2(z)+ch_3(z)$ at $z=\mathrm{i}$ must be one of

$$\bigl\{0, \kappa_{\mathrm{i}}^{(2)}-\kappa_{\mathrm{i}}^{(1)}, \kappa_{\mathrm{i}}^{(3)}-\kappa_{\mathrm{i}}^{(1)}\bigr\} \tag{2.10}$$

for any $(a,b,c)\in\mathbb{C}^3$. Consider the function

$$\begin{aligned} h_1+\mathrm{i}h_2-h_3 &= z^2\bigl(f_0+f_1E_2+f_2E_2^2\bigr)+\alpha z(f_1+2f_2E_2)+\alpha^2 f_2 \\ &\quad + 2\mathrm{i}z\bigl(f_0+f_1E_2+f_2E_2^2\bigr)+\mathrm{i}\alpha(f_1+2f_2E_2)-\bigl(f_0+f_1E_2+f_2E_2^2\bigr) \\ &= (z+\mathrm{i})^2 f_0+(z+\mathrm{i})((z+\mathrm{i})E_2+\alpha)f_1+((z+\mathrm{i})E_2+\alpha)^2 f_2. \end{aligned}$$

We compute that $z=\mathrm{i}(1+w)/(1-w)$ and hence

$$z+\mathrm{i}=\frac{2\mathrm{i}}{1-w}, \qquad \frac{\mathrm{d}w}{\mathrm{d}z}=\frac{2\mathrm{i}}{(z+\mathrm{i})^2}=\frac{(1-w)^2}{2\mathrm{i}}. \tag{2.11}$$

Also, since $E_2=\frac{1}{2\pi\mathrm{i}}\mathrm{d}\log\Delta(z)/\mathrm{d}z$ and the expansion of $\Delta(z)$ is of the form

$$\Delta(z)=(1-w)^{12}\sum_{n=0}^{\infty} d_{2n}w^{2n},$$

we find that

$$\begin{aligned} E_2(z) &= -\frac{(1-w)^2}{4\pi}\left(-\frac{12}{1-w}+\frac{\sum 2nd_{2n}w^{2n-1}}{\sum d_{2n}w^{2n}}\right) \\ &= \frac{3}{\pi}(1-w)-\frac{(1-w)^2}{4\pi}\sum_{n=0}^{\infty} d'_{2n+1}w^{2n+1} \end{aligned} \tag{2.12}$$

for some power series $\sum d'_{2n+1}w^{2n+1}$. It follows that, by (2.11),

$$(z+\mathrm{i})E_2(z)+\frac{6}{\pi\mathrm{i}}=\frac{1-w}{2\pi\mathrm{i}}\sum_{n=0}^{\infty} d'_{2n+1}w^{2n+1}.$$

From this, we see that

$$h_1+\mathrm{i}h_2-h_3=(1-w)^{k-2}\sum_{n=0}^{\infty} e_{2n+1}w^{2n+1}$$

for some e_j. This, together with (2.10), implies that either $\kappa_{\mathrm{i}}^{(2)}-\kappa_{\mathrm{i}}^{(1)}$ or $\kappa_{\mathrm{i}}^{(3)}-\kappa_{\mathrm{i}}^{(1)}$ is odd. This proves the assertion (3) that $\bigl\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\bigr\}\equiv\{0,0,1\} \mod 2$.

The proof of (4) is similar. We consider the function

$$h_1-\overline{\rho}h_2+\overline{\rho}^2h_3=(z-\overline{\rho})^2f_0+(z-\overline{\rho})((z-\overline{\rho})E_2+\alpha)f_1+((z-\overline{\rho})E_2+\alpha)^2f_2.$$

Setting $w=(z-\rho)/(z-\overline{\rho})$, we have $z=(\rho-\overline{\rho}w)/(1-w)$,

$$z-\overline{\rho}=\frac{\sqrt{3}\mathrm{i}}{1-w}, \qquad \frac{\mathrm{d}w}{\mathrm{d}z}=\frac{(1-w)^2}{\sqrt{3}\mathrm{i}},$$

and

$$(z-\overline{\rho})E_2(z)+\frac{6}{\pi \mathrm{i}}=\frac{1-w}{2\pi \mathrm{i}}\sum_{n=0}^{\infty}d'_{3n+2}w^{3n+2} \tag{2.13}$$

(since the expansion of $\Delta(z)$ is $(1-w)^{12}\sum d_{3n}w^{3n}$ for some d_{3n}). When $k\equiv 1 \bmod 3$, the expansions of f_j are of the form $f_0=(1-w)^k\sum a_{3n+1}w^{3n+1}$, $f_1=(1-w)^{k-2}\sum b_{3n+2}w^{3n+2}$, and $f_2=(1-w)^{k-4}\sum c_{3n}w^{3n}$. Therefore, the expansion of $h_1-\overline{\rho}h_2+\overline{\rho}^2h_3$ is of the form

$$h_1-\overline{\rho}h_2+\overline{\rho}^2h_3=(1-w)^{k-2}\sum_{n=0}^{\infty}e_{3n+1}w^{3n+1},$$

which implies that either $\kappa_\rho^{(2)}-\kappa_\rho^{(1)}$ or $\kappa_\rho^{(3)}-\kappa_\rho^{(1)}$ is congruent to 1 modulo 3. Since the sum of $\kappa_\rho^{(j)}$ is 3, we deduce that the set $\{\kappa_\rho^{(1)},\kappa_\rho^{(2)},\kappa_\rho^{(3)}\}$ is congruent to $\{0,1,2\}$ modulo 3. Likewise, when $k\equiv 2 \bmod 3$, we can show that $h_1-\overline{\rho}h_2+\overline{\rho}^2h_3=(1-w)^{k-2}\sum e_{3n+2}w^{3n+2}$ and obtain the same conclusion.

For the case $k\equiv 0 \bmod 3$, we need to make the computation more precise. Let

$$\begin{aligned}
f_0(z)&=(1-w)^k\sum_{n=0}^{\infty}a_{3n}w^{3n},\\
f_1(z)&=(1-w)^{k-2}\sum_{n=0}^{\infty}b_{3n+1}w^{3n+1},\\
f_2(z)&=(1-w)^{k-4}\sum_{n=0}^{\infty}c_{3n+2}w^{3n+2}
\end{aligned}$$

be the expansions of f_j. A computation similar to (2.12) yields

$$E_2(z)=\frac{2\sqrt{3}}{\pi}(1-w)-\frac{(1-w)^2}{2\pi\sqrt{3}}\sum_{n=0}^{\infty}d'_{3n+2}w^{3n+2}$$

and hence

$$\begin{aligned}
f(z)&=(1-w)^k\sum_{n=0}^{\infty}a_{3n}w^{3n}\\
&\quad+(1-w)^{k-1}\left(\frac{2\sqrt{3}}{\pi}-\frac{1-w}{2\pi\sqrt{3}}\sum_{n=0}^{\infty}d'_{3n+2}w^{3n+2}\right)\left(\sum_{n=0}^{\infty}b_{3n+1}w^{3n+1}\right)\\
&\quad+(1-w)^{k-2}\left(\frac{2\sqrt{3}}{\pi}-\frac{1-w}{2\pi\sqrt{3}}\sum_{n=0}^{\infty}d'_{3n+2}w^{3n+2}\right)^2\left(\sum_{n=0}^{\infty}c_{3n+2}w^{3n+2}\right).
\end{aligned}$$

On the other hand, by (2.13), we have

$$\begin{aligned}
&h_1-\overline{\rho}h_2+\overline{\rho}^2h_3\\
&\quad=(1-w)^{k-2}\left(-3\sum_{n=0}^{\infty}a_{3n}w^{3n}+\frac{\sqrt{3}}{2\pi}\left(\sum_{n=0}^{\infty}d'_{3n+2}w^{3n+2}\right)\left(\sum_{n=0}^{\infty}b_{3n+1}w^{3n+1}\right)\right.\\
&\qquad\left.-\frac{1}{4\pi^2}\left(\sum_{n=0}^{\infty}d'_{3n+2}w^{3n+2}\right)^2\left(\sum_{n=0}^{\infty}c_{3n+2}w^{3n+2}\right)\right).
\end{aligned}$$

We then check that the expansion of $h_1 - \overline{\rho} h_2 + \overline{\rho}^2 h_3 + 3f$ is of the form

$$(1-w)^{k-2}\sum_{n=0}^{\infty}\left(e_{3n+1}w^{3n+1}+e_{3n+2}w^{3n+2}\right),$$

which again implies that either $\kappa_\rho^{(2)} - \kappa_\rho^{(1)}$ or $\kappa_\rho^{(3)} - \kappa_\rho^{(1)}$ is not congruent to 0 modulo 3 and hence $\{\kappa_\rho^{(1)}, \kappa_\rho^{(2)}, \kappa_\rho^{(3)}\} \equiv \{0,1,2\} \bmod 3$. This completes the proof. ■

3 The MODE on $\mathrm{SL}(2,\mathbb{Z})$

The purpose of this section is to prove Theorems 1.3 and 1.4, and Corollary 1.5. Let $Q_2(z)$ and $Q_3(z) - \frac{1}{2}Q_2(z)$ be meromorphic modular forms on $\mathrm{SL}(2,\mathbb{Z})$ of weight 4 and 6 respectively, i.e.,

$$Ly := y'''(z) + Q_2(z)y'(z) + Q_3(z)y(z) = 0, \qquad z \in \mathbb{H} \tag{3.1}$$

is a MODE. In this section, we use the notations $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $R = ST = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ such that $S^2 = R^3 = -I_2$. Suppose the MODE (3.1) has regular singularities at $\{z_1, \dots, z_m\} \sqcup \{\mathrm{i}, \rho, \infty\} \bmod \mathrm{SL}(2,\mathbb{Z})$, where $\rho = \left(-1+\sqrt{3}\mathrm{i}\right)/2$ is the fixed point of R.

3.1 Proof of Theorem 1.4(1)

First we want to give the proof of Theorem 1.4(1), which is long and will be separated into several lemmas. For this purpose, throughout this section we always assume that

(S1) $\kappa_\infty^{(1)}, \kappa_{\mathrm{i}}^{(1)}, \kappa_{z_j}^{(1)} \in \frac{1}{3}\mathbb{Z}_{\le 0}$ and $\kappa_\rho^{(1)} \in \mathbb{Z}_{\le 0}$ such that

$$\ell := -2 - 12\kappa_\infty^{(1)} - 4\kappa_\rho^{(1)} - 6\kappa_{\mathrm{i}}^{(1)} - 12\sum_{j=1}^{m}\kappa_{z_j}^{(1)} \in \mathbb{Z}.$$

(Note $\ell \in \mathbb{Z}$ implies $\ell \in 2\mathbb{Z}$.) Furthermore, $m_z^{(j)} := \kappa_z^{(j+1)} - \kappa_z^{(j)} - 1 \in \mathbb{Z}_{\ge 0}$ for $z \in \{z_1, \dots, z_m\} \sqcup \{\mathrm{i}, \rho\}$ and $j = 1, 2$, and $m_\infty^{(j)} := \kappa_\infty^{(j+1)} - \kappa_\infty^{(j)} \in \mathbb{Z}_{\ge 0}$ for $j = 1, 2$.

(S2) ODE (3.1) is apparent at any singular point $z \in \{z_1, \dots, z_m\} \sqcup \{\mathrm{i}, \rho\}$.

(S3) $\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\} \equiv \{0, 0, 1\} \bmod 2$.

Remark 3.1. Note that (S1) and (S2) are equivalent to the assumptions (H1)–(H3) and $\kappa_\rho^{(1)} \in \mathbb{Z}_{\le 0}$ in Theorem 1.3, while (S3) is needed to obtain quasimodular forms as stated in Theorem 1.3(1). In view of Theorem 1.1, the above assumptions (S1)–(S3) are necessary for the validity of Theorem 1.4(1). We will prove below that they are also sufficient.

Set $t_j := E_4(z_j)^3/E_6(z_j)^2 \notin \{0, 1, \infty\}$ and $F_j(z) := E_4(z)^3 - t_j E_6(z)^2$. Then this modular form $F_j(z)$ has only one simple zero at z_j, up to $\mathrm{SL}(2,\mathbb{Z})$-equivalence. Define

$$F(z) := \Delta(z)^{-\kappa_\infty^{(1)}} E_4(z)^{-\kappa_\rho^{(1)}} E_6(z)^{-\kappa_{\mathrm{i}}^{(1)}} \prod_{j=1}^{m} F_j(z)^{-\kappa_{z_j}^{(1)}}.$$

Then $F(z)^3$ is a modular form of weight $3(\ell+2)$.

Clearly for any solution $y(z)$ of (3.1),

$$\hat{y}(z) := F(z)y(z)$$

is single-valued and holomorphic on $\mathbb{H}$. Furthermore, its order at $z \in \{z_1, \dots, z_m\} \sqcup \{\mathrm{i}, \rho\}$ is one of $\{0, m_z^{(1)} + 1, m_z^{(1)} + m_z^{(2)} + 2\}$, and at ∞ is one of $\{0, m_\infty^{(1)}, m_\infty^{(1)} + m_\infty^{(2)}\}$.

Fix a fundamental system of solutions $Y(z) = (y_1(z), y_2(z), y_3(z))^t$ of (3.1) and let $\hat{Y}(z) := F(z)Y(z)$. Then for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, there is a matrix $\hat{\rho}(\gamma) \in \mathrm{GL}(3,\mathbb{C})$ such that

$$\big(\hat{Y}\big|_\ell \gamma\big)(z) = \hat{\rho}(\gamma)\hat{Y}(z). \tag{3.2}$$

This is a lifting of the Bol representation. We will use freely the notation $\hat{\gamma} = \hat{\rho}(\gamma)$ just for convenience.

Lemma 3.2. *There holds* $\det \hat{\rho}(\gamma) = 1$ *for any* $\gamma \in \mathrm{SL}(2,\mathbb{Z})$*. That is,* $\hat{\rho}$ *is a group homomorphism from* $\mathrm{SL}(2,\mathbb{Z})$ *to* $\mathrm{SL}(3,\mathbb{C})$*.*

Proof. The proof is similar to that of [25, Lemma 4.2], where the second-order MODE was studied. Let

$$W(z) = \det \begin{pmatrix} y_1 & y_1' & y_1'' \\ y_2 & y_2' & y_2'' \\ y_3 & y_3' & y_3'' \end{pmatrix}, \qquad \hat{W}(z) = \det \begin{pmatrix} \hat{y}_1 & \hat{y}_1' & \hat{y}_1'' \\ \hat{y}_2 & \hat{y}_2' & y_2'' \\ \hat{y}_3 & \hat{y}_3' & \hat{y}_3'' \end{pmatrix}.$$

Then $W(z) \equiv C$ is a nonzero constant. By (3.2) we have

$$\big(\hat{W}\big|_{3(\ell+2)}\gamma\big)(z) = \det \hat{\rho}(\gamma)\hat{W}(z).$$

Since $\hat{W}(z) = F(z)^3 W(z) = CF(z)^3$, we also have

$$\hat{W}\big|_{3(\ell+2)}\gamma = C\big(F^3\big|_{3(\ell+2)}\gamma\big) = \frac{\big(F^3\big|_{3(\ell+2)}\gamma\big)(z)}{F(z)^3}\hat{W}(z).$$

Thus

$$\det \hat{\rho}(\gamma) = \frac{\big(F^3\big|_{3(\ell+2)}\gamma\big)(z)}{F(z)^3}.$$

This proves $\det \hat{\rho}(\gamma) = 1$ because $F(z)^3$ is a modular form of weight $3(\ell+2)$ on $\mathrm{SL}(2,\mathbb{Z})$. ■

Remark that under our assumption $\kappa_\infty^{(1)}, \kappa_{\mathrm{i}}^{(1)}, \kappa_{z_j}^{(1)} \in \frac{1}{3}\mathbb{Z}_{\le 0}$ and $\kappa_\rho^{(1)} \in \mathbb{Z}_{\le 0}$, $y(z)^3$ is a single-valued and meromorphic function on $\mathbb{H}$ for any solution $y(z)$ of (3.1).

Lemma 3.3. *Under the assumptions* $(S1)$–$(S3)$*, there is at least one solution* $y(z)$ *of* (3.1) *such that* $y(z)^3$ *is not a meromorphic modular form of weight* -6*.*

Proof. Suppose the conclusion is not true, namely $y(z)^3$ is a meromorphic modular form of weight -6 for any solution $y(z)$. Then by the well-known valence formula for modular forms (see, e.g., [30]), we obtain

$$\frac{\mathrm{ord}_{\mathrm{i}}\,(y^3)}{2} + \frac{\mathrm{ord}_\rho\,(y^3)}{3} \equiv \frac{1}{2} \mod \mathbb{Z},$$

so $\mathrm{ord}_{\mathrm{i}}\,(y^3)$ is odd. Since $\mathrm{ord}_{\mathrm{i}}\,(y^3)$ can be chosen as any one of $3\kappa_{\mathrm{i}}^{(1)}$, $3\kappa_{\mathrm{i}}^{(2)}$, $3\kappa_{\mathrm{i}}^{(3)}$, these three numbers are all odd, clearly a contradiction with our assumption (S3). ■

Lemma 3.4. *Under the assumptions* $(S1)$–$(S3)$*,* (3.1) *is completely not apparent at* ∞*.*

To prove Lemma 3.4, we need the following well-known lemma due to Beukers and Heckman [2].

Lemma 3.5 ([2])**.** *Let $n \geq 2$ and $H \subset \mathrm{GL}(n,\mathbb{C})$ be a subgroup generated by two matrices A, B such that $\operatorname{rank}(A-B) \leq 1$. Then H acts irreducibly on $\mathbb{C}^n$ if and only if A and B have distinct eigenvalues.*

Let $\{a_1,\dots,a_n\}$ and $\{b_1,\dots,b_n\}$ be the eigenvalues of A and B respectively. The following lemma, due to Levelt, is to recover A and B by their eigenvalues. See [2] for a proof.

Lemma 3.6 (cf. [2])**.** *Suppose that $\operatorname{rank}(A-B) = 1$ and $a_1,\dots,a_n$, $b_1,\dots,b_n$ are all nonzero complex numbers with $a_i \neq b_j$ for any i, j. Then up to a common conjugation in $\mathrm{GL}(n,\mathbb{C})$, A and B can be uniquely determined by*

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_n \\ 1 & 0 & \cdots & 0 & -A_{n-1} \\ 0 & 1 & \cdots & 0 & -A_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_n \\ 1 & 0 & \cdots & 0 & -B_{n-1} \\ 0 & 1 & \cdots & 0 & -B_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -B_1 \end{pmatrix},$$

where A_j's and B_j's are given by

$$\prod_{j=1}^{n}(t-a_j) = t^n + A_1 t^{n-1} + \cdots + A_n,$$
$$\prod_{j=1}^{n}(t-b_j) = t^n + B_1 t^{n-1} + \cdots + B_n.$$

Corollary 3.7. *Let A, B be 3×3 matrices such that $\operatorname{rank}(A-B) \leq 1$. Suppose that $A^2 = I_3$ and the eigenvalues of A are $\{1,-1,-1\}$. Then A, B have common eigenvalues and so the group H generated by A and B acts on $\mathbb{C}^3$ reducibly.*

Proof. This corollary is trivial if $\operatorname{rank}(A-B) = 0$, i.e., $A = B$. So we may assume $\operatorname{rank}(A-B) = 1$. Assume by contradiction that A and B have no common eigenvalues, then by $(t-1)(t+1)^2 = t^3 + t^2 - t - 1$, we see from Lemma 3.6 that A is conjugate to

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

and so $A^2 \neq I_3$, a contradiction with the assumption $A^2 = I_3$.

Thus A and B have common eigenvalues, and it follows from Lemma 3.5 that H acts on $\mathbb{C}^3$ reducibly. ■

Proof of Lemma 3.4. Suppose that the statement is not true. Then $\operatorname{rank}\big(\hat{T} - I_3\big) \leq 1$ (note $\hat{T} = I_3$ if and only if ∞ is apparent). Then $\hat{R} = \hat{S}\hat{T}$ implies $\operatorname{rank}\big(\hat{S} - \hat{R}\big) \leq 1$.

By $\hat{R}^3 = \hat{\rho}\big(R^3\big) = \hat{\rho}(-I_2) = (-1)^\ell I_3 = I_3$ and $\det \hat{R} = 1$, we have either $\hat{R} = \lambda I_3$ for some $\lambda^3 = 1$ or $\hat{R}$ is conjugate to $\operatorname{diag}\big(1,\varepsilon,\varepsilon^2\big)$ where $\varepsilon = \mathrm{e}^{2\pi\mathrm{i}/3}$. Similarly, by $\hat{S}^2 = \hat{\rho}\big(S^2\big) = \hat{\rho}(-I_2) = I_3$ and $\det \hat{S} = 1$, we have either $\hat{S} = I_3$ or $\hat{S}$ is a conjugate of $\operatorname{diag}(1,-1,-1)$.

If $\hat{S} = I_3$, then by $\operatorname{rank}\big(\hat{S} - \hat{R}\big) \leq 1$ we obtain $\hat{R} = I_3$. This implies that for any solution $y(z)$ of (3.1), $\hat{y}(z)^3 = F(z)^3 y(z)^3$ is a modular form of weight 3ℓ and so $y(z)^3$ is a meromorphic modular form of weight -6, a contradiction with Lemma 3.3.

Thus $\hat{S}$ is a conjugate of $\operatorname{diag}(1,-1,-1)$. If $\hat{R} = \lambda I_3$ for some $\lambda^3 = 1$, then by $\lambda \neq -1$ we obtain

$$1 \geq \operatorname{rank}\big(\hat{S} - \hat{R}\big) = \operatorname{rank}(\operatorname{diag}(1-\lambda, -1-\lambda, -1-\lambda)) \geq 2,$$

a contradiction.

So $\hat{R}$ is conjugate to $\operatorname{diag}\left(1,\varepsilon,\varepsilon^2\right)$. By Corollary 3.7, there is a subspace $V \subsetneqq \mathbb{C}^3$ which is invariant under the actions $\hat{S}$ and $\hat{R}$. If $\dim V = 2$, then there is an invertible matrix P such that

$$P\hat{S}P^{-1} = \begin{pmatrix} A_1 & 0 \\ * & a_1 \end{pmatrix}, \qquad P\hat{R}P^{-1} = \begin{pmatrix} B_1 & 0 \\ * & b_1 \end{pmatrix},$$

where A_1 and B_1 are 2×2 matrices. This implies

$$\operatorname{rank}\begin{pmatrix} A_1 - B_1 & 0 \\ * & a_1 - b_1 \end{pmatrix} = \operatorname{rank}\left(\hat{R} - \hat{S}\right) \le 1. \tag{3.3}$$

Note $a_1 \in \{1,-1\}$ and $b_1 \in \left\{1,\varepsilon,\varepsilon^2\right\}$. If $a_1 \neq b_1$, then (3.3) implies $A_1 = B_1$, namely $\hat{S}$ and $\hat{R}$ have two common eigenvalues, a contradiction. So $a_1 = b_1 = 1$. Then the eigenvalues of A_1 are $\{-1,-1\}$, so $A_1 = -I_2$. Similarly, B_1 is conjugate to $\operatorname{diag}\left(\varepsilon,\varepsilon^2\right)$. Thus

$$1 \ge \operatorname{rank}(A_1 - B_1) = \operatorname{rank}\left(\operatorname{diag}\left(-1-\varepsilon,-1-\varepsilon^2\right)\right) = 2,$$

a contradiction. So $\dim V = 1$, which implies the existence of an invertible matrix P such that

$$P\hat{S}P^{-1} = \begin{pmatrix} a_1 & 0 \\ * & A_1 \end{pmatrix}, \qquad P\hat{R}P^{-1} = \begin{pmatrix} b_1 & 0 \\ * & B_1 \end{pmatrix},$$

where A_1 and B_1 are 2×2 matrices. Clearly the same argument as (3.3) also yields a contradiction. This completes the proof. ∎

Let $\hat{y}_3(z) := \hat{y}_+(z) = F(z)y_+(z)$, $\hat{y}_1(z) := (\hat{y}_3\big|_\ell S)(z)$ and $\hat{y}_2(z) := (\hat{y}_3\big|_\ell R)(z)$, and $y_j(z) := \hat{y}_j(z)/F(z)$ for $j = 1,2,3$.

Lemma 3.8. *Under the assumptions (S1)–(S3), $\hat{y}_1(z)$, $\hat{y}_2(z)$ and $\hat{y}_3(z)$ are linearly independent and $\hat{y}_1(z)$ can be written as*

$$\hat{y}_1(z) = \beta z^2 \hat{y}_3(z) + z\hat{m}_1^*(z) + \hat{m}_2(z), \tag{3.4}$$

where $\beta \neq 0$ is a constant and $\hat{m}_1^*(z)$, $\hat{m}_2(z)$ are of the form

$$\frac{\hat{m}_1^*(z)}{F(z)} = m_1^*(z) = q^{\kappa_\infty^{(2)}} \sum_{j\ge 0} c_{j,1} q^j, \qquad \frac{\hat{m}_2(z)}{F(z)} = m_2(z) = q^{\kappa_\infty^{(1)}} \sum_{j\ge 0} c_{j,2} q^j. \tag{3.5}$$

Proof. Under our assumption, Lemma 3.4 says that ∞ is completely not apparent, so it follows from Remark A.9 that (3.1) has a basis of solutions of the form $(y_-, y_\perp, y_+)$, where y_+, $y_\perp$, and y_- are given by (A.19) and (A.20).

Step 1. We show that $\hat{y}_1(z)$ is linearly independent with $\hat{y}_3(z)$.

Suppose not, i.e., there is some constant $\alpha \neq 0$ such that

$$(\hat{y}_3\big|_\ell S)(z) = \hat{y}_1(z) = \alpha \hat{y}_3(z).$$

Then with respect to $(Fy_-, Fy_\perp, \hat{y}_3(z))^t$, we have

$$\hat{\rho}(S) = \begin{pmatrix} S_1 & * \\ 0 & \alpha \end{pmatrix},$$

where S_1 is a 2×2 matrix. Then $\hat{\rho}(S)^2 = I_3$ implies $S_1^2 = I_2$ and $\alpha^2 = 1$, i.e., $\alpha = \pm 1$.

On the other hand, it follows from the expressions of $(y_-, y_\perp, y_+)$ in Remark A.9 that with respect to $(Fy_-, Fy_\perp, \hat{y}_3(z))^t$,

$$\hat{\rho}(T) = \begin{pmatrix} 1 & 2 & * \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

so

$$\hat{\rho}(R) = \hat{\rho}(S)\hat{\rho}(T) = \begin{pmatrix} S_1 & * \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 2 & * \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

From this and $\hat{\rho}(R)^3 = I_3$, we obtain

$$(S_1 R_1)^3 = I_2, \qquad \text{where} \quad R_1 := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \tag{3.6}$$

and $\alpha^3 = 1$, so $\alpha = 1$. Then $\det S_1 = \det \hat{\rho}(S) = 1$, which together with $S_1^2 = I_2$ easily implies $S_1 = \pm I_2$, and then (3.6) yields $\pm\left(\begin{smallmatrix} 1 & 6 \\ 0 & 1 \end{smallmatrix}\right) = I_2$, a contradiction.

Step 2. By Step 1, there exists $(\beta, \delta) \neq (0, 0)$ and ϵ such that

$$\hat{y}_1 = \beta F y_- + \delta F y_\perp + \epsilon \hat{y}_3. \tag{3.7}$$

We claim that $\beta \neq 0$ and (3.4) holds.

Assume by contradiction that $\beta = 0$, i.e., $\hat{y}_1 = \delta F y_\perp + \epsilon \hat{y}_3$ with $\delta \neq 0$. Then with respect to $(Fy_-, \hat{y}_1, \hat{y}_3)$ we have

$$\hat{\rho}(T) = \begin{pmatrix} 1 & \frac{2}{\delta} & a \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{for some } a \in \mathbb{C},$$

and

$$\hat{\rho}(S) = \begin{pmatrix} -1 & b & b \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{for some } b \in \mathbb{C},$$

where we used $\hat{\rho}(S)^2 = I_3$, $\det \hat{\rho}(S) = 1$, $\hat{y}_1(z) = (\hat{y}_3\big|_\ell S)(z)$ and

$$(\hat{y}_1\big|_\ell S)(z) = (\hat{y}_3\big|_\ell(-I_2))(z) = (-1)^{-\ell}\hat{y}_3(z) = \hat{y}_3(z). \tag{3.8}$$

Thus

$$\hat{\rho}(R) = \hat{\rho}(S)\hat{\rho}(T) = \begin{pmatrix} -1 & b - \frac{2}{\delta} & b + b\delta - a \\ 0 & 0 & 1 \\ 0 & 1 & \delta \end{pmatrix}.$$

But this implies that -1 is an eigenvalue of $\hat{\rho}(R)$, a contradiction with $\hat{\rho}(R)^3 = I_3$. This proves $\beta \neq 0$ and so it follows from the expression of $(y_-, y_\perp, y_+)$ in Remark A.9 that (3.4) and (3.5) hold.

Step 3. We show that $\hat{y}_1(z)$, $\hat{y}_2(z)$ and $\hat{y}_3(z)$ are linearly independent.

In fact, we see from $R = ST$ that $\hat{y}_2(z) = (\hat{y}_+\big|_\ell R)(z) = (\hat{y}_1\big|_\ell T)(z) = \hat{y}_1(z+1)$, from which and (3.4) we obtain

$$\hat{y}_2 = \hat{y}_1 + 2\beta z \hat{y}_3 + \hat{m}_1^* + \beta \hat{y}_3. \tag{3.9}$$

By (3.7) and (3.9) we have

$$\begin{pmatrix}\hat{y}_1\\ \hat{y}_2\\ \hat{y}_3\end{pmatrix}=A\begin{pmatrix}Fy_-\\ Fy_\perp\\ \hat{y}_3\end{pmatrix}\qquad\text{with}\quad A=\begin{pmatrix}\beta & \delta & \epsilon\\ \beta & 2\beta+\delta & *\\ 0 & 0 & 1\end{pmatrix}.$$

Since $\det A=2\beta^2\neq 0$, we obtain that $\hat{y}_1(z)$, $\hat{y}_2(z)$ and $\hat{y}_3(z)$ are also linearly independent. This completes the proof. ■

Recalling (3.4), we set $\hat{m}_1(z)$ and $\hat{m}_0(z)$ to be

$$\hat{m}_1^*(z)=\hat{m}_1(z)+\frac{\pi \mathrm{i}}{3}\hat{m}_2(z)E_2(z), \tag{3.10}$$

$$\hat{y}_3(z)=\left(\frac{\pi \mathrm{i}}{6}\right)^2\hat{m}_2(z)E_2(z)^2+\frac{\pi \mathrm{i}}{6}\hat{m}_1(z)E_2(z)+\hat{m}_0(z). \tag{3.11}$$

The following result can be seen as the converse statement of Theorem 2.2.

Theorem 3.9. *Under the assumptions $(S1)$–$(S3)$, the following hold.*

(a) $\beta=1$.

(b) *$\hat{m}_j(z)$ are meromorphic modular forms of weight $\ell+2-2j$ for $j=0,1,2$, that is, $\hat{y}_3(z)$ is a quasimodular form of weight $\ell+2$ with depth 2.*

Proof. (a) By Lemma 3.8, we can take $\hat{Y}(z)=(\hat{y}_1(z),\hat{y}_2(z),\hat{y}_3(z))^t$ to be a basis and let $\hat{T}$, $\hat{S}$, $\hat{R}$ denote the associated matrices $\hat{\rho}(T)$, $\hat{\rho}(S)$ and $\hat{\rho}(R)$ of the Bol representation. Recalling (3.8) that $(\hat{y}_3\big|_\ell S)=\hat{y}_1(z)$ and $(\hat{y}_1\big|_\ell S)=\hat{y}_3(z)$, we have

$$\hat{S}=\begin{pmatrix}0 & 0 & 1\\ \lambda & -1 & \lambda\\ 1 & 0 & 0\end{pmatrix},\qquad \text{for some } \lambda\in\mathbb{C}, \tag{3.12}$$

where $\hat{S}^2=I_3$ is used. Note from $SR=S^2T=-T$ that

$$\hat{y}_1\big|_\ell R=\hat{y}_3\big|_\ell SR=\hat{y}_3\big|_\ell(-T)=\hat{y}_3,$$

and from $R^2=T^{-1}S$ that

$$\hat{y}_2\big|_\ell R=\hat{y}_3\big|_\ell R^2=\hat{y}_3\big|_\ell\big(T^{-1}S\big)=\hat{y}_3\big|_\ell S=\hat{y}_1,$$

so

$$\hat{R}=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}. \tag{3.13}$$

Therefore,

$$\hat{T}=\hat{S}^{-1}\hat{R}=\hat{S}\hat{R}=\begin{pmatrix}0 & 0 & 1\\ \lambda & -1 & \lambda\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ -1 & \lambda & \lambda\\ 0 & 0 & 1\end{pmatrix}. \tag{3.14}$$

On the other hand, by (3.9) we have

$$\hat{y}_2\big|_\ell T=\hat{y}_1\big|_\ell T+2\beta z\hat{y}_3+\hat{m}_1^*+3\beta\hat{y}_3=-\hat{y}_1+2\hat{y}_2+2\beta\hat{y}_3.$$

Hence (3.14) yields $\lambda = 2$ and $\beta = 1$. This proves (a). In particular, it is easy to see from (3.12)–(3.14) that the representation $\hat{\rho}$ is irreducible, i.e., there is no proper nontrivial subspace of $\mathbb{C}\hat{y}_1 + \mathbb{C}\hat{y}_2 + \mathbb{C}\hat{y}_3$ that is invariant under $\hat{\rho}(\mathrm{SL}(2,\mathbb{Z}))$.

(b) To prove (b), we note that (3.4) and (3.9) become

$$\hat{y}_1(z) = z^2\hat{y}_3(z) + z\hat{m}_1^*(z) + \hat{m}_2(z), \tag{3.15}$$

$$\hat{y}_2 = \hat{y}_1 + 2z\hat{y}_3 + \hat{m}_1^* + \hat{y}_3, \tag{3.16}$$

which implies that

$$y^*(z) := 2zy_3(z) + m_1^*(z)$$

is also a solution of (3.1) and

$$\hat{y}_2 = \hat{y}_1 + \hat{y}^* + \hat{y}_3, \qquad \text{where} \quad \hat{y}^* = Fy^* = 2z\hat{y}_3 + \hat{m}_1^*.$$

By (3.12) and $\lambda = 2$, we have

$$2\left(\tfrac{-1}{z}\right)\hat{y}_1 + \hat{m}_1^*\big|_\ell S = \hat{y}^*\big|_\ell S = (\hat{y}_2 - \hat{y}_1 - \hat{y}_3)\big|_\ell S = \hat{y}_1 - \hat{y}_2 + \hat{y}_3 = -\hat{y}^* = -2z\hat{y}_3 - \hat{m}_1^*.$$

From here and (3.15), we obtain

$$z(\hat{m}_1^*\big|_\ell S)(z) = z\hat{m}_1^*(z) + 2\hat{m}_2(z). \tag{3.17}$$

On the other hand, by (3.15),

$$\hat{y}_3 = \hat{y}_1\big|_\ell S = \left(\tfrac{-1}{z}\right)^2\hat{y}_1 + \left(\tfrac{-1}{z}\right)(\hat{m}_1^*\big|_\ell S) + \hat{m}_2\big|_\ell S,$$

which implies

$$\hat{y}_1 = z^2\hat{y}_3 + z(\hat{m}_1^*\big|_\ell S) - z^2(\hat{m}_2\big|_\ell S).$$

Again by (3.15), we have

$$z(\hat{m}_1^*\big|_\ell S)(z) = z\hat{m}_1^*(z) + \hat{m}_2(z) + z^2(\hat{m}_2\big|_\ell S)(z). \tag{3.18}$$

Thus (3.17) and (3.18) imply $\hat{m}_2\big|_{\ell-2} S = z^2(\hat{m}_2\big|_\ell S) = \hat{m}_2$. This proves that $\hat{m}_2(z)$ is a modular form of weight $\ell - 2$.

Recalling $E_2\big|_2 S = E_2 + 6/\pi \mathrm{i} z$, it follows from (3.10) and (3.17) that

$$\begin{aligned} z\left(\hat{m}_1 + \frac{\pi \mathrm{i}}{3}\hat{m}_2 E_2\right) + 2\hat{m}_2 = z(\hat{m}_1^*\big|_\ell S) &= z\left\{\hat{m}_1\big|_\ell S + \frac{\pi \mathrm{i}}{3}(\hat{m}_2\big|_{\ell-2} S)\left(E_2 + \frac{6}{\pi \mathrm{i} z}\right)\right\} \\ &= z(\hat{m}_1\big|_\ell S) + \frac{\pi \mathrm{i} z}{3}\hat{m}_2 E_2 + 2\hat{m}_2, \end{aligned}$$

which yields $\hat{m}_1\big|_\ell S = \hat{m}_1$. This proves that $\hat{m}_1(z)$ is a modular form of weight ℓ.

Finally, to prove the modularity of $\hat{m}_0(z)$, we use (3.15) and (3.11) to obtain

$$\begin{aligned} z^2\hat{y}_3 + z\left(\hat{m}_1 + \frac{\pi \mathrm{i}}{3}\hat{m}_2 E_2\right) + \hat{m}_2 &= \hat{y}_1 = \hat{y}_3\big|_\ell S \\ &= \left(\frac{\pi \mathrm{i} z}{6}\right)^2 (\hat{m}_2\big|_{\ell-2} S)\left(E_2 + \frac{6}{\pi \mathrm{i} z}\right)^2 + \frac{\pi \mathrm{i} z^2}{6}(\hat{m}_1\big|_\ell S)\left(E_2 + \frac{6}{\pi \mathrm{i} z}\right) + \hat{m}_0\big|_\ell S \\ &= z^2\hat{y}_3 + z\left(\hat{m}_1 + \frac{\pi \mathrm{i}}{3}\hat{m}_2 E_2\right) + \hat{m}_2 + \hat{m}_0\big|_\ell S - z^2\hat{m}_0, \end{aligned}$$

which implies $\hat{m}_0\big|_{\ell+2} S = z^{-2}(\hat{m}_0\big|_\ell S) = \hat{m}_0$. This proves that $\hat{m}_0(z)$ is a modular form of weight $\ell + 2$. The proof is complete. ■

Clearly the above arguments imply Theorem 1.4(1).

3.2 Proofs of Theorems 1.3 and 1.4, and Corollary 1.5

In this section, we complete the proof of Theorems 1.3 and 1.4, and Corollary 1.5. First we need the following general observation.

Lemma 3.10. *Let (S1)–(S2) hold. Then the eigenvalues of $\hat{\rho}(R)$ are precisely*

$$\mathrm{e}^{-\frac{\pi \mathrm{i}}{3}(\ell+2\kappa)}, \qquad \kappa \in \left\{0, \kappa_\rho^{(2)} - \kappa_\rho^{(1)}, \kappa_\rho^{(3)} - \kappa_\rho^{(1)}\right\}, \tag{3.19}$$

and the eigenvalues of $\hat{\rho}(S)$ are precisely

$$i^{-\ell-2\kappa}, \qquad \kappa \in \left\{0, \kappa_\mathrm{i}^{(2)} - \kappa_\mathrm{i}^{(1)}, \kappa_\mathrm{i}^{(3)} - \kappa_\mathrm{i}^{(1)}\right\}. \tag{3.20}$$

Proof. Under our assumption (S2) that ρ is apparent, it follows from Remark 5.2 and Lemma 5.5 below that any solution $y(z)$ of (3.1) has an expansion of the form

$$\frac{1}{(1-w)^2}\sum_{n=0}^{\infty} a_n w^{n+\kappa}, \qquad w = w(z) = \frac{z-\rho}{z-\overline{\rho}}$$

at ρ, where $a_0 \neq 0$ and $\kappa \in \left\{\kappa_\rho^{(j)} : j = 1,2,3\right\}$. Hence, $\hat{y}(z)$ has an expansion of the form

$$(1-w)^\ell \sum_{n=0}^{\infty} b_n w^{n+\kappa}, \qquad \kappa \in \left\{0, \kappa_\rho^{(2)} - \kappa_\rho^{(1)}, \kappa_\rho^{(3)} - \kappa_\rho^{(1)}\right\}$$

with $b_0 \neq 0$. Recalling that $\rho = \left(-1+\sqrt{3}\mathrm{i}\right)/2$ is a fixed point of $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, we denote

$$\zeta := c\rho + d = \rho + 1 = \mathrm{e}^{\pi \mathrm{i}/3}.$$

Then a direct computation gives

$$w(Rz) = \zeta^{-2} w(z), \qquad 1 - w(Rz) = \zeta^{-1}(z+1)(1-w(z)),$$

so

$$\left(\hat{y}\big|_\ell R\right)(z) = \zeta^{-\ell}(1-w)^\ell \sum_{n=0}^{\infty} b_n \left(\zeta^{-2} w\right)^{n+k}.$$

Therefore, $\hat{y}(z)$ is an eigenfunction of $\hat{\rho}(R)$ if and only if the series expansion of $\hat{y}(z)$ is of the form

$$(1-w)^\ell \sum_{n=0}^{\infty} b_{3n} w^{3n+\kappa}, \qquad \kappa \in \left\{0, \kappa_\rho^{(2)} - \kappa_\rho^{(1)}, \kappa_\rho^{(3)} - \kappa_\rho^{(1)}\right\}$$

and the corresponding eigenvalue is $\zeta^{-\ell-2\kappa} = \mathrm{e}^{-\frac{\pi \mathrm{i}}{3}(\ell+2\kappa)}$. Note from $\hat{\rho}(R)^3 = (-1)^\ell I_3 = I_3$ that $\hat{\rho}(R)$ can be diagonizalied. We claim that the eigenvalues of $\hat{\rho}(R)$ are precisely those in (3.19).

Indeed, for any $\kappa \in \left\{0, \kappa_\rho^{(2)} - \kappa_\rho^{(1)}, \kappa_\rho^{(3)} - \kappa_\rho^{(1)}\right\}$, we define

$$N_\kappa := \#\left\{\tilde{\kappa} \in \left\{0, \kappa_\rho^{(2)} - \kappa_\rho^{(1)}, \kappa_\rho^{(3)} - \kappa_\rho^{(1)}\right\} \,\big|\, \zeta^{-\ell-2\tilde{\kappa}} = \zeta^{-\ell-2\kappa}\right\}.$$

Clearly (3.19) holds if $N_\kappa = 3$ for some κ (and so for all κ). So we only consider the case $N_\kappa \in \{1,2\}$ for all κ. Assume by contradiction that there are $N_\kappa + 1 \in \{2,3\}$ linearly independent eigenfunctions

$$\hat{y}_j = (1-w)^\ell \sum_{n=0}^{\infty} b_{j,3n} w^{3n+\kappa}, \qquad b_{1,0} = b_{2,0} = b_{N_\kappa+1,0} \neq 0, \qquad 1 \leq j \leq N_\kappa + 1,$$

corresponding to the same eigenvalue $\zeta^{-\ell-2\kappa}$. Then

$$\hat{y}_1 - \hat{y}_2 = (1-w)^\ell \sum_{n=n_0}^{\infty} (b_{1,3n} - b_{2,3n}) w^{3n+\kappa}$$

is also an eigenfunction of $\zeta^{-\ell-2\kappa}$, where $n_0 \geq 1$ is the smallest integer such that $b_{1,3n_0} - b_{2,3n_0} \neq 0$. This implies $\kappa, \kappa+3n_0 \in \{0, \kappa_\rho^{(2)} - \kappa_\rho^{(1)}, \kappa_\rho^{(3)} - \kappa_\rho^{(1)}\}$, already a contradiction if $N_\kappa = 1$. If $N_\kappa = 2$, then by using the linear combination of $\hat{y}_1$, $\hat{y}_2$, $\hat{y}_3$, there is another $n_1 \geq 1$ satisfying $n_1 \neq n_0$ such that $\kappa, \kappa+3n_0, \kappa+3n_1 \in \{0, \kappa_\rho^{(2)} - \kappa_\rho^{(1)}, \kappa_\rho^{(3)} - \kappa_\rho^{(1)}\}$, again a contradiction with $N_\kappa = 2$. Thus, for any κ, the dimension of eigenfunctions of $\zeta^{-\ell-2\kappa}$ is at most N_κ. This implies the assertion (3.19).

The proof of (3.20) is similar and is omitted here. ■

Note that if (S3) does not hold, it follows from $\kappa_{\mathrm{i}}^{(1)} + \kappa_{\mathrm{i}}^{(2)} + \kappa_{\mathrm{i}}^{(3)} = 3$ that $\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\} \equiv \{1,1,1\} \bmod 2$. We have

Theorem 3.11. *Let (S1)–(S2) hold and suppose* $\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\} \equiv \{1,1,1\} \bmod 2$. *Then* $12|\ell$ *and for any solution* $y(z)$ *of* (3.1), $\hat{y}(z)$ *is a modular form of weight* ℓ. *In particular, the representation* $\hat{\rho}$ *is trivial.*

Proof. By (3.20) and $\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\} \equiv \{1,1,1\} \bmod 2$, we see that the eigenvalues of $\hat{S}$ are all the same, so we see from $\hat{S}^2 = I_3$ that $\hat{S} = I_3$. Consequently, $\hat{T} = \hat{R}$ and then $\hat{T}^3 = \hat{R}^3 = I_3$. Since the eigenvalues of $\hat{T}$ are $\{1,1,1\}$, we obtain $\hat{R} = \hat{T} = I_3$, i.e., the representation $\hat{\rho}$ is trivial and $\hat{y}(z)$ is a modular form of weight ℓ for any solution $y(z)$. Furthermore, it follows from Lemma 3.10 that $\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}\ell} = \mathrm{i}^{-\ell} = 1$, so $\ell \equiv 0 \bmod 12$. ■

Proof of Theorems 1.3 and 1.4. Theorem 1.3 follows from Theorems 3.9 and 3.11. ■

Proof of Corollary 1.5. Under the assumptions (H1)–(H3) and $\kappa_\rho^{(1)} \in \mathbb{Z}$, we have $\kappa_\rho^{(j)} \in \mathbb{Z}$ for all j. Together with $\kappa_\rho^{(1)} + \kappa_\rho^{(2)} + \kappa_\rho^{(3)} = 3$, we have either $\kappa_\rho^{(1)} \equiv \kappa_\rho^{(2)} \equiv \kappa_\rho^{(3)} \bmod 3$ or $\{\kappa_\rho^{(1)}, \kappa_\rho^{(2)}, \kappa_\rho^{(3)}\} \equiv \{0,1,2\} \bmod 3$.

First suppose $\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\} \equiv \{0,0,1\} \bmod 2$. Then Theorem 3.9 holds, in particular, $\hat{R} \neq I_3$ and the eigenvalues can not be all the same. This together with (3.19) imply that $\kappa_\rho^{(1)} \equiv \kappa_\rho^{(2)} \equiv \kappa_\rho^{(3)} \bmod 3$ is impossible, so

$$\{\kappa_\rho^{(1)}, \kappa_\rho^{(2)}, \kappa_\rho^{(3)}\} \equiv \{0,1,2\} \mod 3. \tag{3.21}$$

Conversely, suppose (3.21) holds. If $\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\} \equiv \{1,1,1\} \bmod 2$, then Theorem 3.11 implies $\hat{R} = I_3$, which together with (3.19) imply $\kappa_\rho^{(1)} \equiv \kappa_\rho^{(2)} \equiv \kappa_\rho^{(3)} \bmod 3$, a contradiction with (3.21). Thus $\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\} \equiv \{0,0,1\} \bmod 2$. ■

Remark 3.12. We note the under the assumption that the eigenvalues of $\hat{\rho}(T)$ are all 1, Proposition 2.5 and Corollary to Theorem 2.9 of [31] and results of [32] imply that $\hat{\rho}$ is irreducible if and only if the eigenvalues of $\hat{\rho}(S)$ and $\hat{\rho}(R)$ are 1, -1, -1 and 1, $\mathrm{e}^{2\pi\mathrm{i}/3}$, $\mathrm{e}^{-2\pi\mathrm{i}/3}$, respectively. Our Theorem 1.3 shows that the irreducibility property of $\hat{\rho}$ is solely determined by the local exponents at i. This link between the results of [31, 32] and Theorem 1.3 is provided by Lemma 3.10. In other words, one may also use results of [31, 32] and Lemma 3.10 to give an alternative proof of Theorem 1.3. Our approach has the advantage that it directly shows that $\hat{y}_+(z)$ is a quasimodular form of depth 2. (Note that Westbury's paper [32] does not seem to be easily available. We refer the reader to the introduction section of [22] for a quick review of Westbury's results.)

4 Reducibility and SU(3) Toda systems on $\mathrm{SL}(2,\mathbb{Z})$

In view of Theorem 3.11 or equivalently Theorem 1.4(2), it is natural to ask whether the converse statement holds or not. The purpose of this section is to establish such a converse statement and apply it to the SU(3) Toda system. Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ commensurable with $\mathrm{SL}(2,\mathbb{Z})$. In general, there are at least three sources of modular forms and quasimodular forms that will give rise to third-order MODEs on Γ:

(i) If $f(z) \in \widetilde{\mathfrak{M}}_k^{\le 2}(\Gamma,\chi)$, then $f(z)/\sqrt[3]{W_f(z)}$ satisfies a third-order MODE on Γ. This case has been studied in Section 2.

(ii) If $f(z) = f_1(z)\phi(z) + f_0(z) \in \widetilde{\mathfrak{M}}_k^{\le 1}(\Gamma,\chi_1)$ and $g(z) \in \mathfrak{M}_{k-1}(\Gamma,\chi_2)$ with $\chi_1(-I_2)\chi_2(-I_2) = -1$, then a similar argument as Theorem 2.2 shows that

$$f(z)/\sqrt[3]{W_{f,g}(z)}, \qquad (zf+\alpha f_1)/\sqrt[3]{W_{f,g}(z)} \qquad \text{and} \qquad g(z)/\sqrt[3]{W_{f,g}(z)}$$

are solutions of some third-order MODE on Γ. Here

$$W_{f,g}(z) = \det\begin{pmatrix} f & f' & f'' \\ zf+\alpha f_1 & (zf+\alpha f_1)' & (zf+\alpha f_1)'' \\ g & g' & g'' \end{pmatrix}.$$

(iii) If $f(z) \in \mathfrak{M}_k(\Gamma,\chi_1)$, $g(z) \in \mathfrak{M}_k(\Gamma,\chi_2)$, and $h(z) \in \mathfrak{M}_k(\Gamma,\chi_3)$ for some characters χ_j of Γ, then $f(z)/\sqrt[3]{W_{f,g,h}(z)}$, $g(z)/\sqrt[3]{W_{f,g,h}(z)}$, $h(z)/\sqrt[3]{W_{f,g,h}(z)}$ are solutions of some third-order MODE on Γ. Here

$$W_{f,g,h}(z) = \det\begin{pmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{pmatrix}. \tag{4.1}$$

To simplify the situation, we impose the condition that the values of $\chi_j(T)$ in the second and the third cases are all the same (so that $\rho(T)$ has only one eigenvalue with multiplicity 3). In the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, this condition implies that χ_j are all the same, say, $\chi_j = \chi$ for all j, so Case (ii) will not occur. Moreover, in Case (iii), we can divide f, g, h by an eta-power $\eta(z)^m$ satisfying $\mathrm{e}^{2\pi \mathrm{i} m/24} = \chi(T)$. The differential equation corresponding to $f(z)/\eta(z)^m$ is the same as that corresponding to $f(z)$. Thus, in the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, we may assume that χ is trivial.

Lemma 4.1. *Let $f,g,h \in \mathfrak{M}_k(\mathrm{SL}(2,\mathbb{Z}))$ be three linearly independent modular forms of weight k on $\mathrm{SL}(2,\mathbb{Z})$. Define $W_{f,g,h}(z)$ by (4.1). Then $W_{f,g,h}$ is a modular form of weight $3(k+2)$ on $\mathrm{SL}(2,\mathbb{Z})$.*

Proof. Let $F(z) = (f(z),g(z),h(z))^t$, which satisfies $F(\gamma z) = (cz+d)^k F(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. Then the assertion follows from the basic properties of the determinant function. ■

Theorem 4.2. *Let $f,g,h \in \mathfrak{M}_k(\mathrm{SL}(2,\mathbb{Z}))$ be linearly independent modular forms and $\mathcal{L}y = 0$ be the differential equation satisfied by $f/\sqrt[3]{W_{f,g,h}}$, $g/\sqrt[3]{W_{f,g,h}}$, and $h/\sqrt[3]{W_{f,g,h}}$. Then $Q_2(z)$ and $Q_3(z) - \frac{1}{2}Q_2'(z)$ are meromorphic modular forms of weight 4 and 6 respectively. Furthermore, $(H1)$–$(H3)$ hold and ∞ is also apparent.*

Proof. The proof is similar as that of Theorem 2.2. The only difference is that ∞ is also apparent because $f/\sqrt[3]{W_{f,g,h}}$, $g/\sqrt[3]{W_{f,g,h}}$ and $h/\sqrt[3]{W_{f,g,h}}$ are linearly independent solutions. ■

Proposition 4.3. *Let $f,g,h \in \mathfrak{M}_k(\mathrm{SL}(2,\mathbb{Z}))$ be linearly independent modular forms and $\mathcal{L}y = 0$ be the differential equation satisfied by $f/\sqrt[3]{W_{f,g,h}}$, $g/\sqrt[3]{W_{f,g,h}}$, and $h/\sqrt[3]{W_{f,g,h}}$. Then the local exponents of $\mathcal{L}y = 0$ at the elliptic points i and $\rho = \big(-1+\sqrt{3}\mathrm{i}\big)/2$ satisfy*

1) $\kappa_{\mathrm{i}}^{(2)}-\kappa_{\mathrm{i}}^{(1)},\kappa_{\mathrm{i}}^{(3)}-\kappa_{\mathrm{i}}^{(1)}\equiv 0 \bmod 2$, *and*

2) $\kappa_{\rho}^{(j)}\in\mathbb{Z}$ *for all* j, *and* $\kappa_{\rho}^{(1)}\equiv\kappa_{\rho}^{(2)}\equiv\kappa_{\rho}^{(3)} \bmod 3$.

Proof. Note that every solution $y(z)$ of $\mathcal{L}y=0$ can be written as

$$(af(z)+bg(z)+ch(z))/\sqrt[3]{W_{f,g,h}(z)}$$

for some $a,b,c\in\mathbb{C}$. As a, b, and c vary, the order of $y(z)$ at i (respectively, ρ) will go through all possible local exponents of $\mathcal{L}y=0$ at i (respectively, ρ). Since

$$\frac{\mathrm{ord}_{\mathrm{i}}(af+bg+cz)}{2}+\frac{\mathrm{ord}_{\rho}(af+bg+cz)}{3}\equiv\frac{\mathrm{ord}_{\mathrm{i}}(f)}{2}+\frac{\mathrm{ord}_{\rho}(f)}{3}\mod\mathbb{Z},$$

so

$$\mathrm{ord}_{\mathrm{i}}(af+bg+cz)-\mathrm{ord}_{\mathrm{i}}(f)\equiv 0\mod 2,$$
$$\mathrm{ord}_{\rho}(af+bg+cz)-\mathrm{ord}_{\rho}(f)\equiv 0\mod 3$$

hold for any $(a,b,c)\neq(0,0,0)$. From here and

$$\kappa_{\mathrm{i}}^{(j)}+\tfrac{1}{3}\,\mathrm{ord}_{\mathrm{i}}\,W_{f,g,h}(z)\in\{\mathrm{ord}_{\mathrm{i}}(af+bg+cz)\,|\,(a,b,c)\neq(0,0,0)\},$$
$$\kappa_{\rho}^{(j)}+\tfrac{1}{3}\,\mathrm{ord}_{\rho}\,W_{f,g,h}(z)\in\{\mathrm{ord}_{\rho}(af+bg+cz)\,|\,(a,b,c)\neq(0,0,0)\}$$

for all j, we obtain the assertion (1) and $\kappa_{\rho}^{(2)}-\kappa_{\rho}^{(1)},\kappa_{\rho}^{(3)}-\kappa_{\rho}^{(1)}\equiv 0 \bmod 3$. This together with $\kappa_{\rho}^{(1)}+\kappa_{\rho}^{(2)}+\kappa_{\rho}^{(3)}=3$ imply $\kappa_{\rho}^{(j)}\in\mathbb{Z}$ for all j and so the assertion (2) holds. ■

The above result is precisely the converse statement of Theorem 1.4(2).

Example 4.4. Recall that the smallest weight k such that $\dim\mathfrak{M}_k(\mathrm{SL}(2,\mathbb{Z}))=3$ is 24. Let $f(z)=E_4(z)^6$, $g(z)=E_4(z)^3\Delta(z)$, and $h(z)=\Delta(z)^2$, which form a basis for $\mathfrak{M}_{24}(\mathrm{SL}(2,\mathbb{Z}))$. To determine the differential equation

$$\mathcal{L}y:=D_q^3y(z)+Q(z)D_qy(z)+\left(\frac{1}{2}D_qQ(z)+R(z)\right)y(z)=0$$

satisfied by $f/\sqrt[3]{W_{f,g,h}}$, $g/\sqrt[3]{W_{f,g,h}}$, and $h/\sqrt[3]{W_{f,g,h}}$, we use Ramanujan's identities

$$D_qE_2=\frac{E_2^2-E_4}{12},\qquad D_qE_4=\frac{E_2E_4-E_6}{3},\qquad D_qE_6=\frac{E_2E_6-E_4^2}{2},$$

and compute that $W_{f,g,h}(z)=cE_4(z)^6E_6(z)^3\Delta(z)^3$ for some nonzero number c. Noticing that $W_{f,g,h}(z)$ has a zero of order 3 at i and a zero of order 6 at ρ, we know that $\kappa_{\mathrm{i}}^{(1)}=-1$, $\kappa_{\rho}^{(1)}=-2$, which, by Proposition 4.3, implies that $\kappa_{\mathrm{i}}^{(2)}=1$, $\kappa_{\mathrm{i}}^{(3)}=3$, $\kappa_{\rho}^{(2)}=1$, and $\kappa_{\rho}^{(3)}=4$. In other words, the indicial equations at i and at ρ are $(x+1)(x-1)(x-3)=0$ and $(x+2)(x-1)(x-4)=0$, respectively. Also, we have $\mathrm{ord}_\infty f-\frac{1}{3}\mathrm{ord}_\infty W_{f,g,h}=-1$, $\mathrm{ord}_\infty g-\frac{1}{3}\mathrm{ord}_\infty W_{f,g,h}=0$, and $\mathrm{ord}_\infty h-\frac{1}{3}\mathrm{ord}_\infty W_{f,g,h}=1$, which implies that the indicial equation at ∞ is $(x+1)x(x-1)$. Therefore, according to Lemmas 5.7, 5.8, and 5.9, the meromorphic modular forms $Q(z)$ and $R(z)$ in $\mathcal{L}y$ are

$$Q(z)=-E_4(z)-\frac{3}{4}\frac{E_4(z)(E_4(z)^3-E_6(z)^2)}{E_6(z)^2}+\frac{8}{9}\frac{E_4(z)^3-E_6(z)^2}{E_4(z)^2}$$

(note that this can also be computed directly using (2.5)) and

$$R(z) = s_{\rm i}^{(1)} \frac{E_4(z)^3 - E_6(z)^2}{E_6(z)}$$

for some complex number $s_{\rm i}^{(1)}$. Using the apparentness condition at i, we can show that $s_{\rm i}^{(1)} = 0$. In other words, the differential equation is $D_q^3 y(z) + Q(z) D_q y(z) + \frac{1}{2} D_q Q(z) y(z) = 0$. We remark that the reason why the differential equation is of this special form is due to the fact that it is the symmetric square of some second order MODE.

It is worth to point out that Theorem 4.2 can be applied to construct solutions of the SU(3) Toda system

$$\begin{cases} \Delta v_1 + 2\mathrm{e}^{v_1} - \mathrm{e}^{v_2} = 4\pi \sum_{k=1}^{N} n_{1,k} \delta_{p_k}, \\ \Delta v_2 + 2\mathrm{e}^{v_2} - \mathrm{e}^{v_1} = 4\pi \sum_{k=1}^{N} n_{2,k} \delta_{p_k} \end{cases} \quad \text{in } \mathbb{R}^2,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator and δ_p denotes the Dirac measure at p. We always use the complex variable $w = x_1 + \mathrm{i} x_2$. Then the Laplace operator $\Delta = 4\partial_{w\bar{w}}$.

As in Theorem 4.2, we let $f, g, h \in \mathfrak{M}_k(\mathrm{SL}(2,\mathbb{Z}))$ be linearly independent modular forms and

$$y'''(z) + Q_2(z) y'(z) + Q_3(z) y(z) = 0, \qquad z \in \mathbb{H} \tag{4.2}$$

be the MODE satisfied by $y_1(z) := f(z)/\sqrt[3]{W_{f,g,h}}$, $y_2(z) := g(z)/\sqrt[3]{W_{f,g,h}}$, and $y_3(z) := h(z)/\sqrt[3]{W_{f,g,h}}$. Denote the set of regular singular points of (4.2) modulo $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ by $\mathcal{S} = \{z_1, \dots, z_m, \mathrm{i}, \rho\}$. For each $z \in \mathcal{S}$, it follows from Theorem 4.2 that there are $m_z^{(1)}, m_z^{(2)} \in \mathbb{Z}_{\geq 0}$ such that the local exponents of (4.2) at γz are the same as those at z and are given by

$$\kappa_z^{(1)} = -\frac{2m_z^{(1)} + m_z^{(2)}}{3}, \qquad \kappa_z^{(2)} = \kappa_z^{(1)} + m_z^{(1)} + 1, \qquad \kappa_z^{(3)} = \kappa_z^{(2)} + m_z^{(2)} + 1$$

for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Similarly, there are $m_\infty^{(1)}, m_\infty^{(2)} \in \mathbb{N}$ such that the local exponents of (4.2) at the cusp ∞ are given by

$$\kappa_\infty^{(1)} = -\frac{2m_\infty^{(1)} + m_\infty^{(2)}}{3}, \qquad \kappa_\infty^{(2)} = \kappa_\infty^{(1)} + m_\infty^{(1)}, \qquad \kappa_\infty^{(3)} = \kappa_\infty^{(2)} + m_\infty^{(2)}.$$

Given any $\lambda, \mu > 0$, we define

$$\begin{aligned} \mathrm{e}^{-U_{1;\lambda,\mu}(z)} &:= \frac{1}{4}\big(\lambda^2\mu^{-1}|y_1|^2 + \mu^2\lambda^{-1}|y_2|^2 + \lambda^{-1}\mu^{-1}|y_3|^2\big), \\ \mathrm{e}^{-U_{2;\lambda,\mu}(z)} &:= \frac{1}{4}\big[\lambda\mu|W(y_1,y_2)|^2 + \lambda^{-2}\mu|W(y_2,y_3)|^2 + \lambda\mu^{-2}|W(y_3,y_1)|^2\big], \end{aligned}$$

where $W(y_i, y_j) := y_i' y_j - y_j' y_i$. Note that $\mathrm{e}^{-U_{k;\lambda,\mu}(z)}$ is single-valued for any $z \in \mathbb{H}$ and $0 < \mathrm{e}^{-U_{k;\lambda,\mu}(z)} < \infty$ as long as $z \notin \mathrm{SL}(2,\mathbb{Z})\mathcal{S}$. We have

Lemma 4.5. *Given any $\lambda, \mu > 0$, there holds*

$$\begin{cases} \Delta U_{1;\lambda,\mu} + \mathrm{e}^{2U_{1;\lambda,\mu} - U_{2;\lambda,\mu}} = 0, \\ \Delta U_{2;\lambda,\mu} + \mathrm{e}^{2U_{2;\lambda,\mu} - U_{1;\lambda,\mu}} = 0 \end{cases} \qquad \textit{in } \mathbb{H} \setminus (\mathrm{SL}(2,\mathbb{Z})\mathcal{S}). \tag{4.3}$$

Proof. The proof can be easily adopted from [7, 23, 24]; we sketch the proof here for the reader's convenience. Given any $\lambda, \mu > 0$, we define

$$\mathcal{W}_{\lambda,\mu} := \begin{pmatrix} \lambda^{\frac{3}{2}} y_1 & \mu^{\frac{3}{2}} y_2 & y_3 \\ \lambda^{\frac{3}{2}} y_1' & \mu^{\frac{3}{2}} y_2' & y_3' \\ \lambda^{\frac{3}{2}} y_1'' & \mu^{\frac{3}{2}} y_2'' & y_3'' \end{pmatrix}.$$

Since the Wroksian $W(y_1, y_2, y_3) = 1$, we have $\det \mathcal{W}_{\lambda,\mu} = (\lambda\mu)^{\frac{3}{2}}$. Define a positive definite matrix

$$R_{\lambda,\mu} := (\lambda\mu)^{-1} \mathcal{W}_{\lambda,\mu} \overline{\mathcal{W}_{\lambda,\mu}}^T.$$

then $\det R_{\lambda,\mu} = 1$. For $1 \le m \le 3$, we let $R_{\lambda,\mu;m}$ denote the leading principal minor of $R_{\lambda,\mu}$ of dimension m. Since $y_j(z)$ is holomorphic in $\mathbb{H} \setminus (\mathrm{SL}(2,\mathbb{Z})\mathcal{S})$, a direct computation leads to (see, e.g., [24])

$$R_{\lambda,\mu;m}(\partial_{z\bar{z}} R_{\lambda,\mu;m}) - (\partial_z R_{\lambda,\mu;m})(\partial_{\bar{z}} R_{\lambda,\mu;m}) = R_{\lambda,\mu;m-1} R_{\lambda,\mu;m+1}, \qquad m = 1, 2, \tag{4.4}$$

for $z \in \mathbb{H} \setminus (\mathrm{SL}(2,\mathbb{Z})\mathcal{S})$, where $R_{\lambda,\mu;0} := 1$.

On the other hand,

$$\frac{1}{4} R_{\lambda,\mu;1} = \frac{1}{4} (\lambda\mu)^{-1} \big(\lambda^3 |y_1|^2 + \mu^3 |y_2|^2 + |y_3|^2\big) = \mathrm{e}^{-U_{1;\lambda,\mu}(z)}.$$

Define $\mathrm{e}^{-V_{\lambda,\mu}(z)} := \frac{1}{4} R_{\lambda,\mu;2}$. we will prove that $\mathrm{e}^{-V_{\lambda,\mu}(z)} = \mathrm{e}^{-U_{2;\lambda,\mu}(z)}$.

Note that $R_{\lambda,\mu;3} = \det R_{\lambda,\mu} = 1$. Letting $m = 1$ in (4.4) leads to (note $0 < \mathrm{e}^{-U_{1;\lambda,\mu}}, \mathrm{e}^{-V_{\lambda,\mu}} < +\infty$ in $\mathbb{H} \setminus (\mathrm{SL}(2,\mathbb{Z})\mathcal{S})$)

$$\begin{aligned} 4\mathrm{e}^{-V_{\lambda,\mu}} &= R_{\lambda,\mu;2} = 16\big[\mathrm{e}^{-U_{1;\lambda,\mu}} \big(\partial_{z\bar{z}} \mathrm{e}^{-U_{1;\lambda,\mu}}\big) - \big(\partial_z \mathrm{e}^{-U_{1;\lambda,\mu}}\big)\big(\partial_{\bar{z}} \mathrm{e}^{-U_{1;\lambda,\mu}}\big)\big] \\ &= -16\mathrm{e}^{-2U_{1;\lambda,\mu}} \partial_{z\bar{z}} U_{1;\lambda,\mu} = -4\mathrm{e}^{-2U_{1;\lambda,\mu}} \Delta U_{1;\lambda,\mu} \quad \text{in } \mathbb{H} \setminus (\mathrm{SL}(2,\mathbb{Z})\mathcal{S}), \end{aligned} \tag{4.5}$$

and letting $m = 2$ in (4.4) leads to

$$\begin{aligned} 4\mathrm{e}^{-U_{1;\lambda,\mu}} &= R_{\lambda,\mu;1} = 16\big[\mathrm{e}^{-V_{\lambda,\mu}} (\partial_{z\bar{z}} \mathrm{e}^{-V_{\lambda,,\mu}}) - \big(\partial_z \mathrm{e}^{-V_{\lambda,\mu}}\big)\big(\partial_{\bar{z}} \mathrm{e}^{-V_{\lambda,\mu}}\big)\big] \\ &= -16\mathrm{e}^{-2V_{\lambda,\mu}} \partial_{z\bar{z}} V_{\lambda,\mu} = -4\mathrm{e}^{-2V_{\lambda,\mu}} \Delta V_{\lambda,\mu} \quad \text{in } \mathbb{H} \setminus (\mathrm{SL}(2,\mathbb{Z})\mathcal{S}). \end{aligned}$$

Furthermore, we insert $\mathrm{e}^{-U_{1;\lambda,\mu}} = \frac{1}{4} \sum |a_j y_j|^2$ (where $a_1 = \lambda\mu^{-1/2}$, $a_2 = \mu\lambda^{-1/2}$ and $a_3 = (\lambda\mu)^{-1/2}$) into (4.5), which leads to

$$\begin{aligned} \frac{1}{4}\mathrm{e}^{-V_{\lambda,\mu}} &= \mathrm{e}^{-U_{1;\lambda,\mu}} \big(\partial_{z\bar{z}} \mathrm{e}^{-U_{1;\lambda,\mu}}\big) - \big(\partial_z \mathrm{e}^{-U_{1;\lambda,\mu}}\big)\big(\partial_{\bar{z}} \mathrm{e}^{-U_{1;\lambda,\mu}}\big) \\ &= \frac{1}{16} \Big[\Big(\sum |a_j y_j|^2\Big)\Big(\sum |a_j y_j'|^2\Big) - \Big(\sum a_j^2 y_j' \overline{y_j}\Big)\Big(\sum a_j^2 y_j \overline{y_j'}\Big)\Big] \\ &= \frac{1}{16} \big[|W(a_1 y_1, a_2 y_2)|^2 + |W(a_2 y_2, a_3 y_3)|^2 + |W(a_3 y_3, a_1 y_1)|^2\big], \end{aligned}$$

so $\mathrm{e}^{-V_{\lambda,\mu}(z)} = \mathrm{e}^{-U_{2;\lambda,\mu}(z)}$. This proves that $(U_{1;\lambda,\mu}(z), U_{2;\lambda,\mu}(z))$ solves the Toda system (4.3). ■

Now any $\tilde{z} \in \mathcal{S}$, it follows from the local behavior of y_j's that near $\gamma\tilde{z}$,

$$\begin{aligned} U_{1;\lambda,\mu}(z) &= -2\kappa_{\tilde{z}}^{(1)} \ln |z - \gamma\tilde{z}| + O(1), \\ U_{2;\lambda,\mu}(z) &= -2\big(2 - \kappa_{\tilde{z}}^{(3)}\big) \ln |z - \gamma\tilde{z}| + O(1). \end{aligned}$$

Similarly, at the cusp ∞, we have

$$U_{1;\lambda,\mu}(z) = -2\kappa_\infty^{(1)} \ln|q| + O(1),$$
$$U_{2;\lambda,\mu}(z) = 2\kappa_\infty^{(3)} \ln|q| + O(1),$$

where $q = \mathrm{e}^{2\pi \mathrm{i} z}$. Since f, g, h, $W_{f,g,h}$ are modular forms of weights k, k, k and $3(k+2)$ respectively, we easily obtain

$$|y_j(\gamma z)|^2 = \frac{|y_j(z)|^2}{|cz+d|^4}, \qquad |W(y_i,y_j)(\gamma z)|^2 = \frac{|W(y_i,y_j)(z)|^2}{|cz+d|^4}$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, so

$$U_{j;\lambda,\mu}(\gamma z) = U_{j;\lambda,\mu}(z) + 4\ln|cz+d|, \qquad j=1,2.$$

Now we define

$$(u_{1;\lambda,\mu}, u_{2;\lambda,\mu}) := (2U_{1;\lambda,\mu} - U_{2;\lambda,\mu}, 2U_{2;\lambda,\mu} - U_{1;\lambda,\mu}).$$

Then we have

$$\begin{cases} \Delta u_{1;\lambda,\mu} + 2\mathrm{e}^{u_{1;\lambda,\mu}} - \mathrm{e}^{u_{2;\lambda,\mu}} = 0, \\ \Delta u_{2;\lambda,\mu} + 2\mathrm{e}^{u_{2;\lambda,\mu}} - \mathrm{e}^{u_{1;\lambda,\mu}} = 0 \end{cases} \qquad \text{in } \mathbb{H} \setminus (\mathrm{SL}(2,\mathbb{Z})\mathcal{S}),$$

and near $\gamma\tilde{z}$,

$$u_{1;\lambda,\mu}(z) = 2m_{\tilde{z}}^{(1)} \ln|z-\gamma\tilde{z}| + O(1), \qquad u_{2;\lambda,\mu}(z) = 2m_{\tilde{z}}^{(2)} \ln|z-\gamma\tilde{z}| + O(1),$$

while at the cusp ∞,

$$u_{1;\lambda,\mu}(z) = 2m_\infty^{(1)} \ln|q| + O(1), \qquad u_{2;\lambda,\mu}(z) = 2m_\infty^{(2)} \ln|q| + O(1).$$

Therefore, $(u_{1;\lambda,\mu}, u_{2;\lambda,\mu})$ is a solution of the following SU(3) Toda system

$$\begin{cases} \Delta u_1 + 2\mathrm{e}^{u_1} - \mathrm{e}^{u_2} = 4\pi \sum_\gamma \left(m_{\mathrm{i}}^{(1)} \delta_{\gamma i} + m_\rho^{(1)} \delta_{\gamma\rho} + \sum_{j=1}^m m_{z_j}^{(1)} \delta_{\gamma z_j} \right) & \text{on } \mathbb{H}, \\ \Delta u_2 + 2\mathrm{e}^{u_2} - \mathrm{e}^{u_1} = 4\pi \sum_\gamma \left(m_{\mathrm{i}}^{(2)} \delta_{\gamma i} + m_\rho^{(2)} \delta_{\gamma\rho} + \sum_{j=1}^m m_{z_j}^{(2)} \delta_{\gamma z_j} \right) & \text{on } \mathbb{H}, \\ u_k(z) = 2m_\infty^{(k)} \ln|q| + O(1) \quad \text{as } \operatorname{Im} z \to \infty, \\ u_j(\gamma z) = u_j(z) + 4\ln|cz+d|, \quad \forall \gamma \in \mathrm{SL}(2,\mathbb{Z}). \end{cases} \tag{4.6}$$

Now consider the modular function $w\colon \mathbb{H} \to \mathbb{C}$ defined by

$$w = w(z) := \frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2}.$$

It is well known that $w(z)$ is holomorphic, surjective and

$$w(\mathrm{i}) = 1, \qquad w(\rho) = 0, \qquad w(\infty) = \infty.$$

A direct computation gives

$$w'(z) = -2\pi\mathrm{i} \frac{E_4(z)^2 E_6(z)}{E_4(z)^3 - E_6(z)^2}.$$

Denote $p_j := w(\gamma z_j)$. Then all points of $\{p_1,\dots,p_m,1,0\}\subset\mathbb{C}$ are distinct. Now we define $(v_1(w),v_2(w))$ for $w\in\mathbb{C}$ by

$$u_k(z) = v_k(w(z)) + 2\ln|w'(z)|, \qquad z\in\mathbb{H}.$$

Since $w(\gamma z)=w(z)$ gives $w'(\gamma z)=(cz+d)^2w'(z)$, it follows from $u_k(\gamma z)=u_k(z)+4\ln|cz+d|$ that $v_k(w)$ is well-defined for $w\in\mathbb{C}$. Now outside $\{p_1,\dots,p_m,1,0\}$, we have

$$\begin{aligned}\Delta u_k(z) &= 4\partial_{z\bar z}u_k(z) = 4\partial_{w\bar w}v_k(w)|w'(z)|^2 = |w'(z)|^2\Delta v_k(w)\\ &= \mathrm{e}^{u_{k'}(z)} - 2\mathrm{e}^{u_k(z)} = |w'(z)|^2\big(\mathrm{e}^{v_{k'}(w)} - 2\mathrm{e}^{v_k(w)}\big),\end{aligned}$$

so

$$\Delta v_k(w) + 2\mathrm{e}^{v_k(w)} - \mathrm{e}^{v_{k'}(w)} = 0,$$

where $\{k,k'\}=\{1,2\}$. Furthermore, since $w'(z_j)\neq 0$, we have that at $w=p_j=w(z_j)$,

$$v_k(w) = u_k(z) - 2\ln|w'(z)| = 2m_{z_j}^{(k)}\ln|z-z_j| + O(1) = 2m_{z_j}^{(k)}\ln|w-p_j| + O(1).$$

At $w=w(\mathrm{i})=1$, since $\mathrm{ord}_{\mathrm{i}}(w-1)=2$ and $\mathrm{ord}_{\mathrm{i}}\,w'=1$, we have

$$\begin{aligned}v_k(w) &= u_k(z) - 2\ln|w'(z)| = 2\big(m_{\mathrm{i}}^{(k)}-1\big)\ln|z-\mathrm{i}| + O(1)\\ &= \big(m_{\mathrm{i}}^{(k)}-1\big)\ln|w-1| + O(1).\end{aligned}$$

At $w=w(\rho)=0$, since $\mathrm{ord}_\rho\,w=3$ and $\mathrm{ord}_\rho\,w'=2$, we have

$$\begin{aligned}v_k(w) &= u_k(z) - 2\ln|w'(z)| = 2\big(m_\rho^{(k)}-2\big)\ln|z-\rho| + O(1)\\ &= \frac{2\big(m_\rho^{(k)}-2\big)}{3}\ln|w| + O(1).\end{aligned}$$

At $w=w(\infty)=\infty$, since

$$w(z) = Cq^{-1}(1+O(q)), \qquad w'(z) = -2\pi\mathrm{i}Cq^{-1}(1+O(q)),$$

where $q=\mathrm{e}^{2\pi\mathrm{i}z}$ and $C\neq 0$ is a constant, so

$$\begin{aligned}v_k(w) &= u_k(z) - 2\ln|w'(z)| = 2\big(m_\infty^{(k)}+1\big)\ln|q| + O(1)\\ &= -2\big(m_\infty^{(k)}+1\big)\ln|w| + O(1).\end{aligned}$$

Therefore, (4.6) is equivalent to

$$\begin{cases}\Delta v_1 + 2\mathrm{e}^{v_1} - \mathrm{e}^{v_2} = 4\pi\dfrac{m_{\mathrm{i}}^{(1)}-1}{2}\delta_1 + 4\pi\dfrac{m_\rho^{(1)}-2}{3}\delta_0 + 4\pi\displaystyle\sum_{j=1}^m m_{z_j}^{(1)}\delta_{p_j} \ \text{ in } \mathbb{R}^2,\\ \Delta v_2 + 2\mathrm{e}^{v_2} - \mathrm{e}^{v_1} = 4\pi\dfrac{m_{\mathrm{i}}^{(2)}-1}{2}\delta_1 + 4\pi\dfrac{m_\rho^{(2)}-2}{3}\delta_0 + 4\pi\displaystyle\sum_{j=1}^m m_{z_j}^{(2)}\delta_{p_j} \ \text{ in } \mathbb{R}^2,\\ v_k(w) = -2\big(m_\infty^{(k)}+1\big)\ln|w| + O(1) \quad \text{as } |w|\to\infty.\end{cases} \tag{4.7}$$

Note from Proposition 4.3 that

$$m_{\mathrm{i}}^{(k)} + 1 = \kappa_{\mathrm{i}}^{(k+1)} - \kappa_{\mathrm{i}}^{(k)} \equiv 0 \bmod 2 \qquad \text{and} \qquad m_\rho^{(k)} + 1 = \kappa_\rho^{(k+1)} - \kappa_\rho^{(k)} \equiv 0 \bmod 3,$$

so $\frac{m_{\mathrm{i}}^{(k)}-1}{2}\in\mathbb{Z}_{\geq 0}$ and $\frac{m_\rho^{(2)}-2}{3}\in\mathbb{Z}_{\geq 0}$ for $k=1,2$. In conclusion, starting from any given linearly independent modular forms $f,g,h\in\mathfrak{M}_k(\mathrm{SL}(2,\mathbb{Z}))$, we can construct a two-parametric family of solutions to certain Toda system (4.7):

Theorem 4.6. *$(v_{1;\lambda,\mu}(w), v_{2;\lambda,\mu}(w))$ defined by*

$$v_{k;\lambda,\mu}(w(z)) = u_{k;\lambda,\mu}(z) - 2\ln|w'(z)|, \qquad z\in\mathbb{H}$$

are a two-parametric family of solutions of the $\mathrm{SU}(3)$ *Toda system* (4.7)*, where $\lambda,\mu>0$ can be arbitrary.*

5 Polynomial systems derived from the conditions (H1)–(H3)

In view of Theorem 1.3 proved in Section 3, a natural question is whether given a prescribed set of singular points and the local exponents at singularities and at the cusps, there exist MODEs (1.1) satisfying the conditions (H1)–(H3). We will see in this section that this problem of existence is equivalent to that of solving a certain system of polynomial equations. Note that in view of Theorems 1.1 and 4.2 such a MODE (1.1) exists for certain sets of data.

5.1 Solution expansions for MODEs

Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ commensurable with $\mathrm{SL}(2,\mathbb{Z})$ and (1.1) be a MODE on Γ. To verify the apparentness of a singular point z_0 of (1.1), we use the classical Frobenius method. However, since (1.1) is modular, it will be more convenient that all functions are expanded in terms of $\widetilde{w}$ as introduced in [25] rather than $z - z_0$.

Fix $z_0 \in \mathbb{H}$, we let $w = (z - z_0)/(z - \overline{z}_0)$ and

$$\widetilde{w} = \frac{w}{1 + Aw}, \qquad \text{with} \quad A = \frac{4\pi\phi^*(z_0)\operatorname{Im} z_0}{\alpha}, \tag{5.1}$$

where $\phi(z)$ is the quasimodular form of weight 2 and depth 1 on Γ, i.e.,

$$\left(\phi\big|_2\gamma\right)(z) = \phi(z) + \frac{\alpha_0 c}{2\pi\mathrm{i}(cz+d)}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

for some nonzero complex number α_0 (note $\alpha_0 = 2\pi\mathrm{i}\alpha$ for α given in (2.1)), and $\phi^*(z) := \phi(z) + \frac{\alpha_0}{2\pi\mathrm{i}(z-\bar{z})}$. Clearly $\phi^*(z)$ satisfies

$$\left(\phi^*\big|_2\gamma\right)(z) = \phi^*(z), \qquad \gamma \in \Gamma.$$

The advantage of the expansion in terms of $\widetilde{w}$ is the following result.

Proposition 5.1 ([25, Propositions A.4 and A.7])**.** *Let $f(z)$ be a meromorphic modular form of weight k on Γ. Then $f(z)$ admits an expansion of the form*

$$f(z) = (1 - (1+A)\widetilde{w})^k \sum_{n=n_0}^{\infty} a_n(-4\pi(\operatorname{Im} z_0)\widetilde{w})^n.$$

Furthermore, if z_0 is an elliptic point with the stabilizer subgroup Γ_{z_0} of order N, then $a_n = 0$ whenever $k + 2n \not\equiv 0 \bmod N$.

Remark 5.2. Note that when $f(z)$ is a holomorphic modular form, the coefficients a_n in the series can be expressed in terms of the Serre derivatives of $f(z)$. See [25, Proposition A.4] for the precise statement.

Also note that when z_0 is an elliptic point, we have $A = 0$ and hence $\widetilde{w} = w$. This is because ϕ^* transforms like a modular form of weight 2, but any modular form of weight 2 will vanish at every elliptic point.

By Proposition 5.1, we can write

$$Q(z) := \frac{Q_2(z)}{(2\pi\mathrm{i})^2} = (1 - (1+A)\widetilde{w})^4 \sum_{n=-2}^{\infty} a_n(-4\pi(\operatorname{Im} z_0)\widetilde{w})^n,$$

$$R(z) := \frac{Q_3(z) - \frac{1}{2}Q_2'(z)}{(2\pi\mathrm{i})^3} = (1 - (1+A)\widetilde{w})^6 \sum_{n=-3}^{\infty} b_n(-4\pi(\operatorname{Im} z_0)\widetilde{w})^n.$$

Then (1.1) is equivalent to

$$D_q^3 y(z) + Q(z) D_q y(z) + \left(\frac{1}{2} D_q Q(z) + R(z) \right) y(z) = 0, \tag{5.2}$$

where $D_q := q\frac{\mathrm{d}}{\mathrm{d}q} = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z}$. In particular, Proposition 5.1 yields

Corollary 5.3. *Suppose* $-I_2 \in \Gamma$ *and let* z_0 *be the elliptic point of order* e *of* Γ. *Then* $N = 2e$ *and so* $a_n = 0$ *if* $n \not\equiv e - 2 \bmod e$ *and* $b_n = 0$ *if* $n \not\equiv e - 3 \bmod e$.

For later usage, we recall Bol's identity in the following form.

Lemma 5.4. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{C})$ *and* r *be a positive integer. Set*

$$w = \gamma z = (az + b)/(cz + d).$$

Then for a $(r+1)$*-th differentiable function* $g(z)$*, we have*

$$\frac{\mathrm{d}^{r+1}}{\mathrm{d}z^{r+1}} \left(\frac{(\det \gamma)^{r/2}}{(a - cw)^r} g(w) \right) = \frac{(a - cw)^{r+2}}{(\det \gamma)^{r/2+1}} \frac{\mathrm{d}^{r+1}}{\mathrm{d}w^{r+1}} g(w).$$

Proof. Bol's identity states that

$$\left(y\big|_{-r}\gamma\right)^{(r+1)}(z) = \left(y^{(r+1)}\big|_{r+2}\gamma\right)(z).$$

Noticing that

$$a - cw = a - c\frac{az + b}{cz + d} = \frac{\det \gamma}{cz + d},$$

we find that the factor $(\det \gamma)^{1/2}/(cz + d)$ appearing in the slash operator can be written as

$$\frac{(\det \gamma)^{1/2}}{cz + d} = \frac{a - cw}{(\det \gamma)^{1/2}}, \tag{5.3}$$

which yields the version of Bol's identity stated in the lemma. ■

To apply Frobenius' method by using the expansion in Proposition 5.1, we need the following result.

Lemma 5.5. *Let* $Q(z)$ *and* $R(z)$ *be meromorphic modular forms of weight* 4 *and* 6*, respectively, on* Γ. *Assume that* $\widetilde{Q}(x) = \sum_{n \geq n_0} a_n x^n$ *and* $\widetilde{R}(x) = \sum_{n \geq n_0} b_n x^n$ *are the power series such that*

$$\begin{aligned} Q(z) &= (1 - (1 + A)\widetilde{w})^4 \widetilde{Q}(-4\pi(\operatorname{Im} z_0)\widetilde{w}), \\ R(z) &= (1 - (1 + A)\widetilde{w})^6 \widetilde{R}(-4\pi(\operatorname{Im} z_0)\widetilde{w}). \end{aligned} \tag{5.4}$$

Then

$$y(z) = \frac{1}{(1 - (1 + A)\widetilde{w})^2} \sum_{n=0}^{\infty} c_n (-4\pi(\operatorname{Im} z_0)\widetilde{w})^{n+\alpha} \tag{5.5}$$

is a solution of (5.2) *if and only if the series* $\widetilde{y}(x) = \sum_{n=0}^{\infty} c_n x^{n+\alpha}$ *satisfies*

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} \widetilde{y}(x) + \widetilde{Q}(x) \frac{\mathrm{d}}{\mathrm{d}x} \widetilde{y}(x) + \left(\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} \widetilde{Q}(x) + \widetilde{R}(x) \right) \widetilde{y}(x) = 0. \tag{5.6}$$

Proof. Let

$$\gamma = \begin{pmatrix} -4\pi \operatorname{Im} z_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \begin{pmatrix} 1 & -z_0 \\ 1 & -\overline{z}_0 \end{pmatrix} = \begin{pmatrix} -4\pi \operatorname{Im} z_0 & (4\pi \operatorname{Im} z_0) z_0 \\ 1 + A & -Az_0 - \overline{z}_0 \end{pmatrix}$$

with $\det \gamma = -4\pi (\operatorname{Im} z_0)(z_0 - \overline{z}_0)$. Let $x = \gamma z = -4\pi(\operatorname{Im} z_0)\widetilde{w}$. Note that if we write γ as $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$\frac{(a - cx)^2}{\det \gamma} = 2\pi \mathrm{i}(1 - (1+A)\widetilde{w})^2.$$

Thus, by (5.3), $y(z)$ and $\widetilde{y}(x)$ are related by

$$y(z) = 2\pi\mathrm{i}\big(\widetilde{y}\big|_{-2}\gamma\big)(z).$$

Hence, applying Lemma 5.4 with $r = 2$, we obtain

$$D_q^3 y(z) = \frac{1}{(2\pi\mathrm{i})^2} \frac{\mathrm{d}^3}{\mathrm{d}z^3} \big(\widetilde{y}\big|_{-2}\gamma\big)(z) = (1 - (1+A)\widetilde{w})^4 \frac{\mathrm{d}^3}{\mathrm{d}x^3} \widetilde{y}(x).$$

Also, a direct computation yields, by (5.3),

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\mathrm{d}\gamma z}{\mathrm{d}z} = \frac{\det \gamma}{(cz+d)^2} = \frac{(a-cx)^2}{\det \gamma} = 2\pi\mathrm{i}(1 - (1+A)\widetilde{w})^2,$$

and

$$\frac{\mathrm{d}\widetilde{w}}{\mathrm{d}z} = -\frac{1}{4\pi \operatorname{Im} z_0} \frac{\mathrm{d}x}{\mathrm{d}z} = \frac{(1 - (1+A)\widetilde{w})^2}{2\mathrm{i} \operatorname{Im} z_0}.$$

Hence,

$$\begin{aligned} D_q y(z) &= \frac{1}{2\pi\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{(1 - (1+A)\widetilde{w})^2} \widetilde{y}(x) \right) \\ &= -\frac{1+A}{2\pi (\operatorname{Im} z_0)(1 - (1+A)\widetilde{w})} \widetilde{y}(x) + \frac{\mathrm{d}}{\mathrm{d}x} \widetilde{y}(x), \end{aligned}$$

and

$$\begin{aligned} D_q Q(z) &= \frac{1}{2\pi\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z} \left((1 - (1+A)\widetilde{w})^4 \widetilde{Q}(x) \right) \\ &= (1 - (1+A)\widetilde{w})^6 \left(\frac{1+A}{\pi (\operatorname{Im} z_0)(1 - (1+A)\widetilde{w})} \widetilde{Q}(x) + \frac{\mathrm{d}}{\mathrm{d}x} \widetilde{Q}(x) \right). \end{aligned}$$

Putting everything together, we find that

$$\begin{aligned} & D_q^3 y(z) + Q(z) D_q y(z) + \left(\frac{1}{2} D_q(z) + R(z) \right) y(z) \\ &\quad = (1 - (1+A)\widetilde{w})^4 \left(\frac{\mathrm{d}^3}{\mathrm{d}x^3} \widetilde{y}(x) + \widetilde{Q}(x) \frac{\mathrm{d}}{\mathrm{d}x} \widetilde{y}(x) + \left(\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} \widetilde{Q}(x) + \widetilde{R}(x) \right) \widetilde{y}(x) \right). \end{aligned}$$

Thus, $y(z)$ is a solution of (5.2) if and only if $\widetilde{y}(x)$ is a solution of (5.6). ∎

We will see from Lemma 5.10 below that Corollary 5.3 and Lemma 5.5 can be applied to prove that (1.1) or equivalently (5.2) is apparent at elliptic points of order $e \geq 3$. This is a great advantage of using expansions in terms of $\widetilde{w}$ rather than $z - z_0$.

Remark 5.6. In practice, the power series $\widetilde{Q}(x)$ and $\widetilde{R}(x)$ can be computed using Proposition A.4 of [25]. For example, for the Eisenstein series $E_4(z)$ and $E_6(z)$ on $\mathrm{SL}(2,\mathbb{Z})$, we find that

$$E_4(z) = (1-(1+A)\widetilde{w})^4 \left(B - \frac{1}{3}Cu + \frac{5}{72}B^2u^2 - \frac{5}{432}BCu^3 + \cdots\right),$$
$$E_6(z) = (1-(1+A)\widetilde{w})^6 \left(C - \frac{1}{2}B^2u + \frac{7}{48}BCu^2 + \cdots\right), \tag{5.7}$$

where $u = -4\pi(\operatorname{Im} z_0)\widetilde{w}$, $B = E_4(z_0)$, and $C = E_6(z_0)$.

5.2 Existence of $Q(z)$ and $R(z)$

In this section we shall discuss the criterion of the existence of meromorphic modular forms $Q(z)$ and $R(z)$ of weight 4 and 6, respectively, on $\mathrm{SL}(2,\mathbb{Z})$, such that the differential equation (5.2) is Fuchsian and apparent throughout $\mathbb{H}$ with prescribed local exponents at singularities and at cusps.

Throughout the section, we let z_j, $j = 1,\dots,m$, be $\mathrm{SL}(2,\mathbb{Z})$-inequivalent point on $\mathbb{H}$, none of which is an elliptic point. We let $\mathrm{i} = \sqrt{-1}$ and $\rho = \left(-1+\sqrt{3}\mathrm{i}\right)/2$ be the unique elliptic point of order 2 and 3 of $\mathrm{SL}(2,\mathbb{Z})$, respectively. For $z = z_j$, i, or ρ, we assume that $\kappa_z^{(1)} < \kappa_z^{(2)} < \kappa_z^{(3)}$, are rational numbers in $\frac{1}{3}\mathbb{Z}$ such that $\kappa_z^{(1)} + \kappa_z^{(2)} + \kappa_z^{(3)} = 3$ and $\kappa_z^{(2)} - \kappa_z^{(1)} \in \mathbb{Z}$ (and hence $\kappa_z^{(3)} - \kappa_z^{(1)} \in \mathbb{Z}$). When $z = \mathrm{i}$, we further assume that

$$\left\{3\kappa_{\mathrm{i}}^{(1)}, 3\kappa_{\mathrm{i}}^{(2)}, 3\kappa_{\mathrm{i}}^{(3)}\right\} \equiv \{0,0,1\} \bmod 2. \tag{5.8}$$

Also, when $z = \rho$, we note that the assumptions on $\kappa_\rho^{(j)}$ above imply that $\kappa_\rho^{(j)} \in \mathbb{Z}$ for all j. We further assume that

$$\left\{\kappa_\rho^{(1)}, \kappa_\rho^{(2)}, \kappa_\rho^{(3)}\right\} \equiv \{0,1,2\} \quad \bmod 3. \tag{5.9}$$

Finally, for the cusp ∞ of $\mathrm{SL}(2,\mathbb{Z})$, we let $\kappa_\infty^{(1)} \le \kappa_\infty^{(2)} \le \kappa_\infty^{(3)}$ be three rational numbers in $\frac{1}{3}\mathbb{Z}$ such that $\kappa_\infty^{(1)} + \kappa_\infty^{(2)} + \kappa_\infty^{(3)} = 0$ and $\kappa_\infty^{(2)} - \kappa_\infty^{(1)}, \kappa_\infty^{(3)} - \kappa_\infty^{(2)} \in \mathbb{Z}$. We shall consider the problem whether there exist meromorphic modular forms $Q(z)$ and $R(z)$ of weight 4 and 6, respectively, on $\mathrm{SL}(2,\mathbb{Z})$, such that (3.1) is Fuchsian and apparent throughout $\mathbb{H}$ and the local exponents κ's are given as above.

Lemma 5.7. *Let notations* i *and* ρ *be as above. Then meromorphic modular forms* $Q(z)$ *and* $R(z)$ *of weight* 4 *and* 6*, respectively, on* $\mathrm{SL}(2,\mathbb{Z})$ *that have poles of order at most* 2 *and* 3*, respectively, at points* $\mathrm{SL}(2,\mathbb{Z})$*-equivalent to* z_j*,* i*, or* ρ *and are holomorphic at other points and cusps are of the form*

$$Q(z) = r_\infty E_4(z) + r_{\mathrm{i}}^{(2)} \frac{E_4(z)\Delta_0(z)}{E_6(z)^2} + r_\rho^{(2)} \frac{\Delta_0(z)}{E_4(z)^2}$$
$$+ \sum_{j=1}^{m} \left(r_{z_j}^{(2)} \frac{E_4(z)\Delta_0(z)^2}{F_j(z)^2} + r_{z_j}^{(1)} \frac{E_4(z)\Delta_0(z)}{F_j(z)} \right),$$
$$R(z) = s_\infty E_6(z) + s_{\mathrm{i}}^{(3)} \frac{\Delta_0(z)^2}{E_6(z)^3} + s_{\mathrm{i}}^{(1)} \frac{\Delta_0(z)}{E_6(z)}$$
$$+ s_\rho^{(3)} \frac{E_6(z)\Delta_0(z)}{E_4(z)^3} + \sum_{j=1}^{m}\sum_{k=1}^{3} s_{z_j}^{(k)} \frac{E_6(z)\Delta_0(z)^k}{F_j(z)^k}, \tag{5.10}$$

where $\Delta_0(z) = 1728\Delta(z) = (E_4(z)^3 - E_6(z)^2)$ *and* $F_j(z) = E_4(z)^3 - t_j E_6(z)^2$ *with* $t_j = E_4(z_j)^3/E_6(z_j)^2$.

Proof. By Corollary 5.3, there are no meromorphic modular forms of weight 4 having a pole at i or ρ with a nonzero residue. Also, the order of a meromorphic modular form of weight 6 on $\mathrm{SL}(2,\mathbb{Z})$ at i is necessarily odd, while that at ρ is congruent to 0 modulo 3. Thus, we can take $r_{\mathrm{i}}^{(2)}$, $r_\rho^{(2)}$, $r_{z_j}^{(2)}$, $r_{z_j}^{(1)}$ such that

$$Q(z) - r_{\mathrm{i}}^{(2)}\frac{E_4(z)\Delta_0(z)}{E_6(z)^2} - r_\rho^{(2)}\frac{\Delta_0(z)}{E_4(z)^2} - \sum_{j=1}^{m}\left(r_{z_j}^{(2)}\frac{E_4(z)\Delta_0(z)^2}{F_j(z)^2} + r_{z_j}^{(1)}\frac{E_4(z)\Delta_0(z)}{F_j(z)}\right)$$

is a holomorphic modular form of weight 4 on $\mathrm{SL}(2,\mathbb{Z})$, so it must be a multiple of $E_4(z)$. The proof for $R(z)$ is similar. ■

We first determine the indicial equations of (5.2) at ∞, i, ρ, and z_j, $j=1,\dots,m$.

Lemma 5.8. *Suppose that $Q(z)$ and $R(z)$ are meromorphic modular forms given by (5.10). Then the indicial equation of (5.2) at the cusp ∞ is*

$$x^3 + r_\infty x + s_\infty = 0.$$

Proof. It is clear that $Q(z) = r_\infty + O(q)$ and $R(z) = s_\infty + O(q)$. Assume that there is a solution of (5.2) of the form $y(z) = q^\alpha(1+O(q))$, $\alpha\in\mathbb{R}$. We compute that

$$\begin{aligned}&D_q^3 y(z) + Q(z)D_q y(z) + \left(\frac{1}{2}D_q Q(z) + R(z)\right)y(z)\\ &\qquad = \alpha^3 q^\alpha + r_\infty \alpha q^\alpha + s_\infty q^\alpha + O\big(q^{\alpha+1}\big),\end{aligned}$$

from which we see that the indicial equation at ∞ is $x^3 + r_\infty x + s_\infty = 0$. ■

We now consider the indicial equation of (5.2) at a point in $\mathbb{H}$. Let z_0 be one of z_j, i, or ρ and $\widetilde{w}$ be defined by (5.1) with $\phi^*(z) = E_2^*(z) := E_2(z) + 6/(\pi\mathrm{i}(z-\overline{z}))$. Recall that in Lemma 5.5 we have proved that if $\widetilde{Q}(x)$ and $\widetilde{R}(x)$ are the Laurent series in x such that (5.4) holds. Then

$$y(z;\alpha) = \frac{1}{(1-(1+A)\widetilde{w})^2}\sum_{n=0}^{\infty} c_n(\alpha)(-4\pi(\operatorname{Im} z_0)\widetilde{w})^{n+\alpha}, \qquad c_0(\alpha) = 1,$$

is a solution of (5.2) if and only if the series $\widetilde{y}(x;\alpha) = \sum_{n=0}^{\infty} c_n(\alpha)x^{n+\alpha}$ satisfies

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\widetilde{y}(x;\alpha) + \widetilde{Q}(x)\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{y}(x;\alpha) + \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{Q}(x) + \widetilde{R}(x)\right)\widetilde{y}(x;\alpha) = 0. \tag{5.11}$$

Let

$$\widetilde{Q}(x) = \sum_{n=-2}^{\infty} a_n x^n, \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{Q}(x) + \widetilde{R}(x) = \sum_{n=-3}^{\infty} b_n x^n, \tag{5.12}$$

where each a_n and b_n is linear in the parameters r's and s's. Then following the computation in Appendix A, we see that (5.11) is equivalent to

$$R_n(\alpha) := f(\alpha+n)c_n(\alpha) + \sum_{k=0}^{n-1}[(\alpha+k)a_{n-k-2} + b_{n-k-3}]c_k(\alpha) = 0 \tag{5.13}$$

for all $n\geq 0$, where $f(t) := t(t-1)(t-2) + a_{-2}t + b_{-3}$. In particular, $R_0(\alpha)\colon f(\alpha)=0$ is the indicial equation at z_0, i.e.,

$$f(t) = \big(t-\kappa_{z_0}^{(1)}\big)\big(t-\kappa_{z_0}^{(2)}\big)\big(t-\kappa_{z_0}^{(3)}\big).$$

Using (5.7), we can work out the coefficients a_{-2} and b_{-3}.

Lemma 5.9. *Suppose that $Q(z)$ and $R(z)$ are meromorphic modular forms given by (5.10). Then the indicial equations of (5.2) at* i, ρ, *and* z_j, $j=1,\dots,m$, *are*

$$\begin{aligned}
&x(x-1)(x-2)+4r_{\mathrm{i}}^{(2)}x-4r_{\mathrm{i}}^{(2)}-8s_{\mathrm{i}}^{(3)}=0,\\
&x(x-1)(x-2)-9r_{\rho}^{(2)}x+9r_{\rho}^{(2)}+27s_{\rho}^{(3)}=0,\\
&x(x-1)(x-2)+\frac{r_{z_j}^{(2)}}{t_j}x-\frac{r_{z_j}^{(2)}}{t_j}+\frac{s_{z_j}^{(3)}}{t_j^2}=0,
\end{aligned}$$

respectively.

Proof. For the elliptic point i, we know that $E_6(\mathrm{i})=0$. Hence, using (5.7), we find that

$$\widetilde{Q}(x)=\frac{4r_{\mathrm{i}}^{(2)}}{x^2}+\dots,\qquad \widetilde{R}(x)=-\frac{8s_{\mathrm{i}}^{(2)}}{x^3}+\cdots.$$

Therefore, we have $a_{-2}=4r_{\mathrm{i}}^{(2)}$ and $b_{-3}=-4r_{\mathrm{i}}^{(2)}-8s_{\mathrm{i}}^{(2)}$. Similarly, for the elliptic point ρ, using $E_4(\rho)=0$ and (5.7) again, we find that $a_{-2}=-9r_{\rho}^{(2)}$ and $b_{-3}=9r_{\rho}^{(2)}+27s_{\rho}^{(3)}$. For the point z_j, letting $B=E_4(z_j)$ and $C=E_6(z_j)$, we compute that

$$\begin{aligned}
\frac{\Delta_0(z)}{(1-(1+A)\widetilde{w})^{12}}&=\big(B^3-C^2\big)+O(\widetilde{w})=C^2(t_j-1)+O(\widetilde{w}),\\
\frac{F_j(z)}{(1-(1+A)\widetilde{w})^{12}}&=\left(B-\frac{1}{3}Cu+O\big(\widetilde{w}^2\big)\right)^3-t_j\left(C-\frac{1}{2}B^2u+O\big(\widetilde{w}^2\big)\right)^2\\
&=(t_j-1)B^2Cu+O\big(\widetilde{w}^2\big),
\end{aligned}$$

where $u=-4\pi(\operatorname{Im} z_j)\widetilde{w}$. It follows that

$$\widetilde{Q}(x)=r_{z_j}^{(2)}\frac{(t_j-1)^2BC^4}{(t_j-1)^2B^4C^2x^2}+\dots=\frac{r_{z_j}^{(2)}}{t_jx^2}+\dots,$$

and

$$\widetilde{R}(x)=s_{z_j}^{(3)}\frac{(t_j-1)^3C^7}{(t_j-1)^3B^6C^3}+\dots=\frac{s_{z_j}^{(3)}}{t_j^2x^3}+\cdots.$$

Therefore, $a_{-2}=r_{z_j}^{(2)}/t_j$ and $b_{-3}=-r_{z_j}^{(2)}/t_j+s_{z_j}^{(3)}/t_j^2$. This proves the lemma. ■

The two lemmas show that the parameters r_∞, s_∞, $r_{\mathrm{i}}^{(2)}$, $s_{\mathrm{i}}^{(3)}$, $r_{\rho}^{(2)}$, $s_{\rho}^{(3)}$, $r_{z_j}^{(2)}$, and $s_{z_j}^{(3)}$, $j=1,\dots,m$, solely depend on the local exponents κ's. The remaining parameters are

$$\begin{cases} r_{z_j}^{(1)}, s_{z_j}^{(2)}, s_{z_j}^{(1)}, & \text{for } j=1,\dots,m,\\ s_{\mathrm{i}}^{(1)}. \end{cases}$$

That is, the number of remaining parameters is $3m+1$. We now show that the apparentness condition will impose $3m+1$ polynomial constraints on the remaining parameters. For convenience, we let $\mathbf{r}$ and $\mathbf{s}$ denote $r_{z_1}^{(1)},\dots,r_{z_m}^{(1)}$ and $s_{\mathrm{i}}^{(1)}, s_{z_1}^{(2)}, s_{z_1}^{(1)},\dots,s_{z_m}^{(2)}, s_{z_m}^{(1)}$, respectively.

Let $z_0\in\{\mathrm{i},\rho,z_1,\dots,z_m\}$. Assume that α is one of the local exponents $\kappa_{z_0}^{(k)}$. We observe that one can recursively determine $c_n(\alpha)$ using (5.13) as long as $f(\alpha+n)\neq 0$. Thus, there always exists a solution $\widetilde{y}\big(x;\kappa_{z_0}^{(3)}\big)$ with local exponent $\kappa_{z_0}^{(3)}$; see, e.g., Lemma A.2. For the exponent $\alpha=\kappa_{z_0}^{(2)}$, because

$$f\big(\alpha+\kappa_{z_0}^{(3)}-\kappa_{z_0}^{(2)}\big)=f\big(\kappa_{z_0}^{(3)}\big)=0,$$

we can only solve for $c_n(\alpha)$ up to $n = \kappa_{z_0}^{(3)} - \kappa_{z_0}^{(2)} - 1$. At $n' = \kappa_{z_0}^{(3)} - \kappa_{z_0}^{(2)}$, the relation $R_{n'}(\alpha)$ becomes

$$\sum_{k=0}^{n'-1} [(\alpha+k)a_{n'-k-2} + b_{n'-k-3}]c_k(\alpha) = 0. \tag{5.14}$$

If $c_k(\alpha)$, $k = 0, \dots, n'-1$, do not satisfy this relation, then there is no solution with exponent $\kappa_{z_0}^{(2)}$. If $c_k(\alpha)$, $k = 0, \dots, n'-1$, satisfy this relation, we then can choose $c_{n'}(\alpha)$ to be any number and solve for $c_n(\alpha)$ recursively and obtain a solution $\widetilde{y}\big(x; \kappa_{z_0}^{(2)}\big)$ with local exponent $\kappa_{z_0}^{(2)}$. Likewise, for the local exponent $\alpha = \kappa_{z_0}^{(1)}$, there will be two conditions (5.14) corresponding to $n' = \kappa_{z_0}^{(2)} - \kappa_{z_0}^{(1)}$ and $n' = \kappa_{z_0}^{(3)} - \kappa_{z_0}^{(1)}$ that $c_k(\alpha)$ must satisfy. Thus, there are three polynomial equations

$$P_{z_0,k_1,k_2}(\mathbf{r}, \mathbf{s}) = 0, \qquad (k_1, k_2) \in \{(1,2), (1,3), (2,3)\}$$

that $\mathbf{r}$ and $\mathbf{s}$ need to satisfy. However, we will see in a moment that when z_0 is an elliptic point, some or all of the three polynomial equations hold trivially.

Lemma 5.10. *Assume that z_0 is an elliptic point of order e of $\mathrm{SL}(2,\mathbb{Z})$. Let $(k_1, k_2) \in \{(1,2), (1,3), (2,3)\}$. If $\kappa_{z_0}^{(k_2)} - \kappa_{z_0}^{(k_1)} \not\equiv 0 \bmod e$, then the polynomial $P_{z_0,k_1,k_2}(\mathbf{r}, \mathbf{s})$ is identically zero. In particular, under our assumptions (5.8) and (5.9), the differential equation (5.2) is apparent at ρ for any $\mathbf{r}$ and $\mathbf{s}$, and also there is at most one pair (k_1, k_2) such that $P_{\mathrm{i},k_1,k_2}(\mathbf{r}, \mathbf{s}) \neq 0$.*

Proof. By Corollary 5.3, the coefficients a_n in the expansion of $\widetilde{Q}(x)$ vanish whenever $n \not\equiv -2 \bmod e$. Likewise, the coefficients b_n in $\widetilde{R}(x)$ vanish whenever $n \not\equiv -3 \bmod e$. Using these facts, we find that the condition (5.13) reduces to

$$f(\alpha+n)c_n(\alpha) + \sum_{k \equiv n \bmod e,\, k \le n-1} [(\alpha+k)a_{n-k-2} + b_{n-k-3}]c_k(\alpha) = 0. \tag{5.15}$$

Then we can easily prove by induction up to $n = n'-1$ that $c_n(\alpha) = 0$ whenever $n \not\equiv 0 \bmod e$. Now if $n' \not\equiv 0 \bmod e$, then (5.14) automatically holds because every summand is 0 due to the facts that $a_{n'-k-2}$ and $b_{n'-k-3}$ are nonzero only when $k \equiv n' \bmod e$ and $c_k(\alpha)$ is nonzero only when $k \equiv 0 \bmod e$, but k cannot be congruent to n' and 0 at the same time. This proves the lemma. ■

In the following, we let $P_{\mathrm{i}}(\mathbf{r}, \mathbf{s})$ denote the only nonzero polynomial $P_{\mathrm{i},k_1,k_2}(\mathbf{r}, \mathbf{s})$ in the lemma. The discussion above shows that there are $3m+1$ polynomial equations $P_{\mathrm{i}}(\mathbf{r}, \mathbf{s}) = 0$, $P_{z_j,k_1,k_2}(\mathbf{r}, \mathbf{s}) = 0$, $j = 1, \dots, m$, $(k_1, k_2) \in \{(1,2), (1,3), (2,3)\}$ in $3m+1$ variables such that (5.2) is Fuchsian and apparent throughout $\mathbb{H}$ and all $\mathrm{SL}(2,\mathbb{Z})$-inequivalent singularities belong to $\{\mathrm{i}, \rho, z_1, \dots, z_m\}$ with the given local exponents if and only if the parameters $\mathbf{r}$ and $\mathbf{s}$ are common roots of the polynomials. We now consider the degree of these polynomials.

Proposition 5.11. *We have*

$$\deg P_{z_j,k_1,k_2}(\mathbf{r}, \mathbf{s}) = \begin{cases} \kappa_{z_j}^{(k_2)} - \kappa_{z_j}^{(k_1)}, & \text{if } (k_1,k_2) = (1,2) \text{ or } (2,3), \\ \kappa_{z_j}^{(3)} - \kappa_{z_j}^{(1)} - 1, & \text{if } (k_1,k_2) = (1,3), \end{cases}$$

and

$$\deg P_{\mathrm{i}}(\mathbf{r}, \mathbf{s}) = \big(\kappa_{\mathrm{i}}^{(k_2)} - \kappa_{\mathrm{i}}^{(k_1)}\big)/2,$$

where (k_1,k_2) *is the unique pair such that* $\kappa_{\mathrm{i}}^{(k_2)}-\kappa_{\mathrm{i}}^{(k_1)}\equiv 0 \bmod 2$. *To be more precise, for a polynomial* $P(\mathbf{r},\mathbf{s})$ *in* $\mathbf{r}$ *and* $\mathbf{s}$, *we let* $\mathrm{LT}(P)$ *denote the sum of the terms of highest degree in* P. *Then, up to nonzero scalars, we have*

$$\mathrm{LT}(P_{z_j,k_1,k_2}(\mathbf{r},\mathbf{s}))=\prod_{k=1}^{\kappa_{z_j}^{(k_2)}-\kappa_{z_j}^{(k_1)}}\big(\big(\kappa_{z_j}^{(k_1)}+k-3/2\big)d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)}\big)$$

for $(k_1,k_2)=(1,2)$ *or* $(2,3)$, *and*

$$\mathrm{LT}(P_{z_j,1,3}(\mathbf{r},\mathbf{s}))=\big(d_3s_{z_j}^{(1)}+d_1'r_{z_1}^{(1)}+\cdots+d_m'r_{z_m}^{(1)}\big)\times\prod_{k=1,\,k\neq\kappa_{z_j}^{(2)}-\kappa_{z_j}^{(1)},\kappa_{z_j}^{(2)}-\kappa_{z_j}^{(1)}+1}\big(\big(\kappa_{z_j}^{(k_1)}+k-3/2\big)d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)}\big),$$

where $d_1,d_2,d_3,d_1',\dots,d_m'$ *are complex numbers with* $d_1,d_2,d_3\neq 0$.

Also, $\mathrm{LT}(P_{\mathrm{i}}(\mathbf{r},\mathbf{s}))$ *is a product of* $\big(\kappa_{\mathrm{i}}^{(k_2)}-\kappa_{\mathrm{i}}^{(k_1)}\big)/2$ *linear sums in* $s_{\mathrm{i}}^{(1)}$ *and* $\mathbf{r}$ *with the coefficients of* $s_{\mathrm{i}}^{(1)}$ *being nonzero.*

Proof. By Lemma 5.7, each of a_n and b_n in (5.12) is a linear combination of r's and s's. The key observation here is that only $r_{z_j}^{(2)}$ (which has been determined by the local exponents and is regarded as a constant) appears in a_{-2}, only $r_{z_j}^{(2)}$ and $r_{z_j}^{(1)}$ appear in a_{-1}, only $r_{z_j}^{(2)}$ and $s_{z_j}^{(3)}$ (which has been determined by the local exponents and is regarded as a constant) appear in b_{-3}, only $r_{z_j}^{(2)}$, $r_{z_j}^{(1)}$, $s_{z_j}^{(3)}$, and $s_{z_j}^{(2)}$ appear in b_{-2}, and only $r_{z_j}^{(2)}$, $r_{z_j}^{(1)}$, $s_{z_j}^{(3)}$, $s_{z_j}^{(2)}$, and $s_{z_j}^{(1)}$ appear in b_{-1}. In particular, we have

$$\mathrm{LT}(a_{-1})=d_1r_{z_j}^{(1)},\qquad \mathrm{LT}(b_{-2})=-\frac{1}{2}d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)},\tag{5.16}$$

where d_1 and d_2 are the leading coefficients in the series expansions of $E_4(z)\Delta(z)/F_j(z)$ and $E_6(z)\Delta(z)^2/F_j(z)^2$ in $\widetilde{w}$ (and hence are nonzero). Consider (5.14) for the cases $(\alpha,n')=\big(\kappa_{z_j}^{(2)},\kappa_{z_j}^{(3)}-\kappa_{z_j}^{(2)}\big)$ and $(\alpha,n')=\big(\kappa_{z_j}^{(1)},\kappa_{z_j}^{(2)}-\kappa_{z_j}^{(1)}\big)$ first. From (5.13) and (5.16), we can easily show inductively that, up to $n=n'-1$,

$$\begin{aligned}\mathrm{LT}(c_n(\alpha))&=\prod_{k=1}^{n}\left(-\frac{\mathrm{LT}((\alpha+k-1)a_{-1}+b_{-2})}{f(\alpha+k)}\right)\\&=\prod_{k=1}^{n}\left(-\frac{(\alpha+k-3/2)d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)}}{f(\alpha+k)}\right),\end{aligned}\tag{5.17}$$

where d_1 and d_2 are the two nonzero complex numbers in (5.16), and hence

$$\mathrm{LT}(P_{z_j,k_1,k_2}(\mathbf{r},\mathbf{s}))=\prod_{k=1}^{\kappa_{z_0}^{(k_2)}-\kappa_{z_0}^{(k_1)}}\left(-\frac{(\alpha+k-3/2)d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)}}{f(\alpha+k)}\right)$$

for $(k_1,k_2)=(1,2)$ or $(2,3)$.

We now consider (5.14) for the remaining case $(\alpha,n')=\big(\kappa_{z_j}^{(1)},\kappa_{z_j}^{(3)}-\kappa_{z_j}^{(1)}\big)$. Let $n''=\kappa_{z_j}^{(2)}-\kappa_{z_j}^{(1)}$. Up to $n=n''-1$, the terms of highest degree in $c_n(\alpha)$ is given by (5.17). Since $P_{z_j,1,2}(\mathbf{r},\mathbf{s})=0$ is assumed to hold, (5.13) holds for $n=n''$ for arbitrary $c_{n''}(\alpha)$. Here, we simply choose $c_{n''}(\alpha)$ to be 0. Then we have, by (5.13)

$$c_{n''+1}(\alpha)=-\frac{1}{f(\alpha+n''+1)}\sum_{k=0}^{n''-1}[(\alpha+k)a_{n''-k-1}+b_{n''-k-2}]c_k(\alpha),$$

and hence

$$\mathrm{LT}(c_{n''+1}(\alpha)) = -\frac{1}{f(\alpha+n''+1)}\,\mathrm{LT}((\alpha+n''-1)a_0+b_{-1})\,\mathrm{LT}(c_{n''-1}(\alpha)),$$

which, by (5.17), is equal to

$$-\frac{\mathrm{LT}((\alpha+n''-1)a_0+b_{-1})}{f(\alpha+n''+1)}\prod_{k=1}^{n''-1}\left(-\frac{(\alpha+k-3/2)d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)}}{f(\alpha+k)}\right).$$

Then using (5.13) we can inductively show that for n with $n''+1\le n\le n'-1$,

$$\begin{aligned}\mathrm{LT}(c_n(\alpha)) &= \mathrm{LT}(c_{n''+1}(\alpha))\prod_{k=n''+2}^{n}\left(-\frac{\mathrm{LT}((\alpha+k-1)a_{-1}+b_{-2})}{f(\alpha+k)}\right)\\ &= \mathrm{LT}(c_{n''+1}(\alpha))\prod_{k=n''+2}^{n}\left(-\frac{(\alpha+k-3/2)d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)}}{f(\alpha+i)}\right).\end{aligned}$$

Thus, for $(k_1,k_2)=(1,3)$, we have

$$\begin{aligned}\mathrm{LT}(P_{z_j,1,3}(\mathbf{r},\mathbf{s})) &= \mathrm{LT}(c_{n''+1}(\alpha))\prod_{k=n''+2}^{n'}\left(-\frac{(\alpha+k-3/2)d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)}}{f(\alpha+k)}\right)\\ &= -\frac{\mathrm{LT}((\alpha+n''-1)a_0+b_{-1})}{f(\alpha+n''+1)}\\ &\quad\times\prod_{k=1,\,k\neq n'',n''+1}^{n'}\left(-\frac{(\alpha+k-3/2)d_1r_{z_j}^{(1)}+d_2s_{z_j}^{(2)}}{f(\alpha+k)}\right).\end{aligned}$$

Note that $\mathrm{LT}((\alpha+n''-1)a_0+b_{-1})$ is of the form

$$d_3s_{z_j}^{(1)}+d'_1r_{z_1}^{(1)}+\cdots+d'_mr_{z_m}^{(1)},$$

where d_3, $d'_1,\dots,d'_m$ are complex numbers with $d_3\neq 0$.

We next consider the case where $z_0=\mathrm{i}$ is the elliptic point of order 2. There exists a unique pair of k_1 and k_2 with $k_1<k_2$ such that $\kappa_{\mathrm{i}}^{(k_2)}-\kappa_{\mathrm{i}}^{(k_1)}\equiv 0 \bmod 2$. We let $\alpha=\kappa_{\mathrm{i}}^{(k_1)}$ and $n'=\kappa_{\mathrm{i}}^{(k_2)}-\kappa_{z_0}^{(k_1)}$. We have seen earlier that $c_n(\alpha)\neq 0$ only when $2|n$. Also, from (5.15), we can inductively show that

$$\mathrm{LT}(c_n(\alpha)) = \prod_{k=1}^{n/2}\left(-\frac{\mathrm{LT}((\alpha+2k-2)a_0+b_{-1})}{f(\alpha+2k)}\right)$$

and

$$\mathrm{LT}(P_{\mathrm{i}}(\mathbf{r},\mathbf{s})) = \prod_{k=1}^{n'/2}\left(-\frac{\mathrm{LT}((\alpha+2k-2)a_0+b_{-1})}{f(\alpha+2k)}\right).$$

Noting that $\mathrm{LT}((\alpha+2k-2)a_0+b_{-1}))=ds_{\mathrm{i}}^{(1)}+$ (a linear sum in $\mathbf{r}$) for some nonzero complex number d (which is the leading coefficient of the expansion of $\Delta(z)/E_6(z)$ at i in $\widetilde{w}$), we conclude that $\mathrm{LT}(P_{\mathrm{i}}(\mathbf{r},\mathbf{s})$ is a product of $\left(\kappa_{\mathrm{i}}^{(k_2)}-\kappa_{\mathrm{i}}^{(k_1)}\right)/2$ linear sums in $s_{\mathrm{i}}^{(1)}$ and $\mathbf{r}$ with all coefficients of $s_{\mathrm{i}}^{(1)}$ nonzero. This completes the proof. ■

Remark 5.12. The proposition suggests that when the local exponents κ's are fixed, for generic points z_j, the number of pairs (Q,R) of modular forms such that (5.2) satisfies the conditions (S1)–(S3) is

$$\frac{\kappa_{\mathrm{i}}^{(k_2)}-\kappa_{\mathrm{i}}^{(k_1)}}{2}\prod_{j=1}^{m}\big(\kappa_{z_j}^{(2)}-\kappa_{z_j}^{(1)}\big)\big(\kappa_{z_j}^{(3)}-\kappa_{z_j}^{(2)}\big)\big(\kappa_{z_j}^{(3)}-\kappa_{z_j}^{(1)}-1\big),$$

where k_1 and k_2 are the integers such that $\kappa_{\mathrm{i}}^{(k_2)}-\kappa_{\mathrm{i}}^{(k_1)}\in 2\mathbb{N}$. (Notice that if $\big(\kappa_z^{(1)},\kappa_z^{(2)},\kappa_z^{(3)}\big)=(0,1,2)$ for all $z\in\mathbb{H}$, this number is 1, as expected.) However, because the polynomials have intersection of positive dimension at infinity in general, we are not able to use the Bézout theorem to obtain this conclusion. (Even the existence of (Q,R) with an arbitrary set of given data is not established yet.) We leave this problem for future study.

6 Extremal quasimodular forms

We note that by using some results in Section 5, Theorem 2.2 can be improved in some special case. More precisely, the main result of this section is Theorem 6.2 below, which states that $Q_2(z)$, $Q_3(z)$ in Theorem 2.2 can be explicitly written down in the case $f(z)$ is an extremal quasimodular form on $\Gamma=\mathrm{SL}(2,\mathbb{Z})$.

Definition 6.1 ([18])**.** A quasimodular form $f\in\widetilde{\mathfrak{M}}_k^{\le r}(\mathrm{SL}(2,\mathbb{Z}))$ is said to be extremal if its vanishing order at ∞ is equal to $\dim\widetilde{\mathfrak{M}}_k^{\le r}(\mathrm{SL}(2,\mathbb{Z}))-1$. We say f is normalized if its leading Fourier coefficient is 1.

Pellarin [28] proved that if $r\le 4$, then a normalized extremal quasimodular form in $\widetilde{\mathfrak{M}}_k^{\le r}(\mathrm{SL}(2,\mathbb{Z}))$ exists and is unique.

Theorem 6.2. *Let $f(z)$ be an extremal quasimodular form of weight k and depth 2 on $\mathrm{SL}(2,\mathbb{Z})$ and*

$$D_q^3y(z)+Q(z)D_qy(z)+\left(\frac{1}{2}D_qQ(z)+R(z)\right)y(z)=0 \tag{6.1}$$

be the differential equation satisfied by $f(z)/\sqrt[3]{W_f(z)}$ as derived in Section 2.

(i) If $k\equiv 0 \bmod 4$, then

$$Q(z)=-\frac{k^2}{48}E_4(z),\qquad R(z)=-\frac{k^3}{864}E_6(z).$$

(ii) If $k\equiv 2 \bmod 4$, then

$$Q(z)=-\frac{(k-2)^2}{48}E_4(z)-\frac{1}{3}\frac{E_4(z)\big(E_4(z)^3-E_6(z)^2\big)}{E_6(z)^2},$$

and

$$R(z)=-\frac{(k-2)^3}{864}E_6(z)+\frac{5}{54}\frac{\big(E_4(z)^3-E_6(z)^2\big)^2}{E_6(z)^3}+\frac{12-(k-2)^2}{144}\frac{E_4(z)^3-E_6(z)^2}{E_6(z)}.$$

Remark 6.3. We note that the differential equation in the case $k \equiv 0 \bmod 4$ is equivalent to the following equation studied by Kaneko and Koike [18, Theorem 3.1]:

$$D_q^3 f - \frac{k}{4} E_2 D_q^2 f + \frac{k(k-1)}{4} D_q E_2 D_q f - \frac{k(k-1)(k-2)}{24} D_q^2 E_2 f = 0.$$

Indeed, by letting $f(z) = \Delta(z)^{\frac{k}{12}} y(z)$, a direct computation shows that $y(z)$ solves

$$D_q^3 y - \frac{k^2}{48} E_4(z) D_q y - \frac{2k^2}{12^3} \left(3E_2(z)E_4(z) + (k-3)E_6(z)\right) y = 0.$$

To prove Theorem 6.2, we need the following general lemma, in which the quasimodular form $f(z)$ is not assumed to be extremal.

Lemma 6.4. *Assume that*

$$f(z) = f_0(z) + f_1(z)E_2(z) + f_2(z)E_2(z)^2 \in \widetilde{\mathfrak{M}}_k^{\le 2}(\mathrm{SL}(2,\mathbb{Z})),$$
$$f_j(z) \in \mathfrak{M}_{k-2j}(\mathrm{SL}(2,\mathbb{Z})).$$

Let

$$g(z) = f_1(z) + 2f_2(z)E_2(z), \qquad h(z) = f_2(z), \qquad m = \min(\mathrm{ord}_\infty f, \mathrm{ord}_\infty g, \mathrm{ord}_\infty h).$$

Let $\kappa_\infty^{(1)} \le \kappa_\infty^{(2)} \le \kappa_\infty^{(3)}$ *be the local exponents of* (2.4) *at* ∞.

(i) *If* $\mathrm{ord}_\infty f = m$, *then* $\mathrm{ord}_\infty W_f = 3m$ *and* $\kappa_\infty^{(j)} = 0$ *for all* j.

(ii) *If* $\mathrm{ord}_\infty g = m$ *and* $\mathrm{ord}_\infty f \ne m$, *then* $\mathrm{ord}_\infty W_f = \mathrm{ord}_\infty f + 2\,\mathrm{ord}_\infty g$ *and* $\kappa_\infty^{(1)} = \kappa_\infty^{(2)} = (\mathrm{ord}_\infty g - \mathrm{ord}_\infty f)/3$ *and* $\kappa_\infty^{(3)} = 2(\mathrm{ord}_\infty f - \mathrm{ord}_\infty g)/3$.

(iii) *If* $\mathrm{ord}_\infty h < \mathrm{ord}_\infty f \le \mathrm{ord}_\infty g$, *then* $\mathrm{ord}_\infty W_f = 2\,\mathrm{ord}_\infty f + \mathrm{ord}_\infty h$ *and* $\kappa_\infty^{(1)} = 2(\mathrm{ord}_\infty h - \mathrm{ord}_\infty f)/3$ *and* $\kappa_\infty^{(2)} = \kappa_\infty^{(3)} = (\mathrm{ord}_\infty f - \mathrm{ord}_\infty h)/3$.

(iv) *If* $\mathrm{ord}_\infty h < \mathrm{ord}_\infty g < \mathrm{ord}_\infty f$, *then* $\mathrm{ord}_\infty W_f = \mathrm{ord}_\infty f + \mathrm{ord}_\infty g + \mathrm{ord}_\infty h$ *and* $\kappa_\infty^{(1)} = \mathrm{ord}_\infty h - \frac{1}{3}\mathrm{ord}_\infty W_f$, $\kappa_\infty^{(2)} = \mathrm{ord}_\infty g - \frac{1}{3}\mathrm{ord}_\infty W_f$, *and* $\kappa_\infty^{(3)} = \mathrm{ord}_\infty f - \frac{1}{3}\mathrm{ord}_\infty W_f$.

Proof. Let $r = \frac{1}{3}\mathrm{ord}_\infty W_f$. Since up to scalars, $g_3(z) = f(z)/\sqrt[3]{W_f(z)}$ is the unique solution of (2.4) without logarithmic singularity near ∞, according to Frobenius' method for complex ordinary differential equations (see, e.g., Appendix A), we must have $\kappa_\infty^{(3)} = \mathrm{ord}_\infty f - r$. Likewise, because $g_2(z) = (2zf(z) + \alpha g(z))/\sqrt[3]{W_f(z)}$ and $g_1(z) = \left(z^2 f(z) + \alpha z g(z) + \alpha^2 h(z)\right)/\sqrt[3]{W_f(z)}$ are the other two linearly independent solutions of (2.4), we have

$$\kappa_\infty^{(2)} = \min(\mathrm{ord}_\infty f, \mathrm{ord}_\infty g) - r, \qquad \kappa_\infty^{(1)} = \min(\mathrm{ord}_\infty f, \mathrm{ord}_\infty g, \mathrm{ord}_\infty h) - r.$$

Analyzing case by case, we obtain the claimed conclusions. ■

Proof of Theorem 6.2. First of all, recall that

$$\dim \widetilde{\mathfrak{M}}_k^{\le 2}(\mathrm{SL}(2,\mathbb{Z})) = 1 + \left\lfloor \frac{k}{4} \right\rfloor. \tag{6.2}$$

Let $f(z) = f_0(z) + f_1(z)E_2(z) + f_2(z)E_2(z)^2$, $f_j \in \mathfrak{M}_{k-2j}(\mathrm{SL}(2,\mathbb{Z}))$, be an extremal quasimodular form in $\widetilde{\mathfrak{M}}_k^{\le 2}(\mathrm{SL}(2,\mathbb{Z}))$. Note that $f_j(z)$ cannot have a common zero on $\mathbb{H}$. To see this, say, assume that $f_j(z)$ has a common zero at z_0. Let $F(z)$ be a modular form of weight k'

with a simple zero at z_0 and nonvanishing elsewhere. Then $f(z)/F(z) \in \widetilde{\mathfrak{M}}^{\le 2}_{k-k'}(\mathrm{SL}(2,\mathbb{Z}))$ has order $\lfloor k/4 \rfloor$ at ∞, which is impossible by (6.2) and the facts that $k' \ge 4$ and that extremal quasimodular forms of depth 2 exist for any weight and are unique up to scalars. Therefore, $f_j(z)$ have no common zeros on $\mathbb{H}$.

Now according to Pellarin's argument [28], one has $\operatorname{ord}_\infty W_f = \operatorname{ord}_\infty f = \lfloor k/4 \rfloor$. Hence, we have

$$W_f(z) = \begin{cases} c\Delta(z)^{k/4}, & \text{if } k \equiv 0 \bmod 4, \\ c\Delta(z)^{(k-2)/4}E_6(z), & \text{if } k \equiv 2 \bmod 4, \end{cases}$$

for some nonzero complex number c. Also, by Lemma 6.4, we must have $\operatorname{ord}_\infty(f_1 + 2f_2E_2) = 0$ and the local exponents at ∞ must be $-r/3$, $-r/3$, and $2r/3$, where $r = \lfloor k/4 \rfloor$. In other words, the indicial equation of (2.4) at ∞ is

$$x^3 - \frac{r^2}{3}x - \frac{2r^3}{27} = 0. \tag{6.3}$$

Consider first the case $k \equiv 0 \bmod 4$. In this case, since $\Delta(z)$ has no zeros on $\mathbb{H}$, (2.4) has no singularities on $\mathbb{H}$. Hence, $Q(z)$ is a multiple of $E_4(z)$, while $R(z)$ is a multiple of $E_6(z)$. In view of (6.3) and Lemma 5.8, we see that

$$Q(z) = -\frac{k^2}{48}E_4(z), \qquad R(z) = -\frac{k^3}{864}E_6(z).$$

We now consider the case $k \equiv 2 \bmod 4$. In this case, $W_f(z) = c\Delta(z)^{(k-2)/4}E_6(z)$ has a simple zero at i. Thus, the local exponents of (2.4) at i are $-1/3$, $2/3$, and $8/3$ since the differences must be positive integers and the sum must be equal to 3, and the indicial equation at i is

$$x^3 - 3x^2 + \frac{2}{3}x + \frac{16}{27} = 0. \tag{6.4}$$

We will use this information, together with the apparentness property, to determine $Q(z)$ and $R(z)$.

First of all, according to Lemma 5.7, $Q(z)$ is of the form

$$Q(z) = r_\infty E_4(z) + r_{\mathrm{i}}^{(2)} \frac{E_4(z)\big(E_4(z)^3 - E_6(z)^2\big)}{E_6(z)^2},$$

while $R(z)$ is of the form

$$R(z) = s_\infty E_6(z) + s_{\mathrm{i}}^{(3)} \frac{\big(E_4(z)^3 - E_6(z)^2\big)^2}{E_6(z)^3} + s_{\mathrm{i}}^{(1)} \frac{E_4(z)^3 - E_6(z)^2}{E_6(z)}$$

for some complex numbers r_∞, $r_{\mathrm{i}}^{(2)}$, s_∞, $s_{\mathrm{i}}^{(3)}$, and $s_{\mathrm{i}}^{(1)}$. The parameters r_∞ and s_∞ are determined by the local exponents at ∞. As in the case $k \equiv 0 \bmod 4$, we find that $r_\infty = -\frac{(k-2)^2}{48}$ and $s_\infty = -\frac{(k-2)^3}{864}$. We now determine the other parameters.

By (6.4) and Lemma 5.9, we have $r_{\mathrm{i}}^{(2)} = -\frac{1}{3}$ and $s_{\mathrm{i}}^{(3)} = \frac{5}{54}$. To determine the remaining parameter $s_{\mathrm{i}}^{(1)}$, we let $w = (z-\mathrm{i})/(z+\mathrm{i})$ and recall that, by (5.7),

$$E_4(z) = (1-w)^4\left(B + \frac{5}{72}B^2u^2 + \frac{5}{6912}B^3u^4 + \cdots\right)$$

and

$$E_6(z) = (1-w)^6\left(-\frac{1}{2}B^2u - \frac{7}{432}B^3u^3 - \frac{7}{17280}B^4u^5 + \cdots\right),$$

where $u = -4\pi w$ and $B = E_4(\mathrm{i})$. (Note that $E_6(z_0)$ and the constant A in Lemma 5.5 are both 0 when $z_0 = \mathrm{i}$.) Then the power series $\widetilde{Q}(x)$ and $\widetilde{R}(x)$ such that $Q(z) = (1-w)^4\widetilde{Q}(-4\pi w)$ and $R(z) = (1-w)^6\widetilde{R}(-4\pi w)$ are

$$\widetilde{Q}(x) = \frac{4r_{\mathrm{i}}^{(2)}}{x^2} + \left(r_\infty - \frac{4r_{\mathrm{i}}^{(2)}}{27}\right)B + \cdots = -\frac{4}{3x^2} + \left(-\frac{(k-2)^2}{48} + \frac{4}{81}\right)B + \cdots$$

and

$$\widetilde{R}(x) = -\frac{8s_{\mathrm{i}}^{(3)}}{x^3} + \left(-2s_{\mathrm{i}}^{(1)} + \frac{13}{9}s_{\mathrm{i}}^{(3)}\right)\frac{B}{x} + \cdots = -\frac{20}{27x^3} + \left(-2s_{\mathrm{i}}^{(1)} + \frac{65}{486}\right)\frac{B}{x} + \cdots,$$

respectively. By Lemma 5.5, the series (5.5) with $c_0 = 1$ is a solution of (6.1) if and only if the power series $\widetilde{y}(x) = \sum_{n=0}^{\infty} c_n x^{n+\alpha}$ satisfies (5.6). Consider $\widetilde{y}(x)$ with $\alpha = 2/3$. The coefficients c_n need to satisfy

$$\begin{aligned}&\sum_{n=0}^{\infty} c_n(n+2/3)(n-1/3)(n-4/3)x^{n-7/3}\\ &\quad + \left(-\frac{4}{3x^2} + \left(\frac{4}{81} - \frac{(k-2)^2}{48}\right)B + \cdots\right)\sum_{n=0}^{\infty} c_n(n+2/3)x^{n-1/3}\\ &\quad + \left(\frac{16}{27x^3} + \left(\frac{65}{486} - 2s_{\mathrm{i}}^{(1)}\right)\frac{B}{x} + \cdots\right)\sum_{n=0}^{\infty} c_n x^{n+2/3} = 0.\end{aligned}$$

Considering the coefficients of $x^{-1/3}$, we find that $s_{\mathrm{i}}^{(1)} = \frac{12-(k-2)^2}{144}$. This completes the proof of the theorem. ∎

A The solution structure of third order ODE

In this appendix, we apply Frobenius' method to study the solution structure for

$$\mathcal{L}y := \frac{\mathrm{d}^3}{\mathrm{d}x^3}y(x) + \mathcal{Q}(x)\frac{\mathrm{d}}{\mathrm{d}x}y(x) + \mathcal{R}(x)y(x) = 0. \tag{A.1}$$

See, e.g., [15, 16] for detailed expositions of Frobenius' method. The following arguments are known to experts in this field. However, since we can not find a suitable reference, we would like to provide all necessary details for later usage.

Suppose 0 is a regular singular point of (A.1) with three local exponents

$$\kappa_1, \qquad \kappa_2 = \kappa_1 + m_1, \qquad \kappa_3 = \kappa_2 + m_2, \qquad \text{where} \quad m_1, m_2 \in \mathbb{Z}_{\geq 0},$$

Since the exponent differences are all integers, there might be logarithmic singularities for solutions of (A.1), or more precisely, the local expansion of some solutions at $x = 0$ might contains $\ln x$ terms or even $(\ln x)^2$ terms. This leads us to give the following definition.

Definition A.1.

(1) We say (A.1) is apparent at $x = 0$ if all solutions have no logarithmic singularities at $x = 0$. Otherwise (A.1) is called not apparent at $x = 0$.

(2) If (A.1) is not apparent at $x = 0$ and the local expansion of some solutions contains $(\ln x)^2$ terms, we say (A.1) is completely not apparent at $x = 0$.

Since 0 is a regular singular point of (A.1), both $x^2\mathcal{Q}(x)$ and $x^3\mathcal{R}(x)$ are holomorphic at $x=0$, so we may write

$$\mathcal{Q}(x) = \sum_{n=-2}^{\infty} a_n x^n, \qquad \mathcal{R}(x) = \sum_{n=-3}^{\infty} b_n x^n.$$

Let

$$y(x;\alpha) = x^{\alpha}\sum_{n=0}^{\infty} c_n(\alpha)x^n, \qquad \text{where} \quad c_0(\alpha)=1.$$

Then

$$\begin{aligned}\mathcal{L}y(x;\alpha) &= \sum_{n=0}^{\infty}\left(f(\alpha+n)c_n(\alpha) + \sum_{k=0}^{n-1}[(\alpha+k)a_{n-k-2} + b_{n-k-3}]c_k(\alpha)\right)x^{n+\alpha-3}\\ &=: x^{\alpha-3}\sum_{n=0}^{\infty} R_n(\alpha)x^n,\end{aligned} \tag{A.2}$$

where

$$f(s) := s(s-1)(s-2) + sa_{-2} + b_{-3} = \prod_{j=1}^{3}(s-\kappa_j),$$

i.e., $f(s)=0$ is the indicial equation of (A.1) at $x=0$. Note $R_0(\alpha)=f(\alpha)$.

For any α satisfying $|\alpha-\kappa_3|<1/2$, we have

$$f(\alpha+n)\neq 0 \qquad \text{for any } n\geq 1,$$

so by letting

$$R_n(\alpha) = f(\alpha+n)c_n(\alpha) + \sum_{k=0}^{n-1}[(\alpha+k)a_{n-k-2} + b_{n-k-3}]c_k(\alpha) = 0, \qquad n\geq 1,$$

we see that $c_n(\alpha)$ can be uniquely solved for any $n\geq 1$ such that

$$\mathcal{L}y(x;\alpha) = x^{\alpha-3}f(\alpha), \qquad \text{for any } |\alpha-\kappa_3|<1/2. \tag{A.3}$$

Note that $c_n(\alpha)\in\mathbb{C}(\alpha)$ is a rational function of α for any $n\geq 1$.

In particular, letting $\alpha=\kappa_3$ in (A.3) leads to $\mathcal{L}y(x;\kappa_3)=0$, so

Lemma A.2.

$$y(x;\kappa_3) = x^{\kappa_3}\sum_{n=0}^{\infty} c_n(\kappa_3)x^n$$

is always a solution of (A.1) *with the local exponent* κ_3*.*

By (A.3), we have

$$\mathcal{L}\frac{\partial y(x;\alpha)}{\partial\alpha} = x^{\alpha-3}f(\alpha)\ln x + x^{\alpha-3}f'(\alpha),$$

so

$$\mathcal{L}\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_3} = x^{\kappa_3-3}f'(\kappa_3). \tag{A.4}$$

Similarly,

$$\mathcal{L}\frac{\partial^2 y(x;\alpha)}{\partial\alpha^2}\Big|_{\alpha=\kappa_3} = 2x^{\kappa_3-3}f'(\kappa_3)\ln x + x^{\kappa_3-3}f''(\kappa_3). \tag{A.5}$$

Note that

$$\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_3} = (\ln x)y(x;k_3) + x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n, \tag{A.6}$$

$$\frac{\partial^2 y(x;\alpha)}{\partial\alpha^2}\Big|_{\alpha=\kappa_3} = (\ln x)^2 y(x;k_3) + 2(\ln x)x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n + x^{\kappa_3}\sum_{n=1}^{\infty} c_n''(\kappa_3)x^n. \tag{A.7}$$

Theorem A.3. *If $\kappa_1 = \kappa_2 = \kappa_3$, then $\frac{\partial y(x;\alpha)}{\partial\alpha}|_{\alpha=\kappa_3}$ and $\frac{\partial^2 y(x;\alpha)}{\partial\alpha^2}|_{\alpha=\kappa_3}$ given in (A.6) and (A.7) are the other two linearly independent solutions of (A.1), namely (A.1) is completely not apparent at $x = 0$.*

Proof. Since $f(s) = (s-\kappa_3)^3$, we have $f'(\kappa_3) = f''(\kappa_3) = 0$, so this theorem follows from (A.4) and (A.5). ∎

Next we consider the case $\kappa_1 < \kappa_2 = \kappa_3$, i.e., $m_1 > 0$, $m_2 = 0$ and $f(s) = (s-\kappa_1)(s-\kappa_3)^2$. Then $f'(\kappa_3) = 0$ and $f''(\kappa_3) \neq 0$, so $\frac{\partial y(x;\alpha)}{\partial\alpha}|_{\alpha=\kappa_3}$ given in (A.6) is the second solution of (A.1), and (A.5) becomes

$$\mathcal{L}\frac{\partial^2 y(x;\alpha)}{\partial\alpha^2}\Big|_{\alpha=\kappa_3} = x^{\kappa_3-3}f''(\kappa_3) \neq 0. \tag{A.8}$$

On the other hand, $f(\kappa_1 + n) \neq 0$ for $n \in \mathbb{N}\setminus\{m_1\}$. Thus by letting $c_{m_1}(\kappa_1) = 0$ and $R_n(\kappa_1) = 0$ for any $n \in \mathbb{N}\setminus\{m_1\}$ in $\mathcal{L}y(x;\kappa_1)$ (see (A.2) with $\alpha = \kappa_1$), we see that $c_n(\kappa_1)$ can be uniquely solved for any $n \in \mathbb{N}\setminus\{m_1\}$ such that

$$\mathcal{L}y(x;\kappa_1) = x^{\kappa_1-3}R_{m_1}(\kappa_1)x^{m_1} = R_{m_1}(\kappa_1)x^{\kappa_3-3}, \tag{A.9}$$

where

$$R_{m_1}(\kappa_1) = \sum_{k=0}^{m_1-1}[(\kappa_1+k)a_{m_1-k-2} + b_{m_1-k-3}]c_k(\kappa_1)$$

is a constant. Thus we obtain

Theorem A.4. *Suppose $\kappa_1 < \kappa_2 = \kappa_3$. Then $\frac{\partial y(x;\alpha)}{\partial\alpha}|_{\alpha=\kappa_3}$ given in (A.6) is the second solution of (A.1). Furthermore,*

(1) *If $R_{m_1}(\kappa_1) = 0$, then*

$$y(x;\kappa_1) = x^{\kappa_1}\sum_{n=0}^{\infty} c_n(\kappa_1)x^n$$

is the third solution of (A.1) that has the local exponent κ_1.

(2) *If $R_{m_1}(\kappa_1) \neq 0$, then (A.8) and (A.9) imply that*

$$y_3(x) := \frac{\partial^2 y(x;\alpha)}{\partial\alpha^2}\Big|_{\alpha=\kappa_3} - \frac{f''(\kappa_3)}{R_{m_1}(\kappa_1)}y(x;\kappa_1)$$

$$
\begin{aligned}
&= (\ln x)^2 y(x;k_3) + 2(\ln x)x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n \\
&\quad + x^{\kappa_3}\sum_{n=1}^{\infty} c_n''(\kappa_3)x^n - \frac{f''(\kappa_3)}{R_{m_1}(\kappa_1)}y(x;\kappa_1)
\end{aligned}
$$

is the third solution of (A.1) *that corresponds to the local exponent* κ_1*, namely* (A.1) *is completely not apparent at* $x = 0$*.*

The remaining case is $\kappa_1 \le \kappa_2 < \kappa_3$, i.e., $m_2 = \kappa_3 - \kappa_2 > 0$. Then for any α satisfying $|\alpha - \kappa_2| < 1/2$, we have

$$f(\alpha + n) \neq 0 \qquad \text{for any } n \in \mathbb{N} \setminus \{m_2\},$$

and

$$f(\alpha + n) = 0 \qquad \text{for } n \ge 1 \text{ if and only if } \alpha = \kappa_2 \text{ and } n = m_2,$$

so by letting $c_{m_2}(\alpha) = 0$ and $R_n(\alpha) = 0$ for any $n \in \mathbb{N} \setminus \{m_2\}$ in $\mathcal{L}y(x;\alpha)$ (see (A.2)), we see that $c_n(\alpha)$ can be uniquely solved for any $n \in \mathbb{N} \setminus \{m_2\}$ such that

$$\mathcal{L}y(x;\alpha) = x^{\alpha-3}f(\alpha) + x^{\alpha+m_2-3}R_{m_2}(\alpha), \qquad \text{for } |\alpha - \kappa_2| < 1/2, \tag{A.10}$$

where

$$R_{m_2}(\alpha) = \sum_{k=0}^{m_2-1}[(\alpha+k)a_{m_2-k-2} + b_{m_2-k-3}]c_k(\alpha) \in \mathbb{C}(\alpha),$$

because $c_k(\alpha) \in \mathbb{C}(\alpha)$ for any $n \in \mathbb{N} \setminus \{m_2\}$.

Similarly as before, it follows from (A.10) that

$$\mathcal{L}y(x;\kappa_2) = x^{\kappa_3-3}R_{m_2}(\kappa_2), \tag{A.11}$$

$$\mathcal{L}\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} = x^{\kappa_2-3}f'(\kappa_2) + x^{\kappa_3-3}R_{m_2}'(\kappa_2) + (\ln x)x^{\kappa_3-3}R_{m_2}(\kappa_2), \tag{A.12}$$

where

$$\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} = (\ln x)y(x;\kappa_2) + x^{\kappa_2}\sum_{n=1}^{\infty}c_n'(\kappa_2)x^n.$$

Theorem A.5. *Suppose* $\kappa_1 = \kappa_2 < \kappa_3$*.*

(1) *If* $R_{m_2}(\kappa_2) = 0$*, then* (A.11) *implies that* $y(x;\kappa_2)$ *is the second solution of* (A.1)*, and*

$$
\begin{aligned}
y_3(x) &:= \frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} - \frac{R_{m_2}'(\kappa_2)}{f'(\kappa_3)}\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_3} \\
&= (\ln x)y(x;\kappa_2) + x^{\kappa_2}\sum_{n=1}^{\infty}c_n'(\kappa_2)x^n \\
&\quad - \frac{R_{m_2}'(\kappa_2)}{f'(\kappa_3)}\left((\ln x)y(x;\kappa_3) + x^{\kappa_3}\sum_{n=1}^{\infty}c_n'(\kappa_3)x^n\right)
\end{aligned}
$$

is the third solution of (A.1)*.*

(2) *If $R_{m_2}(\kappa_2) \neq 0$, then* (A.4) *and* (A.11) *imply that*

$$\begin{aligned}
&\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_3} - \frac{f'(\kappa_3)}{R_{m_2}(\kappa_2)}y(x;\kappa_2) \\
&\quad = (\ln x)y(x;\kappa_3) + x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n - \frac{f'(\kappa_3)}{R_{m_2}(\kappa_2)}y(x;\kappa_2)
\end{aligned}$$

is the second solution of (A.1)*, and*

$$\begin{aligned}
y_3(x) := {}& \frac{\partial^2 y(x;\alpha)}{\partial\alpha^2}\Big|_{\alpha=\kappa_3} - \frac{2f'(\kappa_3)}{R_{m_2}(k_2)}\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} \\
&- \frac{f''(\kappa_3)R_{m_2}(\kappa_2) - 2f'(\kappa_3)R_{m_2}'(\kappa_2)}{R_{m_2}(\kappa_2)^2}y(x;\kappa_2) \\
= {}& (\ln x)^2 y(x;k_3) + 2(\ln x)x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n + x^{\kappa_3}\sum_{n=1}^{\infty} c_n''(\kappa_3)x^n \\
&- \frac{2f'(\kappa_3)}{R_{m_2}(k_2)}\left((\ln x)y(x;\kappa_2) + x^{\kappa_2}\sum_{n=1}^{\infty} c_n'(\kappa_2)x^n\right) \\
&- \frac{f''(\kappa_3)R_{m_2}(\kappa_2) - 2f'(\kappa_3)R_{m_2}'(\kappa_2)}{R_{m_2}(\kappa_2)^2}y(x;\kappa_2).
\end{aligned}$$

is the third solution of (A.1)*, namely* (A.1) *is completely not apparent at $x = 0$.*

Proof. Since $\kappa_1 = \kappa_2 < \kappa_3$, i.e., $m_1 = 0$, $m_2 > 0$ and $f(s) = (s-\kappa_2)^2(s-\kappa_3)$, so $f'(\kappa_2) = 0$ and $f'(\kappa_3) \neq 0$.

(1) Note that (A.12) becomes

$$\mathcal{L}\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} = x^{\kappa_3-3}R_{m_2}'(\kappa_2),$$

we see from (A.4) that $\mathcal{L}y_3 = 0$.

(2) Note that (A.12) becomes

$$\mathcal{L}\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} = x^{\kappa_3-3}R_{m_2}'(\kappa_2) + (\ln x)x^{\kappa_3-3}R_{m_2}(\kappa_2). \tag{A.13}$$

Then (A.5), (A.11) and (A.13) together imply $\mathcal{L}y_3 = 0$. ■

Finally, we consider the last case $\kappa_1 < \kappa_2 < \kappa_3$, i.e., $m_1 > 0$ and $m_2 > 0$. Then $f'(\kappa_j) \neq 0$ for all j. Since $f(\kappa_1 + n) = 0$ for $n \geq 1$ if and only if $n \in \{m_1, m_1+m_2\}$, by letting $c_{m_1}(\kappa_1) = c_{m_1+m_2} = 0$ and $R_n(\kappa_1) = 0$ for any $n \in \mathbb{N}\setminus\{m_1, m_1+m_2\}$ in $\mathcal{L}y(x;\kappa_1)$ (see (A.2) with $\alpha = \kappa_1$), we see that $c_n(\kappa_1)$ can be uniquely solved for any $n \in \mathbb{N}\setminus\{m_1, m_1+m_2\}$ such that

$$\mathcal{L}y(x;\kappa_1) = R_{m_1}(\kappa_1)x^{\kappa_2-3} + R_{m_1+m_2}(\kappa_1)x^{\kappa_3-3}, \tag{A.14}$$

where

$$\begin{aligned}
R_{m_1}(\kappa_1) &= \sum_{k=0}^{m_1-1}[(\kappa_1+k)a_{m_1-k-2} + b_{m_1-k-3}]c_k(\kappa_1), \\
R_{m_1+m_2}(\kappa_1) &= \sum_{k=0}^{m_1+m_2-1}[(\kappa_1+k)a_{m_1+m_2-k-2} + b_{m_1+m_2-k-3}]c_k(\kappa_1),
\end{aligned}$$

are constants.

Theorem A.6. *Suppose $\kappa_1 < \kappa_2 < \kappa_3$ and $R_{m_2}(\kappa_2) = 0$. Then (A.11) implies that $y(x;\kappa_2)$ is the second solution of (A.1). Furthermore,*

(1) *If $R_{m_1}(\kappa_1) = R_{m_1+m_2}(\kappa_1) = 0$, then (A.14) implies that $y(x;\kappa_3)$ is the third solution of (A.1), namely 0 is an apparent singularity of (A.1).*

(2) *If $R_{m_1}(\kappa_1) = 0$ and $R_{m_1+m_2}(\kappa_1) \neq 0$, then*

$$\begin{aligned} y_3(x) &:= \frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_3} - \frac{f'(\kappa_3)}{R_{m_1+m_2}(\kappa_1)} y(x;\kappa_1) \\ &= (\ln x) y(x;\kappa_3) + x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n - \frac{f'(\kappa_3)}{R_{m_1+m_2}(\kappa_1)} y(x;\kappa_1) \end{aligned}$$

is the third solution of (A.1).

(3) *If $R_{m_1}(\kappa_1) \neq 0$, then*

$$\begin{aligned} y_3(x) &:= \frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} - \frac{f'(\kappa_2)}{R_{m_1}(k_1)} y(x;\kappa_1) \\ &\quad - \frac{R_{m_1}(\kappa_1)R'_{m_2}(\kappa_2) - f'(\kappa_2)R_{m_1+m_2}(\kappa_1)}{f'(\kappa_3)R_{m_1}(\kappa_1)} \frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_3} \\ &= (\ln x) y(x;\kappa_2) + x^{\kappa_2}\sum_{n=1}^{\infty} c_n'(\kappa_2)x^n - \frac{f'(\kappa_2)}{R_{m_1}(k_1)} y(x;\kappa_1) - \\ &\quad \frac{R_{m_1}(\kappa_1)R'_{m_2}(\kappa_2) - f'(\kappa_2)R_{m_1+m_2}(\kappa_1)}{f'(\kappa_3)R_{m_1}(\kappa_1)} \left((\ln x)y(x;\kappa_3) + x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n \right) \end{aligned}$$

is the third solution of (A.1).

Proof. Note that (A.12) becomes

$$\mathcal{L}\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} = x^{\kappa_2-3}f'(\kappa_2) + x^{\kappa_3-3}R'_{m_2}(\kappa_2). \tag{A.15}$$

(2) Note that (A.14) becomes

$$\mathcal{L}y(x;\kappa_1) = R_{m_1+m_2}(\kappa_1)x^{\kappa_3-3} \neq 0. \tag{A.16}$$

From here and (A.4), we easily obtain $\mathcal{L}y_3 = 0$.

(3) Similarly, it is easy see from (A.4), (A.14) and (A.15) that $\mathcal{L}y_3 = 0$. ■

Similarly, we can obtain

Theorem A.7. *Suppose $\kappa_1 < \kappa_2 < \kappa_3$ and $R_{m_2}(\kappa_2) \neq 0$. Then (A.4) and (A.11) imply that*

$$\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_3} - \frac{f'(\kappa_3)}{R_{m_2}(\kappa_2)} y(x;\kappa_2) = (\ln x)y(x;\kappa_3) + x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n - \frac{f'(\kappa_3)}{R_{m_2}(\kappa_2)} y(x;\kappa_2)$$

is the second solution of (A.1). Furthermore,

(1) *If $R_{m_1}(\kappa_1) = R_{m_1+m_2}(\kappa_1) = 0$, then (A.14) implies that $y(x;\kappa_3)$ is the third solution of (A.1).*

(2) *If $R_{m_1}(\kappa_1) = 0$ and $R_{m_1+m_2}(\kappa_1) \neq 0$, then (A.11) and (A.16) imply that*

$$y(x;\kappa_1) - \frac{R_{m_1+m_2}(\kappa_1)}{R_{m_2}(\kappa_2)} y(x;\kappa_2)$$

is the third solution of (A.1).

(3) *If $R_{m_1}(\kappa_1)\neq 0$, then* (A.5), (A.11), (A.12) *and* (A.14) *imply that*

$$\begin{aligned} y_3(x) &:= \frac{\partial^2 y(x;\alpha)}{\partial\alpha^2}\Big|_{\alpha=\kappa_3} - \frac{2f'(\kappa_3)}{R_{m_2}(\kappa_2)}\frac{\partial y(x;\alpha)}{\partial\alpha}\Big|_{\alpha=\kappa_2} + C_1 y(x;\kappa_1) - C_2 y(x;\kappa_2) \\ &= (\ln x)^2 y(x;\kappa_3) + 2(\ln x)x^{\kappa_3}\sum_{n=1}^{\infty} c_n'(\kappa_3)x^n + x^{\kappa_3}\sum_{n=1}^{\infty} c_n''(\kappa_3)x^n \\ &\quad - \frac{2f'(\kappa_3)}{R_{m_2}(k_2)}\left((\ln x)y(x;\kappa_2) + x^{\kappa_2}\sum_{n=1}^{\infty} c_n'(\kappa_2)x^n\right) + C_1 y(x;\kappa_1) - C_2 y(x;\kappa_2) \end{aligned}$$

is the third solution of (A.1), *where*

$$\begin{aligned} C_1 &:= \frac{2f'(\kappa_3)f'(\kappa_2)}{R_{m_2}(\kappa_2)R_{m_1}(\kappa_1)}, \\ C_2 &:= \frac{1}{R_{m_2}(\kappa_2)}\left[f''(\kappa_3) - \frac{2f'(\kappa_3)R'_{m_2}(\kappa_2)}{R_{m_2}(\kappa_2)} + \frac{2f'(\kappa_3)f'(\kappa_2)R_{m_1+m_2}(\kappa_1)}{R_{m_2}(\kappa_2)R_{m_1}(\kappa_1)}\right]. \end{aligned}$$

In particular, (A.1) *is completely not apparent at $x = 0$.*

Remark A.8. It follows from Theorem A.6(1) that 0 can be apparent only for the case $\kappa_1 < \kappa_2 < \kappa_3$.

Remark A.9. Clearly all the above arguments work when we study whether the regular singularity ∞ is apparent or not for

$$y'''(z) + Q_2(z)y'(z) + Q_3(z)y(z) = 0, \qquad z\in\mathbb{H}, \tag{A.17}$$

when the local exponents $\kappa_\infty^{(1)} \le \kappa_\infty^{(2)} \le \kappa_\infty^{(3)}$ satisfy $\kappa_\infty^{(j)} - \kappa_\infty^{(j-1)} \in \mathbb{Z}$. Since $Q_j(z)$'s have Fourier expansions in terms of $q_N = \mathrm{e}^{\frac{2\pi \mathrm{i} z}{N}}$ (where N is the width of the cusp ∞ on Γ and $N = 1$ for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$), this is equivalent to whether the regular singularity $q_N = 0$ is apparent or not for

$$\left(q_N\frac{\mathrm{d}}{\mathrm{d}q_N}\right)^3 y + \frac{N^2}{(2\pi\mathrm{i})^2}Q_2 q_N\frac{\mathrm{d}}{\mathrm{d}q_N}y + \frac{N^3}{(2\pi\mathrm{i})^3}Q_3 y = 0. \tag{A.18}$$

All the above statements are true for (A.18) in terms of q_N. In particular, (A.17) or equivalently (A.18) always has a solution of the form

$$y_+(z) := q_N^{\kappa_\infty^{(3)}}\sum_{n=0}^{\infty} c_n(\kappa_\infty^{(3)})q_N^n, \quad c_0 = 1, \tag{A.19}$$

and (A.17) is completely not apparent at $z = \infty$ or equivalently (A.18) is completely not apparent at $q_N = 0$ if and only if the local expansion of some solutions in terms of q_N contains the term $z^2 y_+(z)$ because of $\ln q_N = 2\pi\mathrm{i}z/N$. More precisely, if (A.17) is completely not apparent at $z = \infty$, then it follows from Theorems A.3, A.4(2), A.5(2) and Theorem A.7(3) that (A.17) has two solutions of the following form

$$y_-(z) := z^2 y_+(z) + z\eta_1(z) + \eta_2(z), \qquad y_\perp(z) := z y_+(z) + \eta_3(z), \tag{A.20}$$

such that $(y_-, y_\perp, y_+)^t$ is a basis of solutions, where

$$\eta_1(z) = q_N^{\kappa_\infty^{(2)}}\sum_{n=0}^{\infty} c_{n,1}q_N^n, \qquad \eta_2(z) = q_N^{\kappa_\infty^{(1)}}\sum_{n=0}^{\infty} c_{n,2}q_N^n, \qquad \eta_3(z) = q_N^{\kappa_\infty^{(2)}}\sum_{n=0}^{\infty} c_{n,3}q_N^n.$$

Note that $c_{0,j} = 0$ may happen for any j; see Theorem A.3 for example.

Acknowledgements

We would like to thank the referees for many valuable comments and pointing out some references. The research of Z. Chen was supported by NSFC (No. 12071240).

References

[1] Arike Y., Kaneko M., Nagatomo K., Sakai Y., Affine vertex operator algebras and modular linear differential equations, *Lett. Math. Phys.* **106** (2016), 693–718.

[2] Beukers F., Heckman G., Monodromy for the hypergeometric function ${}_nF_{n-1}$, *Invent. Math.* **95** (1989), 325–354.

[3] Bol G., Invarianten linearer differentialgleichungen, *Abh. Math. Sem. Univ. Hamburg* **16** (1949), 1–28.

[4] Chen Z., Kuo T.-J., Lin C.-S., The geometry of generalized Lamé equation, I, *J. Math. Pures Appl.* **127** (2019), 89–120, arXiv:1708.05306.

[5] Chen Z., Kuo T.-J., Lin C.-S., The geometry of generalized Lamé equation, II: Existence of pre-modular forms and application, *J. Math. Pures Appl.* **132** (2019), 251–272, arXiv:1807.07745.

[6] Chen Z., Kuo T.-J., Lin C.-S., The geometry of generalized Lamé equation, III: One to one of the Riemann–Hilbert correspondnece, *Pure Appl. Math. Q.* **17** (2021), 1619–1668, arXiv:2009.00840.

[7] Chen Z., Lin C.-S., On number and evenness of solutions of the SU(3) Toda system on flat tori with non-critical parameters, *J. Differ. Geom.*, to appear, arXiv:2109.11721.

[8] Choie Y., Lee M.H., Jacobi-like forms, pseudodifferential operators, and quasimodular forms, *Springer Monographs in Mathematics*, Springer, Cham, 2019.

[9] Franc C., Mason G., Hypergeometric series, modular linear differential equations and vector-valued modular forms, *Ramanujan J.* **41** (2016), 233–267, arXiv:1503.05519.

[10] Franc C., Mason G., Classification of some vertex operator algebras of rank 3, *Algebra Number Theory* **14** (2020), 1613–1668, arXiv:1905.07500.

[11] Franc C., Mason G., Constructions of vector-valued modular forms of rank four and level one, *Int. J. Number Theory* **16** (2020), 1111–1152, arXiv:1810.09408.

[12] Gaberdiel M.R., Keller C.A., Modular differential equations and null vectors, *J. High Energy Phys.* **2008** (2008), no. 9, 079, 29 pages, arXiv:0804.0489.

[13] Grabner P.J., Quasimodular forms as solutions of modular differential equations, *Int. J. Number Theory* **16** (2020), 2233–2274, arXiv:2002.02736.

[14] Hampapura H.R., Mukhi S., Two-dimensional RCFT's without Kac–Moody symmetry, *J. High Energy Phys.* **2016** (2016), no. 7, 138, 19 pages, arXiv:1605.03314.

[15] Henrici P., Applied and computational complex analysis. Vol. 2. Special functions – integral transforms – asymptotics – continued fractions, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1991.

[16] Hille E., Ordinary differential equations in the complex domain, Dover Publications, Inc., Mineola, NY, 1997.

[17] Kaneko M., Koike M., On modular forms arising from a differential equation of hypergeometric type, *Ramanujan J.* **7** (2003), 145–164, arXiv:math.NT/0206022.

[18] Kaneko M., Koike M., On extremal quasimodular forms, *Kyushu J. Math.* **60** (2006), 457–470.

[19] Kaneko M., Nagatomo K., Sakai Y., The third order modular linear differential equations, *J. Algebra* **485** (2017), 332–352.

[20] Kaneko M., Zagier D., A generalized Jacobi theta function and quasimodular forms, in The Moduli Space of Curves (Texel Island, 1994), *Progr. Math.*, Vol. 129, Birkhäuser Boston, Boston, MA, 1995, 165–172.

[21] Kawasetsu K., Sakai Y., Modular linear differential equations of fourth order and minimal $\mathcal{W}$-algebras, *J. Algebra* **506** (2018), 445–488, arXiv:1803.02022.

[22] Le Bruyn L., Bulk irreducibles of the modular group, *J. Algebra Appl.* **15** (2016), 1650006, 14 pages.

[23] Lin C.-S., Nie Z., Wei J., Toda systems and hypergeometric equations, *Trans. Amer. Math. Soc.* **370** (2018), 7605–7626, arXiv:1610.03194.

[24] Lin C.-S., Wei J., Ye D., Classification and nondegeneracy of SU$(n+1)$ Toda system with singular sources, *Invent. Math.* **190** (2012), 169–207, arXiv:1111.0390.

[25] Lin C.-S., Yang Y., Quasimodular forms and modular differential equations which are not apparent at cusps: I, arXiv:2103.04890.

[26] Mason G., Vector-valued modular forms and linear differential operators, *Int. J. Number Theory* **3** (2007), 377–390.

[27] Mathur S.D., Mukhi S., Sen A., On the classification of rational conformal field theories, *Phys. Lett. B* **213** (1988), 303–308.

[28] Pellarin F., On extremal quasi-modular forms after Kaneko and Koike (with an appendix by Gabriele Nebe), *Kyushu J. Math.* **74** (2020), 401–413, arXiv:1910.11668.

[29] Sebbar A., Saber H., Automorphic Schwarzian equations, *Forum Math.* **32** (2020), 1621–1636, arXiv:2002.00493.

[30] Serre J.-P., A course in arithmetic, *Graduate Texts in Mathematics*, Vol. 7, Springer-Verlag, New York – Heidelberg, 1973.

[31] Tuba I., Wenzl H., Representations of the braid group B_3 and of $\mathrm{SL}(2,\mathbf{Z})$, *Pacific J. Math.* **197** (2001), 491–510, arXiv:math.RT/9912013.

[32] Westbury B., On the character varieties of the modular group, Preprint, Nottingham University, 1995.

[33] Zagier D., Elliptic modular forms and their applications, in The 1-2-3 of Modular Forms, *Universitext*, Springer, Berlin, 2008, 1–103.

[34] Zhu Y., Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–302, arXiv:math.RT/9912013.