

Classical Superintegrable Systems in a Magnetic Field that Separate in Cartesian Coordinates

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Abstract. We consider superintegrability in classical mechanics in the presence of magnetic fields. We focus on three-dimensional systems which are separable in Cartesian coordinates. We construct all possible minimally and maximally superintegrable systems in this class with additional integrals quadratic in the momenta. Together with the results of our previous paper [*J. Phys. A: Math. Theor.* **50** (2017), 245202, 24 pages], where one of the additional integrals was by assumption linear, we conclude the classification of three-dimensional quadratically minimally and maximally superintegrable systems separable in Cartesian coordinates. We also describe two particular methods for constructing superintegrable systems with higher-order integrals.

Key words: integrability; superintegrability; higher-order integrals; magnetic field

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1 Introduction

In this paper we investigate superintegrability of three-dimensional systems that separate in Cartesian coordinates in the presence of a magnetic field. We say that a mechanical system is superintegrable if it is Liouville integrable and possesses additional independent integrals of motion. Depending on their number we distinguish minimal superintegrability when only one additional integral is present, and maximal superintegrability when the number of additional integrals is the maximal possible, i.e., equal to the number of degrees of freedom minus one. (In three spatial dimensions there is no other possibility.)

The study of superintegrability with magnetic fields was initiated in [5] and subsequently followed in both two spatial dimensions [3, 4, 24, 25] and three spatial dimensions [2, 14, 15, 16, 17]; relativistic version of the problem was recently considered too, cf. [11]. Separability of three-dimensional systems with magnetic fields was considered in the papers [1, 27]. Particular planar two-body systems, e.g., Coulomb, in perpendicular constant magnetic field were also studied from the point of solvability and superintegrability, see, e.g., [8, 28, 29, 30].

It turns out that the presence of magnetic field significantly increases the complexity of both calculations and structure of these systems. E.g., contrary to the case without magnetic field separability in orthogonal coordinates and integrability with integrals at most quadratic in the momenta are no longer equivalent, namely separability is stronger and implies the existence of at least one integral linear in the momenta. Similarly, the explicit construction of superintegrable systems and their classification become much harder when magnetic fields are present.

In the present paper we attempt to approach the problem from a different viewpoint. We exploit the fact that in certain situations the three-dimensional system can be rewritten as effectively a two-dimensional one without magnetic field, thus generalizing the principal idea of [15]. In other cases we show that the existence of a quadratic integral necessarily implies the existence of an integral in a particular simpler form, which makes our calculations tractable. When the results of the present paper and [14, 16] are viewed together, they provide an exhaustive list of three-dimensional quadratically minimally and maximally superintegrable systems with magnetic fields separable in Cartesian coordinates.

We shall investigate the superintegrability of the system defined on the phase space \mathbb{R}^6 , with the canonical coordinates (\vec{x}, \vec{p}) , by

$$H(\vec{x}, \vec{p}) = \frac{1}{2}((p_1^A)^2 + (p_2^A)^2 + (p_3^A)^2) + W(\vec{x}), \quad (1.1)$$

where $W(\vec{x})$ denotes the so called electrostatic or effective potential, p_j^A are the covariant expressions for the momenta

$$p_j^A = p_j + A_j(\vec{x}), \quad j = 1, 2, 3, \quad (1.2)$$

and $A_j(\vec{x})$ are the components of the vector potential. The magnetic field $\vec{B}(\vec{x})$ is related to $\vec{A}(\vec{x})$ through

$$\vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x}).$$

Newtonian equations of motion and thus also the physical dynamics are gauge invariant, i.e., depend only on $B(\vec{x})$ and $\nabla W(\vec{x})$. However, in the Hamiltonian formulation gauge transformations can be seen as canonical transformations (cf. [12, Problem 11.25]), namely they alter the Hamiltonian, the corresponding Hamilton's equations of motion and the Hamilton–Jacobi equation in a prescribed way. Separation of variables in the Hamilton–Jacobi equation is related to a specific choice of the coordinate system and is not preserved under canonical transformations – on the contrary, one looks for a suitable canonical transformation such that the system becomes separable after it. Since we are interested in systems that separate in Cartesian coordinates, we find it preferable to work in a suitably chosen fixed gauge adapted to the separation.

Furthermore, we will sometimes use canonical transformations to reduce to cyclic coordinates corresponding to integrals. Also in this perspective, it is helpful to fix an appropriate gauge. However, the final results, in particular the superintegrable systems found shall be given in the gauge covariant form, so to express them in the most general way.

In gauge dependent form the Hamiltonian (1.1) reads

$$H(\vec{x}, \vec{p}) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + A_1(\vec{x})p_1 + A_2(\vec{x})p_2 + A_3(\vec{x})p_3 + V(\vec{x}), \quad (1.3)$$

where the gauge dependent “scalar” potential $V(\vec{x})$, i.e., the momentum-free term in (1.3), is related to the gauge invariant electrostatic potential $W(\vec{x})$ via

$$V(\vec{x}) = W(\vec{x}) + \frac{1}{2}|\vec{A}(\vec{x})|^2.$$

There are only two cases in which the system (1.3) separates in Cartesian coordinates [1, 27], up to a canonical permutation of the variables. Let us write them in both gauge dependent and gauge covariant form:

Case I

$$V(\vec{x}) = V_1(x_1) + V_2(x_2), \quad \vec{A}(\vec{x}) = (0, 0, u_1(x_2) - u_2(x_1)), \quad (1.4)$$

therefore

$$\vec{B}(\vec{x}) = (u'_1(x_2), u'_2(x_1), 0), \quad W(\vec{x}) = V_1(x_1) + V_2(x_2) - \frac{1}{2}(u_1(x_2) - u_2(x_1))^2. \quad (1.5)$$

Case II

$$V(\vec{x}) = V_1(x_1), \quad \vec{A}(\vec{x}) = (0, u_3(x_1), -u_2(x_1)), \quad (1.6)$$

thus

$$\vec{B}(\vec{x}) = (0, u'_2(x_1), u'_3(x_1)), \quad W(\vec{x}) = V_1(x_1) - \frac{1}{2}(u_3(x_1)^2 + u_2(x_1)^2). \quad (1.7)$$

In these two cases the system admits two Cartesian-type integrals, related to the separation of variables:

$$\begin{aligned} X_1 &= (p_1^A)^2 - 2(u_2(x_1)(p_3^A - u_1(x_2) + u_2(x_1)) - V_1(x_1)) = p_1^2 - 2(u_2(x_1)p_3 - V_1(x_1)), \\ X_2 &= (p_2^A)^2 + 2(u_1(x_2)(p_3^A - u_1(x_2) + u_2(x_1)) + V_2(x_2)) = p_2^2 + 2(u_1(x_2)p_3 + V_2(x_2)) \end{aligned} \quad (1.8)$$

for (1.4) and

$$X_1 = p_2^A - u_3(x_1) = p_2, \quad X_2 = p_3^A - u_2(x_1) = p_3 \quad (1.9)$$

for (1.6).

Remark. $X_0 = p_3^A - u_1(x_2) + u_2(x_1) = p_3$ is another integral of (1.5), though dependent on the Hamiltonian and (1.8).

Minimal superintegrability due to the existence of another first-order integral has been studied in [14, 16]. Here we investigate the conditions for the existence of an additional integral of order at least two for the systems (1.5), (1.7). We give an exhaustive list of systems for which an additional second-order integral exists, and are able to answer the question on the existence of higher-order integrals in special cases.

Sections 2 and 3 present two propositions for finding out whether certain classes of systems are superintegrable by reducing to a two-dimensional (2D) problem without magnetic field. In this way we also construct families of systems with higher-order integrals. Next, in Section 4 we address the problem of second-order superintegrability. The determining equations for second-order integrals are given in gauge covariant form, together with their compatibility conditions. In Section 5 we give a necessary condition for second-order superintegrability, which is used in Sections 6 and 7 to simplify the structure of the integral for the classes (1.5), (1.7), respectively. With these simplifications at hand, the determining equations for the integral can be solved. In Section 9.1 we list the superintegrable systems so found; their explicit derivation is rather technical and tedious and we review it in Appendices A, B and C. The special case in which the magnetic field is constant and the functions V_j in (1.5) and (1.7) are at most quadratic polynomials is studied in Section 8. Finally, in Section 9.2 we discuss the approaches to construction of higher-order integrals.

2 Minimal superintegrability for Case I when all the integrals commute with one linear momentum

Let us consider the natural Hamiltonian systems on the phase space (x_1, x_2, p_1, p_2) , for $\kappa \in \mathbb{R}$, $\kappa \neq 0$

$$\mathcal{H}_0^\kappa(x_1, x_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + \kappa(u_1(x_2) - u_2(x_1)) + V_1(x_1) + V_2(x_2). \quad (2.1)$$

For the sake of clarity let us refer here to the Hamiltonian of Case I as to \mathcal{H} . Since p_3 is an integral of motion for (1.4), by setting $p_3 = \kappa$, $\mathcal{H}_0^\kappa = \mathcal{H}(x, y, z, p_1, p_2, \kappa) - \frac{1}{2}\kappa^2$. Both systems have a pair of second-order integrals corresponding to separation: X_j as in (1.8) for \mathcal{H} and clearly

$$\mathcal{I}_j^\kappa = X_j(x_1, x_2, p_1, p_2, \kappa), \quad j = 1, 2$$

for (2.1).

If (1.6) possesses any additional integral X_3 independent of the variable x_3 , then $\mathcal{I}_3^\kappa(x_1, x_2, p_1, p_2) = X_3(x_1, x_2, p_1, p_2, \kappa)$ would be an integral for (2.1). And vice versa, any additional integral \mathcal{I}_3^κ of (2.1), would correspond to an integral X_3 of (1.6), obtained by simply replacing κ by p_3 , i.e., $X_3(x_1, x_2, x_3, p_1, p_2, p_3) = \mathcal{I}_3^{p_3}(x_1, x_2, p_1, p_2)$. Indeed,

$$\{\mathcal{H}, X_3\} = \sum_{i=1}^2 \left(\frac{\partial \mathcal{H}_0^{p_3}}{\partial x_i} \frac{\partial \mathcal{I}_3^{p_3}}{\partial p_i} - \frac{\partial \mathcal{I}_3^{p_3}}{\partial x_i} \frac{\partial \mathcal{H}_0^{p_3}}{\partial p_i} \right) + \frac{1}{2} \{p_3^2, X_3\} = 0,$$

where $\{, \}$ is the Poisson bracket on the phase space \mathbb{R}^6 . The right hand side of the equality is zero since both \mathcal{H} and X_3 do not depend on x_3 and $\mathcal{I}_3^{p_3}$ is an integral of $\mathcal{H}_0^{p_3}$. Thus, we arrive at the following immediate conclusion

Proposition 2.1. *Let us consider the Hamiltonian system defined by (1.1) on the phase space $(x_1, x_2, x_3, p_1, p_2, p_3)$ with magnetic field and effective potential as in (1.5). Such system admits an additional independent integral I_3 such that $\{I_3, p_3\} = 0$ if and only if (2.1) is superintegrable on the phase space (x_1, x_2, p_1, p_2) .*

Therefore all the systems of the form (1.5) that are minimally superintegrable, with an additional integral independent of Cartesian coordinate, can be deduced from 2D natural superintegrable systems of the form (2.1). And vice versa, every superintegrable system in two degrees of freedom can be extended to a minimally superintegrable system in three degrees of freedom with magnetic field. Superintegrable systems of the form (2.1) have been widely studied. In particular they have been completely classified for integrals up to third order [22]. Concerning higher-order integrals, many examples are known, including the harmonic oscillator and the caged oscillator [10, 26], and a wide class of so called exotic potentials [6, 7, 20].

2.1 Example: extension of 2D second-order superintegrable systems

Table 1 contains all three-dimensional systems that can be proven to be (at least) minimally quadratically superintegrable by applying Proposition 2.1 to 2D superintegrable systems that separate in Cartesian coordinates and have integrals at most quadratic. The list of 2D systems is taken from [22], from which we consider only the systems on real phase space. To obtain the most general family of systems (and recalling that the Hamiltonian must depend linearly on κ), we renamed all the parameters as $c_j = a_j\kappa + b_j$, a_j not all vanishing, then set $p_3 = \kappa$ and applied Proposition 2.1. The third integral, leading to superintegrability, can then be found from the integral \mathcal{I}_3 of the 2D system, by substituting $c_j = a_j p_3 + b_j$. Since the dependence on the constants c_j is linear, the order of the so obtained integral remains quadratic.

2.2 Example: a family of higher-order superintegrable systems from the 2D caged oscillator

Let us consider the two-dimensional caged anisotropic oscillator

$$\mathcal{H}_0 = \frac{1}{2}(p_1^2 + p_2^2) + \omega(\ell^2 x_1^2 + m^2 x_2^2) + \frac{\alpha}{x_1^2} + \frac{\beta}{x_2^2} \quad (2.2)$$

Table 1. 3D (at least) minimally quadratically superintegrable extensions of 2D quadratically superintegrable systems that separate in Cartesian coordinates. For the reader's convenience, we give the Hamiltonian expressed in the gauge choice (1.4), but also the functions u_j and V_j that allow to find the magnetic field \vec{B} and potential W as in the more general gauge invariant form (1.5). In the integrals, L_3 denotes angular momentum on the plane, $L_3 = x_1 p_2 - x_2 p_1$.

	2D system and its third integral	3D system
\mathcal{E}_1 :	$\mathcal{H}_0 = \frac{1}{2}(p_1^2 + p_2^2) + c_1(x_1^2 + x_2^2) + \frac{c_2}{x_1} + \frac{c_3}{x_2}$ $\mathcal{I}_3 = L_3^2 + 2\left(c_2 \frac{x_2^2}{x_1^2} + c_3 \frac{x_1^2}{x_2^2}\right)$	$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (a_1(x_1^2 + x_2^2) + \frac{a_2}{x_1} + \frac{a_3}{x_2})p_3 + b_1(x_1^2 + x_2^2) + \frac{b_2}{x_1} + \frac{b_3}{x_2}$ $u_1(x_2) = a_1 x_2^2 + \frac{a_3}{x_2}, \quad u_2(x_1) = -a_1 x_1^2 - \frac{a_2}{x_1}$ $V_1(x_1) = b_1 x_1^2 + \frac{b_2}{x_1}, \quad V_2(x_2) = b_1 x_2^2 + \frac{b_3}{x_2}$
\mathcal{E}_2 :	$\mathcal{H}_0 = \frac{1}{2}(p_1^2 + p_2^2) + c_1(4x_1^2 + x_2^2) + c_2 x_1 + \frac{c_3}{x_2}$ $\mathcal{I}_3 = p_2 L_3 - x_2^2(2c_1 x_1 + \frac{c_2}{2}) + 2c_3 \frac{x_1}{x_2}$	$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (a_1(4x_1^2 + x_2^2) + a_2 x_1 + \frac{a_3}{x_2})p_3 + b_1(4x_1^2 + x_2^2) + b_2 x_1 + \frac{b_3}{x_2}$ $u_1(x_2) = a_1 x_2^2 + \frac{a_3}{x_2}, \quad u_2(x_1) = -4a_1 x_1^2 - a_2 x_1$ $V_1(x_1) = 4b_1 x_1^2 + b_2 x_1, \quad V_2(x_2) = b_1 x_2^2 + \frac{b_3}{x_2}$
\mathcal{E}_3 :	$\mathcal{H}_0 = \frac{1}{2}(p_1^2 + p_2^2) + c_1(x_1^2 + x_2^2) + c_2 x_1 + c_3 x_2$ $\mathcal{I}_3 = p_1 p_2 + 2c_1 x_1 x_2 + c_2 x_2 + c_3 x_1$	$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (a_1(x_1^2 + x_2^2) + a_2 x_1 + a_3 x_2)p_3 + b_1(x_1^2 + x_2^2) + b_2 x_1 + b_3 x_2$ $u_1(x_2) = a_1 x_2^2 + a_3 x_2, \quad u_2(x_1) = -a_1 x_1^2 - a_2 x_1$ $V_1(x_1) = b_1 x_1^2 + b_2 x_1, \quad V_2(x_2) = b_1 x_2^2 + b_3 x_2$

for $\omega \in \mathbb{R} \setminus \{0\}$, ℓ , m nonvanishing integers and $\alpha, \beta \in \mathbb{R}$. The system is well known to be superintegrable if $\frac{\ell}{m}$ rational [10, 26]. A first straightforward extension to a 3D superintegrable system is given by

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (\ell^2 x_1^2 + m^2 x_2^2)p_3 + \frac{\alpha}{x_1^2} + \frac{\beta}{x_2^2}, \quad (2.3)$$

that can be transformed into (2.2) by simply reducing $p_3 = \omega$.

A more general extension can be constructed as in the previous example. Let us set

$$\ell^2 = \ell_1 \kappa + \ell_2, \quad \alpha = \alpha_1 \kappa + \alpha_2, \quad m^2 = m_1 \kappa + m_2, \quad \beta = \beta_1 \kappa + \beta_2. \quad (2.4)$$

The system (2.2) can then be seen as the 2D reduction of

$$\begin{aligned} \mathcal{H} = & \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \left(\omega(\ell_1 x_1^2 + m_1 x_2^2) + \frac{\alpha_1}{x_1^2} + \frac{\beta_1}{x_2^2} \right) p_3 \\ & + \omega(\ell_2 x_1^2 + m_2 x_2^2) + \frac{\alpha_2}{x_1^2} + \frac{\beta_2}{x_2^2}, \end{aligned} \quad (2.5)$$

by substituting $p_3 = \kappa$. We obtain in this way the three-dimensional integrable system (2.5) that becomes superintegrable when the frequency ratio of (2.2) (where (2.4) has to be taken into account) is a rational number, i.e., when

$$\frac{\ell_1 p_3 + \ell_2}{m_1 p_3 + m_2} = \frac{\ell^2}{m^2}, \quad \frac{\ell}{m} \in \mathbb{Q}, \quad (2.6)$$

for every possible value of the phase space variable p_3 . Equivalently, (2.6) can be written as

$$(m^2 \ell_1 - \ell^2 m_1) p_3 + m^2 \ell_2 - m_2 \ell^2 = 0, \quad \frac{\ell}{m} \in \mathbb{Q}.$$

The above equation contains a polynomial in p_3 that must be identically zero. This is possible only when the coefficient of each power of p_3 vanishes. Namely, when

$$\frac{\ell_1}{m_1} = \frac{\ell_2}{m_2} = \frac{\ell^2}{m^2}, \quad \frac{\ell}{m} \in \mathbb{Q}. \quad (2.7)$$

Thus, the family of systems (2.5) is superintegrable if and only its parameters satisfy (2.7) (and in that case also (2.2) is superintegrable). For $\ell_j = m_j = 0$ for some j (not both $j = 1, 2$), the previous condition reduces to

$$\frac{\ell_j}{m_j} = \frac{\ell^2}{m^2}, \quad \frac{\ell}{m} \in \mathbb{Q}.$$

For $\alpha_1 = \beta_1 = \ell_2 = m_2 = 0$ we have the simpler system (2.3). The case $\alpha_j = \beta_j = 0$, $\ell_j = m_j = \pm 1$, $j = 1, 2$ was studied in [14] and it is shown there to be quadratically minimally superintegrable, with the fourth independent integral (besides the two Cartesian ones) inherited from the 2D caged oscillator, of first order. In the more general case (2.5), the order of the fourth integral can be arbitrarily high, depending on the value of $\frac{\ell}{m}$. Notice that all the systems in Table 1 are contained in the family (2.5), except the systems \mathcal{E}_2 and \mathcal{E}_3 for the special case $a_1 = b_1 = 0$ (i.e., $c_1 = 0$), in which the linear terms in the space variables cannot be eliminated by translation, due to the absence of quadratic terms.

3 Maximal superintegrable class canonically conjugated to natural 2D systems

Let us consider the system whose magnetic field and effective potential read

$$\vec{B}(\vec{x}) = (0, \gamma, 0), \quad \gamma \in \mathbb{R} \setminus \{0\} \quad (3.1)$$

and

$$W(\vec{x}) = V(x_2), \quad (3.2)$$

respectively. This system can be written in the form (1.4), with the gauge chosen as

$$\vec{A}(\vec{x}) = (0, 0, -\gamma x_1).$$

Its Hamiltonian reads

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \gamma x_1 p_3 + \frac{\gamma^2}{2} x_1^2 + V(x_2). \quad (3.3)$$

Actually by a different choice of the gauge and a canonical permutation of the variables x_1 and x_2 we see that the system belongs also to Case II. The Hamiltonian (3.3) admits three independent first-order integrals [14]

$$I_1 = p_1 - \gamma x_3, \quad I_2 = p_3, \quad I_3 = 2l_2 + \gamma(x_1^2 - x_3^2). \quad (3.4)$$

Out of them, we can construct two Cartesian-type integrals,

$$X_1 = I_1^2 + \gamma I_3, \quad X_2 = 2H - I_1^2 - I_2^2 - \gamma X_3. \quad (3.5)$$

The system can be reduced to two degrees of freedom through the following canonical transformation

$$x_1 = X + \frac{P_3}{\gamma}, \quad x_2 = Y, \quad x_3 = Z + \frac{1}{\gamma} P_1, \quad p_j = P_j, \quad j = 1, 2, 3, \quad (3.6)$$

with the second type generating function

$$G(\vec{x}, \vec{P}) = \left(x_1 - \frac{1}{\gamma} P_3 \right) P_1 + x_2 P_2 + x_3 P_3.$$

The Hamiltonian in the new coordinates reads

$$\mathcal{K}(\vec{X}, \vec{P}) = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}\gamma^2 X^2 + V(Y), \quad (3.7)$$

i.e., it is effectively in two degrees of freedom and without magnetic field. This system (3.7) has two cyclic coordinates in the full phase space (\vec{X}, \vec{P}) , namely Z and P_3 , that are therefore both integrals. Expressed in the original variables, these integrals correspond to p_3 and $\frac{I_1}{\gamma}$ as in (3.4). Moreover (3.7) separates in the Cartesian coordinates (X, Y) , and the corresponding Cartesian-type integrals, \mathcal{I}_1 , \mathcal{I}_2 , once written in the original coordinates, provide (3.5). Thus we have

Proposition 3.1. *The system with the magnetic field (3.1) and potential (3.2) is maximally superintegrable if and only if (3.7), seen as a system in two degrees of freedom on the phase space (X, Y, P_1, P_2) has one additional integral of motion, besides \mathcal{I}_1 , \mathcal{I}_2 , and independent of them.*

Therefore the problem of maximal superintegrability of (3.3) has been reduced to the two-dimensional problem of superintegrability of (3.7). In particular, all the potentials $V(Y)$ that make (3.7) superintegrable give (by simply replacing $Y = x_2$) the effective potentials that render (3.3) superintegrable.

The cases

$$V(Y) = \frac{c}{Y^2} + \frac{\gamma^2 Y^2}{8}, \quad (3.8)$$

and

$$V(Y) = \frac{\gamma^2}{2} Y^2, \quad (3.9)$$

that correspond to 3D superintegrable systems with additional second-order integral have already been found in [14] with a different approach.

All the potentials $V(Y)$ that lead to second and third-order superintegrability in 2D have been classified [22]. If we focus on second-order integrals, they are listed in Table 1. The systems that can be obtained from it, after applying the transformation (3.6) and are still quadratically superintegrable are given by (3.8), and

$$V(Y) = \frac{\gamma^2}{2} Y^2 + cY,$$

that, since $\gamma \neq 0$, can be reduced to (3.9) by translation in Y .

However, higher-order superintegrable systems can be generated, e.g., from

$$V(Y) = \frac{c}{Y^2} + \frac{\gamma^2 Y^2}{2}, \quad c \geq 0. \quad (3.10)$$

The additional integral of (3.7) is second order and reads (see Table 1)

$$\mathcal{X}_4 = \mathcal{L}_3^2 + 2c \frac{X^2}{Y^2}.$$

Here \mathcal{L}_3 denotes the third component of the angular momentum with respect to the coordinates (X, Y, Z, P_1, P_2, P_3) . Inverting the transformation (3.6), it gives the fourth-order integral

$$X_4 = \frac{1}{\gamma^2} \left((p_2^A p_3^A + \gamma p_1^A x_2)^2 + 2c \frac{(p_3^A)^2}{x_2^2} \right).$$

Actually, by polynomial combinations with the other integrals, it can be reduced to the third order one

$$\begin{aligned} X_5 = & 2\gamma p_2^A p_3^A l_3^A + \gamma^2 \left(x_1^2 (p_2^A)^2 + x_2^2 \left((p_3^A)^2 - (p_1^A)^2 \right) \right) + 2\gamma \frac{x_1}{x_2} (\gamma^2 x_2^4 + 2c) p_3^A \\ & + \gamma^2 \frac{x_1^2}{x_2^2} (\gamma^2 x_2^4 + 2c), \end{aligned}$$

that cannot be further reduced to lower order by using any of the integrals (3.4) nor (3.5).

A more general 3D infinite family of maximally superintegrable system, including the previous cases (3.8) and (3.10) and the one found in [15], corresponds to the caged oscillator

$$V(Y) = \frac{c}{Y^2} + \frac{m^2}{\ell^2} \gamma^2 Y^2, \quad \ell, m \in \mathbb{N} \quad (3.11)$$

If we compare it with (2.5), we see that for $\gamma^2 = \omega l_2^2$, $\alpha_2 = 0$, $\beta_2 = c$ and m_2 satisfying (2.7) the two obtained 3D families would have the same scalar potential. However, the magnetic fields differ, rendering (3.11) maximally superintegrable, while (2.5) – as far as we can see – is only minimally superintegrable.

4 Second-order integrals

Any second-order integral of motion we can write

$$X = \sum_{j=1}^3 h_j(\vec{x}) p_j^A p_j^A + \sum_{j,k,l=1}^3 \frac{1}{2} |\epsilon_{jkl}| n_j(\vec{x}) p_k^A p_l^A + \sum_{j=1}^3 s_j(\vec{x}) p_j^A + m(\vec{x}), \quad (4.1)$$

where ϵ_{jkl} is the completely antisymmetric tensor with $\epsilon_{123} = 1$.

The condition that the Poisson bracket

$$\{a(\vec{x}, \vec{p}), b(\vec{x}, \vec{p})\} = \sum_{j=1}^3 \left(\frac{\partial a}{\partial x_j} \frac{\partial b}{\partial p_j} - \frac{\partial b}{\partial x_j} \frac{\partial a}{\partial p_j} \right)$$

of the integral (4.1) with the Hamiltonian (1.1) vanishes

$$\{H, X\} = 0$$

seen as a polynomial in the momenta leads to the determining equations for the unknown functions h_j , n_j , s_j , $j = 1, 2, 3$ and m in the integral. Order by order (from the third to the zeroth) they read (cf. [16]):

$$\begin{aligned} \partial_{x_1} h_1 &= 0, & \partial_{x_2} h_1 &= -\partial_{x_1} n_3, & \partial_{x_3} h_1 &= -\partial_{x_1} n_2, \\ \partial_{x_1} h_2 &= -\partial_{x_2} n_3, & \partial_{x_2} h_2 &= 0, & \partial_{x_3} h_2 &= -\partial_{x_2} n_1, \\ \partial_{x_1} h_3 &= -\partial_{x_3} n_2, & \partial_{x_2} h_3 &= -\partial_{x_3} n_1, & \partial_{x_3} h_3 &= 0, \\ \nabla \cdot \vec{n} &= 0, \end{aligned} \quad (4.2)$$

$$\begin{aligned} \partial_{x_1} s_1 &= n_2 B_2 - n_3 B_3, \\ \partial_{x_2} s_2 &= n_3 B_3 - n_1 B_1, \\ \partial_{x_3} s_3 &= n_1 B_1 - n_2 B_2, \\ \partial_{x_2} s_1 + \partial_{x_1} s_2 &= n_1 B_2 - n_2 B_1 + 2(h_1 - h_2) B_3, \\ \partial_{x_3} s_1 + \partial_{x_1} s_3 &= n_3 B_1 - n_1 B_3 + 2(h_3 - h_1) B_2, \\ \partial_{x_2} s_3 + \partial_{x_3} s_2 &= n_2 B_3 - n_3 B_2 + 2(h_2 - h_3) B_1, \end{aligned} \quad (4.3)$$

$$\begin{aligned} \partial_{x_1} m &= 2h_1 \partial_{x_1} W + n_3 \partial_{x_2} W + n_2 \partial_{x_3} W + s_3 B_2 - s_2 B_3, \\ \partial_{x_2} m &= n_3 \partial_{x_1} W + 2h_2 \partial_{x_2} W + n_1 \partial_{x_3} W + s_1 B_3 - s_3 B_1, \\ \partial_{x_3} m &= n_2 \partial_{x_1} W + n_1 \partial_{x_2} W + 2h_3 \partial_{x_3} W + s_2 B_1 - s_1 B_2, \end{aligned} \quad (4.4)$$

$$\vec{s} \cdot \nabla W = 0. \quad (4.5)$$

The equations (4.2) prescribe that the functions h_j , n_j are such that the highest-order terms in the integral (4.1) are linear combinations of products of the generators p_1 , p_2 , p_3 , l_1 , l_2 , l_3 of the Euclidean group, where $l_j = \sum_{k,l} \epsilon_{jkl} x_k p_l$ [16]. Explicitly, in terms of the expressions (1.2),

we have

$$X = \sum_{i,j: i < j} \alpha_{ij} l_i^A l_j^A + \sum_{i,j} \beta_{ij} p_i^A p_j^A + \sum_{i,j: i < j} \gamma_{ij} p_i^A p_j^A + \sum_{j=1}^3 s_j(\vec{x}) p_j^A + m(\vec{x}), \quad (4.6)$$

where $l_j^A = \sum_{k,l} \epsilon_{jkl} x_k p_l^A$. By subtracting the Hamiltonian and the two Cartesian integrals we can a priori set $\gamma_{11} = \gamma_{22} = \gamma_{33} = 0$. There are compatibility conditions on equations (4.3), consequence of the following conditions on the derivatives of the functions s_j , namely,

$$\partial_{x_2}^2 \partial_{x_1} s_1 + \partial_{x_1}^2 \partial_{x_2} s_2 = \partial_{x_1} \partial_{x_2} (\partial_{x_2} s_1 + \partial_{x_1} s_2),$$

$$\begin{aligned}
\partial_{x_3}^2 \partial_{x_1} s_1 + \partial_{x_1}^2 \partial_{x_3} s_3 &= \partial_{x_1} \partial_{x_3} (\partial_{x_3} s_1 + \partial_{x_1} s_3), \\
\partial_{x_3}^2 \partial_{x_2} s_2 + \partial_{x_2}^2 \partial_{x_3} s_3 &= \partial_{x_2} \partial_{x_3} (\partial_{x_3} s_2 + \partial_{x_2} s_3), \\
\partial_{x_1} \partial_{x_3} (\partial_{x_2} s_1 + \partial_{x_1} s_2) &= 2\partial_{x_2} \partial_{x_3} (\partial_{x_1} s_1) - \partial_{x_1} \partial_{x_2} (\partial_{x_3} s_1 + \partial_{x_1} s_3) + \partial_{x_1}^2 (\partial_{x_3} s_2 + \partial_{x_2} s_3), \\
\partial_{x_2} \partial_{x_3} (\partial_{x_2} s_1 + \partial_{x_1} s_2) &= 2\partial_{x_1} \partial_{x_3} (\partial_{x_2} s_2) - \partial_{x_1} \partial_{x_2} (\partial_{x_3} s_2 + \partial_{x_2} s_3) + \partial_{x_2}^2 (\partial_{x_3} s_1 + \partial_{x_1} s_3), \\
\partial_{x_2} \partial_{x_3} (\partial_{x_3} s_1 + \partial_{x_1} s_3) &= 2\partial_{x_1} \partial_{x_2} (\partial_{x_3} s_3) - \partial_{x_1} \partial_{x_3} (\partial_{x_3} s_2 + \partial_{x_2} s_3) + \partial_{x_3}^2 (\partial_{x_2} s_1 + \partial_{x_1} s_2). \tag{4.7}
\end{aligned}$$

These translate into compatibility conditions on the magnetic field and the constants in the coefficients of the second-order terms. Further compatibility constraints come from (4.4), consequence of

$$\partial_{x_i} \partial_{x_j} m = \partial_{x_j} \partial_{x_i} m, \quad i, j = 1, 2, 3, \quad i \neq j. \tag{4.8}$$

5 A necessary condition for second-order superintegrability

Both classes of systems that separate in Cartesian coordinates have at least one first-order integral and it is always possible to choose a gauge so that such integral reads as one of the linear momenta. To fix the ideas, let us work in such a gauge choice and assume that the constant momentum is p_3 . If a second-order integral X exists, then $K_1 = \{X, p_3\}$ is still an integral at most of second order or a constant. Since the highest-order terms in X are as in (4.6), they can be at most quadratic in x_3 . This means that if K_1 is quadratic in the momenta, its second-order terms are at most linear in x_3 , since $K_1 = \{X, p_3\} = \frac{\partial X}{\partial x_3}$. Thus, $K_2 = \{K_1, p_3\}$ can be, as above, either an integral at most quadratic or a constant. If K_2 is again quadratic, $K_3 = \{K_2, p_3\}$ can be now at most linear in the momenta, since the highest-order terms in K_2 do not depend on x_3 . Therefore, we can conclude that if a second-order independent integral X exists, then necessarily there must exist a second-order integral (which could be X itself) such that $\{X, p_3\}$ is at most linear in the momenta. In general, for a conserved momentum p_j , the result is the same, it is enough to replace x_3 by x_j in the argument above. Thus, we obtain the following

Proposition 5.1. *Let the system defined by H as in (1.1) separate in Cartesian coordinates and have a quadratic integral \mathcal{I} independent of the Cartesian integrals. Then there exists a second-order integral X , not necessarily different from \mathcal{I} , such that $\{X, p_j\}$ is a polynomial expression in the momenta of at most first order, for some j .*

Thus, to answer the question on the existence of an additional second-order integral for the class of systems we are considering here, we can start by answering the simpler question on the existence of the necessary integral X that satisfies the above property. This is done in the following Sections 6 and 7 and Appendices A, B, C.

Since we found that the special case in which the magnetic field is constant and the functions V_j are second-order polynomials in the respective variables appears several times in the computation therein, we discuss it at once in the separate Section 8.

6 Quadratic superintegrability in Case I

We start with the class of systems in (1.5). To fix the ideas, let us choose a gauge as in (1.4) and assume that there exists a quadratic independent integral \mathcal{I} . Thus, by Proposition 5.1 there exist another quadratic integral X such that $\{X, p_3\}$ is at most first order as a polynomial in the momenta. Here we consider only the case in which the two Cartesian-type integrals do not reduce to first-order integrals. In case one of them does, then the system is at the intersection of Case I and Case II (up to a permutation of indices) and it is treated at once in Section 7.

Moreover, we assume there does not exist a linear integral, other than p_3 . If it exists, the corresponding systems can be found in [14], where there is a complete study of quadratically superintegrable systems with Cartesian integrals and one independent first-order integral. We can have several cases:

- (i) $\{X, p_3\}$ is at most linear and not vanishing. Thus the only possibility of finding something new is in assuming that $\{X, p_3\}$ is a dependent integral or a constant (we excluded the case there is an independent first-order integral). We therefore look for a quadratic integral X such that

$$\partial_{x_3} X = \{X, p_3\} = c_1 p_3 + c_0, \quad c_j \in \mathbb{R}, \quad (6.1)$$

and c_j not both vanishing, $j = 0, 1$.

- (ii) $\{X, p_3\} = 0$ and there exist no quadratic integral independent of the Cartesian integrals and commuting with p_3 . Then X is trivial, in the sense that it depends on the Cartesian integrals and p_3 . However, to have a quadratic superintegrable system, a quadratic integral \mathcal{I} as in Proposition 5.1 must exist. Without loss of generality, we can assume $X = \{\mathcal{I}, p_3\}$ with

$$\{\mathcal{I}, p_3\} = a_0 p_3^2 + a_1 X_1 + a_2 X_2 + c_1 p_3 + c_0, \quad (6.2)$$

where X_1 and X_2 are as in (1.8), $a_0, a_1, a_2, c_0, c_1 \in \mathbb{R}$, not all a_j vanishing, otherwise we are in the previous point i).

- (iii) $\{X, p_3\} = 0$ and X is independent of the Cartesian integrals. Since X commutes with p_3 , it satisfies the assumptions of Proposition 2.1. Thus, the corresponding systems can be found in Table 1. If an additional quadratic independent integral exists, then its Poisson bracket with p_3 cannot vanish. This is a consequence of the fact that the 2D system (2.1) cannot have more than 3 independent integrals. However, as in the previous point, there could exist a quadratic independent integral \mathcal{I} such that $\{\mathcal{I}, p_3\}$ depends on the others, namely

$$\{\mathcal{I}, p_3\} = a_0 p_3^2 + a_1 X_1 + a_2 X_2 + a_3 X_3 + c_1 p_3 + c_0, \quad (6.3)$$

where $a_0, a_1, a_2, a_3, c_0, c_1 \in \mathbb{R}$ and not all a_j are vanishing (otherwise we are in case (i)), X_1, X_2 as in (1.8) and $X_3 = X$.

Let us investigate the possibilities for X_3 in (6.3). Its highest-order terms should come from a Poisson bracket of the quadratic terms of \mathcal{I} with p_3 , i.e., their derivatives with respect to x_3 . Moreover, by assumption X_3 does not depend on x_3 . Thus, its second-order terms can arise only by taking derivatives of a second-order polynomial that contains terms of the form $p_i \cdot l_j$, $i = 1, 2, 3$, $j = 1, 2$. By computing their Poisson bracket with p_3 , we see that the only outcome (for an integral X_3 independent of X_1 and X_2) is in terms of the type $p_i p_\ell$, $i \neq \ell$. Looking at the integrals of the of 2D systems in Table 1, and the dependent integrals obtained by their Poisson bracket with the Cartesian integrals, we see that the only possibility is (8.4) below.

Now that we outlined all the possibilities, we need to solve the determining equations (4.3)–(4.5), for the different cases. For this, it is necessary to work in gauge covariant setting.

The conditions (6.1), (6.2) and (6.3) can be written together as (we can now set $a_3 = 0$):

$$\partial_{x_3} X = a_0 (p_3^A - u_1(x_2) + u_2(x_1))^2 + a_1 X_1 + a_2 X_2 + c_1 (p_3^A - u_1(x_2) + u_2(x_1)) + c_0, \quad (6.4)$$

where with an abuse in the notation we denoted \mathcal{I} as X (the unknown independent integral we are looking for), with $a_j, c_j \in \mathbb{R}$ and not all vanishing. For $a_j = 0$, $j = 0, 1, 2$ we are in case (i).

Equation (6.4) implies the following values for the second-order terms of X as in (4.6):

$$\begin{aligned} \alpha_{11} = \alpha_{22} = \alpha_{12} = \alpha_{13} = \alpha_{23} = \beta_{31} = \beta_{32} = 0, \quad \beta_{11} = \beta_{22}, \\ a_0 = 0, \quad a_1 = \beta_{12}, \quad a_2 = -\beta_{21}. \end{aligned} \quad (6.5)$$

Moreover, since $\vec{p} \cdot \vec{L} = 0$ we can set $\beta_{22} = 0$ (and consequently also $\beta_{11} = 0$).

Concerning the lower-order terms, by integrating the right-hand side of (6.4), we obtain the following restriction on the structure of X :

$$\begin{aligned} s_j &= S_j(x_1, x_2), \quad j = 1, 2, \\ s_3 &= S_3(x_1, x_2) - (2\beta_{12}u_2(x_2) + 2\beta_{21}u_1(x_2) - c_1)x_3, \\ m_3 &= c_0x_3 + (u_1(x_2) - u_2(x_1))((2\beta_{12}u_2(x_1) + 2\beta_{21}u_1(x_2) - c_1)x_3 \\ &\quad + (2\beta_{12}V_1(x_1) - 2\beta_{21}V_2(x_2))x_3 + M(x_1, x_2)). \end{aligned} \quad (6.6)$$

With this simplifications at hand, we can solve equations (4.3)–(4.5).

Let us assume that a_1 and a_2 in (6.4) are not both zero; e.g., let it be $a_1 \neq 0$. Then we can shift both the potential $V_1(x_1)$ and the third component of the vector potential by a constant, thus absorbing the constants c_0 and c_1 . Similarly, if $a_2 \neq 0$ we could use X_2 . Therefore, by (6.5), we see that if either $\beta_{12} \neq 0$ or $\beta_{21} \neq 0$, we can proceed in the solution of (4.3)–(4.5) as if $c_1 = c_0 = 0$. We obtain that no new superintegrable system can be found in this case. The details of the computation are in Appendix B.

For $\beta_{12} = \beta_{21} = 0$ we find it convenient to start from (4.3), in which the third equation simplifies to

$$(\beta_{33}x_1 + \gamma_{23})u_1'(x_2) + (\beta_{33}x_2 - \gamma_{13})u_2'(x_1) - c_1 = 0. \quad (6.7)$$

The above equation could be trivially satisfied for some of the functions u_j or not. This determines a major splitting in the computation. For the details see Appendix A, the resulting list of systems is given in the conclusions, Section 9.1.

7 Quadratic superintegrability in Case II

For the class of systems (1.7) we can choose a gauge so that there are two mutually orthogonal conserved linear momenta. Let us assume that they are p_2 and p_3 as in (1.9). As above, we assume there exists an independent quadratic integral. Thus, by Proposition 5.1 we can have two possibilities:

- (i) there exists a quadratic integral X such that $\{X, p_2\} = \{X, p_3\} = 0$. Then X is an integral of the reduced system obtained from (1.7) by setting the conserved momenta to constants, i.e., function of the 1-dimensional Hamiltonian. Thus, it is dependent on the Hamiltonian and the conserved momenta. The only hope to find something interesting is to look for a quadratic integral \mathcal{I} such that $\{\mathcal{I}, p_j\} = X$ for some j .
- (ii) There exists a quadratic integral X such that $\{X, p_j\}$ is linear and not vanishing for at least one p_j , $j = 2, 3$. Without loss of generality we can assume that $\{X, p_3\} \neq 0$, otherwise we permute the coordinates x_2 and x_3 .

Let us set $j = 3$ in both cases and with an abuse of notation let us rename \mathcal{I} in case (i) as X . Thus, we look for a quadratic integral X such that

$$\{X, p_3\} = 2a_0(H - X_1^2 - X_2^2) + a_1X_1^2 + a_2X_2^2 + a_3X_1X_2 + c_0 + c_1X_1 + c_2X_2, \quad (7.1)$$

X_1, X_2 as in (1.9). For $a_j = 0$, $j = 1, \dots, 4$, we have case (ii).

Equation (7.1) implies the following conditions on the coefficients of the higher-order terms of the integral, expressed as in (4.6) (again, we use the condition $\vec{p} \cdot \vec{L} = 0$)

$$\begin{aligned} \alpha_{11} = \alpha_{22} = \alpha_{12} = \alpha_{13} = \alpha_{23} = \beta_{11} = \beta_{22} = \beta_{32} = 0, \\ a_0 = \beta_{12}, \quad a_1 = -\beta_{21}, \quad a_2 = 0, \quad a_3 = -\beta_{31}. \end{aligned} \quad (7.2)$$

Moreover, by subtracting $X_1 X_2$ from X , we can set $\gamma_{23} = 0$.

Still as a consequence of (7.1), we have further conditions on the coefficients of the lower-order terms

$$\begin{aligned} s_1 = S_1(x_1, x_2), \quad s_2 = S_2(x_1, x_2) + (2(\beta_{12} + \beta_{21})u_3(x_1) - \beta_{31}u_2(x_1) + c_1)x_3, \\ s_3 = S_3(x_1, x_2) + z(\beta_{31}u_3(x_1) - 2\beta_{12}u_2(x_1) + c_2) \end{aligned}$$

and

$$\begin{aligned} m = M(x_1, x_2) - ((2\beta_{12} + \beta_{21})u_3(x_1)^2 - \beta_{31}u_2(x_1)u_3(x_1) \\ + c_1u_3(x_1) - c_2u_2(x_1) + 2\beta_{12}u_2(x_1)^2 - 2\beta_{12}V_1(x_1) - c_0)x_3. \end{aligned}$$

With these simplifications at hand, we are able to solve the determining equations (4.2)–(4.5).

Let us perform the substitution

$$u_j(x_1) = U'_j(x_1), \quad j = 2, 3. \quad (7.3)$$

Since u_j are defined in (1.5) up to addition of arbitrary constants and U_j is defined as in (7.3), in the following we can set to zero all the coefficient of first and zero-order powers of x_1 in the solutions for U_j .

From (4.3) we find

$$\begin{aligned} S_1(x_1, x_2) &= s_1(x_2) + \beta_{12}U_2(x_1) + (\beta_{13} - 2\alpha_{33}x_2)U_3(x_1) - (\beta_{12}x_1 + \beta_{33}x_2 - \gamma_{13})U'_2(x_1) \\ &\quad + (2\alpha_{33}x_1x_2 - \beta_{13}x_1 + \beta_{23}x_2 - \gamma_{12})U'_3(x_1), \\ S_2(x_1, x_2) &= s_2(x_1) - \left(\alpha_{33}x_1x_2^2 - \beta_{13}x_1x_2 + \frac{1}{2}\beta_{23}x_2^2 - \gamma_{12}x_2 \right) U''_3(x_1), \\ S_3(x_1, x_2) &= s_3(x_1) + c_1x_2 + \beta_{31}x_2U'_2(x_1) - 2(\beta_{12} + \beta_{21})x_2U'_3(x_1) \\ &\quad + \left(\alpha_{33}x_1x_2^2 - \beta_{13}x_1x_2 + \frac{1}{2}\beta_{23}x_2^2 - \gamma_{12}x_2 \right) U''_2(x_1) \\ &\quad - \left(\beta_{12}x_1x_2 + \frac{1}{2}\beta_{33}x_2^2 - \gamma_{13}x_2 \right) U''_3(x_1), \end{aligned}$$

where U_j and s_ℓ must satisfy the third, fourth and fifth equation of (4.3). Let us continue by considering the third of these equations, namely

$$U''_2(x_1)(\beta_{12}x_1 + \beta_{33}x_2 - \gamma_{13}) + 2\beta_{12}U'_2(x_1) - \beta_{31}U'_3(x_1) - c_2 = 0, \quad (7.4)$$

together with the compatibility conditions (4.7). The first one is trivially satisfied, while the remaining five read

$$\begin{aligned} \beta_{33}U''_2(x_1) &= 0, \\ (2\alpha_{33}x_1 + \beta_{23})U''_2(x_1) + 6\alpha_{33}U''_2(x_1) - \beta_{33}U'''_3(x_1) &= 0, \\ (\beta_{12}x_1 + \beta_{33}x_2 - \gamma_{13})U''_2(x_1) + 4\beta_{12}U''_2(x_1) - \beta_{31}U'''_3(x_1) &= 0, \\ -(2\alpha_{33}x_1x_2 - \beta_{13}x_1 + \beta_{23}x_2 - \gamma_{12})U''_3(x_1) + 4(\beta_{13} - 2\alpha_{33}x_2)U'''_3(x_1) - \beta_{21}U''_2(x_1) &= 0, \end{aligned}$$

$$(8\alpha_{33}x_2 - 4\beta_{13} - \beta_{31})U_2'''(x_1) - (4\beta_{12} + \beta_{21})U_3'''(x_1) \quad (7.5)$$

$$+ (2\alpha_{33}x_1x_2 - \beta_{13}x_1 + \beta_{23}x_2 - \gamma_{12})U_2^{(4)}(x_1) - (\beta_{12}x_1 + \beta_{33}x_2 - \gamma_{13})U_3^{(4)}(x_1) = 0.$$

We can have different subcases according to whether the equations (7.4), (7.5) are trivially satisfied for some of the functions U_j or not. This determines a major splitting in the computation. The details are given in Appendix C.

8 Constant magnetic field and second-order polynomial potentials

Let us consider the particular case in which the magnetic field is constant

$$\vec{B}(\vec{x}) = (a_1, a_2, 0) \quad (8.1)$$

and in (1.5) we have

$$V_1(x) = v_{11}x_1 + v_{12}x_1^2, \quad V_2(x_2) = v_{21}x_2 + v_{22}x_2^2, \quad u_1 = a_1x_2, \quad u_2 = -a_2x_1.$$

This system appears in various branches of calculation in the appendices; thus we find it practical to discuss it separately here.

Notice that since the magnetic field is constant, by rotation around x_3 -axis we could reduce it to the case in which it is aligned with one of the Cartesian axis. However, the system would no longer separate in the corresponding rotated Cartesian coordinates, therefore we prefer not to perform such a rotation.

Let us also point out that if $V_1(x_1) = 0$, for constant magnetic field a rotation around x_2 brings the system (1.5) into (1.7). Thus, what we will deduce in the following for $V_1 = 0$ applies also for (1.7).

8.1 v_{12} and v_{22} both not vanishing

Let us assume v_{12} and v_{22} are both not vanishing. Then by the translation of the coordinate system we can set $v_{11} = v_{21} = 0$ without loss of generality.

Then, similarly to Section 3, we can reduce to a natural Hamiltonian system through canonical transformations. Namely, let us take as the generating function

$$G = \left(x_1 - \frac{a_2P_3}{2v_{12}}\right)P_1 + \left(\frac{a_1P_3}{2v_{22}} + x_2\right)P_2 + x_3P_3, \quad (8.2)$$

so that $p_j = P_j$, $j = 1, 2, 3$ and

$$x_1 = X + \frac{a_2P_3}{2v_{12}}, \quad x_2 = Y - \frac{a_1P_3}{2v_{22}}, \quad x_3 = Z + \frac{a_2v_{22}P_1 - a_1v_{12}P_2}{2v_{12}v_{22}}.$$

After the transformation, with gauge chosen as in (1.4), the Hamiltonian reads

$$H = \frac{1}{2} \left(P_1^2 + P_2^2 + \left(1 - \frac{a_1^2}{2v_{22}} - \frac{a_2^2}{2v_{12}}\right) P_3^2 \right) + v_{12}X^2 + v_{22}Y^2.$$

If $\frac{a_1^2}{2v_{22}} + \frac{a_2^2}{2v_{12}} \neq 1$ we can, by a canonical transformation

$$P_3 = \frac{1}{\lambda} \tilde{P}_3, \quad Z = \lambda \tilde{Z}, \quad \lambda^2 = \left| 1 - \frac{a_1^2}{2v_{22}} - \frac{a_2^2}{2v_{12}} \right|$$

scale the P_3^2 term to have the Hamiltonian of the form

$$H = \frac{1}{2}(P_1^2 + P_2^2 \pm \tilde{P}_3^2) + v_{12}X^2 + v_{22}Y^2.$$

The system can therefore be reduced to a system determined by a two-dimensional, possibly inverted, anisotropic harmonic oscillator and free motion along the Z -direction. The original 3D system is minimally superintegrable if and only if the corresponding 2D oscillator is superintegrable as a system in the (X, Y, P_1, P_2) space. If $v_{12} = v_{22}$ we have a special case of the system \mathcal{E}_3 in Table 1. If $\frac{v_{12}}{v_{22}} \in \mathbb{Q}$, $\frac{v_{12}}{v_{22}} \neq 1$ we have a higher-order integral when expressed in the variables x_j, p_j .

If $\frac{a_1^2}{2v_{22}} + \frac{a_2^2}{2v_{12}} = 1$, the coordinate P_3 becomes cyclic and its conjugated variable Z is an independent constant of motion. In this case the system (1.5) becomes at least minimally superintegrable. It is maximally superintegrable if and only if its reduction on the (X, Y, P_1, P_2) space is superintegrable. Indeed, we have reduced to the system (3.7) for $V(Y) = v_{22}Y^2$ and $2v_{12} = \gamma^2 = a_2^2$. Its maximally superintegrable exception is included in the family of systems (3.11).

8.2 $v_{22} = 0$ and v_{12} not vanishing

In this case by translation in x_1 we can still set $v_{11} = 0$. Then by a canonical transformation such that

$$x_1 = X + \frac{a_2}{2v_{12}}P_3, \quad x_2 = Y, \quad x_3 = Z + \frac{a_2}{2v_{12}}P_1, \quad p_j = P_j, \quad j = 1, 2, 3,$$

we obtain the system

$$H = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + v_{12}X^2 + a_1P_3Y + v_{21}Y.$$

If $a_1 = 0$ we have reduced to a natural system. By reducing the integral P_3 we have a 2D system that, to our knowledge, is not superintegrable.

If $a_1 \neq 0$ we have reduced to the case with magnetic field aligned along one axis. The effective potential of the so obtained system reads

$$W = v_{12}X^2 + v_{21}Y - (a_1^2Y^2)/2. \tag{8.3}$$

Thus, by the translation

$$Y \rightarrow Y + \frac{v_{21}}{a_1^2}$$

we can eliminate the linear term from the effective potential. A shift of the vector potential by a constant, i.e.,

$$A_3 \rightarrow A_3 - \frac{v_{21}}{a_1},$$

gives

$$H = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + v_{12}X^2 + a_1P_3Y.$$

Thus, without loss of generality, we can set $v_{21} = 0$. By plugging (8.3) and (8.1) with these simplifications into the determining equations for a second-order integral as in (4.6), we find that they have no solution.

8.3 $v_{12} = v_{22} = 0$

We have a subcase of the system \mathcal{E}_3 in Table 1. Thus, the system admits a second-order integral. With the gauge chosen as in (1.4), that integral reads

$$X_3 = p_1 p_2 + (v_{11} - a_2 p_3) x_2 + (v_{21} + a_1 p_3) x_1.$$

Equivalently, in gauge covariant form, we have

$$X_3 = p_1^A p_2^A + (a_1 x_1 - a_2 x_2) p_3^A - (a_1^2 + a_2^2) x_1 x_2 + a_1 a_2 (x_1^2 + x_2^2) + v_{11} x_2 + v_{21} x_1,$$

corresponding to the fact that the system actually separates in any rotated system of Cartesian coordinates, since the Hamiltonian is linear in the space variables. Without altering the structure of the Cartesian-type integrals, we can therefore by rotation align the magnetic field along one Cartesian axis, let us say the x_2 -axis. Thus, without loss of generality, let us assume $a_1 = 0$. The determining equations for an additional second-order integral can be solved. We find for $v_{11} = 0$ one maximally superintegrable system:

$$\vec{B}(\vec{x}) = (0, a_2, 0), \quad W(\vec{x}) = v_{21} x_2 - \frac{1}{2} a_2^2 x_1^2, \quad (8.4)$$

with the integral

$$\begin{aligned} X_4 &= 3p_3^A l_1^A - p_1^A l_3^A - \frac{3v_{21}}{a_2} l_2^A + a_2 x_1 x_2 p_3^A + 3a_2 x_1 l_1^A + v_{21} x_1^2 + a_2^2 x_1^2 x_2 \\ &= 3p_3 l_1 - p_1 l_3 - \frac{3}{a_2} (3v_{21} l_2 + 2a_2^2 x_1 x_2 p_3 + 2a_2 v_{21} x_1^2). \end{aligned}$$

9 Conclusions

Let us summarize our results. We have provided an exhaustive determination of quadratically superintegrable systems which separate in the Cartesian coordinates with magnetic field. In addition, we have found classes of systems minimally and maximally superintegrable with higher-order integrals. We list them below for reader's convenience.

9.1 Superintegrable systems with second-order integrals

We have constructed an exhaustive list of quadratically superintegrable systems with nonvanishing magnetic field which separate in Cartesian coordinates. Under the assumption that there is no independent first-order integral other than the Cartesian ones (in that case we refer the reader to our previous work [14, 16]) we have found 8 classes of minimally superintegrable systems, among which one contains a quadratically maximally superintegrable subclass, cf. (8.4).

For brevity, we write here the magnetic field, the electrostatic potential and the leading order terms in the integral(s) together with the reference to the equation in which the system was introduced. We refer the reader to the relevant formulas therein encoding the complete information about the integral(s).

Case I, i.e., the magnetic field and potential are of the form (1.5) and the Cartesian integrals as in (1.8). The superintegrable systems read

(a)

$$\vec{B}(\vec{x}) = (a e^{bx_2}, c, 0), \quad W(\vec{x}) = a \left(w + \frac{c}{b} x_1 \right) e^{bx_2} - \frac{a^2}{2b^2} e^{2bx_2},$$

$$X_3 = p_1^A p_3^A + \dots, \text{ cf. (A.11).}$$

(b)

$$\begin{aligned}\vec{B}(\vec{x}) &= 2 \left(a_1 x_2 - \frac{a_3}{x_2^3}, -a_1 x_1 + \frac{a_2}{x_1^3}, 0 \right), \\ W(\vec{x}) &= -\frac{1}{2} a_1^2 (x_1^2 + x_2^2)^2 - \frac{a_2^2}{2x_1^4} - \frac{a_3^2}{2x_2^4} - a_1 \left(a_2 \frac{x_2^2}{x_1} + a_3 \frac{x_1^2}{x_2} \right) \\ &\quad - \frac{a_2 a_3}{x_2^2 x_1^2} + \frac{b_3}{x_2} + b_1 (x_1^2 + x_2^2) + \frac{b_2}{x_1^2},\end{aligned}$$

$$X_3 = (l_3^A)^2 + \dots, \text{ cf. Table 1.}$$

(c)

$$\begin{aligned}\vec{B}(\vec{x}) &= \left(2 \left(a_1 x_2 - \frac{a_3}{x_2^3} \right), -8a_1 x_1 - a_2, 0 \right), \\ W(\vec{x}) &= -\frac{a_1^2}{2} (4x_1^2 + x_2^2)^2 - \frac{a_2^2}{2} x_1^2 - \frac{a_3^2}{2x_1^4} - a_2 a_3 \frac{x_1}{x_2^2} \\ &\quad - a_1 a_2 x_1 (4x_1^2 + x_2^2) - 4a_1 a_3 \frac{x_1^2}{x_2^2} + \frac{b_3}{x_2} + b_1 (4x_1^2 + x_2^2) + b_2 x_1,\end{aligned}$$

$$X_3 = p_2^A l_3^A + \dots, \text{ cf. Table 1.}$$

(d)

$$\begin{aligned}\vec{B}(\vec{x}) &= (2a_1 x_2 + a_3, -2a_1 x_1 - a_2, 0), \\ W(\vec{x}) &= -\frac{a_1^2}{2} (x_1^2 + x_2^2)^2 - \frac{a_3^2}{2} x_2^2 - \frac{a_2^2}{2} x_1^2 - a_2 a_1 x_1 (x_1^2 + x_2^2) \\ &\quad - a_2 a_3 x_1 x_2 - a_1 a_3 x_2 (x_1^2 + x_2^2) + b_1 (x_1^2 + x_2^2) + b_2 x_1 + b_3 x_2,\end{aligned}$$

$X_3 = p_1^A p_2^A + \dots$, cf. Table 1. When $a_1 = a_3 = 0$ and $b_1 = b_2 = 0$ the system becomes maximally superintegrable, with the additional integral of the form $X_4 = p_1^A l_3^A - 3p_3^A l_1^A + \dots$, cf. (8.4).

Case II, i.e., the magnetic field and potential are of the form (1.7) and the Cartesian integrals as in (1.9). The superintegrable systems read

(a)

$$\vec{B}(\vec{x}) = (0, a e^{bx_1}, 0), \quad W(\vec{x}) = w x_1 + c e^{bx_1} - \frac{1}{2} \frac{a^2}{b^2} e^{2bx_1},$$

$$X_3 = p_1^A p_2^A - b p_3^A l_1^A + \dots, \text{ cf. (C.7),}$$

(b)

$$\vec{B}(\vec{x}) = (0, a(b-2)x_1^{b-3}, 0), \quad W(\vec{x}) = -\frac{a^2 x_1^{2(b-2)}}{2} + a(b-2) c x_1^{b-2} + \frac{w}{x_1^2},$$

$$X_3 = p_1^A l_3^A - b p_3^A l_1^A + \dots, \text{ cf. (C.9),}$$

(c)

$$\vec{B}(\vec{x}) = \left(0, \frac{a}{x_1}, 0 \right), \quad W(\vec{x}) = -\frac{1}{2} a^2 (\ln |x_1|)^2 + b \ln |x_1| + \frac{w}{x_1^2},$$

$$X_3 = 2p_1^A l_3^A - p_3^A l_1^A + \dots, \text{ cf. (C.10),}$$

(d)

$$\vec{B}(\vec{x}) = \left(0, 0, \frac{a}{x_1^3}\right), \quad W(\vec{x}) = -\frac{ab \ln|x_1|}{x_1^2} - \frac{a^2}{8x_1^4} + \frac{w}{x_1^2},$$

$$X_3 = p_1^A l_2^A + \dots, \text{ cf. (C.18).}$$

Our approach also demonstrates that any quadratically maximally superintegrable system with magnetic field which separates in Cartesian coordinates would necessarily appear at the intersection of the presented classes. Given the different structure of the magnetic field in each of the cases we find only few potential candidates. One is the intersection of Case I.d and Case II.b which, as we already observed, leads to the maximally superintegrable system (8.4). Another is the system Case I.a which for $c = 0$ reduces to the system Case II.a (upon interchange of the x_1 and x_2 coordinates and momenta). However, the integral X_3 of Case I.a when $c = 0$ becomes a function of the two first-order Cartesian integrals, i.e., it is not independent anymore. Last but not least, the systems Case I.b, Case I.c, Case II.b and Case II.d (after a permutation of coordinates) overlap for $a_1 = a_2 = 0$ (Case I.b/c) and $b = 0$ (Case II.b/d) but the integrals again turn out to be dependent.

Thus we conclude that no other quadratically maximally superintegrable systems separating in Cartesian coordinates other than (8.4) and the ones found in [14] exist.

9.2 Superintegrable systems with higher-order integrals

Above we have provided a complete answer to the problem of quadratic superintegrability for the considered classes of systems (1.5) and (1.7). As we have seen, maximal superintegrability via at most quadratic integrals is very rare in the presence of magnetic field, as opposed to numerous purely scalar maximally superintegrable systems discussed, e.g., in [9, 13]. Thus one should consider also the possible existence of higher-order integrals. However, these are computationally very difficult to find. In this paper we have presented two propositions, namely Propositions 2.1 and 3.1 which can be used to construct three-dimensional maximally superintegrable systems with magnetic field out of two-dimensional scalar ones. In particular, Proposition 3.1 states that a system with

$$\vec{B}(\vec{x}) = (0, \gamma, 0), \quad \gamma \neq 0, \quad W(\vec{x}) = V(x_2),$$

is maximally superintegrable if and only if the two-dimensional system with the Hamiltonian

$$\mathcal{K}(\vec{X}, \vec{P}) = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}\gamma^2 X^2 + V(Y)$$

is superintegrable (where V is the same function of a single variable).

Using Proposition 3.1 we have arrived at an explicit example of maximally superintegrable system with

$$\vec{B}(\vec{x}) = (0, \gamma, 0), \quad W(\vec{x}) = \frac{c}{x_2^2} + \frac{m^2}{\ell^2}\gamma^2 x_2^2, \quad \ell, m \in \mathbb{N}, \quad c \in \mathbb{R},$$

cf. (3.11), with three first-order integrals (3.4) and an additional integral coming from the integral of two-dimensional caged oscillator through the change of variables (3.6).

Similarly, Proposition 2.1 led us to minimally superintegrable systems (2.5)

$$\vec{B}(\vec{x}) = 2 \left(\omega m_1 x_2 - \frac{\beta_1}{x_2^3}, -\omega \ell_1 x_1 + \frac{\alpha_1}{x_1^3}, 0 \right),$$

$$W(\vec{x}) = -\frac{\omega^2}{2}(\ell_1 x_1^2 + m_1 x_2^2)^2 + \omega \left(\ell_2 x_1^2 + m_2 x_2^2 - \alpha_1 m_1 \frac{x_2^2}{x_1^2} - \beta_1 \ell_1 \frac{x_1^2}{x_2^2} \right) \\ + \frac{\alpha_2}{x_1^2} + \frac{\beta_2}{x_2^2} - \frac{1}{2} \left(\frac{\alpha_1}{x_1^2} + \frac{\beta_1}{x_2^2} \right)^2, \quad \frac{l_1}{m_1} = \frac{l_2}{m_2} = \frac{l^2}{m^2}, \quad l, m \in \mathbb{Z}.$$

Systems I.b and I.c of Section 9.1 with $a_1 \neq 0$ are special subcases of it when the integral X_3 becomes second order one.

Another class of minimally superintegrable systems with

$$\vec{B}(\vec{x}) = (a_1, a_2, 0), \quad W(\vec{x}) = v_{12}x_1^2 + v_{22}x_2^2 - \frac{1}{2}(a_2x_1 + a_1x_2)^2, \\ \frac{v_{12}}{v_{22}} \in \mathbb{Q}, \quad v_{12}, v_{22} \neq 0$$

can be constructed out of anisotropic harmonic oscillator in two dimensions through the canonical transformation (8.2).

Of course, more efficient and widely applicable tools for construction of higher-order superintegrable systems are needed. Given the recent rapid progress on a similar problem for scalar potentials [7, 18, 19, 21, 23] (see also references in [22]) we hope that in foreseeable future we will be able to report on further development also in the case with magnetic field.

A Solution of the determining equations in Case I for $\beta_{12} = \beta_{21} = 0$

We consider here $\beta_{12} = \beta_{21} = 0$. Thus c_0 and c_1 cannot both vanish, since by assumption the right-hand side of (6.4) is not identically zero and, by (6.5), $a_j = 0$, $j = 1, 2$. There are several subcases, according to whether equation (6.7) is trivially satisfied or not. For the reader's convenience, let us type it here again

$$(\beta_{33}x_1 + \gamma_{23})u_1'(x_2) + (\beta_{33}x_2 - \gamma_{13})u_2'(x_1) - c_1 = 0. \quad (\text{A.1})$$

We can have

- (a) the above equation is not trivially satisfied for u_2 nor for u_1 , thus $\beta_{33} \neq 0$ or $\beta_{33} = 0$ and both γ_{13}, γ_{23} not vanishing;
- (b) equation (A.1) is trivially satisfied for u_2 but not for u_1 , thus $\beta_{33} = \gamma_{13} = 0$ and $\gamma_{23} \neq 0$;
- (c) equation (A.1) is trivially satisfied for u_1 but not for u_2 , thus $\beta_{33} = \gamma_{23} = 0$ and $\gamma_{13} \neq 0$;
- (d) equation (A.1) is trivially satisfied for both u_j , thus $c_1 = \beta_{33} = \gamma_{23} = \gamma_{13} = 0$.

Notice that Case (b) can be reduced to Case (c), and viceversa, by a canonical exchange of p_1 with p_2 . Thus both cases are recovered by Appendix A.3. Case (a) is splitted for convenience into Appendices A.1 and A.2, while Case (d) is treated in Appendix A.4.

A.1 $\beta_{33} \neq 0$

In this case, by translation in x_1 and x_2 , we can set $\gamma_{13} = \gamma_{23} = 0$.

By taking second derivatives of (A.1) with respect to x_1 and x_2 and by equating them to zero, we find that

$$u_1(x_2) = \frac{a_{12}}{2}x_2^2 + a_{11}x_2, \quad u_2(x_1) = -\frac{a_{12}}{2}x_1^2 + a_{21}x_1, \quad a_{ij} \in \mathbb{R}. \quad (\text{A.2})$$

By imposing then that (A.2) solves (A.1), we obtain a polynomial expression in x_1 and x_2 which must vanish

$$c_1 - (a_{11}x_1 - a_{21}x_2)\beta_{33} = 0.$$

We conclude $a_{11} = a_{21} = c_1 = 0$.

Solving the remaining equations in (4.3) we find the functions S_j . Next we look at the first-order equations (4.4), and in particular at the third one, for m_{x_3} . We proceed as above to solve it for V_j . By considering its second-order derivatives with respect to x_1 and x_2 and by equating them to zero, we find that also the functions V_j must be second-order polynomials. Then we plug such solution back into the equation for m_{x_3} and find a polynomial in x_1 and x_2 that must vanish. This implies $c_0 = 0$. Thus $\{X_3, p_3\} = 0$. Nothing new can be found in this case.

A.2 $\beta_{33} = 0, \gamma_{13} \neq 0, \gamma_{23} \neq 0$

Proceeding as above we find from (A.1) that

$$c_1 = (a_1\gamma_{23} - a_2\gamma_{13})$$

and

$$u_1(x_2) = a_1x_2, \quad u_2(x_1) = a_2x_1, \quad a_{ij} \in \mathbb{R}, \quad (\text{A.3})$$

where $|a_1|^2 + |a_2|^2 \neq 0$. The solution (A.3) implies that the magnetic field is constant. By substituting (A.3) into the compatibility conditions (4.7) for the magnetic field, they imply $\alpha_{33} = 0$. The second-order equations (4.3) can now be solved and give

$$\begin{aligned} S_1(x_1, x_2) &= a_2\gamma_{13}x_1 + (S - a_1\gamma_{13}a_2\gamma_{23})x_2 + s_{11}, & S_2(x_1, x_2) &= -Sx_1 - a_1\gamma_{23}x_2 + s_{21}, \\ S_3(x_1, x_2) &= \frac{1}{2}(a_1\beta_{13}x_1^2 + a_2\beta_{23}x_2^2) - a_2\gamma_{13}x_2 + (a_2\beta_{13}x_2 + a_1\gamma_{13})x_1, & S, s_{ij} &\in \mathbb{R}, \end{aligned}$$

together with the condition

$$\beta_{13}a_2 = \beta_{23}a_1.$$

We now look at the first-order equations (4.4), starting with the equation for m_{x_3} . Its first-order derivatives with respect to x_1 and x_2 imply that each V_j is a second-order polynomial in its variable, $j = 1, 2$. This case is discussed in Section 8.

A.3 $\beta_{33} = \gamma_{23} = 0, \gamma_{13} \neq 0$

In this case equation (A.1) is trivially satisfied for u_1 and as an equation for u_2 implies

$$u_2(x_1) = -\frac{c_1x_1}{\gamma_{13}}. \quad (\text{A.4})$$

By solving the second-order equations (4.3) for s_j , $j = 1, 2, 3$, we find

$$\begin{aligned} S_1(x_1, x_2) &= -c_1x_1 - Sx_2 - \gamma_{13}u_1(x_2) + s_{11}, & S_2(x_1, x_2) &= Sx_1 + s_{21}, \\ S_3(x_1, x_2) &= \frac{c_1x_2(2\alpha_{33}x_1x_2 - 2\beta_{13}x_1 - \frac{1}{2}\beta_{23}x_2 + \gamma_{12})}{\gamma_{13}} \\ &\quad + x_1 \left(x_1 \left(-\alpha_{33}x_2 + \frac{1}{2}\beta_{13} \right) - \beta_{23}x_2 + \gamma_{12} \right) u_1'(x_2), \end{aligned}$$

where $S, s_{ij} \in \mathbb{R}$ and u_1 has to solve the remaining compatibility conditions (4.7) (some are already satisfied by the conditions imposed on the constant $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ and (A.4)):

$$\frac{6c_1\alpha_{33}}{\gamma_{13}} - \beta_{23}x_2u_1^{(3)}(x_2) + \gamma_{12}u_1^{(3)}(x_2) - 4\beta_{23}u_1''(x_2) = 0, \quad (\text{A.5})$$

$$(2\alpha_{33}x_2 - \beta_{13})u_1''(x_2) + 6\alpha_{33}u_1'(x_2) = 0. \quad (\text{A.6})$$

The second derivative with respect to x_1 of the third first-order equation in (4.4) implies

$$V_1(x_1) = v_{11}x_1 + v_{12}x_1^2, \quad v_{ij} \in \mathbb{R}. \quad (\text{A.7})$$

To proceed, we must solve the equations (A.5)–(A.6) for u_1 . There can be several subcases.

A.3.1 $\alpha_{33} = \beta_{13} = \beta_{23} = \gamma_{12} = 0$

In this case (A.5)–(A.6) are trivially satisfied. Let us consider the second set of compatibility conditions (4.8) coming from the first-order equations, which simplify to

$$\frac{c_1^2}{\gamma_{13}} - 2v_{12}\gamma_{13} - Su_1'(x_2) = 0, \quad (\text{A.8})$$

$$c_1 \left(\frac{S}{\gamma_{13}} + u_1'(x_2) \right) - s_{21}u_1''(x_2) = 0. \quad (\text{A.9})$$

Let us first assume $u_1'' \neq 0$. If $c_1 = 0$, then $S = v_{12} = s_{21} = 0$ and $u_2(x_1) = 0$. Equation (4.5) implies that then u_1 is a constant, in contradiction with the assumption $u_1'' \neq 0$. Thus, let us consider the case u_1'' and c_1 both not zero. From equation (A.9) we obtain $S = 0$ and

$$u_1(x_2) = \frac{a_1 s_{21}}{c_1} e^{\frac{c_1 x_2}{s_{21}}}, \quad a_1 \in \mathbb{R}$$

together with

$$v_{12} = \frac{c_1^2}{2\gamma_{13}^2}$$

that we substitute into (A.7).

The solution of the compatibility constraints (A.8)–(A.9) assures that a solution of (4.4) for m exists. Thus, let us look at the zero-order equation (4.5), which in this case reduces to

$$c_1 v_{11} x_1 - \frac{a_1 s_{21} e^{\frac{c_1 x_2}{s_{21}}} (4v_{11}\gamma_{13}^2 + c_1 s_{11})}{c_1 \gamma_{13}} + s_{11} v_{11} + s_{21} V_2'(x_2) = 0. \quad (\text{A.10})$$

Therefore $v_{11} = 0$ and (A.7) simplifies to

$$V_1(x_1) = \frac{c_1^2 x_1^2}{2\gamma_{13}^2}.$$

Let us notice that if $s_{21} = 0$ we have constant magnetic field along the x_2 direction and arbitrary V_2 . Thus, we obtain the class of systems already studied in Section 3. Otherwise, if s_{21} is not zero, by solving (A.10) we find

$$V_2(x_2) = \frac{a_1 s_{11} s_{21} e^{\frac{c_1 x_2}{s_{21}}}}{c_1 \gamma_{13}}.$$

Thus, we arrive at the system determined by

$$\vec{B}(\vec{x}) = (ae^{bx_2}, c, 0), \quad W(\vec{x}) = a \left(w + \frac{c}{b}x_1 \right) e^{bx_2} - \frac{a^2}{2b^2} e^{2bx_2}, \quad (\text{A.11})$$

where we relabelled the integration constants as $a_1 = a$, $c_1 = -\gamma_{13}c$, $s_{11} = \gamma_{13}bw$, $s_{21} = -\frac{c\gamma_{13}}{b}$, with $a, b, c, w \in \mathbb{R}$ such that a , b , and c are not vanishing. The system (A.11) is (at least) minimally superintegrable, with the integral X_3 as in (4.6), where all the coefficients of the second-order terms are zero except γ_{13} , and S_j and m_3 of equation (6.6) are given by

$$\begin{aligned} S_1(x_1, x_2) &= \gamma_{13} \left(wb - e^{bx_2} \frac{a}{b} + cx_1 \right), & S_2(x_1, x_2) &= -\gamma_{13} \frac{c}{b}, & S_3(x_1, x_2) &= 0, \\ m_3(\vec{x}) &= \gamma_{13}c \left(e^{bx_2} \frac{a}{b} - cx_1 - wb \right) x_3. \end{aligned}$$

It remains to be considered the case in which $u_1'' = 0$, therefore $u_1(x_2) = a_1x_2$, the magnetic field is constant. If $c_1 = 0$, from the zero-order equation (4.5) we get that either V_2 is a second-order polynomial (not of interest here since V_1 is already a second-order polynomial, too), or $S = s_{21} = v_{11} = 0$ and $v_{12} = 0$ from (A.8). This gives $V_1(x_1) = 0$. From the first-order equations (4.4) we find $c_0 = 0$. Nothing new here.

If c_1 is not zero, the compatibility condition (A.8), (A.9) together with the zero-order equation (4.5) imply that either V_1 and V_2 are both second-order polynomials or $V_1 = 0$ and the magnetic field is constant and aligned along the x_2 -axis. Both these cases are of no interest in this section since they are considered elsewhere, in Section 8 and Appendix C.

A.3.2 $\alpha_{33} = 0$, $\beta_{13} \neq 0$

By translation in x_1 we can assume $\gamma_{12} = 0$.

The compatibility condition (A.5) reduces to

$$\begin{aligned} -\beta_{23}(x_2u_1^{(3)}(x_2) + 4u_1''(x_2)) &= 0, \\ \beta_{13}u_1''(x_2) &= 0. \end{aligned} \quad (\text{A.12})$$

Since β_{13} is not zero, (A.12) gives

$$u_1(x_2) = a_1x_2, \quad a_1 \in \mathbb{R}.$$

The compatibility conditions (4.8) for m_3 imply that V_2 is a second-order polynomial. This case is of no interest here and it is studied in Section 8.

A.3.3 $\alpha_{33} = 0$, $\beta_{13} = 0$, $\beta_{23} \neq 0$

Still, by translation in x_1 , we can set $\gamma_{12} = 0$. Equations (4.3) imply

$$u_1(x_2) = \frac{a_2}{6x_2^2}.$$

In order to have nonvanishing magnetic field and nonvanishing Poisson bracket in (6.1), we find from equations (4.4) and (4.5) that we must have $S = 0$, $a_2 = 0$ and $c_0 = \frac{c_1s_{11}}{\gamma_{13}}$. Looking for a solution of (4.5) for V_2 not second or lower-order polynomial we find

$$\vec{B}(\vec{x}) = \left(0, -\frac{c_1}{\gamma_{13}}, 0 \right), \quad V_1(x_1) = \frac{c_1^2x_1^2}{2\gamma_{13}^2}, \quad V_2(x_2) = \frac{c_1^2x_2^2}{8\gamma_{13}^2} - \frac{v_{21}}{2x_2^2},$$

which is the already known system (3.8).

A.3.4 $\alpha_{33} = \beta_{13} = \beta_{23} = 0, \gamma_{12} \neq 0$

The compatibilities (A.5)–(A.6) reduce to the sole equation

$$\gamma_{12}u_1^{(3)}(x_2) = 0,$$

therefore

$$u_1(x_2) = a_1x_2 + a_2x_2^2, \quad a_j \in \mathbb{R}.$$

The compatibility conditions (4.8), together with the zero-order equation, imply that either the magnetic field is constant and V_2 is a second-order polynomial, or $V_1(x_1) = u_2(x_1) = 0$ and X_2 reduces to a first-order integral. Both these cases are of no interest here.

A.3.5 $\alpha_{33} \neq 0$

By translation in x_1 and x_2 we can set $\beta_{13} = \beta_{23} = 0$. Thus, (A.6) gives

$$u_1(x_2) = \frac{a}{2x_2^2}, \quad a \in \mathbb{R}.$$

By plugging this solution into (4.3) we obtain that it must be $c_1 = 0$ and therefore by (A.5) $\gamma_{12} = 0$. By looking for a solution of the equations (4.8) and (4.4) we find that also $c_0 = 0$, i.e., $\{X, p_3\} = 0$, and this case is of no interest here.

A.4 $\beta_{33} = \gamma_{23} = \gamma_{13} = c_1 = 0$

In this case equation (A.1) is trivially satisfied for both u_j . The remaining second-order equations give for S_j :

$$\begin{aligned} S_1(x_1, x_2) &= Sx_2 + s_{11}, & S_2(x_1, x_2) &= -Sx_1 + s_{21}, \\ S_3(x_1, x_2) &= -\frac{x_1}{2}(2\alpha_{33}x_1x_2 - \beta_{13}x_1 + 2\beta_{23}x_2 - 2\gamma_{12})u_1'(x_2) \\ &\quad - 2x_2(\alpha_{33}x_2 - \beta_{13})u_2(x_1) + S_{31}(x_2), \quad S, s_{ij} \in \mathbb{R}, \end{aligned}$$

where $S_{31}(x_2)$ must solve

$$\begin{aligned} &\left(\alpha_{33}x_1^2x_2 - \frac{\beta_{13}}{2}x_1^2 + \beta_{23}x_1x_2 - \gamma_{12}x_1 \right) u_1''(x_2) + 3x_1(\alpha_{33}x_1 + \beta_{23})u_1'(x_2) \\ &\quad - ((\beta_{13} - 2\alpha_{33}x_2)x_1 - \beta_{23}x_2 + \gamma_{12})u_2'(x_1) \\ &\quad + 2(2\alpha_{33}x_2 - \beta_{13})u_2(x_1) - S'_{31}(x_2) = 0 \end{aligned}$$

and u_j satisfy the compatibilities (4.7), that in this case reduce to

$$\begin{aligned} &(2\alpha_{33}x_1x_2 - \beta_{13}x_1 + \beta_{23}x_2 - \gamma_{12})u_1'''(x_2) - 4(2\alpha_{33}x_1 + \beta_{23})u_1''(x_2) \\ &\quad + (2\alpha_{33}x_1 + \beta_{23})u_2''(x_1) + 6\alpha_{33}u_2'(x_1) = 0, \end{aligned} \tag{A.13}$$

$$\begin{aligned} &6\alpha_{33}u_1'(x_2) + (2\alpha_{33}x_2 - \beta_{13})(u_1'(x_2) + 8u_2'(x_1)) \\ &\quad - (2\alpha_{33}x_1x_2 - \beta_{13}x_1 + \beta_{23}x_2 - \gamma_{12})u_2'''(x_1) = 0. \end{aligned} \tag{A.14}$$

As in the above Appendix A.3, to solve (A.13)–(A.14) we have to distinguish several subcases.

A.4.1 $\alpha_{33} \neq 0$

By translation in x_1 and x_2 we can eliminate β_{13} and β_{23} . By taking the third-order derivatives $\partial_{x_1}^2 \partial_{x_2}$, $\partial_{x_2}^2 \partial_{x_1}$ of (A.13) and (A.14), respectively we get

$$u_2(x_1) = a_{22}x_1^2 - \frac{a_{23}}{x_1^2} + a_{21}x_1, \quad u_1(x_2) = a_{12}x_2^2 - \frac{a_{13}}{x_2^2} + a_{11}x_2, \quad a_{ij} \in \mathbb{R}.$$

By plugging the above solution into the third condition in (4.4) we see that, as a polynomial in x_1 and x_2 , it can be vanishing only if $c_0 = 0$. Since also $c_1 = 0$, there is nothing new here.

A.4.2 $\alpha_{33} = 0, \beta_{13} \neq 0$

By translation in x_1 we can assume $\gamma_{12} = 0$. By considering the first-order derivative of (A.13) with respect to x_2 and of (A.14) with respect to x_1 , we find

$$u_1(x_2) = a_{11}x_2 + a_{12}x_2^2 + a_{13}x_2^3, \quad u_2(x_1) = a_{21}x_1 + a_{22}x_1^2 + \frac{a_{23}}{x_1^2}, \quad a_{ij} \in \mathbb{R}. \quad (\text{A.15})$$

We plug (A.15) into (A.13)–(A.14) and we obtain the conditions $a_{23} = a_{13} = 0$ and

$$a_{22}\beta_{23} = 0.$$

If $\beta_{23} \neq 0$, then $a_{22} = 0$. The equation for m_{x_3} in (4.4), together with (4.8), implies that V_j are second-order polynomials and $a_{12} = 0$, i.e., the magnetic field is constant. This case is of no interest here, cf. Section 8.

If $\beta_{23} = 0$ we again consider the third equation in (4.4), together with (4.8) and their first order and second order mixed derivatives with respect to x_1 and x_2 . If a_{22} is not vanishing, we can find a solution for both V_j which is not a second-order polynomial (or lower) but only assuming that $c_0 = 0$. Therefore nothing new arises in this case.

If $a_{22} = 0$, we can find a solution for V_j only if the magnetic field is vanishing, or $c_0 = 0$, or the solution is equivalent to the system (3.8). Anyway, not of interest here (we recall that in Appendix A.4 we have $c_1 = 0$).

A.4.3 $\alpha_{33} = \beta_{13} = 0, \beta_{23} \neq 0$

We can eliminate γ_{12} by translation in x_2 , thus setting $\gamma_{12} = 0$.

The compatibility conditions (A.13), (A.14) imply that

$$u_1(x_2) = a_{12}x_2^2 + \frac{a_{11}}{x_2^2}, \quad u_2(x_1) = -4a_{12}x_1^2 + a_{21}x_1, \quad a_{ij} \in \mathbb{R}.$$

By imposing that also the remaining equations (4.3), (4.4) and (4.5) are satisfied we find that either $c_0 = 0$ which implies $\{X, p_3\} = 0$ in contradiction with our assumption, or we find $a_{11} = a_{12} = 0$ and

$$V_1(x_1) = \frac{1}{2}a_{21}^2x_1^2, \quad V_2(x_2) = \frac{a_{21}^2}{8}x_2^2 - \frac{v_{21}}{2x_2^2},$$

i.e., a system equivalent to (3.8).

A.4.4 $\alpha_{33} = \beta_{13} = \beta_{23} = 0, \gamma_{12} \neq 0$

The compatibility conditions (A.13), (A.14) imply that

$$u_1(x_2) = a_{12}x_2^2 + a_{11}x_2, \quad u_2(x_1) = -a_{12}x_1^2 + a_{21}x_1, \quad a_{ij} \in \mathbb{R}.$$

By imposing that also the remaining equations are satisfied we find that necessarily $a_{12} = 0$ and

$$V_1(x_1) = \frac{1}{2}(a_{11}^2 + a_{21}^2)x_1^2 + v_{12}x_1, \quad V_2(x_2) = \frac{1}{2}(a_{11}^2 + a_{21}^2)x_2^2 + \frac{a_{21}v_{12}}{a_{11}}x_2.$$

Thus this case was already studied in Section 8.

B Solution of the determining equations in Case I for β_{12}, β_{21} not both vanishing

Here we distinguish between the cases

- (a) β_{12} and β_{21} both not vanishing;
- (b) $\beta_{12} \neq 0$ and $\beta_{21} = 0$. Notice that, by a canonical permutation of the variables, this case is equivalent to $\beta_{21} \neq 0$ and $\beta_{12} = 0$.

Case (a) is treated in the following Appendix B.1. Case (b) follows in Appendix B.2.

B.1 $\beta_{12} \neq 0, \beta_{21} \neq 0$

Since β_{12} and β_{21} are both nonvanishing, by translation in x_1 and x_2 we can set $\gamma_{13} = \gamma_{23} = 0$.

Let us start from the compatibility conditions (4.7), that read

$$\begin{aligned} &(\beta_{33}x_1 + \beta_{21}x_2)u_1'''(x_2) + 4\beta_{21}u_1''(x_2) = 0, \\ &\beta_{33}(u_1''(x_2) + u_2''(x_1)) = 0, \\ &(2\alpha_{33}x_1x_2 - \beta_{13}x_1 + \beta_{23}x_2 - \gamma_{12})u_1'''(x_2) + 4(2\alpha_{33}x_1 + \beta_{23})u_1''(x_2) \\ &\quad + (2\alpha_{33}x_1 + \beta_{23})u_2''(x_1) + 6\alpha_{33}u_2'(x_1) = 0, \\ &(\beta_{12}x_1 + \beta_{33}x_2)u_2'''(x_1) + 4\beta_{12}u_2''(x_1) = 0, \\ &\beta_{21}u_2''(x_1) + \beta_{12}u_1''(x_2) = 0, \\ &(2\alpha_{33}x_1x_2 - \beta_{13}x_1 + \beta_{23}x_2 - \gamma_{12})u_2'''(x_1) - 4(\beta_{13} - 2\alpha_{33}x_2)u_2''(x_1) \\ &\quad - (\beta_{13} - 2\alpha_{33}x_2)u_1''(x_2) + 6\alpha_{33}u_1'(x_2) = 0. \end{aligned} \tag{B.1}$$

The above equations could be trivially satisfied for u_j or not, depending on the constants $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$. This determines a splitting in the computation.

B.1.1 $\beta_{33} \neq 0$

The second equation in (B.1) implies

$$u_1(x_2) = \frac{a_{12}}{2}x_2^2 + a_{11}x_2, \quad u_2(x_1) = -\frac{a_{12}}{2}x_1^2 + a_{21}x_1, \quad a_{ij} \in \mathbb{R}.$$

By imposing that the above solution satisfies the remaining compatibilities (B.1) and that the magnetic field does not vanish, we find that $a_{12} = \alpha_{33} = 0$, therefore the magnetic field is constant. The third second-order equation simplifies to

$$(\beta_{33}a_{11} + 3\beta_{12}a_{21})x_1 + (3\beta_{21}a_{11} + \beta_{33}a_{21})x_2 = c_1,$$

which must hold for all values of x_1 and x_2 . Without loss of generality, we can assume that one component of the magnetic fields is not vanishing, e.g., a_{11} . Thus the above equation implies

$$\beta_{33} = -3\frac{\beta_{12}a_{21}}{a_{11}}, \quad \beta_{21} = \frac{a_{21}^2}{a_{11}^2}\beta_{12}, \quad c_1 = 0. \quad (\text{B.2})$$

With this assumption the first and second-order equations (4.3) can be solved for S_1 and S_2 . We plug the so found solution into the third equation in (4.4) and take its third-order derivatives $\partial_{x_1}^2 \partial_{x_2}$ and $\partial_{x_2}^2 \partial_{x_1}$. In this way we obtain the condition

$$\beta_{12}a_{21}V_j'''(x_j) = 0, \quad j = 1, 2.$$

Since the magnetic field is already constant, we do not consider here solutions for V_j in the form of at most quadratic polynomials. Thus necessarily $a_{21} = 0$. However, from (B.2) we have $\beta_{21} = 0$, which violates our assumption for this subcase.

B.1.2 $\beta_{33} = 0$

Equations (B.1) can be solved for nonvanishing magnetic field only if $\alpha_{33} = 0$. In this case we find the solution

$$u_1(x_2) = a_1x_2, \quad u_2(x_1) = a_2x_1, \quad a_j \in \mathbb{R},$$

corresponding to constant magnetic field. The third second-order equation then reduces to

$$a_1\beta_{21}x_2 + a_2\beta_{12}x_1 = c_1,$$

which must hold for all x_1, x_2 . Since β_{12} and β_{21} are assumed to be not vanishing in this section, we conclude that there is no solution for nonvanishing magnetic field.

B.2 $\beta_{12} \neq 0, \beta_{21} = 0$

Again, we start by the compatibilities (B.1), now simplified by the condition $\beta_{21} = 0$. We can proceed as above, considering first the case $\beta_{33} \neq 0$. Then, by translation in x_1 and x_2 we can assume $\gamma_{23} = \gamma_{13} = 0$. We proceed as in Appendix B.1.1 with the simplification $\beta_{21} = 0$. Equation (B.2) implies that the magnetic field has to vanish.

Therefore, we continue in the following by assuming $\beta_{33} = 0$. Since $\beta_{12} \neq 0$, by translation in x_1 we can still set $\gamma_{13} = 0$. The computation then splits into two major subcases, according to whether $\gamma_{23} = 0$ or not.

B.2.1 $\gamma_{23} \neq 0$

The third second-order equation simplifies to

$$2\beta_{12}u_2(x_1) + \gamma_{23}u_1'(x_2) + \beta_{12}x_1u_2'(x_1) = c_1. \quad (\text{B.3})$$

By solving for u_j we find

$$u_1(x_2) = a_1x_2, \quad u_2(x_1) = \frac{a_2}{2x_1^2} + \frac{c_1 - a_1\gamma_{23}}{2\beta_{12}}, \quad a_j \in \mathbb{R}.$$

The remaining second-order equations can be solved for S_j only under the condition $a_1\beta_{23} = a_2\beta_{23} = a_2\gamma_{12} = a_1\alpha_{33} = 0$. We find

$$S_1(x_1, x_2) = -s_{21}x_2 - \frac{a_2\beta_{12}}{x_1} + s_{12},$$

$$S_2(x_1, x_2) = \frac{a_1\beta_{12}}{2}x_1^2 - a_1\gamma_{23}x_2 + s_{21}x_1 + \frac{a_2\gamma_{23}}{2x_1^2} + s_{22},$$

$$S_3(x_1, x_2) = \frac{\beta_{13}a_1}{2}x_1^2 + \gamma_{12}a_1x_1 - \alpha_{33}a_2\frac{x_2^2}{x_1^2} + \beta_{13}a_2\frac{x_2}{x_1^2}.$$

Equations (4.4) imply the following constraint on our integration constants: $a_2s_{21} = 0$. The compatibility conditions (4.8) can be solved for V_j and together with (4.5) imply $a_2 = 0$. In order to have nonvanishing magnetic field we thus must have $\alpha_{33} = \beta_{23} = 0$. Equation (4.5) further implies $s_{21} = s_{12} = 0$ together with

$$V_1(x_1) = \frac{1}{8}a_1^2x_1^2 - \frac{v_1}{2x_1^2},$$

$$V_2(x_2) = \frac{1}{2}a_1^2x_2^2 + \left(a_1^2\gamma_{23}^2 - a_1c_1\gamma_{23} - 2c_0\beta_{12}\right)\frac{x_2}{2\gamma_{23}\beta_{12}} + \frac{c_0x_2}{\gamma_{23}},$$

i.e., we arrived at a system equivalent to (3.8) (after translation in x_2).

B.2.2 $\gamma_{23} = 0$

Equation (B.3) does not contain u_1 anymore, while for u_2 implies

$$u_2(x_1) = \frac{a_2}{x_1^2} + \frac{c_1}{2\beta_{12}}, \quad a_2 \in \mathbb{R}.$$

From the last but one equation in (B.1) we see that

$$u_1(x_2) = a_1x_2, \quad a_1 \in \mathbb{R},$$

while the remaining equations (B.1) read

$$a_2\beta_{23} = 0, \quad \alpha_{33}a_1x_1^5 - 4a_2(\beta_{23}x_2 - \gamma_{12}) = 0, \quad (\text{B.4})$$

i.e., $a_2\beta_{23} = \alpha_{33}a_1 = a_2\gamma_{12} = 0$.

The second-order equations (4.3) can be solved for S_1 and S_2 . We find

$$S_1(x_1, x_2) = Sx_2 - \frac{2a_2\beta_{12}}{x_1} + s_{11}, \quad S_2(x_1, x_2) = -Sx_1 + \frac{1}{2}a_1\beta_{12}x_1^2 + s_{21}, \quad s_{ij} \in \mathbb{R}.$$

The remaining second-order equations for S_3 imply that $\beta_{23} = 0$, otherwise they would imply that the magnetic field must vanish.

To proceed, we need to solve (B.4). We have to distinguish between the cases $a_2 = 0$ and $a_2 \neq 0$.

Let us start by assuming $a_2 = 0$. The compatibility conditions (4.8) together with the zero-order equation (4.5) can be solved only if $\alpha_{33} = \gamma_{12} = c_1 = 0$ and $S = s_{11} = s_{21} = 0$. However, the resulting system is equivalent to the already known maximally superintegrable system (3.8), up to a permutation of the canonical variables.

If $a_2 \neq 0$, then from (B.4) we have $\gamma_{12} = 0$ and $\alpha_{33}a_1 = 0$. Under these two conditions, also the remaining second-order equations (4.3) can be solved for S_3 . We find

$$S_3(x_1, x_2) = \frac{1}{2}a_1\beta_{13}x_1^2 + 2a_2 \left(\beta_{13}\frac{x_2}{x_1^2} - \alpha_{33}\frac{x_2^2}{x_1^2} \right).$$

The compatibility conditions for the first-order equations (4.8) can be solved for V_1 . They admit a solution for V_2 only if $\alpha_{33} = \beta_{13} = 0$ (and in this way also the remaining (B.4) is satisfied). After solving for V_1 and imposing the previous condition, we see that they are satisfied for

any V_2 . To find V_2 we look at the zero-order equation, in which we impose all the conditions we obtained till now and the solutions found for V_1 and S_j . We obtain in this way a polynomial in x_1 (whose some coefficients contains equations for V_2) that must vanish. By collecting the different powers of x_1 and impose that they are all equal to zero, we arrive at the condition $a_2\beta_{12} = 0$, which cannot be satisfied in the case we are considering here. Thus, no new system can be found.

C Solution of the determining equations for Case II

Let us start by some preliminary considerations. By taking second-order derivatives with respect to x_2 of (4.4) and (4.5) we obtain the conditions

$$s_1''(x_2)U_2''(x_1) = 0 \quad \text{and} \quad s_1''(x_2)W'(x_1) = 0.$$

If $s_1'' \neq 0$, we therefore have $U_2'' = W' = 0$. As a consequence, new solutions can arise here only for $U_3''' \neq 0$ (i.e., for nonconstant magnetic field). If so, (7.5) imply $\beta_{33} = \beta_{31} = 0$. However then the compatibility conditions (4.8) cannot be solved for $s_1'' \neq 0$.

Thus, necessarily $s_1'' = 0$, which gives

$$s_1(x_2) = s_{11} + s_{12}x_2. \tag{C.1}$$

Let us proceed by looking at (7.5) in case $\beta_{33} \neq 0$. Then $U_2''' = 0$ and the third equation in (7.5) can be solved for U_3 . Then the remaining equations imply that the magnetic field and the potential W are constant, which is not of interest here. Thus, necessarily $\beta_{33} = 0$ and (C.1) holds.

C.1 $\alpha_{33} \neq 0$

By translation in both x_1 and x_2 we can set $\beta_{13} = \beta_{23} = 0$. The conditions (7.5) and (4.3) can be solved. We find

$$U_2(x_1) = \frac{a_1}{2x_1}, \quad U_3(x_1) = \frac{a_2}{2x_1}, \quad s_3(x_1) = 0, \quad s_2(x_1) = -s_{12}x_1, \quad a_1, a_2, s_{12} \in \mathbb{R},$$

together with $c_2 = \beta_{21} = \beta_{31} = \gamma_{13} = \gamma_{12} = 0$. This give the solution for S_j , $j = 1, 2, 3$. The compatibility conditions (4.8) can then be solved for W and by imposing also (4.5) give

$$W(x_1) = -\frac{a}{8x_1^4} + \frac{w}{x_1^2}, \quad a, w \in \mathbb{R}, \tag{C.2}$$

together with $s_{11} = s_{12} = \beta_{12} = a_2 = 0$, where $a \equiv a_1$. The magnetic field is

$$\vec{B}(\vec{x}) = (0, ax_1^{-3}, 0), \quad a \in \mathbb{R}. \tag{C.3}$$

The first-order equations (4.8) give

$$M(x_1, x_2) = \alpha_{33} \left(\frac{2wx_2^2}{x_1^2} - \frac{a^2x_2^2}{2x_1^4} \right)$$

with the condition $c_0 = c_1 = 0$. This system is a special case of the systems in Table 1, namely it can be expressed in the forms both \mathcal{E}_1 and \mathcal{E}_2 . The integral constructed here is included in the ones coming from \mathcal{E}_1 and \mathcal{E}_2 , i.e., the system is only minimally quadratically superintegrable.

From now on we continue our search by assuming $\alpha_{33} = 0$.

C.2 $\alpha_{33} = 0, \beta_{23} \neq 0$

By translation in x_2 we can set $\gamma_{12} = 0$. Then the second and fourth equation in (7.5) imply

$$U_2(x_1) = a_{11}x_1^2, \quad U_3(x_1) = a_{21}x_1^2 + a_{22}x_1^3, \quad a_{ij} \in \mathbb{R}.$$

From the compatibility conditions (4.8) and the zero-order equation (4.5) we easily see that the only possibility is polynomial potential at most quadratic and $a_{22} = 0$ (and $a_{21}a_{11} = 0$), i.e., constant magnetic field. Thus, this is not of interest here.

C.3 $\alpha_{33} = \beta_{23} = 0, \beta_{31} \neq 0$

Equation (7.4) reads

$$(\beta_{12}x_1 - \gamma_{13})U_2''(x_1) + 2\beta_{12}U_2'(x_1) - \beta_{31}U_3'(x_1) - c_2 = 0,$$

which, integrated by x_1 and neglecting integration constants (which do not affect the magnetic field), gives

$$U_3(x_1) = \frac{1}{\beta_{31}}((\beta_{12}x_1 - \gamma_{13})U_2'(x_1) + \beta_{12}U_2(x_1) + c_2x_1). \quad (\text{C.4})$$

Conditions (4.8) together with (4.4) and (4.5) seen as a polynomial in x_2 imply that unless $s_{12} = 0$ the magnetic field vanishes. Since this is of no interest here, we continue in the following by assuming $s_{12} = 0$. From the second-order equations (4.3), considered as polynomials in x_2 , we see that necessarily $s_3(x_1) = s_2(x_1) = 0$. We shall distinguish several subcases.

C.3.1 $\beta_{12} = \gamma_{13} = 0$

Let us start by the easiest case in which $\beta_{12} = \gamma_{13} = 0$.

By (C.4) we have here $U_3'' = 0$. Thus U_2'' cannot vanish, otherwise we have no magnetic field. From equations (4.3) we see that necessarily $\beta_{21} = 0$. At this point, all (4.3) are solved, except for the condition

$$(\beta_{13}x_1 + \gamma_{12})U_2'''(x_1) + (3\beta_{13} + \beta_{31})U_2''(x_1) = 0. \quad (\text{C.5})$$

The zero-order equation reads

$$\left(s_{11} + \frac{c_2\gamma_{12}}{\beta_{31}}\right)W'(x_1) = 0.$$

For $s_{11} \neq -\frac{c_2\gamma_{12}}{\beta_{31}}$ the above condition implies vanishing potential $W(x_1)$. From (4.4) we see that then also U_2'' is constant. Thus, this solution is of no interest here.

Thus, let us set $s_{11} = -\frac{c_2\gamma_{12}}{\beta_{31}}$. In this case the third equation in (4.4) is solved for $c_0 = -\frac{c_1c_2}{\beta_{31}}$. The remaining first-order equations can be solved for M and give

$$M(x_1, x_2) = (\beta_{13}x_1 + \gamma_{12})x_2W'(x_1)$$

for $W(x_1)$ satisfying

$$(W''(x_1) + U_2''(x_1)^2)(\beta_{13}x_1 + \gamma_{12}) + (c_1 - \beta_{31}U_2'(x_1))U_2''(x_1) + 3\beta_{13}W'(x_1) = 0. \quad (\text{C.6})$$

Recall that U_2 needs to satisfy (C.5), whose solution depends on the constants involved there. We can have

(a) $\beta_{13} = 0$. If $\gamma_{12} \neq 0$, (C.5) is solved by

$$U_2(x_1) = \frac{a}{b^2} e^{bx_1}, \quad a, b \in \mathbb{R}, \quad b = -\frac{\beta_{31}}{\gamma_{12}} \in \mathbb{R},$$

giving a new superintegrable system

$$\vec{B}(\vec{x}) = (0, ae^{bx_1}, 0), \quad W(\vec{x}) = wx_1 + ce^{bx_1} - \frac{1}{2} \frac{a^2}{b^2} e^{2bx_1}, \quad a, w \in \mathbb{R}, \quad (\text{C.7})$$

where $c = -\frac{ac_1\gamma_{12}}{\beta_{31}^2} \in \mathbb{R}$.

For $\gamma_{12} = 0$, equation (C.5) reduces to $\beta_{31}U_2''(x_1) = 0$, implying $U_2''(x_1) = 0$ and therefore no magnetic field.

(b) $\beta_{31} \neq -\beta_{13} \neq \frac{1}{2}\beta_{31}$, $\beta_{13} \neq 0$. By translation in x_1 we can set $\gamma_{12} = 0$. After the substitution $\beta_{31} = -b\beta_{13}$, the solution of (C.5) in this case reads

$$U_2(x_1) = \frac{ax_1^b}{b-1}, \quad a \in \mathbb{R}, \quad b \neq 0, 1, 2. \quad (\text{C.8})$$

By solving (C.6) for W , we arrive at a new system determined by

$$\vec{B}(\vec{x}) = (0, a(b-2)x_1^{b-3}, 0), \quad W(\vec{x}) = -\frac{a^2x_1^{2(b-2)}}{2} + a(b-2)cx_1^{b-2} + \frac{w}{x_1^2}, \quad (\text{C.9})$$

where $a, b, w, c = \frac{c_1}{b(2-b)\beta_{13}} \in \mathbb{R}$, $b \neq 0, 1, 2$.

Notice that in the limit $b \rightarrow 0$ we reduce (C.9) to the system determined by (C.2) and (C.3), that therefore can be seen as a special case of (C.9) (which is already known to have two second-order integrals, with highest-order terms $(l_3^A)^2$ and $p_1^A l_3^A$, that are however dependent once also the Cartesian integrals (1.9) and the Hamiltonian are taken into account).

(c) $\beta_{13} = -\frac{1}{2}\beta_{31}$. Since in this case $\beta_{13} = -\frac{1}{2}\beta_{31} \neq 0$, by translation in x_1 we can still set $\gamma_{12} = 0$. Equation (C.5) has solution

$$U_2(x_1) = ax_1 \ln|x_1|, \quad a \in \mathbb{R},$$

that gives the new system

$$\vec{B}(\vec{x}) = (0, ax_1^{-1}, 0), \quad W(\vec{x}) = -\frac{1}{2}a^2(\ln|x_1|)^2 + b \ln|x_1| + \frac{w}{x_1^2}, \quad a, w \in \mathbb{R}, \quad (\text{C.10})$$

where $b = \frac{ac_1}{\beta_{31}} - a^2$.

(d) $\beta_{13} = -\beta_{31}$. As above, we can still set $\gamma_{12} = 0$ by translation. Equation (C.5) admits the solution

$$U_2(x_1) = -a \ln|x_1|, \quad a \in \mathbb{R}. \quad (\text{C.11})$$

By denoting $c = -\frac{c_1 a}{\beta_{13}}$, from (C.6) and (C.11) we obtain

$$\vec{B}(\vec{x}) = (0, ax_1^{-2}, 0), \quad W(\vec{x}) = \frac{c}{x_1} + \frac{w}{x_1^2}, \quad a, c, w \in \mathbb{R}. \quad (\text{C.12})$$

Thus, though (C.8) is singular for $b = 1$ (and we obtain a different solution (C.11) for (C.5) if $b = 1$), indeed such a singularity does not appear in the solution for \vec{B} and W (it is lost by differentiating) and the system (C.9) becomes (C.12) for $b = 1$.

C.3.2 $\gamma_{13} \neq 0, \beta_{12} = 0$

In this case it is convenient to start by looking at the zeroth-order equation (4.5), that simplifies to

$$(\gamma_{13}((\beta_{13}x_1 + \gamma_{12})U_2''(x_1) - (\beta_{13} - \beta_{31}))U_2'(x_1) + s_{11}\beta_{31} + c_2\gamma_{12})W'(x_1) = 0. \quad (\text{C.13})$$

For $W' \neq 0$, we solve the above equation for U_2 . Once we plug the so found solution into the remaining equations we see that there can be several subcases according to different values of the constants involved. However, in most cases, the solution for U_2 is polynomial and at most of first order. By (C.4) also U_3 results into a polynomial of order at most one. Thus in these cases we have vanishing magnetic field. The only exception is the solution

$$\begin{aligned} U_2(x_1) &= \frac{2\gamma_{12}a^2(b_2^2 + \frac{1}{2})e^{b_1x_1} - ((2b_1b_2s_{11} + c_1)a + w_2b_1^2\gamma_{12})b_1x_1}{b_1^2\gamma_{12}a(2b_2^2 + 1)}, \\ \vec{B}(\vec{x}) &= (0, 1, b_2) \cdot ae^{b_1x_1}, \\ W(\vec{x}) &= -\frac{(b_2^2 + 1)}{2b_1^2}a^2e^{2b_1x_1} + w_2e^{b_1x_1} + x_1w_1, \\ M(x_1, x_2) &= -\frac{\gamma_{12}}{b_1}(a^2(b_2^2 + 1)e^{2b_1x_1} - w_2b_1^2e^{b_1x_1} - w_1b_1)x_2, \end{aligned} \quad (\text{C.14})$$

where $s_{12} = 0, \beta_{13} = 0, \beta_{21} = b_1b_2\gamma_{12}, \beta_{31} = -b_1\gamma_{12}, \gamma_{13} = b_2\gamma_{12}, c_0 = \gamma_{12}b_2w_1 + \frac{b_1^3\gamma_{12}w_2^2b_2^3}{a^2(2b_2^2+1)^2} - \frac{b_1b_2(2b_1b_2s_{11}+c_1)w_2}{a(2b_2^2+1)^2} + \frac{(b_1s_{11}-b_2c_1)(b_1b_2s_{11}+b_2^2c_1+c_1)}{b_1\gamma_{12}(2b_2^2+1)^2}$ and $c_2 = \frac{ab_1s_{11}-w_2b_1^2b_2\gamma_{12}-ab_2c_1}{a(2b_2^2+1)}$. By rotation of the coordinate frame around the x_1 -axis, which preserves the existence of the Cartesian integrals p_2 and p_3 , the system (C.14) can be brought to the form (C.7).

For $W' = 0$ the equation (C.13) does not give any condition on U_2 , that is therefore constrained only by the remaining second and first-order equations (4.3), (4.4). However, also in this case we always arrive to a solution for U_2 at most linear in x_1 , except for $U_2 = \frac{1}{b^2}(a_1 \sin(bx_1) + a_2 \cos(bx_1))$ that leads to the superintegrable system already found in [16], whose magnetic field has components

$$B_1 = 0, \quad B_2 = a_1 \sin(bx_1) + a_2 \cos(bx_1), \quad B_3 = a_2 \sin(bx_1) - a_1 \cos(bx_1),$$

where $a_1, a_2 \in \mathbb{R}$ and $b = \frac{\beta_{31}}{\gamma_{13}}$ (which must for this case be also equal to $b = \frac{\beta_{21}}{\gamma_{12}}$).

C.3.3 $\beta_{12} \neq 0$

By translation in x_1 , let us set $\gamma_{13} = 0$. We can proceed as above and, after a long and tedious computation, arrive at two solutions

- if W is not identically zero, we have

$$\begin{aligned} U_2(x_1) &= \frac{1}{b_1 - 1} \left(ax_1^{b_1-1} + \frac{b_1s_{11}}{b_2\beta_{13}} \right), \\ \vec{B}(\vec{x}) &= (0, 1, b_2) \cdot a(b_1 - 2)x_1^{b_1-3}, \\ W(\vec{x}) &= -\frac{a^2}{2}(b_2^2 + 1)x_1^{2(b_1-2)} + w_2x_1^{b_1-2} + \frac{w_1}{x_1^2}, \\ M(x_1, x_2) &= \beta_{13}x_2 \left((b_1 - 2)(w_2x_1^{b_1} - a^2(b_2^2 + 1)x_1^{2(b_1-2)}) - 2\frac{w_1}{x_1^2} \right), \end{aligned} \quad (\text{C.15})$$

where $c_0 = -\frac{c_2}{a}w_2 - \frac{(b_1-1)b_2c_2^2}{b_1^2\beta_{13}}$, $c_1 = -2(b_1-1)\frac{b_2}{b_1}c_2 - \frac{b_1}{a}\beta_{13}w_2$, $s_{12} = 0$, $\beta_{21} = b_1b_2\beta_{13}$, $\beta_{12} = -b_2\beta_{13}$, $\beta_{31} = -b_1\beta_{13}$ and $\gamma_{12} = 0$. By rotation of the coordinate frame around the x_1 -axis, which preserves the existence of the Cartesian integrals p_2 and p_3 , the system (C.15) can be brought to the form (C.9).

- For $W(\vec{x}) = 0$ we find

$$U_2(x_1) = -a \ln |x_1|, \quad \vec{B}(\vec{x}) = (0, 1, b) \cdot \frac{a}{x_1^2}, \quad W(\vec{x}) = 0, \quad M(x_1, x_2) = 0,$$

where $c_0 = 0$, $c_1 = 0$, $s_{11} = -ab\beta_{31}$, $\beta_{13} = -\beta_{31}$, $\beta_{21} = -b\beta_{31}$ and $\gamma_{12} = 0$. However, this solution can be obtained as a limiting case of (C.15) with $b_1 = 1$, $b_2 = b$, $w_2 = 0$ and $w_1 = \frac{a^2(b_2^2+1)}{2}$.

C.4 $\alpha_{33} = \beta_{23} = \beta_{31} = 0$

Equation (7.4) now reads

$$(\beta_{12}x_1 - \gamma_{13})U_2''(x_1) + 2\beta_{12}U_2'(x_1) - c_2 = 0. \quad (\text{C.16})$$

Thus, it does not imply any relationship between U_2 and U_3 . It only gives a condition on U_2 that could be trivially satisfied or not, according to the values of the constants involved. By the same argument used in Appendix C.3, we see that also in this case $s_{12} = 0$ and $s_3(x_1) = s_2(x_1) = 0$.

C.4.1 $\beta_{21} \neq 0$

Let us look at the situation in which $\beta_{21} \neq 0$ and recall that we are looking for an integral X of the form (7.1). Its Poisson bracket with $X_1 = p_2$, i.e., $\{X, p_2\}$, is again an integral, by assumption expressible in terms of the known integrals, similarly to equation (7.1). However, these two expressions can be interchanged under the permutation of x_2 and x_3 ; thus we can reduce the considered problem into an already discussed one. We have that

$$(p_1, p_2, p_3, l_1, l_2, l_3) \rightarrow (p_1, p_3, p_2, -l_1, -l_3, -l_2),$$

while the system (1.7) transforms into

$$W(x_1) \rightarrow W(x_1), \quad (u_2, u_3) \rightarrow (-u_3, u_2). \quad (\text{C.17})$$

Therefore, all the conditions imposed in this section (including $\beta_{21} \neq 0$) on the coefficients of the second-order terms in the integral change into $\alpha_{22} = \beta_{32} = \beta_{21} = \beta_{22} = 0$, $\beta_{31} \neq 0$ and (7.2) turns into $a_0 = -\beta_{13}$, $a_1 = 0$, $a_2 = \beta_{31}$, $a_3 = \beta_{21}$ and

$$\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{23} = \alpha_{33} = \beta_{11} = \beta_{33} = \beta_{23} = 0.$$

Thus we recover all the conditions imposed on our parameters in Appendix C.3. By (C.17) any system belonging into this section would transform into some system already found in Appendix C.3. Therefore in the following we can assume that $\beta_{21} = 0$ without loss of generality.

We distinguish several subcases depending on whether (C.16) is trivially satisfied or not.

C.4.2 $\beta_{21} = 0$, $\beta_{12} \neq 0$

By translation in x_1 we can set $\gamma_{13} = 0$. Equation (C.16) gives

$$U_2(x_1) = \frac{a_1}{2x_1} + \frac{c_2}{2\beta_{12}}x_1, \quad a_1 \in \mathbb{R}.$$

We look at the remaining second-order equations (4.3), that read

$$(\beta_{13}x_1 + \gamma_{12})U_3'''(x_1) + 3\beta_{13}U_3''(x_1) = 0, \quad \beta_{12}x_1^5U_3'''(x_1) + 3\beta_{12}x_1^4U_3''(x_1) - 3\gamma_{12}a_1 = 0.$$

If $\beta_{13} = 0$ and $\gamma_{12} \neq 0$ they have a solution only if $a_1 = 0$ and $U_3 = 0$, i.e., the magnetic field vanishes.

If $\beta_{13} = \gamma_{12} = 0$, the above equations admit solution for

$$U_3(x_1) = \frac{a_2}{x_1}.$$

The zeroth-order equation (4.5) has solution for $W(x_1) = 0$. If $\beta_{12} \neq 0$ the remaining equations (4.4) lead to the solution

$$\begin{aligned} U_2(x_1) &= 0, & U_3(x_1) &= \frac{a}{2x_1}, & \vec{B}(\vec{x}) &= \left(0, 0, \frac{a}{x_1^3}\right), \\ W(\vec{x}) &= -\frac{ab \ln|x_1|}{x_1^2} - \frac{a^2}{8x_1^4} + \frac{w}{x_1^2}, \end{aligned} \quad (\text{C.18})$$

where $a_1 = 0$, $c_0 = 0$, $c_1 = 2\beta_{12}b$, $c_2 = 0$, $s_{11} = 0$, $\beta_{13} = 0$, $\gamma_{12} = 0$ and $M(x_1, x_2) = 0$. This system is new when $b \neq 0$, otherwise it can be turned into a subcase of (C.9) by permutation of x_2 and x_3 .

If $\beta_{13} \neq 0$ we can set $\gamma_{12} = 0$. We have a solution for U_3 given by

$$U_3(x_1) = \frac{4a_2}{\beta_{13}x_1}, \quad b \in \mathbb{R}.$$

Equations (4.4) and (4.5) can be solved for W and M under the conditions $s_{11} = 0$ and $a_2 = -\frac{a_1\beta_{12}}{8}$. We find

$$M(x_1, x_2) = \beta_{13}(b^2 + 1)a^2 \frac{x_2}{2x_1^4} - \beta_{13}c \frac{2 \ln|x_1| - 1}{x_1^2} - 2\beta_{13}w \frac{x_2}{x_1^2}$$

and

$$W(x_1) = -\frac{(b^2 + 1)a^2}{8x_1^4} + \frac{c \ln|x_1| + w}{x_1^2}, \quad w \in \mathbb{R}, \quad (\text{C.19})$$

where $b = -\frac{\beta_{12}}{\beta_{13}} \neq 0$, $a = a_1$, $c = \frac{ac_1}{2\beta_{13}}$ and by using the physically irrelevant shift of the potential by an additive constant we set $c_0 = c_2 = 0$.

The magnetic field reads

$$\vec{B}(\vec{x}) = (0, 1, b) \cdot ax_1^{-3}, \quad a, b \in \mathbb{R}, \quad b \neq 0. \quad (\text{C.20})$$

However, the reference frame can be rotated around x_1 without affecting the fact that p_2 and p_3 are integrals. Such a rotation brings the magnetic field (C.20) and the potential (C.19) and the integral into the form (C.18).

C.4.3 $\beta_{21} = \beta_{12} = 0$, $\gamma_{13} \neq 0$

From equation (C.16) we have

$$U_2(x_1) = -\frac{c_2x_1^2}{2\gamma_{13}}.$$

From the second and zeroth-order equations we get

$$U_3(x_1) = ax_1^2, \quad a \in \mathbb{R}.$$

The first-order equations can be solved for W and give a polynomial solution at most quadratic. Since the magnetic field is constant, nothing of interest can be found here.

C.4.4 $\beta_{21} = \beta_{12} = \gamma_{13} = 0$

Equation (C.16) implies $c_2 = 0$ and it is trivially satisfied for U_2 . The remaining second-order equations read

$$\beta_{13}(3U_j''(x_1) + x_1U_j'''(x_1)) + \gamma_{12}U_j'''(x_1) = 0, \quad j = 2, 3.$$

If $\beta_{13} = 0$ we easily conclude that the only possibility is constant magnetic field and vanishing potential. For $\beta_{13} \neq 0$ we can set $\gamma_{12} = 0$ by translation in x_1 . The above equations imply

$$U_j(x_1) = \frac{a_j}{2x_1}, \quad j = 2, 3.$$

From the first-order equations, seen as polynomials in x_2 , we get the conditions

$$\frac{a_2^2}{x_1^5} + \frac{a_3^2}{x_1^5} - 2W'(x_1) = 0, \quad a_2(2a_3\beta_{13} + 3s_{11}x_1) + c_1a_3x_1^2 = 0,$$

together with two differential equations for M . From the second equation above we see that we have two possibilities for nonvanishing magnetic field: $a_2 = c_1 = 0$ or $a_3 = s_{11} = 0$. In the first case, when imposing also the remaining zero-order equation (4.5) we get that necessarily $a_3 = 0$, i.e., the magnetic field vanishes. In the second case we have the solution

$$W(x_1) = \frac{c \ln|x_1| + w}{x_1^2} - \frac{a^2}{8x_1^4}, \quad a, c, w \in \mathbb{R},$$

with $c = \frac{ac_1}{2\beta_{13}}$, $a = a_2$. The magnetic field reads

$$\vec{B}(\vec{x}) = (0, ax_1^{-3}, 0), \quad a \in \mathbb{R}.$$

By solving the remaining equations for M , we find

$$M(x_1, x_2) = a^2\beta_{13}\frac{x_2}{2x_1^4} - 2\beta_{13}c\ln|x_1|\frac{x_2}{x_1^2} + \beta_{13}(c - 2w)\frac{x_2}{x_1^2}.$$

However, this system is just a special case of (C.19), (C.20) for $b = 0$.

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References

- [1] Benenti S., Chanu C., Rastelli G., Variable separation for natural Hamiltonians with scalar and vector potentials on Riemannian manifolds, *J. Math. Phys.* **42** (2001), 2065–2091.
- [2] Bertrand S., Šnobl L., On rotationally invariant integrable and superintegrable classical systems in magnetic fields with non-subgroup type integrals, *J. Phys. A: Math. Theor.* **52** (2019), 195201, 25 pages, [arXiv:1812.09399](https://arxiv.org/abs/1812.09399).
- [3] Bérubé J., Winternitz P., Integrable and superintegrable quantum systems in a magnetic field, *J. Math. Phys.* **45** (2004), 1959–1973, [arXiv:math-ph/0311051](https://arxiv.org/abs/math-ph/0311051).
- [4] Charest F., Hudon C., Winternitz P., Quasiseparation of variables in the Schrödinger equation with a magnetic field, *J. Math. Phys.* **48** (2007), 012105, 16 pages, [arXiv:math-ph/0502046](https://arxiv.org/abs/math-ph/0502046).
- [5] Dorizzi B., Grammaticos B., Ramani A., Winternitz P., Integrable Hamiltonian systems with velocity-dependent potentials, *J. Math. Phys.* **26** (1985), 3070–3079.

-
- [6] Escobar-Ruiz A.M., López Vieyra J.C., Winternitz P., Fourth order superintegrable systems separating in polar coordinates. I. Exotic potentials, *J. Phys. A: Math. Theor.* **50** (2017), 495206, 35 pages, [arXiv:1706.08655](#).
- [7] Escobar-Ruiz A.M., Winternitz P., Yurduşen I., General N th-order superintegrable systems separating in polar coordinates, *J. Phys. A: Math. Theor.* **51** (2018), 40LT01, 12 pages, [arXiv:1806.06849](#).
- [8] Escobar-Ruiz M.A., Turbiner A.V., Two charges on a plane in a magnetic field: special trajectories, *J. Math. Phys.* **54** (2013), 022901, 15 pages, [arXiv:1208.2995](#).
- [9] Evans N.W., Superintegrability in classical mechanics, *Phys. Rev. A* **41** (1990), 5666–5676.
- [10] Evans N.W., Verrier P.E., Superintegrability of the caged anisotropic oscillator, *J. Math. Phys.* **49** (2008), 092902, 10 pages, [arXiv:0808.2146](#).
- [11] Heinzl T., Ilderton A., Superintegrable relativistic systems in spacetime-dependent background fields, *J. Phys. A: Math. Theor.* **50** (2017), 345204, 14 pages, [arXiv:1701.09168](#).
- [12] Kotkin G.L., Serbo V.G., Collection of problems in classical mechanics, *International Series in Natural Philosophy*, Vol. 31, [Pergamon Press](#), 1971.
- [13] Makarov A.A., Smorodinsky J.A., Valiev K., Winternitz P., A systematic search for nonrelativistic systems with dynamical symmetries, *Nuovo Cimento A* **52** (1967), 1061–1084.
- [14] Marchesiello A., Šnobl L., Superintegrable 3D systems in a magnetic field corresponding to Cartesian separation of variables, *J. Phys. A: Math. Theor.* **50** (2017), 245202, 24 pages.
- [15] Marchesiello A., Šnobl L., An infinite family of maximally superintegrable systems in a magnetic field with higher order integrals, *SIGMA* **14** (2018), 092, 11 pages, [arXiv:1804.03039](#).
- [16] Marchesiello A., Šnobl L., Winternitz P., Three-dimensional superintegrable systems in a static electromagnetic field, *J. Phys. A: Math. Theor.* **48** (2015), 395206, 24 pages, [arXiv:1507.04632](#).
- [17] Marchesiello A., Šnobl L., Winternitz P., Spherical type integrable classical systems in a magnetic field, *J. Phys. A: Math. Theor.* **51** (2018), 135205, 24 pages.
- [18] Marquette I., Superintegrability with third order integrals of motion, cubic algebras, and supersymmetric quantum mechanics. I. Rational function potentials, *J. Math. Phys.* **50** (2009), 012101, 23 pages, [arXiv:0807.2858](#).
- [19] Marquette I., Superintegrability with third order integrals of motion, cubic algebras, and supersymmetric quantum mechanics. II. Painlevé transcendent potentials, *J. Math. Phys.* **50** (2009), 095202, 18 pages, [arXiv:0811.1568](#).
- [20] Marquette I., Sajedi M., Winternitz P., Fourth order superintegrable systems separating in Cartesian coordinates I. Exotic quantum potentials, *J. Phys. A: Math. Theor.* **50** (2017), 315201, 29 pages, [arXiv:1703.09751](#).
- [21] Marquette I., Sajedi M., Winternitz P., Two-dimensional superintegrable systems from operator algebras in one dimension, *J. Phys. A: Math. Theor.* **52** (2019), 115202, 27 pages, [arXiv:1810.05793](#).
- [22] Miller Jr. W., Post S., Winternitz P., Classical and quantum superintegrability with applications, *J. Phys. A: Math. Theor.* **46** (2013), 423001, 97 pages, [arXiv:1309.2694](#).
- [23] Post S., Winternitz P., General N th order integrals of motion in the Euclidean plane, *J. Phys. A: Math. Theor.* **48** (2015), 405201, 24 pages, [arXiv:1501.00471](#).
- [24] Pucacco G., On integrable Hamiltonians with velocity dependent potentials, *Celestial Mech. Dynam. Astronom.* **90** (2004), 111–125.
- [25] Pucacco G., Rosquist K., Integrable Hamiltonian systems with vector potentials, *J. Math. Phys.* **46** (2005), 012701, 25 pages, [arXiv:nlin.SI/0405065](#).
- [26] Rodríguez M.A., Tempesta P., Winternitz P., Reduction of superintegrable systems: the anisotropic harmonic oscillator, *Phys. Rev. E* **78** (2008), 046608, 6 pages, [arXiv:0807.1047](#).
- [27] Shapovalov V.N., Bagrov V.G., Meshkov A.G., Separation of variables in the stationary Schrödinger equation, *Soviet Phys. J.* **15** (1972), 1115–1119.
- [28] Taut M., Two electrons in a homogeneous magnetic field: particular analytical solutions, *J. Phys. A: Math. Gen.* **27** (1994), 1045–1055.
- [29] Taut M., Two particles with opposite charge in a homogeneous magnetic field: particular analytical solutions of the two-dimensional Schrödinger equation, *J. Phys. A: Math. Gen.* **32** (1999), 5509–5515.
- [30] Turbiner A.V., Escobar-Ruiz M.A., Two charges on a plane in a magnetic field: hidden algebra, (particular) integrability, polynomial eigenfunctions, *J. Phys. A: Math. Theor.* **46** (2013), 295204, 15 pages, [arXiv:1303.2345](#).