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Spinorially Twisted Spin Structures. II: Twisted Pure Spinors, Special Riemannian Holonomy and Clifford Monopoles

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Abstract. We introduce a notion of twisted pure spinor in order to characterize, in a unified way, all the special Riemannian holonomy groups just as a classical pure spinor characterizes the special Kähler holonomy. Motivated by certain curvature identities satisfied by manifolds admitting parallel twisted pure spinors, we also introduce the Clifford monopole equations as a natural geometric generalization of the Seiberg–Witten equations. We show that they restrict to the Seiberg–Witten equations in 4 dimensions, and that they admit non-trivial solutions on manifolds with special Riemannian holonomy.

Key words: twisted spinor; pure spinor; parallel spinor; special Riemannian holonomy; Clifford monopole

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1 Introduction

The Berger–Simons' theorem [5, 37] states that the holonomy group of an irreducible non-locally symmetric oriented Riemannian manifold is contained in one of the groups in Table 1 (see [18, 35] for extensive accounts on the theory of Riemannian holonomy).

group	geometry
$\mathrm{SO}(m)$	generic
$\mathrm{U}(m)$	Kähler
$\mathrm{SU}(m)$	Calabi–Yau
$\mathrm{Sp}(m)\mathrm{Sp}(1)$	quaternion-Kähler
$\mathrm{Sp}(m)$	hyper-Kähler
$\mathrm{Spin}(7)$	exceptional
G_2	exceptional

Table 1. Special Riemannian holonomy groups and geometries.

Manifolds whose holonomies are contained in

$$\mathrm{SU}(m),\quad \mathrm{Sp}(m),\quad \mathrm{Spin}(7),\quad G_2$$

are known to be Ricci-flat, Spin and to carry parallel spinors for their classical (untwisted) Spin structures [16, 39]. In the cases of $\mathrm{SU}(m)$, $\mathrm{Spin}(7)$ and G_2, such classical parallel spinors characterize the holonomies [20]. There exist, however, manifolds with holonomies contained in $\mathrm{U}(n)$ and $\mathrm{Sp}(n)\mathrm{Sp}(1)$ which are not Spin, but do admit natural Spin^c and Spin^q structures respectively.

The purpose of this note is to introduce a suitable notion of twisted pure spinor for manifolds admitting spinorially twisted Spin structures [10] in order to give a unified treatment of special Riemannnian holonomies. While there have been other unifying efforts involving the normed division algebras (e.g., [23]), our approach is centered on Clifford algebras and twisted spinors.

The main motivation for our work has been the relationship between (classical) pure spinors and complex structures. More precisely, let $\mathbb{R}^n$ and Δ_n denote the (real and complex) standard representations of $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ respectively. É. Cartan defined (classical) pure spinors in terms of maximal isotropic subspaces [8]. Equivalently, a spinor $\phi \in \Delta_n$ is pure if for every vector $X \in \mathbb{R}^n$ there exists a vector $Y \in \mathbb{R}^n$ satisfying the following equation [20]

$$X \cdot \phi = \mathrm{i} Y \cdot \phi, \tag{1.1}$$

where "$\cdot$" denotes Clifford multiplication. This condition says that the two subspaces $\mathbb{R}^n \cdot \phi$ and $\mathrm{i}\mathbb{R}^n \cdot \phi$ of Δ_n coincide, which allows the transfer of the effect of multiplication by $\mathrm{i} = \sqrt{-1}$ within the complex space Δ_n to the real vector space $\mathbb{R}^n$. Indeed, setting $Y = J(X)$ in (1.1) and a little algebraic manipulation show that J defines a complex structure on $\mathbb{R}^n$. The isotropy group of a pure spinor is isomorphic to $\mathrm{SU}(n/2)$ and, therefore, a spin manifold admits a parallel pure spinor field if and only if it is special Kähler [20]. Furthermore, A. Moroianu proved that a Spin^c manifold admits a (similarly defined) parallel pure spinor field if and only if it is Kähler [25], a result which includes non-Spin non-Ricci-flat Kähler manifolds.

Now recall that the tangent spaces of quaternionic Kähler manifolds and 8-manifolds with $\mathrm{Spin}(7)$ holonomy are representation spaces of $\mathfrak{sp}(1) \cong \mathfrak{spin}(3)$ and $\mathfrak{spin}(7)$ respectively, which can be viewed as restrictions of representations of the even Clifford algebras Cl_3^0 and Cl_7^0 to such subalgebras. Thus, we conjectured that the special Riemannian holonomies must be determined by twisted spinors which, somehow, induce a *transfer* of algebraic structure from an even Clifford algebra to the bundle of endomorphisms of the tangent spaces of the manifold (see [36]). More precisely, let M be a smooth Riemannian manifold and F be an auxiliary Riemannian vector bundle of rank r. Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_r)$ be local orthonormal frames of TM and F respectively, $S(TM)$ and $S(F)$ be the locally defined spinor vector bundles of M and F, and suppose $m \in \mathbb{N}$ is such that the bundle $S(TM) \otimes S(F)^{\otimes m}$ is globally defined. A spinor field $\phi \in \Gamma\big(S(TM) \otimes S(F)^{\otimes m}\big)$ determines maps

$$\begin{aligned} T_xM &\longrightarrow T_xM \cdot \phi_x \subset S(T_xM) \otimes S(F_x)^{\otimes m}, \\ T_xM &\longrightarrow T_xM \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x \subset S(T_xM) \otimes S(F_x)^{\otimes m}, \end{aligned}$$

at $x \in M$, for all $1 \leq k < l \leq r$, where κ_{r*}^m is the induced representation of $\mathfrak{spin}(r)$ on $S(F)^{\otimes m}$. Given a pair $k < l$, we have a projection map

$$\begin{aligned} &T_xM \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x \longrightarrow T_xM \cdot \phi_x, \\ &X \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x, \longmapsto, \sum_{j=1}^n \mathrm{Re}\,\big\langle X \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x, e_j \cdot \phi_x \big\rangle e_j \cdot \phi_x, \end{aligned}$$

which, in turn, gives the map

$$\begin{aligned} &T_xM \longrightarrow T_xM, \\ &X \longmapsto \sum_{j=1}^n \mathrm{Re}\,\big\langle X \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x, e_j \cdot \phi_x \big\rangle e_j. \end{aligned}$$

The conjectured transfer of algebraic structure from the even Clifford algebra Cl_r^0 to $\mathrm{End}(TM)$ must then be encoded in these maps. Thus, we will define twisted pure spinors in such a way that the local 2-forms and endomorphisms

$$\eta_{kl}^{\phi}(X,Y) = \mathrm{Re}\langle X \wedge Y \cdot \kappa_{r*}^{m}(f_k f_l) \cdot \phi, \phi\rangle,$$
$$\hat{\eta}_{kl}^{\phi}(X) = (X \lrcorner \eta_{kl}^{\phi})^{\sharp},$$

$X, Y \in T_xM$, induce a non-trivial representation of Cl_r^0 on T_xM, where $\sharp$ denotes metric dualization of a 1-form. Moreover, by assuming the spinor field to be parallel (given a choice of connection θ on F), the induced almost even-Clifford Hermitian structure will also be parallel, and we will be able to identify the special Riemannian holonomies from Berger's list. Since all of our considerations hinge on the existence of such special spinors, we give explicit representatives for the ranks $r = 3, 7$.

Note that there has been interest on various Clifford-type structures on manifolds (beyond the quaternionic ones) for quite some time (see [1, 2, 3, 4, 6, 7, 9, 11, 13, 14, 15, 17, 21, 22, 26, 27, 30, 31, 32, 33, 34, 38]). The parallel even-Clifford Hermitian structure resulting from our spinorial construction corresponds (with a minor difference) to the notion of a parallel even Clifford structure introduced in [27]. In particular, Moroianu and Semmelmann studied the relationship between parallel even Clifford structures and special Riemannian holonomy groups, with the exception of G_2. While carrying out the relevant curvature calculations (as in [27]), we noticed that the existence of a parallel twisted pure spinor implies an identity between the curvature of the connection θ and a 2-form with values in $\bigwedge^2 F$ associated to the spinor (see below), much in the same way as the self-dual part of the U(1)-connection is related to the 2-form associated to a positive spinor in the Seiberg–Witten monopole equations on 4-manifolds (see [24, 29, 40]). Thus, we introduce the *Clifford monopole equations*.

Let M be a Spin^r manifold with auxiliary bundle $P_{\mathrm{SO}(r)}$ endowed with a connection θ, F the associated Riemannian rank r vector bundle and $m \in \mathbb{N}$ be such that the twisted Dirac operator $\partial\!\!\!/^{\theta}\colon \Gamma\big(S(M)\otimes S(F)^{\otimes m}\big) \longrightarrow \Gamma\big(S(M)\otimes S(F)^{\otimes m}\big)$ is well defined. The *Clifford monopole equations* are

$$\partial\!\!\!/^{\theta}\phi = 0, \qquad \Theta = E\big(\eta^{\phi}\big),$$

where $\Theta \in \Gamma\big(\bigwedge^2 T^*M \otimes \bigwedge^2 F\big)$ is the curvature of θ,

$$\eta^{\phi} = \sum_{1\le k<l\le r} \eta_{kl}^{\phi} \otimes f_{kl} \in \Gamma\big(\textstyle\bigwedge^2 T^*M \otimes \bigwedge^2 F\big)$$

is the 2-form with values in $\bigwedge^2 F$ associated to ϕ, and E is an endomorphism of 2-forms. We will argue that, for suitable choices of the parameters, such equations are a natural geometric generalization of the Seiberg–Witten equations on 4-manifolds. Indeed, we will show that they restrict to the Seiberg–Witten equations on 4-manifolds, and will exhibit non-trivial solutions (involving parallel twisted pure spinors) on manifolds with special Riemannian holonomy. Preliminary work (with A. Quintero and to be published elsewhere) indicates the existence of a smooth compact moduli space which, according to the Mathai–Quillen–Atiyah–Jeffrey formalism, will give raise to a topological quantum field theory. Such topological field theory, at least in dimensions 8, might turn out to be a topological twist of an $N = 2$ supersymmetric theory.

The paper is organized as follows. In Section 2, we recall Clifford algebras, twisted spin groups, representations and structures. In Section 3, we define twisted pure spinors, deduce their relevant properties and show explicit representatives. In Section 4, we characterize the special Riemannian holonomies by the existence of parallel twisted pure spinor fields. In Section 5, we show that the Clifford monopole equations restrict to the Seiberg–Witten equations on 4-manifolds, and exhibit solutions on manifolds with special Riemannian holonomy.

2 Preliminaries

In this section, we recall various concepts from [10, 12, 20] that will be used throughout.

2.1 Clifford algebras, twisted spin groups and representations

2.1.1 Clifford algebra

Let Cl_n denote the Clifford algebra generated by the orthonormal vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

and $\mathbb{C}\mathrm{l}_n = \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. The even Clifford subalgebra Cl_n^0 is defined as the invariant (+1)-subspace of the involution of Cl_r induced by the map $-\mathrm{Id}_{\mathbb{R}^r}$. We have

$$\mathbb{C}\mathrm{l}_n \cong \begin{cases} \mathrm{End}\left(\mathbb{C}^{2^k}\right), & \text{if } n = 2k, \\ \mathrm{End}\left(\mathbb{C}^{2^k}\right) \oplus \mathrm{End}\left(\mathbb{C}^{2^k}\right), & \text{if } n = 2k+1. \end{cases}$$

The space of spinors is defined as

$$\Delta_n := \mathbb{C}^{2^k} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{k \text{ times}}.$$

The map

$$\kappa\colon\ \mathbb{C}\mathrm{l}_n \longrightarrow \mathrm{End}(\Delta_n)$$

is defined to be either the above mentioned isomorphism for n even, or the isomorphism followed by the projection onto the first summand for n odd. In order to make κ explicit, consider the following matrices

$$\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

In terms of the generators $e_1, \ldots, e_n$ of the Clifford algebra, κ can be described explicitly as follows,

$$\begin{aligned}
e_1 &\mapsto \mathrm{Id} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes g_1, \\
e_2 &\mapsto \mathrm{Id} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes g_2, \\
e_3 &\mapsto \mathrm{Id} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes g_1 \otimes T, \\
e_4 &\mapsto \mathrm{Id} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes g_2 \otimes T, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
e_{2k-1} &\mapsto g_1 \otimes T \otimes \cdots \otimes T \otimes T \otimes T, \\
e_{2k} &\mapsto g_2 \otimes T \otimes \cdots \otimes T \otimes T \otimes T,
\end{aligned}$$

and, if $n = 2k+1$,

$$e_{2k+1} \mapsto \mathrm{i}T \otimes T \otimes \cdots \otimes T \otimes T \otimes T.$$

The vectors

$$u_{+1} = \frac{1}{\sqrt{2}}(1, -\mathrm{i}) \qquad \text{and} \qquad u_{-1} = \frac{1}{\sqrt{2}}(1, \mathrm{i}),$$

form a unitary basis of $\mathbb{C}^2$ with respect to the standard Hermitian product. Thus,

$$\mathcal{B} := \{u_{(\varepsilon_1,\ldots,\varepsilon_k)} = u_{\varepsilon_1} \otimes \cdots \otimes u_{\varepsilon_k} \,|\, \varepsilon_j = \pm 1,\ j = 1, \ldots, k\}$$

is a unitary basis of $\Delta_n = \mathbb{C}^{2^k}$ with respect to the naturally induced Hermitian product.

Remark 2.1. We will denote inner and Hermitian products (as well as Riemannian and Hermitian metrics) by the same symbol $\langle\cdot,\cdot\rangle$ trusting that the context will make clear which product is being used.

By means of κ we have Clifford multiplication

$$\begin{aligned} \mu_n\colon\ & \mathbb{R}^n \otimes \Delta_n \longrightarrow \Delta_n, \\ & x \otimes \phi \mapsto \mu_n(x \otimes \phi) = x \cdot \phi := \kappa(x)(\phi). \end{aligned}$$

μ_n is skew-symmetric with respect to the Hermitian product

$$\langle x \cdot \phi_1, \phi_2 \rangle = \langle \mu_n(x \otimes \phi_1), \phi_2 \rangle = -\langle \phi_1, \mu_n(x \otimes \phi_2) \rangle = -\langle \phi_1, x \cdot \phi_2 \rangle, \tag{2.1}$$

and can be extended to a map

$$\begin{aligned} \mu_n\colon\ & \textstyle\bigwedge^*(\mathbb{R}^n) \otimes \Delta_n \longrightarrow \Delta_n, \\ & \omega \otimes \psi \mapsto \omega \cdot \psi. \end{aligned}$$

When n is even, we define the following involution

$$\begin{aligned} & \Delta_n \longrightarrow \Delta_n, \\ & \psi \mapsto (-\mathrm{i})^{\frac{n}{2}} \mathrm{vol}_n \cdot \psi. \end{aligned}$$

The ± 1 eigenspace of this involution is denoted $\Delta_n^\pm$. These spaces have equal dimension and are irreducible representations of $\mathrm{Spin}(n)$.

There exist real or quaternionic structures on the spin representations. A quaternionic structure α on $\mathbb{C}^2$ is given by

$$\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\overline{z}_2 \\ \overline{z}_1 \end{pmatrix},$$

and a real structure β on $\mathbb{C}^2$ is given by

$$\beta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \end{pmatrix}.$$

The real and quaternionic structures γ_n on $\Delta_n = (\mathbb{C}^2)^{\otimes[n/2]}$ are built as follows

$$\begin{aligned} \gamma_n &= (\alpha \otimes \beta)^{\otimes 2k} && \text{if } n = 8k, 8k+1 && \text{(real)}, \\ \gamma_n &= \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k} && \text{if } n = 8k+2, 8k+3 && \text{(quaternionic)}, \\ \gamma_n &= (\alpha \otimes \beta)^{\otimes 2k+1} && \text{if } n = 8k+4, 8k+5 && \text{(quaternionic)}, \\ \gamma_n &= \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k+1} && \text{if } n = 8k+6, 8k+7 && \text{(real)}. \end{aligned}$$

2.1.2 Spin group and representation

The Spin group $\mathrm{Spin}(n) \subset \mathrm{Cl}_n$ is the subset

$$\mathrm{Spin}(n) = \big\{x_1 x_2 \cdots x_{2l-1} x_{2l} \,|\, x_j \in \mathbb{R}^n,\, |x_j| = 1,\, l \in \mathbb{N}\big\},$$

endowed with the product of the Clifford algebra. It is a Lie group and its Lie algebra is

$$\mathfrak{spin}(n) = \mathrm{span}\{e_i e_j \,|\, 1 \le i < j \le n\}.$$

The restriction of κ to $\mathrm{Spin}(n)$ defines the Lie group representation

$$\kappa_n := \kappa|_{\mathrm{Spin}(n)}\colon\ \mathrm{Spin}(n) \longrightarrow \mathrm{GL}(\Delta_n),$$

which is, in fact, special unitary. We have the corresponding Lie algebra representation

$$\kappa_{n_*}\colon\ \mathfrak{spin}(n) \longrightarrow \mathfrak{gl}(\Delta_n).$$

Both representations can be extended to tensor powers $\kappa^m_{n*}\colon \mathfrak{spin}(n) \longrightarrow \mathrm{End}\big(\Delta_n^{\otimes m}\big)$, $m \in \mathbb{N}$, in the usual way. Recall that the Spin group $\mathrm{Spin}(n)$ is the universal double cover of $\mathrm{SO}(n)$, $n \ge 3$. For $n = 2$ we consider $\mathrm{Spin}(2)$ to be the connected double cover of $\mathrm{SO}(2)$. The covering map will be denoted by

$$\lambda_n\colon\ \mathrm{Spin}(n) \to \mathrm{SO}(n) \subset \mathrm{GL}\big(\mathbb{R}^n\big).$$

Its differential is given by $\lambda_{n_*}(e_i e_j) = 2E_{ij}$, where $E_{ij} = e_i^* \otimes e_j - e_j^* \otimes e_i$ is the standard basis of the skew-symmetric matrices, and e^* denotes the metric dual of the vector e. Furthermore, we will abuse the notation and also denote by λ_n the induced representation on the exterior algebra $\bigwedge^* \mathbb{R}^n$. Note that Clifford multiplication μ_n is an equivariant map of $\mathrm{Spin}(n)$ representations.

Now, we summarize some results about real representations of Cl_r^0 in Table 2 (see [20]). Here d_r denotes the dimension of an irreducible representation of Cl_r^0 and v_r the number of distinct irreducible representations. Let $\tilde{\Delta}_r$ denote the irreducible representation of Cl_r^0 for $r \not\equiv 0 \pmod 4$ and $\tilde{\Delta}_r^\pm$ denote the irreducible representations for $r \equiv 0 \pmod 4$.

$r \pmod 8$	d_r	Cl_r^0	$\tilde{\Delta}_r$ / $\tilde{\Delta}_r^\pm$	v_r
1	$2^{\lfloor \frac{r}{2} \rfloor}$	$\mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	1
2	$2^{\frac{r}{2}}$	$\mathbb{C}(d_r/2)$	$\mathbb{C}^{d_r/2}$	1
3	$2^{\lfloor \frac{r}{2} \rfloor + 1}$	$\mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	1
4	$2^{\frac{r}{2}}$	$\mathbb{H}(d_r/4) \oplus \mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	2
5	$2^{\lfloor \frac{r}{2} \rfloor + 1}$	$\mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	1
6	$2^{\frac{r}{2}}$	$\mathbb{C}(d_r/2)$	$\mathbb{C}^{d_r/2}$	1
7	$2^{\lfloor \frac{r}{2} \rfloor}$	$\mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	1
8	$2^{\frac{r}{2} - 1}$	$\mathbb{R}(d_r) \oplus \mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	2

Table 2. Irreducible representations of even Clifford algebras.

Note that the representations are complex for $r \equiv 2, 6 \pmod 8$ and quaternionic for $r \equiv 3, 4, 5 \pmod 8$.

2.1.3 Spinorially twisted spin groups and representations

By using the unit-length complex numbers U(1) or the unit-length quaternions Sp(1), the Spin group has been "twisted" as follows

$$\mathrm{Spin}^c(n) = (\mathrm{Spin}(n) \times \mathrm{U}(1))/\{\pm(1,1)\} = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1),$$
$$\mathrm{Spin}^q(n) = (\mathrm{Spin}(n) \times \mathrm{Sp}(1))/\{\pm(1,1)\} = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{Sp}(1).$$

These groups give rise to the following short exact sequences

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1) \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^q(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{SO}(3) \longrightarrow 1,$$

respectively, which lead to the notions of Spin^c and Spin^q structures [12, 20, 28]. Noticing that $\mathrm{U}(1) = \mathrm{Spin}(2)$ and $\mathrm{Sp}(1) = \mathrm{Spin}(3)$, we are led to define the twisted Spin group $\mathrm{Spin}^r(n)$ as follows

$$\mathrm{Spin}^r(n) = (\mathrm{Spin}(n) \times \mathrm{Spin}(r))/\{\pm(1,1)\} = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{Spin}(r),$$

where $r \in \mathbb{N}$ and $r \geq 2$. $\mathrm{Spin}^r(n)$ also fits into a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^r(n) \xrightarrow{\lambda_n \times \lambda_r} \mathrm{SO}(n) \times \mathrm{SO}(r) \longrightarrow 1,$$

where

$$\begin{aligned} \lambda_n \times \lambda_r \colon \ \mathrm{Spin}^r(n) &\longrightarrow \mathrm{SO}(n) \times \mathrm{SO}(r), \\ [g,h] &\mapsto (\lambda_n(g), \lambda_r(h)). \end{aligned}$$

We will call r the *rank* of the twisting. Note that the groups $\mathrm{Spin}^2(n) = \mathrm{Spin}^c(n)$ and $\mathrm{Spin}^3(n) = \mathrm{Spin}^q(n)$. The Lie algebra of $\mathrm{Spin}^r(n)$ is

$$\mathfrak{spin}^r(n) = \mathfrak{spin}(n) \oplus \mathfrak{spin}(r).$$

Consider the representations

$$\begin{aligned} \kappa_{n,r}^m := \kappa_n \otimes \kappa_r^m \colon \ \mathrm{Spin}^r(n) &\longrightarrow \mathrm{GL}\big(\Delta_n \otimes \Delta_r^{\otimes m}\big), \\ [g,h] &\mapsto \kappa_{n,r}^m([g,h]) = \kappa_n(g) \otimes \kappa_r^m(h), \end{aligned}$$

where $m \in \mathbb{N}$, which are unitary with respect to the Hermitian metric. We will also use the notation

$$[g,h] \cdot (\psi \otimes \varphi) := \kappa_n(g) \otimes \kappa_r^m(h)(\psi \otimes \varphi) = (\kappa_n(g)(\psi)) \otimes \big(\kappa_r^m(h)(\varphi)\big).$$

An element ϕ of $\Delta_n \otimes \Delta_r^{\otimes m}$ will be called a *twisted spinor*, or simply a *spinor*.

Also consider the map

$$\begin{aligned} \mu_n \otimes \mu_r \colon \ \big(\textstyle\bigwedge^* \mathbb{R}^n \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R}^r\big) \otimes_{\mathbb{R}} (\Delta_n \otimes \Delta_r) &\longrightarrow \Delta_n \otimes \Delta_r, \\ (w_1 \otimes w_2) \otimes (\psi \otimes \varphi) &\mapsto (w_1 \otimes w_2) \cdot (\psi \otimes \varphi) = (w_1 \cdot \psi) \otimes (w_2 \cdot \varphi). \end{aligned}$$

As in the untwisted case, $\mu_n \otimes \mu_r$ is an equivariant homomorphism of $\mathrm{Spin}^r(n)$ representations. Note that we can also take tensor products with more copies of Δ_r as follows

$$\begin{aligned} \mu_r^a := \mathrm{Id}_{\Delta_r}^{\otimes a-1} \otimes \mu_r \otimes \mathrm{Id}_{\Delta_r}^{\otimes m-a} \colon \ \textstyle\bigwedge^* \mathbb{R}^r \otimes_{\mathbb{R}} \Delta_r^m &\longrightarrow \Delta_r^m, \\ \beta \otimes (\varphi_1 \otimes \cdots \otimes \varphi_m) &\mapsto \varphi_1 \otimes \cdots \otimes (\mu_r(\beta \otimes \varphi_a)) \otimes \cdots \otimes \varphi_m \\ &= \varphi_1 \otimes \cdots \otimes (\beta \cdot \varphi_a) \otimes \cdots \otimes \varphi_m, \end{aligned}$$

with Clifford multiplication taking place only in the a-th factor. We will also write

$$\mu_r^a(\beta \otimes \varphi_1 \otimes \cdots \otimes \varphi_m) = \mu_r^a(\beta) \cdot (\varphi_1 \otimes \cdots \otimes \varphi_m).$$

Notice that if $(f_1, \dots, f_r)$ is an orthonormal frame of $\mathbb{R}^r$,

$$\kappa_{r*}^m(f_k f_l)(\varphi_1 \otimes \cdots \otimes \varphi_m) = \big(\mu_r^1(f_k f_l) \cdot \varphi_1\big) \otimes \cdots \otimes \varphi_m + \cdots \\ + \varphi_1 \otimes \cdots \otimes (\mu_r^m(f_k f_l) \cdot \varphi_m). \tag{2.2}$$

2.2 Spinorially twisted Spin structures

2.2.1 Spin structures on oriented Riemannian vector bundles

Let F be an oriented Riemannian vector bundle over a smooth manifold M, with $r = \operatorname{rank}(F) \geq 3$. Let $P_{\mathrm{SO}(r)}$ denote the orthonormal frame bundle of F. A Spin structure on F is a principal $\mathrm{Spin}(r)$-bundle $P_{\mathrm{Spin}(r)}$ together with a 2 sheeted covering

$$\Lambda\colon\ P_{\mathrm{Spin}(r)} \longrightarrow P_{\mathrm{SO}(r)},$$

such that $\Lambda(pg) = \Lambda(p)\lambda_r(g)$ for all $p \in P_{\mathrm{Spin}(r)}$, and all $g \in \mathrm{Spin}(r)$, where $\lambda_r\colon \mathrm{Spin}(r) \longrightarrow \mathrm{SO}(r)$ denotes the universal covering map. In the case when $r = \operatorname{rank}(F) = 2$, we set $\lambda_2\colon \mathrm{Spin}(2) \longrightarrow \mathrm{SO}(2)$ to be the connected 2-fold covering of $\mathrm{SO}(2)$. When $r = 1$ a Spin structure is only a 2-fold covering of the base manifold M.

Given a Spin structure $P_{\mathrm{Spin}(r)}$ one can associate a spinor bundle

$$S(F) = P_{\mathrm{Spin}(r)} \times_{\kappa_r} \Delta_r,$$

where Δ_r denotes the standard complex representation of $\mathrm{Spin}(r)$. In fact, one can also associate spinor bundles whose fibers are tensor powers of Δ_r,

$$S(F)^{\otimes m} = P_{\mathrm{Spin}(r)} \times_{\kappa_r^m} \Delta_r^{\otimes m},$$

where $m \in \mathbb{N}$.

2.2.2 Spinorially twisted spin structures on oriented Riemannian manifolds

Definition 2.2. Let M be an oriented n-dimensional Riemannian manifold, $P_{\mathrm{SO}(n)}$ be its principal bundle of orthonormal frames and $r \in \mathbb{N}$, $r \geq 2$. A Spin^r *structure* on M consists of an auxiliary principal $\mathrm{SO}(r)$-bundle $P_{\mathrm{SO}(r)}$ and either

- a principal $\mathrm{Spin}^r(n)$-bundle $P_{\mathrm{Spin}^r(n)}$ together with an equivariant $2:1$ covering map

  $$\Lambda\colon\ P_{\mathrm{Spin}^r(n)} \longrightarrow P_{\mathrm{SO}(n)} \tilde{\times} P_{\mathrm{SO}(r)},$$

  where $\tilde{\times}$ denotes the fibre-product, such that $\Lambda(pg) = \Lambda(p)(\lambda_n \times \lambda_r)(g)$ for all $p \in P_{\mathrm{Spin}^r(n)}$ and $g \in \mathrm{Spin}^r(n)$, where $\lambda_n \times \lambda_r\colon \mathrm{Spin}^r(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{SO}(r)$ denotes the canonical 2-fold cover;
- or a Spin structure on TM.

A manifold M admitting a Spin^r structure will be called a Spin^r *manifold*.

Remark 2.3. There are three possibilities:

- M is a non-Spin Spin^r manifold so that the structure group to be considered is $\mathrm{Spin}^r(n)$.
- M and F are both Spin so that the structure group to be considered is $\mathrm{Spin}(n) \times \mathrm{Spin}(r)$.

- M is Spin and F is not Spin so that the structure groups to be considered is $\mathrm{Spin}(n)\times\mathrm{SO}(r)$.

A Spin^r manifold has associated vector bundles

$$S(M,F,m)=P_{\mathrm{Spin}^r(n)}\times_{\kappa_n\otimes\kappa_r^m}\Delta_n\otimes\Delta_r^{\otimes m},$$

where $m\in\mathbb{N}$ may be odd, arbitrary or even, respectively.

Remark 2.4. One can also consider the case when F is only locally defined, but $\bigwedge^2 F$ is globally defined, as in the case of some almost quaternionic manifolds.

Remark 2.5. There is also a projective case when r is even, i.e., when the auxiliary bundle has structure group $\mathbb{P}\mathrm{SO}(r)=\mathrm{SO}(r)/\{\pm\mathrm{Id}_{r\times r}\}$. There are analogous observations for the structure group and the bundles $S(M,F,m)$ in this case.

Example 2.6 ([10])**.** Let us consider the real Grassmannians of oriented k-dimensional subspaces of $\mathbb{R}^{k+l}$

$$\mathbb{G}\mathrm{r}_k\big(\mathbb{R}^{k+l}\big)=\frac{\mathrm{SO}(k+l)}{\mathrm{SO}(k)\times\mathrm{SO}(l)}.$$

Let $r=ak+bl$, $a,b\in\mathbb{N}$. There exists a homomorphism $\mathrm{SO}(k)\times\mathrm{SO}(l)\to\mathrm{Spin}^r(kl)$ providing a $\mathrm{Spin}^r(kl)$-structure on the real Grassmannian $\mathbb{G}\mathrm{r}_k\big(\mathbb{R}^{k+l}\big)$ if

$$a\equiv l \pmod 2,$$
$$b\equiv k \pmod 2.$$

2.2.3 Covariant derivatives on twisted Spin bundles

Let M be a Spin^r n-dimensional manifold and F its auxiliary Riemannian vector bundle of rank r. Assume F is endowed with a covariant derivative ∇^F (or equivalently, that $P_{\mathrm{SO}(r)}$ is endowed with a connection 1-form θ) and denote by ∇ the Levi-Civita covariant derivative on M. These two derivatives induce the spinor covariant derivative

$$\nabla^\theta\colon\ \Gamma(S(M,F,m))\longrightarrow\Gamma(T^*M\otimes S(M,F,m))$$

given locally by

$$\nabla^\theta(\psi\otimes\varphi)=\mathrm{d}(\psi\otimes\varphi)+\left[\frac{1}{2}\sum_{1\le i<j\le n}\omega_{ji}\otimes e_ie_j\cdot\psi\right]\otimes\varphi$$
$$+\psi\otimes\left[\frac{1}{2}\sum_{1\le k<l\le r}\theta_{lk}\otimes\kappa_{r*}^m(f_kf_l)\cdot\varphi\right],$$

where $\psi\otimes\varphi\in\Gamma(S(M,F,m))$, $(e_1,\dots,e_n)$ and $(f_1,\dots,f_r)$ are a local orthonormal frames of TM and F respectively, ω_{ij} and θ_{kl} are the local connection 1-forms. From now on, we will often omit the upper and lower bounds on the indices, by declaring i and j to be the indices for the local vectors of M, and k and l to be the indices for the local sections of F. For any tangent vectors $X,Y\in\Gamma(TM)$,

$$R^\theta(X,Y)(\psi\otimes\varphi)=\left[\frac{1}{2}\sum_{i<j}\Omega_{ji}(X,Y)e_ie_j\cdot\psi\right]\otimes\varphi$$
$$+\psi\otimes\left[\frac{1}{2}\sum_{k<l}\Theta_{lk}(X,Y)\kappa_{r*}^m(f_kf_l)\cdot\varphi\right], \tag{2.3}$$

where Ω_{ji} and Θ_{lk} are local curvature 2-forms.

For $X, Y \in \Gamma(TM)$ vector fields and $\phi \in \Gamma(S(M, F, m))$ a spinor field, we also have the compatibility of the covariant derivative with Clifford multiplication,

$$\nabla^\theta_X (Y \cdot \phi) = (\nabla_X Y) \cdot \phi + Y \cdot \nabla^\theta_X \phi.$$

2.3 Almost even-Clifford hermitian structures

Definition 2.7. Let $n \in \mathbb{N}$ and $(f_1, \ldots, f_r)$ be an orthonormal basis of $\mathbb{R}^r$.

- A *linear even-Clifford structure of rank* r on $\mathbb{R}^n$ is a homomorphism of associative algebras with unit

  $$\Psi\colon\ \mathrm{Cl}^0_r \longrightarrow \mathrm{End}\left(\mathbb{R}^n\right).$$

- A *linear even-Clifford Hermitian structure of rank* r on the inner product space $\left(\mathbb{R}^n, \langle\, ,\, \rangle\right)$ is a linear even-Clifford structure of rank r such that each bivector $f_k f_l$, $1 \le k < l \le r$, is mapped to a skew-symmetric endomorphism.

Remark 2.8.

- This is also known as the structure of a (left) Cl^0_r-module.
- Note that, for $1 \le k < l \le r$,

  $$(\Psi(f_k f_l))^2 = -\mathrm{Id}_{\mathbb{R}^n}.$$

- Given a linear even-Clifford structure of rank r on $\mathbb{R}^n$, we can average the standard inner product $\langle\, ,\, \rangle$ on $\mathbb{R}^n$ as follows: for $X, Y \in \mathbb{R}^n$,

  $$(X, Y) = \sum_{k=1}^{[r/2]} \left[\sum_{1 \le i_1 < \cdots < i_{2k} < r} \langle \Psi(f_{i_1} \cdots f_{i_{2k}})(X), \Psi(f_{i_1} \cdots f_{i_{2k}})(Y) \rangle \right],$$

  so that the linear even-Clifford structure is Hermitian with respect to the averaged inner product.
- Given a linear even-Clifford Hermitian structure of rank r, the subalgebra $\mathfrak{spin}(r)$ is mapped injectively into the skew-symmetric endomorphisms $\mathrm{End}^-\left(\mathbb{R}^n\right)$.

Definition 2.9. Let $r \ge 2$.

- A *rank* r *almost even-Clifford structure* on a smooth manifold M is a smoothly varying choice of rank r linear even-Clifford structures on the tangent spaces of M.
- A smooth manifold carrying an almost even-Clifford structure will be called an *almost even-Clifford manifold*.
- A *rank* r *almost even-Clifford Hermitian structure* on a Riemannian manifold M is a smoothly varying choice of linear even-Clifford Hermitian structures on the tangent spaces of M.
- A Riemannian manifold carrying an almost even-Clifford Hermitian structure will be called an *almost even-Clifford Hermitian manifold*.

Remark 2.10. Our terminology differs from that of [27]. We have added the words "almost" and "Hermitian" since, in principle, there is no integrability condition on the structure and the compatibility with a Riemannian metric is a separate condition.

3 Twisted spinors

Throughout this section, we will let $(e_1, \dots, e_n)$ and $(f_1, \dots, f_r)$ be orthonormal bases for $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. A linear basis for Cl_r^0 is given by the products $f_{i_1} f_{i_2} \cdots f_{i_{2s}}$, where $\{i_1, i_2, \dots, i_{2s}\} \subset \{1, \dots, r\}$. In order to simplify notation, we will often write $f_{kl} := f_k f_l$.

3.1 2-forms and skew-symmetric endomorphisms associated to a spinor

Lemma 3.1. *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$, $X, Y \in \mathbb{R}^n$, $1 \le a < b < c < d \le n$ and $1 \le k, l \le r$. Then*

$$\mathrm{Re}\left\langle \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi \right\rangle = 0, \tag{3.1}$$
$$\mathrm{Re}\langle X \wedge Y \cdot \phi, \phi \rangle = 0, \tag{3.2}$$
$$\mathrm{Im}\langle X \wedge Y \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi \rangle = 0, \tag{3.3}$$
$$\mathrm{Re}\langle X \cdot \phi, Y \cdot \phi \rangle = \langle X, Y \rangle |\phi|^2, \tag{3.4}$$
$$\mathrm{Re}\left\langle e_a e_b e_c e_d \cdot \kappa_{r*}^m(f_{kl}) \cdot \psi, \psi \right\rangle = 0. \tag{3.5}$$

Proof. By using (2.1) repeatedly

$$\langle \mu_r^a(f_k f_l) \cdot \phi, \phi \rangle = -\overline{\langle \mu_r^a(f_k f_l) \phi, \phi \rangle},$$

so that (3.1) follows from (2.2).

For identity (3.2), recall that for $X, Y \in \mathbb{R}^n$

$$X \wedge Y = X \cdot Y + \langle X, Y \rangle.$$

Thus

$$\langle X \wedge Y \cdot \phi, \phi \rangle = -\overline{\langle X \wedge Y \cdot \phi, \phi \rangle}.$$

Identities (3.3), (3.4) and (3.5) follow similarly. ∎

For any $\xi \in \bigwedge^2 (\mathbb{R}^n)^*$ define $\hat{\xi} \in \mathrm{End}^-\left(\mathbb{R}^n\right)$ as follows

$$X \mapsto \hat{\xi}(X) := (X \lrcorner \xi)^\sharp = \xi(X, \cdot)^\sharp,$$

where $\lrcorner$ denotes contraction and $\sharp$ denotes metric dualization.

Definition 3.2 ([10])**.** Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$, $X, Y \in \mathbb{R}^n$ and $1 \le k, l \le r$.

- Let

$$\eta_{kl}^\phi(X, Y) = \mathrm{Re}\left\langle X \wedge Y \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi \right\rangle$$

  be the real 2-forms associated to the spinor ϕ.

- Define the antisymmetric endomorphisms $\hat{\eta}_{kl}^\phi \in \mathrm{End}^-\left(\mathbb{R}^n\right)$ by

$$X \mapsto \hat{\eta}_{kl}^\phi(X).$$

Remark 3.3.

- For $k \ne l$,

$$\eta_{kl}^\phi = -\eta_{lk}^\phi.$$

- By (3.2),

$$\eta_{kk} \equiv 0.$$

- By (3.3), if $k \neq l$,

$$\eta^{\phi}_{kl}(X,Y) = \left\langle X \wedge Y \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \right\rangle.$$

- For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the spinor $\lambda\phi$ produces the same 2-forms

$$\eta^{\lambda\phi}_{kl} = \eta^{\phi}_{kl}.$$

- Note that, depending on the spinor, such 2-forms can actually be identically zero.

Lemma 3.4 ([10])**.** *Any spinor $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ defines two maps (extended by linearity)*

$$\begin{aligned} \Phi^{\phi} \colon \ \textstyle\bigwedge^2\mathbb{R}^r &\longrightarrow \textstyle\bigwedge^2\mathbb{R}^n, \\ f_{kl} &\mapsto \Phi^{\phi}(f_{kl}) := \eta^{\phi}_{kl}, \end{aligned}$$

and

$$\begin{aligned} \hat{\Phi}^{\phi} \colon \ \textstyle\bigwedge^2\mathbb{R}^r &\longrightarrow \mathrm{End}\left(\mathbb{R}^n\right), \\ f_{kl} &\mapsto \hat{\Phi}^{\phi}(f_{kl}) := \hat{\eta}^{\phi}_{kl}. \end{aligned}$$

3.2 Pure spinors: $r \geq 3$

From now on we shall assume that $r \geq 3$.

Definition 3.5. A (non-zero) spinor $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ is called a *twisted pure spinor* if

$$\left(\eta^{\phi}_{kl} + 2\kappa^m_{r*}(f_{kl})\right) \cdot \phi = 0, \tag{3.6}$$

$$\left(\hat{\eta}^{\phi}_{kl}\right)^2 = -\mathrm{Id}_{\mathbb{R}^n}, \tag{3.7}$$

for all $1 \leq k < l \leq r$.

Remark 3.6.

- The first condition says that the subalgebra span $\left(\eta^{\phi}_{kl} + 2 f_{kl}\right) \subset \mathfrak{spin}(n) \oplus \mathfrak{spin}(r)$ annihilates the twisted pure spinor.
- The second condition ensures that the 2-forms are non-zero and the associated endomorphisms are almost complex structures.
- We will show that the two conditions imply

$$\mathrm{span}\left\{\hat{\eta}^{\phi}_{kl} \in \mathrm{End}\left(\mathbb{R}^n\right) \,|\, 1 \leq k < l \leq r\right\} \cong \mathfrak{spin}(r).$$

Lemma 3.7. *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a twisted pure spinor.*

1. *If $1 \leq i, j, k, l \leq r$ are all different,*

$$\left[\hat{\eta}^{\phi}_{kl}, \hat{\eta}^{\phi}_{ij}\right] = 0. \tag{3.8}$$

2. *If $1 \leq i, j, k \leq r$ are all different,*

$$\left[\hat{\eta}^{\phi}_{ij}, \hat{\eta}^{\phi}_{jk}\right] = -2\hat{\eta}^{\phi}_{ik}. \tag{3.9}$$

Proof. For identity (3.8), suppose $1 \leq i, j, k, l \leq r$ are all different. Notice that in $\mathfrak{spin}(r) \subset \mathrm{Cl}_r^0$,

$$[f_{kl}, f_{ij}] = 0$$

and, since $\kappa_{r*}^m \colon \mathfrak{spin}(r) \subset \mathrm{Cl}_r^0 \longrightarrow \mathrm{End}(\Delta_r^{\otimes m})$ is a Lie algebra homomorphism,

$$0 = \kappa_{r*}^m([f_{kl}, f_{ij}]) = \left[\kappa_{r*}^m(f_{kl}), \kappa_{r*}^m(f_{ij})\right],$$

i.e.,

$$\kappa_{r*}^m(f_{kl})\kappa_{r*}^m(f_{ij}) = \kappa_{r*}^m(f_{ij})\kappa_{r*}^m(f_{kl}).$$

Now recall that, by definition,

$$\eta_{ij}^\phi \cdot \phi = -2\kappa_{r*}^m(f_{ij}) \cdot \phi,$$

which implies

$$\kappa_{r*}^m(f_{kl}) \cdot \eta_{ij}^\phi \cdot \phi = -2\kappa_{r*}^m(f_{kl})\kappa_{r*}^m(f_{ij}) \cdot \phi = -2\kappa_{r*}^m(f_{ij})\kappa_{r*}^m(f_{kl}) \cdot \phi = \kappa_{r*}^m(f_{ij}) \cdot \eta_{kl}^\phi \cdot \phi.$$

By Lemma 3.1,

$$\begin{aligned}
\mathrm{Re}\left\langle e_s \wedge e_t \cdot \eta_{ij}^\phi \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi\right\rangle &= \mathrm{Re}\langle e_s \wedge e_t \cdot \left(\sum_{a<b} \eta_{ij}^\phi(e_a, e_b) e_a \wedge e_b\right) \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi\rangle \\
&= \mathrm{Re} \sum_{a<b} \eta_{ij}^\phi(e_a, e_b)\langle e_s \cdot e_t \cdot e_a \cdot e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi\rangle \\
&= \sum_{s=a<b} \eta_{ij}^\phi(e_s, e_b)\langle e_t \cdot e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi\rangle \\
&\quad + \sum_{t=a<b} \eta_{ij}^\phi(e_t, e_b)(-\langle e_s \cdot e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi\rangle) \\
&\quad + \sum_{a<t=b} \eta_{ij}^\phi(e_a, e_t)\langle e_s \cdot e_a \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi\rangle \\
&\quad + \sum_{a<b=s} \eta_{ij}^\phi(e_a, e_s)(-\langle e_t \cdot e_a \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi\rangle) \\
&= \sum_{s<b} \eta_{ij}^\phi(e_s, e_b)\eta_{kl}^\phi(e_t, e_b) + \sum_{t<b} \eta_{ij}^\phi(e_t, e_b)\left(-\eta_{kl}^\phi(e_s, e_b)\right) \\
&\quad + \sum_{b<t} \eta_{ij}^\phi(e_b, e_t)\eta_{kl}^\phi(e_s, e_b) + \sum_{b<s} \eta_{ij}^\phi(e_b, e_s)\left(-\eta_{kl}^\phi(e_t, e_b)\right) \\
&= -\sum_b \eta_{ij}^\phi(e_s, e_b)\eta_{kl}^\phi(e_b, e_t) + \sum_b \eta_{kl}^\phi(e_s, e_b)\eta_{ij}^\phi(e_b, e_t) \\
&= -\sum_b \left(\hat\eta_{kl}^\phi\right)_{tb}\left(\hat\eta_{ij}^\phi\right)_{bs} + \sum_b \left(\hat\eta_{ij}^\phi\right)_{tb}\left(\hat\eta_{kl}^\phi\right)_{bs} \\
&= \left[\hat\eta_{ij}^\phi, \hat\eta_{kl}^\phi\right]_{ts},
\end{aligned}$$

the entry in row t and column s of the matrix

$$\left[\hat\eta_{kl}^\phi, \hat\eta_{ij}^\phi\right].$$

Analogously,

$$\mathrm{Re}\left\langle e_s \wedge e_t \cdot \eta_{kl}^\phi \cdot \kappa_{r*}^m(f_{ij}) \cdot \phi, \phi\right\rangle = \left[\hat\eta_{kl}^\phi, \hat\eta_{ij}^\phi\right]_{ts}.$$

Thus,

$$[\hat\eta^\phi_{ij},\hat\eta^\phi_{kl}]=[\hat\eta^\phi_{kl},\hat\eta^\phi_{ij}],$$

but by definition of the bracket

$$[\hat\eta^\phi_{kl},\hat\eta^\phi_{ij}]=-[\hat\eta^\phi_{ij},\hat\eta^\phi_{kl}].$$

Hence,

$$[\hat\eta^\phi_{ij},\hat\eta^\phi_{kl}]=0.$$

For identity (3.9), recall that in $\mathfrak{spin}(r)\subset \mathrm{Cl}^0_r$,

$$[f_{ij},f_{jk}]=f_{ij}f_{jk}-f_{jk}f_{ij}=-2f_{ik},$$

so that

$$-2\kappa^m_{r*}(f_{ik})=\kappa^m_{r*}([f_{ij},f_{jk}])=\big[\kappa^m_{r*}(f_{ij}),\kappa^m_{r*}(f_{jk})\big],$$

i.e.,

$$\kappa^m_{r*}(f_{ij})\kappa^m_{r*}(f_{jk})=\kappa^m_{r*}(f_{jk})\kappa^m_{r*}(f_{ij})-2\kappa^m_{r*}(f_{ik}).$$

Now,

$$\eta^\phi_{ij}\cdot\phi=-2\kappa^m_{r*}(f_{ij})\cdot\phi,$$

which implies

$$\begin{aligned}\kappa^m_{r*}(f_{jk})\cdot\eta^\phi_{ij}\cdot\phi&=-2\kappa^m_{r*}(f_{jk})\kappa^m_{r*}(f_{ij})\cdot\phi=-2\big[\kappa^m_{r*}(f_{ij})\kappa^m_{r*}(f_{jk})+2\kappa^m_{r*}(f_{ik})\big]\cdot\phi\\&=\kappa^m_{r*}(f_{ij})\cdot\eta^\phi_{jk}\cdot\phi-4\kappa^m_{r*}(f_{ik})\cdot\phi.\end{aligned}$$

Thus, on the one hand,

$$\begin{aligned}&\mathrm{Re}\,\langle e_s\wedge e_t\cdot\eta^\phi_{ij}\cdot\kappa^m_{r*}(f_{jk})\cdot\phi,\phi\rangle\\&\quad=\mathrm{Re}\,\langle e_s\wedge e_t\cdot\eta^\phi_{jk}\cdot\kappa^m_{r*}(f_{ij})\cdot\phi,\phi\rangle-4\,\mathrm{Re}\,\langle e_s\wedge e_t\cdot\kappa^m_{r*}(f_{ik})\cdot\phi\rangle\\&\quad=\mathrm{Re}\,\langle e_s\wedge e_t\cdot\eta^\phi_{jk}\cdot\kappa^m_{r*}(f_{ij})\cdot\phi,\phi\rangle-4\eta^\phi_{ik}(e_s,e_t).\end{aligned}$$

By the calculation above

$$\begin{aligned}&\mathrm{Re}\,\langle e_s\wedge e_t\cdot\eta^\phi_{ij}\cdot\kappa^m_{r*}(f_{jk})\cdot\phi,\phi\rangle=[\hat\eta^\phi_{ij},\hat\eta^\phi_{jk}]_{ts},\\&\mathrm{Re}\,\langle e_s\wedge e_t\cdot\eta^\phi_{jk}\cdot\kappa^m_{r*}(f_{ij})\cdot\phi,\phi\rangle=[\hat\eta^\phi_{jk},\hat\eta^\phi_{ij}]_{ts},\\&\eta^\phi_{ik}(e_s,e_t)=(\hat\eta^\phi_{ik})_{ts},\end{aligned}$$

so that

$$[\hat\eta^\phi_{ij},\hat\eta^\phi_{jk}]=[\hat\eta^\phi_{jk},\hat\eta^\phi_{ij}]-4\hat\eta^\phi_{ik}=-[\hat\eta^\phi_{ij},\hat\eta^\phi_{jk}]-4\hat\eta^\phi_{ik},$$

and

$$2[\hat\eta^\phi_{ij},\hat\eta^\phi_{jk}]=-4\hat\eta^\phi_{ik}.$$

■

Remark 3.8. For a twisted spinor ϕ satisfying only condition (3.6), the endomorphisms $2\hat{\eta}^{\phi}_{kl}$ satisfy the Lie bracket relations of the Lie algebra $\mathfrak{so}(r)$.

Lemma 3.9. *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a twisted pure spinor. Let $1 \leq i, j, k, l \leq r$ be all different.*

- *The automorphisms $\hat{\eta}^{\phi}_{ij}$ and $\hat{\eta}^{\phi}_{kl}$ commute*

$$\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{kl} = \hat{\eta}^{\phi}_{kl}\hat{\eta}^{\phi}_{ij}. \tag{3.10}$$

- *The automorphisms $\hat{\eta}^{\phi}_{ij}$ and $\hat{\eta}^{\phi}_{jk}$ anticommute*

$$\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk} = -\hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{ij} = -\hat{\eta}^{\phi}_{ik}. \tag{3.11}$$

- *The following identities hold*

$$\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{kl} = -\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{jl} = -\hat{\eta}^{\phi}_{jl}\hat{\eta}^{\phi}_{ik} = \hat{\eta}^{\phi}_{kl}\hat{\eta}^{\phi}_{ij} = \hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{il} = \hat{\eta}^{\phi}_{il}\hat{\eta}^{\phi}_{jk}. \tag{3.12}$$

Proof. Identity (3.10) is the same as (3.8) in Lemma 3.7.

For identity (3.11) recall

$$\left(\hat{\eta}^{\phi}_{ij}\right)^2 = -\mathrm{Id}_{\mathbb{R}^n},$$

and the identity

$$\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk} - \hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{ij} = -2\,\hat{\eta}^{\phi}_{ik}.$$

Compose the last identity on the left and on the right with $\hat{\eta}^{\phi}_{ij}$

$$\left(\hat{\eta}^{\phi}_{ij}\right)^2\hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{ij} - \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk}\left(\hat{\eta}^{\phi}_{ij}\right)^2 = -2\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij},$$

so that

$$-\hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{ij} + \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk} = -2\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij}.$$

Thus,

$$-2\hat{\eta}^{\phi}_{ik} = -2\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij},$$

and

$$\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij} = \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}\left(\hat{\eta}^{\phi}_{ij}\right)^2,$$

i.e.,

$$\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij} = -\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}.$$

Hence,

$$-2\hat{\eta}^{\phi}_{ik} = \left[\hat{\eta}^{\phi}_{ij}, \hat{\eta}^{\phi}_{jk}\right] = \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk} - \hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{ij} = \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk} - \left(-\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk}\right) = 2\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk}.$$

For (3.12), we can see that

$$\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{kl} = \hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{kl} = -\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{jl},$$

and similarly for the remaining identities. ■

Lemma 3.10. *The definition of twisted pure spinor does not depend on the choice of orthonormal basis* $(f_1, \dots, f_r)$ *of* $\mathbb{R}^r$*.*

Proof. Suppose $(f'_1 \dots, f'_r)$ is another orthonormal basis of $\mathbb{R}^r$ so that

$$f'_k = a_{k1} f_1 + \cdots + a_{kr} f_r,$$

for $1 \le k \le r$, and the matrix $A = (a_{kl}) \in \mathrm{SO}(r)$. Recall that

$$\eta^\phi_{kl} = \Phi^\phi(f_{kl}).$$

If we write the left-hand side of (3.6) with respect to the basis $(f'_1, \dots, f'_r)$, we have

$$\begin{aligned}
&\big(\Phi^\phi(f'_{kl}) + 2\kappa^m_{r_*}(f'_{kl})\big) \cdot \phi \\
&\quad = \left(\left(\sum_{s<t} (a_{ks}a_{lt} - a_{kt}a_{ls})\Phi^\phi(f_{st})\right) + 2\kappa^m_{r_*}\left(\sum_{s<t} (a_{ks}a_{lt} - a_{kt}a_{ls}) f_{st}\right)\right) \cdot \phi \\
&\quad = \sum_{s<t} (a_{ks}a_{lt} - a_{kt}a_{ls})(\Phi^\phi(f_{st}) + 2\kappa^m_{r_*}(f_{st})) \cdot \phi = 0.
\end{aligned}$$

In order to simplify notation, let

$$J_{kl} = \hat{\Phi}^\phi(f_{kl}), \qquad J'_{kl} = \hat{\Phi}^\phi(f'_{kl}).$$

Now suppose that the second condition of pure spinor is fulfilled for the frame $(f_1, \dots, f_r)$

$$J^2_{kl} = -\mathrm{Id}_{\mathbb{R}^n}.$$

With respect to an orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$,

$$\begin{aligned}
J'_{kl}(X) &= \sum_{c=1}^n \Phi^\phi(f'_{kl})(X, e_c)e_c = \sum_{c=1}^n \sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls})\Phi^\phi(f_{st})(X, e_c)e_c \\
&= \sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls}) \sum_{c=1}^n \Phi^\phi(f_{st})(X, e_c)e_c \\
&= \sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls}) J_{st}(X).
\end{aligned} \tag{3.13}$$

Since the bases $\{f_{kl} \,|\, 1 \le k < l \le r\}$ and $\{f'_{kl} \,|\, 1 \le k < l \le r\}$ are orthonormal in $\bigwedge^2\mathbb{R}^r$,

$$\begin{aligned}
\delta_{ac}\delta_{bd} = \langle f'_{ab}, f'_{cd}\rangle &= \left\langle \sum_{1\le s<t\le r} (a_{as}a_{bt} - a_{at}a_{bs}) f_{st}, \sum_{1\le u<v\le r} (a_{cu}a_{dv} - a_{cv}a_{du}) f_{uv} \right\rangle \\
&= \sum_{1\le s<t\le r} \sum_{1\le u<v\le r} (a_{as}a_{bt} - a_{at}a_{bs})(a_{cu}a_{dv} - a_{cv}a_{du})\delta_{su}\delta_{tv} \\
&= \sum_{1\le s<t\le r} (a_{as}a_{bt} - a_{at}a_{bs})(a_{cs}a_{dt} - a_{ct}a_{ds}).
\end{aligned} \tag{3.14}$$

By (3.13),

$$J'_{kl} = \sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls}) J_{st},$$

we have

$$\begin{aligned}J'_{kl}J'_{kl} &= \left(\sum_{1\leq s<t\leq r}(a_{ks}a_{lt}-a_{kt}a_{ls})J_{st}\right)\left(\sum_{1\leq u<v\leq r}(a_{ku}a_{lv}-a_{kv}a_{lu})J_{uv}\right)\\ &= \sum_{1\leq s<t\leq r}\sum_{1\leq u<v\leq r}(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ku}a_{lv}-a_{kv}a_{lu})J_{st}J_{uv}.\end{aligned}$$

There are three cases:

(i) the indices s, t, u, v are all different;

(ii) the pairs (s, t) and (u, v) have one, and only one, common entry;

(iii) the pairs (s, t) and (u, v) coincide.

For (i), note that since $s < t$ and $u < v$, we only have the following six summands with those indices, so that

$$\begin{aligned}&(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ku}a_{lv}-a_{kv}a_{lu})J_{st}J_{uv}+(a_{ks}a_{lu}-a_{ku}a_{ls})(a_{kt}a_{lv}-a_{kv}a_{lt})J_{su}J_{tv}\\ &\quad+(a_{ks}a_{lu}-a_{ku}a_{ls})(a_{kt}a_{lv}-a_{kv}a_{lt})J_{su}J_{tv}+(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ku}a_{lv}-a_{kv}a_{lu})J_{st}J_{uv}\\ &\quad+(a_{ks}a_{lv}-a_{kv}a_{ls})(a_{kt}a_{lu}-a_{ku}a_{lt})J_{sv}J_{tu}+(a_{ks}a_{lv}-a_{kv}a_{ls})(a_{kt}a_{lu}-a_{ku}a_{lt})J_{sv}J_{tu}\\ &\;=(2(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ku}a_{lv}-a_{kv}a_{lu})-2(a_{ks}a_{lu}-a_{ku}a_{ls})(a_{kt}a_{lv}-a_{kv}a_{lt})\\ &\quad+2(a_{ks}a_{lv}-a_{kv}a_{ls})(a_{kt}a_{lu}-a_{ku}a_{lt}))J_{st}J_{uv}=0.\end{aligned}$$

For (ii), suppose $s = u$ but $t \neq v$. Now we have two summands

$$\begin{aligned}&(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ks}a_{lv}-a_{kv}a_{ls})J_{st}J_{sv}+(a_{ks}a_{lv}-a_{kv}a_{ls})(a_{ks}a_{lt}-a_{kt}a_{ls})J_{sv}J_{st}\\ &\quad=(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ks}a_{lv}-a_{kv}a_{ls})(J_{st}J_{sv}+J_{sv}J_{st})\\ &\quad=(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ks}a_{lv}-a_{kv}a_{ls})(J_{tv}+J_{vt})\\ &\quad=(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ks}a_{lv}-a_{kv}a_{ls})(J_{tv}-J_{tv})=0,\end{aligned}$$

and similarly for the other cases.

For (iii), we have

$$\sum_{1\leq s<t\leq r}(a_{ks}a_{lt}-a_{kt}a_{ls})^2J_{st}^2=-\mathrm{Id}_{\mathbb{R}^n},$$

where we have used (3.14). ■

Proposition 3.11. *A twisted pure spinor $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ induces a linear even-Clifford Hermitian structure of rank r on $\mathbb{R}^n$*

$$\begin{aligned}\mathrm{Cl}_r^0 &\longrightarrow \mathrm{End}\left(\mathbb{R}^n\right),\\ f_{ij} &\mapsto \hat{\eta}_{ij}^{\phi},\end{aligned}$$

so that

$$\mathbb{R}^n \cong \begin{cases}\mathbb{R}^m\otimes\tilde{\Delta}_r & \text{if } r\not\equiv 0 \pmod 4,\\ \mathbb{R}^{m_1}\otimes\tilde{\Delta}_r^+\oplus\mathbb{R}^{m_2}\otimes\tilde{\Delta}_r^- & \text{if } r\equiv 0 \pmod 4,\end{cases}$$

as a representation of Cl_r^0, for some $m, m_1, m_2 \in \mathbb{N}$, where $\tilde{\Delta}_r$ denotes the (unique) non-trivial real representation of Cl_r^0 if $r \not\equiv 0 \pmod 4$, and $\tilde{\Delta}_r^+$ and $\tilde{\Delta}_r^-$ denote the two non-trivial real representations of Cl_r^0 if $r \equiv 0 \pmod 4$.

Proof. By Lemma 3.9, the map

$$\left(\mathrm{Cl}_r^0\right)^2 \longrightarrow \mathrm{End}^-\left(\mathbb{R}^n\right),$$
$$f_{ij} \mapsto \hat{\eta}_{ij}^{\phi}$$

extends to an algebra homomorphism

$$\mathrm{Cl}_r^0 \longrightarrow \mathrm{End}\left(\mathbb{R}^n\right).$$

Since the matrices $\hat{\eta}_{ij}^{\phi}$ square to $-\mathrm{Id}_{\mathbb{R}^n}$, this representation of Cl_r^0 contains no trivial summands. By [20, Theorem 5.6], we know that the algebra Cl_r^0 has (up to isomorphism) only one or two non-trivial irreducible representations depending on whether $r \not\equiv 0 \pmod 4$ or $r \equiv 0 \pmod 4$ respectively. ■

Remark 3.12. Note that, unlike [27], in our case $\hat{\eta}_{kk}^{\phi} = 0$.

Lemma 3.13. *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a twisted pure spinor and $[g,h] \in \mathrm{Spin}^r(n)$. The spinor $\kappa_{n,r}^m([g,h])(\phi)$ is also a twisted pure spinor.*

Proof. Consider the orthonormal bases

$$(e_1',\dots,e_n') = (\lambda_n(g)(e_1),\dots,\lambda_n(g)(e_n)),$$
$$(f_1',\dots,f_r') = (\lambda_r(h)(f_1),\dots,\lambda_r(h)(f_r)),$$

of $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. We will verify the pure spinor identities for $\varphi := \kappa_{n,r}^m([g,h])(\phi)$ using these bases. Indeed,

$$\begin{aligned}
\Phi^{\varphi}(f_k'f_l')\cdot\varphi &= \sum_{a<b}\langle e_a'e_b'\cdot\kappa_{r*}^m(f_k'f_l')\cdot\varphi,\varphi\rangle e_a'e_b'\cdot\varphi\\
&= \sum_{a<b}\big\langle\lambda_n(g)(e_a)\lambda_n(g)(e_b)\cdot\kappa_{r*}^m(\lambda_r(h)(f_k)\lambda_r(h)(f_l))\cdot\varphi,\varphi\big\rangle\\
&\quad\times\lambda_n(g)(e_a)\lambda_n(g)(e_b)\cdot\varphi\\
&= \sum_{a<b}\big\langle\lambda_n(g)(e_ae_b)\cdot\kappa_{r*}^m(\lambda_r(h)(f_kf_l))\cdot\varphi,\varphi\big\rangle\lambda_n(g)(e_ae_b)\cdot\varphi\\
&= \sum_{a<b}\big\langle\lambda_n\times\lambda_r([g,h])\big(e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\big)\cdot\kappa_{n,r}^m([g,h])(\phi),\kappa_{n,r}^m([g,h])(\phi)\big\rangle\\
&\quad\times\lambda_n(g)(e_ae_b)\cdot\kappa_{n,r}^m([g,h])(\phi)\\
&= \sum_{a<b}\big\langle\kappa_{n,r}^m([g,h])(e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\cdot\phi),\kappa_{n,r}^m([g,h])(\phi)\big\rangle\\
&\quad\times\lambda_n(g)(e_ae_b)\cdot\kappa_{n,r}^m([g,h])(\phi)\\
&= \sum_{a<b}\big\langle e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\cdot\phi,\phi\big\rangle\kappa_{n,r}^m([g,h])(e_ae_b\cdot\phi)\\
&= \kappa_{n,r}^m([g,h])\left(\sum_{a<b}\big\langle e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\cdot\phi,\phi\big\rangle e_ae_b\cdot\phi\right)\\
&= \kappa_{n,r}^m([g,h])\big(\Phi^{\phi}(f_kf_l)\cdot\phi\big)\\
&= \kappa_{n,r}^m([g,h])\big(-2\kappa_{r*}^m(f_kf_l)\cdot\phi\big)\\
&= -2\kappa_{r*}^m(\lambda_r(h)(f_kf_l))\cdot\kappa_{n,r}^m([g,h])(\phi)\\
&= -2\kappa_{r*}^m(f_k'f_l')\cdot\varphi,
\end{aligned}$$

which proves condition (3.6) for φ.

For condition (3.7), consider

$$
\begin{aligned}
\Phi^{\varphi}(f'_k f'_l) &= \sum_{a<b} \left\langle e'_a e'_b \cdot \kappa^m_{r*}(f'_k f'_l) \cdot \varphi, \varphi \right\rangle e'_a e'_b \\
&= \sum_{a<b} \left\langle \lambda_n(g)(e_a)\lambda_n(g)(e_b) \cdot \kappa^m_{r*}(\lambda_r(h)(f_k)\lambda_r(h)(f_l)) \cdot \varphi, \varphi \right\rangle e'_a e'_b \\
&= \sum_{a<b} \left\langle \lambda_n(g)(e_a e_b) \cdot \kappa^m_{r*}(\lambda_r(h)(f_k f_l)) \cdot \varphi, \varphi \right\rangle e'_a e'_b \\
&= \sum_{a<b} \left\langle \lambda_n \times \lambda_r([g,h])(e_a e_b \cdot \kappa^m_{r*}(f_k f_l)) \cdot \kappa^m_{n,r}([g,h])(\phi), \kappa^m_{n,r}([g,h])(\phi) \right\rangle e'_a e'_b \\
&= \sum_{a<b} \left\langle \kappa^m_{n,r}([g,h])(e_a e_b \cdot \kappa^m_{r*}(f_k f_l) \cdot \phi), \kappa^m_{n,r}([g,h])(\phi) \right\rangle e'_a e'_b \\
&= \sum_{a<b} \left\langle e_a e_b \cdot \kappa^m_{r*}(f_k f_l) \cdot \phi, \phi \right\rangle e'_a e'_b \\
&= \sum_{a<b} \Phi^{\phi}(f_k f_l)(e_a, e_b) e'_a e'_b,
\end{aligned}
$$

which means that the matrix representing $\Phi^{\varphi}(f'_k f'_l)$ with respect to the basis $(e'_1, \ldots, e'_n)$ has the same coefficients as the matrix representing $\Phi^{\phi}(f_k f_l)$ with respect to the basis $(e_1, \ldots, e_n)$. Hence,

$$
\left[\Phi^{\varphi}(f'_k f'_l)\right]^2 = -\mathrm{Id}_{\mathbb{R}^n}. \qquad \blacksquare
$$

Lemma 3.14. *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a twisted pure spinor. If $\xi \in \mathfrak{spin}(n)$ is such that*

$$
\xi \cdot \phi = 0,
$$

then

$$
\left[\hat{\xi}, \hat{\eta}_{kl}\right] = 0 \qquad \text{and} \qquad \mathbf{tr}\left(\hat{\xi}\hat{\eta}^{\phi}_{kl}\right) = 0
$$

for all $1 \le k < l \le r$.

Proof. Let

$$
\xi = \sum_{1 \le a < b \le n} \xi(e_a, e_b) e_a e_b.
$$

Note that, if $1 \le s < t \le n$ and $1 \le k < l \le r$,

$$
\begin{aligned}
0 &= \mathrm{Re}\left\langle e_s e_t \cdot \kappa^m_{r^*}(f_k f_l) \cdot \xi \cdot \phi, \phi \right\rangle = \mathrm{Re}\left\langle e_s e_t \cdot \xi \cdot \kappa^m_{r^*}(f_k f_l) \cdot \phi, \phi \right\rangle \\
&= \mathrm{Re}\left\langle e_s e_t \cdot \left(\sum_{1 \le a < b \le n} \xi(e_a, e_b) e_a e_b \right) \cdot \kappa^m_{r^*}(f_k f_l) \cdot \phi, \phi \right\rangle \\
&= \sum_{1 \le a < b \le n} \xi(e_a, e_b) \left\langle e_s e_t \cdot e_a e_b \cdot \kappa^m_{r^*}(f_k f_l) \cdot \phi, \phi \right\rangle \\
&= -\sum_b \left(\hat{\eta}^{\phi}_{kl}\right)_{tb} \hat{\xi}_{bs} + \sum_b \hat{\xi}_{tb} \left(\hat{\eta}^{\phi}_{kl}\right)_{bs} = \left[\hat{\xi}, \hat{\eta}^{\phi}_{kl}\right]_{ts},
\end{aligned}
$$

i.e.,

$$
\left[\hat{\xi}, \hat{\eta}^{\phi}_{kl}\right] = 0.
$$

Since $r \ge 3$, for $q \ne k, l$,

$$
\mathbf{tr}\left(\hat{\xi}\hat{\eta}^{\phi}_{kl}\right) = \mathbf{tr}\left(-\hat{\eta}^{\phi}_{kq}\hat{\xi}\hat{\eta}^{\phi}_{kl}\hat{\eta}^{\phi}_{kq}\right) = -\mathbf{tr}\left(\hat{\eta}^{\phi}_{kq}\hat{\xi}\hat{\eta}^{\phi}_{lq}\right) = -\mathbf{tr}\left(\hat{\xi}\hat{\eta}^{\phi}_{kq}\hat{\eta}^{\phi}_{lq}\right) = -\mathbf{tr}\left(\hat{\xi}\hat{\eta}^{\phi}_{kl}\right). \qquad \blacksquare
$$

Lemma 3.15. *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a twisted pure spinor. The annihilator subalgebra of ϕ in $\mathfrak{spin}(n) \oplus \mathfrak{spin}(r)$ is contained in one of the subalgebras $\tilde{\mathfrak{g}}$ listed in Table 3, where*

$$\widetilde{\mathfrak{spin}}(r) = \mathrm{span}\left(\eta_{kl}^{\phi} + 2f_k f_l\right) \subset \mathfrak{spin}(n) \oplus \mathfrak{spin}(r).$$

$r \pmod 8$	$\tilde{\mathfrak{g}}$
0	$\mathfrak{so}(m_1) \oplus \mathfrak{so}(m_2) \oplus \widetilde{\mathfrak{spin}}(r)$
1, 7	$\mathfrak{so}(m) \oplus \widetilde{\mathfrak{spin}}(r)$
2, 6	$\mathfrak{u}(m) \oplus \widetilde{\mathfrak{spin}}(r)$
3, 5	$\mathfrak{sp}(m) \oplus \widetilde{\mathfrak{spin}}(r)$
4	$\mathfrak{sp}(m_1) \oplus \mathfrak{sp}(m_2) \oplus \widetilde{\mathfrak{spin}}(r)$

Table 3.

Proof. Let

$$\xi \in \mathfrak{spin}(n), \qquad \sigma = \sum_{1\leq k<l\leq r} \sigma_{kl} f_k f_l \in \mathfrak{spin}(r),$$

be such that

$$\left(\xi + \kappa_{r^*}^{m}(\sigma)\right) \cdot \phi = 0.$$

Expanding this identity,

$$\begin{aligned} 0 = \left(\xi + \kappa_{r^*}^{m}(\sigma)\right) \cdot \phi &= \xi \cdot \phi + \sum_{1\leq k<l\leq r} \sigma_{kl} \kappa_{r^*}^{m}(f_k f_l) \cdot \phi \\ &= \xi \cdot \phi + \frac{1}{2} \sum_{1\leq k<l\leq r} \sigma_{kl} 2\kappa_{r^*}^{m}(f_k f_l) \cdot \phi = \xi \cdot \phi - \frac{1}{2} \sum_{1\leq k<l\leq r} \sigma_{kl} \eta_{kl}^{\phi} \cdot \phi \\ &= \left(\xi - \frac{1}{2} \sum_{1\leq k<l\leq r} \sigma_{kl} \eta_{kl}^{\phi}\right) \cdot \phi. \end{aligned}$$

By Lemma 3.14,

$$\hat{\xi} - \frac{1}{2} \sum_{1\leq k<l\leq r} \sigma_{kl} \hat{\eta}_{kl}^{\phi} \in C_{\mathfrak{so}(n)}(\mathfrak{spin}(r))$$

and is orthogonal to $\mathrm{span}(\hat{\eta}_{kl})$. Thus,

$$\xi + \sigma = \left(\xi - \frac{1}{2} \sum_{1\leq k<l\leq r} \sigma_{kl} \eta_{kl}^{\phi}\right) + \left(\frac{1}{2} \sum_{1\leq k<l\leq r} \sigma_{kl} (\eta_{kl}^{\phi} + 2\, f_k f_l)\right) \in \mathfrak{spin}(n) \oplus \mathfrak{spin}(r).$$

The table follows from [2, Theorems 3.1 and 3.2]. ■

We refer the reader to [1, Theorem 3.2] for a description of the (connected components of the identity element of the) corresponding Lie groups.

Lemma 3.16. *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a twisted pure spinor. Every element in the orbit $\widehat{\mathrm{Spin}(r)} \cdot \phi$ induces the same linear even-Clifford Hermitian structure of rank r on $\mathbb{R}^n$, where $\widehat{\mathrm{Spin}(r)}$ denotes the canonical copy of $\mathrm{Spin}(r)$ in $\mathrm{Spin}^r(n)$ given by the elements $[(1, g)]$.*

Proof. Let $g \in \widehat{\mathrm{Spin}(r)} \subset \mathrm{Spin}^r(n)$, i.e., $\lambda_n^r(g) = (1, g_2) \in \mathrm{SO}(n) \times \mathrm{SO}(r)$. Then

$$\begin{aligned}
\hat{\Phi}^{g(\phi)}(f_k f_l)(X) &= \sum_{b=1}^{n} \big\langle X \wedge e_b \cdot \kappa_{r^*}^m(f_k f_l) \cdot g(\phi), g(\phi) \big\rangle e_b \\
&= \sum_{b=1}^{n} \big\langle X \wedge e_b \cdot \kappa_{r^*}^m(g_2(f_k')g_2(f_l')) \cdot g(\phi), g(\phi) \big\rangle e_b \\
&= \sum_{b=1}^{n} \big\langle g(X \wedge e_b \cdot \kappa_{r^*}^m(f_k' f_l') \cdot \phi), g(\phi) \big\rangle e_b \\
&= \sum_{b=1}^{n} \big\langle X \wedge e_b \cdot \kappa_{r^*}^m(f_k' f_l') \cdot \phi, \phi \big\rangle e_b = \hat{\Phi}^{\phi}(f_k' f_l')(X),
\end{aligned}$$

where $g_2(f_k') = f_k$. ■

Lemma 3.17. *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a twisted pure spinor. Let $X \in \mathbb{R}^n$ such that $|X| = 1$. Then, $X \cdot \phi$ is also a twisted pure spinor.*

Proof. Without loss of generality we can assume that $X = e_1$. First, notice that

$$\begin{aligned}
\eta_{kl}^{e_1\cdot\phi} &= \sum_{a<b} \big\langle e_a e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot e_1 \cdot \phi, e_1 \cdot \phi \big\rangle e_a e_b \\
&= \sum_{1<b} \big\langle e_1 e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot e_1 \cdot \phi, e_1 \cdot \phi \big\rangle e_1 e_b + \sum_{2\leq a<b} \big\langle e_a e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot e_1 \cdot \phi, e_1 \cdot \phi \big\rangle e_a e_b \\
&= -\sum_{1<b} \big\langle e_1 e_1 e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, e_1 \cdot \phi \big\rangle e_1 e_b + \sum_{2\leq a<b} \big\langle e_1 e_a e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, e_1 \cdot \phi \big\rangle e_a e_b \\
&= -\sum_{1<b} \big\langle e_1 e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \cdot\phi \big\rangle e_1 e_b + \sum_{2\leq a<b} \big\langle e_a e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \cdot\phi \big\rangle e_a e_b
\end{aligned}$$

has the same coefficients as those in $\eta_{kl}^{e_1\cdot\phi}$ but with some signs changed. More precisely

$$\begin{aligned}
\eta_{kl}^{e_1\cdot\phi}(e_1, e_b) &= -\eta_{kl}^{\phi}(e_1, e_b) \qquad \text{for} \quad 1 \leq b, \\
\eta_{kl}^{e_1\cdot\phi}(e_a, e_b) &= \eta_{kl}^{\phi}(e_a, e_b) \qquad \text{for} \quad 2 \leq a < b.
\end{aligned}$$

Thus

$$\begin{aligned}
\big(\eta_{kl}^{e_1\cdot\phi} + 2\kappa_{r*}^m(f_{kl})\big) \cdot (e_1 \cdot \phi) &= -\left(\sum_{1<b} \big\langle e_1 e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \cdot\phi \big\rangle e_1 e_b\right) \cdot e_1 \cdot \phi \\
&\quad + \left(\sum_{2\leq a<b} \big\langle e_a e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \cdot\phi \big\rangle e_a e_b\right) \cdot e_1 \cdot \phi + 2\kappa_{r*}^m(f_{kl}) \cdot (e_1 \cdot \phi) \\
&= \left(\sum_{1<b} \big\langle e_1 e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \cdot\phi \big\rangle e_1 e_1 e_b\right) \cdot \phi \\
&\quad + \left(\sum_{2\leq a<b} \big\langle e_a e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \cdot\phi \big\rangle e_1 e_a e_b\right) \cdot \phi + 2e_1 \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi \\
&= e_1 \cdot \left(\sum_{1<b} \big\langle e_1 e_b \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \cdot\phi \big\rangle e_1 e_b\right) \cdot \phi
\end{aligned}$$

$$
\begin{aligned}
&+ e_1 \cdot \left(\sum_{2\leq a<b} \big\langle e_a e_b \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \cdot \phi \big\rangle e_a e_b \right) \cdot \phi + e_1 \cdot 2\kappa^m_{r*}(f_{kl}) \cdot \phi \\
&= e_1 \cdot \left(\sum_{a<b} \big\langle e_a e_b \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \cdot \phi \big\rangle e_a e_b \right) \cdot \phi + e_1 \cdot 2\kappa^m_{r*}(f_{kl}) \cdot \phi \\
&= e_1 \cdot \big(\eta^\phi_{kl} + 2\kappa^m_{r*}(f_{kl}) \big) \cdot \phi = 0.
\end{aligned}
$$

Regarding the endomorphism $\hat{\eta}^{e_1\cdot\phi}_{kl}$ we have

$$
\begin{aligned}
\big((\hat{\eta}^{e_1\cdot\phi}_{kl})^2 \big)_{ts} &= \sum_b \big(\hat{\eta}^{e_1\cdot\phi}_{kl}\big)_{tb} \big(\hat{\eta}^{e_1\cdot\phi}_{kl}\big)_{bs} \\
&= \big(\hat{\eta}^{e_1\cdot\phi}_{kl}\big)_{t1} \big(\hat{\eta}^{e_1\cdot\phi}_{kl}\big)_{1s} + \sum_{b\geq 2} \big(\hat{\eta}^{e_1\cdot\phi}_{kl}\big)_{tb} \big(\hat{\eta}^{e_1\cdot\phi}_{kl}\big)_{bs} \\
&= \eta^{e_1\cdot\phi}_{kl}(e_1, e_t)\eta^{e_1\cdot\phi}_{kl}(e_s, e_1) + \sum_{b\geq 2} \eta^{e_1\cdot\phi}_{kl}(e_b, e_t)\eta^{e_1\cdot\phi}_{kl}(e_s, e_b) \\
&= \big(-\eta^{\phi}_{kl}(e_1, e_t)\big)\big(-\eta^{\phi}_{kl}(e_s, e_1)\big) + \sum_{b\geq 2} \eta^{\phi}_{kl}(e_b, e_t)\eta^{\phi}_{kl}(e_s, e_b) \\
&= \sum_b \eta^{\phi}_{kl}(e_b, e_t)\eta^{\phi}_{kl}(e_s, e_b) = -\delta_{ts},
\end{aligned}
$$

i.e.,

$$
\big(\hat{\eta}^{e_1\cdot\phi}_{kl}\big)^2 = -\mathrm{Id}_{\mathbb{R}^n}.
$$

■

3.3 Pure spinors: $r = 2$

We have left out of our discussion the case $r = 2$ due to the following two reasons:

1. The prototypical pure Spin^c spinor is given by $\varphi = u_{1,\ldots,1} \in \Delta_{2n}$. It satisfies the equation

$$
e_{2j-1} \cdot \varphi = \sqrt{-1} e_{2j} \cdot \varphi
$$

for $1 \leq j \leq n$. This means that, as described in the introduction, the orthogonal complex structure determined by φ is the standard one on $\mathbb{R}^{2n}$,

$$
J_0 = \begin{pmatrix} & -1 & & & \\ 1 & & & & \\ & & \ddots & & \\ & & & & -1 \\ & & & 1 & \end{pmatrix}.
$$

Furthermore,

$$
e_{2j-1}e_{2j} \cdot \varphi = \sqrt{-1}\varphi,
$$

so that the associated real 2-form

$$
\eta^\varphi := \sum_{1\leq a<b\leq 2n} \sqrt{-1}\langle e_a e_b \cdot \varphi, \varphi\rangle e_a e_b = -\sum_{a=1}^{n} e_{2a-1}e_{2a}
$$

is such that

$$\eta^{\varphi} \cdot \varphi = -\sum_{a=1}^{n} e_{2a-1}e_{2a} \cdot \varphi = -\sum_{a=1}^{n} \mathrm{i}\varphi = -n\sqrt{-1}\varphi,$$

i.e., the associated 2-form η^{φ} and the spinor φ satisfy

$$\left(\eta^{\varphi} + n\sqrt{-1}\right) \cdot \varphi = 0,$$

which has the coefficient n instead of 2, and describes the $\left(n\sqrt{-1}\right)$-eigenspace of the corresponding Kähler form (see [12]).

2. Recall that Spin(2) is very different from all other spin groups Spin(r), $r \geq 3$, since it is abelian, non-simple and non-simply-connected. All of these differences are somehow reflected by the fact that there are no pure $\mathrm{Spin}^2(2n)$-spinors according to Definition 3.5, except for $n = 2$. Instead, there are spinors satisfying the equations

$$\left(\eta_{12}^{\phi} + n\kappa_2^1(f_{12})\right) \cdot \phi = 0, \tag{3.15}$$

$$\left(\hat{\eta}_{12}^{\phi}\right)^2 = -\mathrm{Id}_{\mathbb{R}^{2n}}, \tag{3.16}$$

with coefficient n instead of 2, just as in the Spin^c description above. However, in this rank, we only need the twisted pure spinor to induce a complex structure, which is dictated by (3.16). Thus (3.15) becomes redundant.

3.4 Existence of pure spinors

In this subsection we present explicit pure spinors for the ranks $r = 3, 7$. Let us define the following maps:

$$\begin{aligned}
G\colon\ \{\pm 1\}^{\times m} &\longrightarrow \{\pm 1\}^{\times 2m},\\
(\varepsilon_1, \ldots, \varepsilon_m) &\longmapsto (\varepsilon_1, \varepsilon_1, \ldots, \varepsilon_m, \varepsilon_m),\\
H\colon\ \{\pm 1\}^{\times m} &\longrightarrow \{0, 1, \ldots, m\},\\
(\varepsilon_1, \ldots, \varepsilon_m) &\longmapsto \sum_{j=1}^{m} \frac{1-\varepsilon_j}{2}.
\end{aligned}$$

Define

$$\{\pm 1\}_j^{\times m} := H^{-1}(j),$$

which is the set of elements in $\{\pm 1\}^{\times m}$ with exactly j entries equal to (-1). Note that

$$\left|\{\pm 1\}_j^{\times m}\right| = \binom{m}{j}.$$

3.4.1 Dimension $n = 4m$, rank $r = 3$

Consider the following spinors

$$\psi_j = \sum_{(\varepsilon_1, \ldots, \varepsilon_m) \in \{\pm 1\}_j^{\times m}} u_{G(\varepsilon_1, \ldots, \varepsilon_m)}, \qquad \varphi_j = \sum_{(\varepsilon_1, \ldots, \varepsilon_m) \in \{\pm 1\}_j^{\times m}} v_{(\varepsilon_1, \ldots, \varepsilon_m)}.$$

The twisted spinor

$$\phi = \sqrt{\frac{3}{m+2}} \frac{1}{\sqrt{m+1}} \sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \psi_j \otimes \varphi_{m-j} \in \Delta_{4m} \otimes \Delta_3^{\otimes m} \tag{3.17}$$

is pure. The 2-forms associated to ϕ are

$$\eta^\phi_{12}=\sum_{j=1}^m(e_{4j-3}e_{4j-2}+e_{4j-1}e_{4j}),\qquad \eta^\phi_{13}=\sum_{j=1}^m(-e_{4j-3}e_{4j-1}+e_{4j-2}e_{4j}),$$
$$\eta^\phi_{23}=\sum_{j=1}^m(-e_{4j-3}e_{4j}-e_{4j-2}e_{4j-1}),$$

which span a copy of $\mathfrak{spin}(3)\in\mathfrak{so}(4m)$. For instance, let us compute

$$\begin{aligned}\eta^\phi_{13}(e_r,e_s)&=\mathrm{Re}\left\langle e_r\wedge e_s\cdot\kappa^m_{3*}(f_{13})\cdot\phi,\phi\right\rangle\\&=\frac{3}{(m+2)(m+1)}\mathrm{Re}\left\langle\sum_{j=0}^m\frac{1}{\binom{m}{j}}\left(e_re_s\cdot\psi_j\otimes\kappa^m_{3*}(f_{13})\cdot\varphi_{m-j}\right),\sum_{j=0}^m\frac{1}{\binom{m}{j}}\psi_j\otimes\varphi_{m-j}\right\rangle.\end{aligned}$$

Consider

$$e_re_s\cdot\psi_j=e_{4k_1-j_1}e_{4k_2-j_2}\cdot\psi_j=\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j}e_{4k_1-j_1}e_{4k_2-j_2}\cdot u_{G(\varepsilon_1,\ldots,\varepsilon_m)},$$

where $4k_1-j_1<4k_2-j_2$, $0\le j_1,\ j_2\le 4$. Define $\left(\varepsilon^0_1,\varepsilon^1_1,\ldots,\varepsilon^0_m,\varepsilon^1_m\right):=(\varepsilon_1,\varepsilon_1,\ldots,\varepsilon_m,\varepsilon_m)$. Note that

$$\begin{aligned}e_{4k-j}\cdot u_{G(\varepsilon_1,\ldots,\varepsilon_m)}&=e_{4k-j}\cdot u_{(\varepsilon^0_1,\varepsilon^1_1,\ldots,\varepsilon^0_m,\varepsilon^1_m)}\\&=-(i)^j(\varepsilon_{m-k+1})^{[\frac{j+1}{2}]}u_{(\varepsilon^0_1,\varepsilon^1_1,\ldots,(-\varepsilon_{m-k+1})^{[\frac{j}{2}]},\ldots,\varepsilon^0_m,\varepsilon^1_m)}.\end{aligned}$$

Thus,

$$\begin{aligned}e_re_s\cdot\psi_j=&\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j}(i)^{j_1+j_2}(\varepsilon_{m-k_1+1})^{[\frac{j_1+1}{2}]}(\varepsilon_{m-k_2+1})^{[\frac{j_2+1}{2}]}\\&\times u_{(\varepsilon^0_1,\varepsilon^1_1,\ldots,(-\varepsilon_{m-k_1+1})^{[\frac{j_1}{2}]},\ldots,(-\varepsilon_{m-k_2+1})^{[\frac{j_2}{2}]},\ldots,\varepsilon^0_m,\varepsilon^1_m)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\kappa^m_{3*}(f_{13})\cdot\varphi_{m-j}&=\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j}}\kappa^m_{3*}(f_{13})\cdot v_{(\varepsilon_1,\ldots,\varepsilon_m)}\\&=\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j}}\left(\sum_{l=1}^m\varepsilon_l\right)v_{(\varepsilon_1,\ldots,-\varepsilon_l,\ldots,\varepsilon_m)}\\&=\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}}(m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)}\\&\quad-\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}}(j+1)v_{(\varepsilon_1,\ldots,\varepsilon_m)}.\end{aligned}$$

For $k_1<k_2$

$$\begin{aligned}&\Big\langle u_{(\varepsilon^0_1,\varepsilon^1_1,\ldots,(-\varepsilon_{m-k_1+1})^{[\frac{j_1}{2}]},\ldots,(-\varepsilon_{m-k_2+1})^{[\frac{j_2}{2}]},\ldots,\varepsilon^0_m,\varepsilon^1_m)}\otimes v_{(\varepsilon_1,\ldots,\varepsilon_m)},\\&\quad u_{(\tilde\varepsilon^0_1,\tilde\varepsilon^1_1,\ldots,\tilde\varepsilon^0_m,\tilde\varepsilon^1_m)}\otimes v_{(\tilde\varepsilon_1,\ldots,\tilde\varepsilon_m)}\Big\rangle=0.\end{aligned}$$

Thus, for $k_1 < k_2$

$$\eta^{\phi}_{13}(e_{4k_1-j_1}, e_{4k_2-j_2}) = 0.$$

Now consider $k_1 = k_2 = k$. In this case

$$\begin{aligned}\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) = \frac{3}{(m+2)(m+1)} &\operatorname{Re}\Big\langle \sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \\ &\times \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j} (i)^{j_1+j_2}(\varepsilon_{m-k+1})^{[\frac{j_1+1}{2}]}(\varepsilon_{m-k+1})^{[\frac{j_2+1}{2}]} \\ &\times u_{(\varepsilon_1^0,\varepsilon_1^1,\ldots,(-\varepsilon_{m-k+1})^{[\frac{j_2}{2}]},\ldots,(-\varepsilon_{m-k+1})^{[\frac{j_1}{2}]},\ldots,\varepsilon_m^0,\varepsilon_m^1)} \\ &\otimes \Big[\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}} (m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)} \\ &- \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}} (j+1)v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big], \\ &\sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j} u_{G(\varepsilon_1,\ldots,\varepsilon_m)} \otimes \sum_{(\varepsilon_1\ldots\varepsilon_m)\in\{\pm1\}^m_{m-j}} v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big\rangle.\end{aligned}$$

We have the following cases:

1. If $[\frac{j_1}{2}] = [\frac{j_2}{2}]$ so that $j_1 + j_2 = 1 \pmod 4$:

$$\begin{aligned}\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) = &\frac{3}{(m+2)(m+1)} \\ &\times \operatorname{Re}\Big\{ i \Big\langle \sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j} \varepsilon_{m-k+1} u_{G(\varepsilon_1,\ldots,\varepsilon_m)} \\ &\otimes \Big[\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}} (m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)} \\ &- \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}} (j+1)v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big], \\ &\sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j} u_{G(\varepsilon_1\ldots\varepsilon_m)} \\ &\otimes \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j}} v_{(\varepsilon_1,\ldots,\varepsilon_m)} \Big\rangle \Big\} = 0.\end{aligned}$$

2. If $[\frac{j_1}{2}] \neq [\frac{j_2}{2}]$:

    i) $j_1 = 2$, $j_2 = 0$ and $j_1 + j_2 = 2$

$$\begin{aligned}\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) = \frac{-3}{(m+2)(m+1)} &\Big\langle \sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \\ &\times \Big[- \sum_{(\varepsilon_1,\ldots,\hat{\varepsilon}_{m-k+1},\ldots,\varepsilon_m)\in\{\pm1\}^{m-1}_{j-1}} u_{G(\varepsilon_1,\ldots,\varepsilon_{m-k}1,\ldots,\varepsilon_m)}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{(\varepsilon_1,\dots,\hat{\varepsilon}_{m-k+1},\dots,\varepsilon_m)\in\{\pm1\}^{m-1}_j} u_{G(\varepsilon_1,\dots,\varepsilon_{m-k-1},\dots,\varepsilon_m)}\Bigg]\\
&\otimes \Bigg[\sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}} (m-(j-1))v_{(\varepsilon_1,\dots,\varepsilon_m)}\\
&- \sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}} (j+1)v_{(\varepsilon_1,\dots,\varepsilon_m)}\Bigg],\\
&\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_j} u_{G(\varepsilon_1,\dots,\varepsilon_m)}\\
&\otimes \sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_{m-j}} v_{(\varepsilon_1,\dots,\varepsilon_m)}\Bigg\rangle\\
= {}&\frac{-3}{(m+2)(m+1)}\Bigg\rangle\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\\
&\times\Bigg\{-\Bigg[\sum_{(\varepsilon_1,\dots,\hat{\varepsilon}_{m-k+1},\dots,\varepsilon_m)\in\{\pm1\}^{m-1}_{j-1}} u_{G(\varepsilon_1,\dots,\varepsilon_{m-k}1,\dots,\varepsilon_m)}\\
&\otimes \sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}} (m-(j-1))v_{(\varepsilon_1,\dots,\varepsilon_m)}\Bigg]\\
&-\Bigg[\sum_{(\varepsilon_1,\dots,\hat{\varepsilon}_{m-k+1},\dots,\varepsilon_m)\in\{\pm1\}^{m-1}_{j}} u_{G(\varepsilon_1,\dots,\varepsilon_{m-k-1},\dots,\varepsilon_m)}\\
&\otimes \sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}} (j+1)v_{(\varepsilon_1,\dots,\varepsilon_m)}\Bigg]\Bigg\},\\
&\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_j} u_{G(\varepsilon_1,\dots,\varepsilon_m)}\\
&\otimes \sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_{m-j}} v_{(\varepsilon_1,\dots,\varepsilon_m)}\Bigg\rangle\\
= {}&\frac{-3}{(m+2)(m+1)}\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\Bigg[-\binom{m-1}{j-1}\binom{m}{m-j+1}\\
&\times\frac{1}{\binom{m}{j-1}}(m-j+1)-\binom{m-1}{j}\binom{m}{m-j-1}\frac{1}{\binom{m}{j+1}}(j+1)\Bigg]\\
= {}&\frac{-3}{m(m+2)(m+1)}\sum_{j=0}^{m}\left(2j^2-2mj-m\right)=1.
\end{aligned}
$$

ii) If $j_1=2$, $j_2=1$ and $j_1+j_2=3$

$$
\begin{aligned}
\eta^{\phi}_{13}(e_{4k-j_1},e_{4k-j_2}) = {}&\frac{3}{(m+2)(m+1)}\,\mathrm{Re}\Bigg\{-\mathrm{i}\Bigg\langle\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\\
&\times\sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm1\}^m_j} u_{G(\varepsilon_1,\dots,-\varepsilon_{m-k+1},\dots,\varepsilon_m)}
\end{aligned}
$$

$$
\begin{aligned}
&\otimes \Big[\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}} (m-(j-1)) v_{(\varepsilon_1,\ldots,\varepsilon_m)} \\
&- \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}} (j+1) v_{(\varepsilon_1,\ldots,\varepsilon_m)} \Big], \\
&\sum_{j=0}^m \frac{1}{\binom{m}{j}} \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j} u_{G(\varepsilon_1,\ldots,\varepsilon_m)} \\
&\otimes \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j}} v_{(\varepsilon_1,\ldots,\varepsilon_m)} \Big\rangle \Big\} = 0.
\end{aligned}
$$

iii) If $j_1 = 3$, $j_2 = 0$ and $j_1 + j_2 = 3$

$$\eta^\phi_{13}(e_{4k-j_1}, e_{4k-j_2}) = 0.$$

iv) If $j_1 = 3$, $j_2 = 1$ and $j_1 + j_2 = 4$

$$\eta^\phi_{13}(e_{4k-j_1}, e_{4k-j_2}) = -1.$$

The spinor ϕ is annihilated by

$$\eta^\phi_{12} + 2\kappa^m_{3*}(f_{12}), \qquad \eta^\phi_{13} + 2\kappa^m_{3*}(f_{13}), \qquad \eta^\phi_{23} + 2\kappa^m_{3*}(f_{23}),$$

and the forms

$$
\begin{aligned}
\beta^1_{ij} &= e_{4i-3}e_{4j-3} + e_{4i-2}e_{4j-2} + e_{4i-1}e_{4j-1} + e_{4i}e_{4j}, \\
\beta^2_{ij} &= e_{4i-3}e_{4j-2} - e_{4i-1}e_{4j}, \\
\beta^3_{ij} &= e_{4i-3}e_{4j-1} + e_{4i-2}e_{4j}, \\
\beta^4_{ij} &= e_{4i-3}e_{4j} - e_{4i-2}e_{4j-1},
\end{aligned}
$$

where $1 \leq i \leq j \leq m$. Thus,

$$
\begin{aligned}
&\mathrm{span}\left(\left\{\beta^s_{ij} \,|\, 1 \leq i \leq j \leq m,\ 1 \leq s \leq 4\right\} \cup \left\{\eta^\phi_{kl} + 2\, f_{kl} \,|\, 1 \leq k < l \leq 3\right\}\right) \\
&\qquad = \mathfrak{sp}(m) \oplus \widetilde{\mathfrak{spin}}(3) \subset \mathfrak{spin}(4m) \oplus \mathfrak{spin}(3)
\end{aligned}
$$

annihilates ϕ, which is consistent with Lemma 3.15. Its isotropy group is

$$\mathrm{Sp}(m)\mathrm{Sp}(1) \equiv \mathrm{Sp}(m)\widetilde{\mathrm{Spin}}(3) \subset \mathrm{Spin}^3(4m).$$

Let us check, for instance,

$$\left(\eta^\phi_{13} + 2\kappa^m_{3*}(f_{13})\right)\phi = \sqrt{\frac{3}{m+2}} \frac{1}{\sqrt{m+1}} \sum_{j=0}^m \frac{1}{\binom{m}{j}} \left\{\eta^\phi_{13} \cdot \psi_j \otimes \varphi_{m-j} + \psi_j \otimes 2\kappa^m_{3*}(f_{13}) \cdot \varphi_{m-j}\right\}.$$

Observe that

$$\eta^\phi_{13} \cdot \psi_j = -2\big[(j+1)\psi_{j+1} + (j-1-m)\psi_{j-1}\big],$$

so that

$$\begin{aligned}\big(\eta_{13}^{\phi}+2\kappa_{3*}^{m}(f_{13})\big)\phi &= \frac{2\sqrt{3}}{\sqrt{(m+2)(m+1)}}\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\\ &\quad\times\big\{\big[(-j-1)\psi_{j+1}-(j-1-m)\psi_{j-1}\big]\otimes\varphi_{m-j}\\ &\quad+\psi_j\otimes\big[(m-j+1)\varphi_{m-j+1}-(j+1)\varphi_{m-j-1}\big]\big\}\\ &= \frac{2\sqrt{3}}{\sqrt{(m+2)(m+1)}}\sum_{j=0}^{m}\left\{\left[\frac{-j}{\binom{m}{j-1}}+\frac{m-j+1}{\binom{m}{j}}\right]\psi_j\otimes\varphi_{m-j+1}\right.\\ &\quad\left.+\left[\frac{-j+m}{\binom{m}{j+1}}-\frac{j+1}{\binom{m}{j}}\right]\psi_j\otimes\varphi_{m-j-1}\right\}=0.\end{aligned}$$

3.4.2 Dimension $n=8$, rank $r=7$

The spinor $\phi_1\in\Delta_8^+\otimes\Delta_7$ given as follows

$$\begin{aligned}\phi_1=\frac{1}{2}\big[&u_{(-1,-1,-1,-1)}\otimes v_{(1,1,1)}-u_{(1,-1,-1,1)}\otimes v_{(1,1,-1)}+u_{(1,-1,1,-1)}\otimes v_{(1,-1,1)}\\ &-u_{(1,1,-1,-1)}\otimes v_{(1,-1,-1)}-u_{(-1,-1,1,1)}\otimes v_{(-1,1,1)}+u_{(-1,1,-1,1)}\otimes v_{(-1,1,-1)}\\ &-u_{(-1,1,1,-1)}\otimes v_{(-1,-1,1)}+u_{(1,1,1,1)}\otimes v_{(-1,-1,-1)}\big]\end{aligned}\tag{3.18}$$

is pure. The 2-forms associated to ϕ_1 are

$$\begin{aligned}
\eta_{12}^{\phi_1}&=e_1e_2-e_3e_4+e_5e_6+e_7e_8, & \eta_{13}^{\phi_1}&=e_1e_3+e_2e_4+e_5e_7-e_6e_8,\\
\eta_{14}^{\phi_1}&=e_1e_4-e_2e_3+e_5e_8+e_6e_7, & \eta_{15}^{\phi_1}&=e_1e_5-e_2e_6-e_3e_7-e_4e_8,\\
\eta_{16}^{\phi_1}&=e_1e_6+e_2e_5+e_3e_8-e_4e_7, & \eta_{17}^{\phi_1}&=e_1e_7-e_2e_8+e_3e_5+e_4e_6,\\
\eta_{23}^{\phi_1}&=-e_1e_4+e_2e_3+e_5e_8+e_6e_7, & \eta_{24}^{\phi_1}&=e_1e_3+e_2e_4-e_5e_7+e_6e_8,\\
\eta_{25}^{\phi_1}&=e_1e_6+e_2e_5-e_3e_8+e_4e_7, & \eta_{26}^{\phi_1}&=-e_1e_5+e_2e_6-e_3e_7-e_4e_8,\\
\eta_{27}^{\phi_1}&=e_1e_8+e_2e_7+e_3e_6-e_4e_5, & \eta_{34}^{\phi_1}&=-e_1e_2+e_3e_4+e_5e_6+e_7e_8,\\
\eta_{35}^{\phi_1}&=e_1e_7+e_2e_8+e_3e_5-e_4e_6, & \eta_{36}^{\phi_1}&=-e_1e_8+e_2e_7+e_3e_6+e_4e_5,\\
\eta_{37}^{\phi_1}&=-e_1e_5-e_2e_6+e_3e_7-e_4e_8, & \eta_{45}^{\phi_1}&=e_1e_8-e_2e_7+e_3e_6+e_4e_5,\\
\eta_{46}^{\phi_1}&=e_1e_7+e_2e_8-e_3e_5+e_4e_6, & \eta_{47}^{\phi_1}&=-e_1e_6+e_2e_5+e_3e_8+e_4e_7,\\
\eta_{56}^{\phi_1}&=e_1e_2+e_3e_4+e_5e_6-e_7e_8, & \eta_{57}^{\phi_1}&=e_1e_3-e_2e_4+e_5e_7+e_6e_8,\\
\eta_{67}^{\phi_1}&=e_1e_4+e_2e_3-e_5e_8+e_6e_7,
\end{aligned}$$

and

$$\mathrm{span}\big\{\hat{\eta}_{kl}^{\phi_1}\,|\,1\le k<l\le 7\big\}=\mathfrak{spin}(7)\subset\mathfrak{so}(8).$$

The subalgebra

$$\mathrm{span}\big\{\eta_{kl}^{\phi_1}+2f_{kl}\,|\,1\le k<l\le 7\big\}=\widetilde{\mathfrak{spin}}(7)\subset\mathfrak{spin}(8)\oplus\mathfrak{spin}(7)$$

annihilates ϕ_1, which is consistent with Lemma 3.15. Its isotropy group is

$$\widetilde{\mathrm{Spin}}(7)\subset\mathrm{Spin}^7(8).$$

Here, we present another twisted pure spinor $\phi_2 = -\mathrm{i}e_1 \cdot \phi_1 \in \Delta_8^- \otimes \Delta_7$ (by Lemma 3.17)

$$\begin{aligned}\phi_2 = \frac{1}{2}\big[&u_{(-1,-1,-1,1)} \otimes v_{(1,1,1)} - u_{(1,-1,-1,-1)} \otimes v_{(1,1,-1)} + u_{(1,-1,1,1)} \otimes v_{(1,-1,1)}\\ &- u_{(1,1,-1,1)} \otimes v_{(1,-1,-1)} - u_{(-1,-1,1,-1)} \otimes v_{(-1,1,1)} + u_{(-1,1,-1,-1)} \otimes v_{(-1,1,-1)}\\ &- u_{(-1,1,1,1)} \otimes v_{(-1,-1,1)} + u_{(1,1,1,-1)} \otimes v_{(-1,-1,-1)}\big].\end{aligned}$$

The 2-forms associated to ϕ_2 are

$$\begin{aligned}
\eta^{\phi_2}_{12} &= -e_1e_2 - e_3e_4 + e_5e_6 + e_7e_8, & \eta^{\phi_2}_{13} &= -e_1e_3 + e_2e_4 + e_5e_7 - e_6e_8,\\
\eta^{\phi_2}_{14} &= -e_1e_4 - e_2e_3 + e_5e_8 + e_6e_7, & \eta^{\phi_2}_{15} &= -e_1e_5 - e_2e_6 - e_3e_7 - e_4e_8,\\
\eta^{\phi_2}_{16} &= -e_1e_6 + e_2e_5 + e_3e_8 - e_4e_7, & \eta^{\phi_2}_{17} &= -e_1e_7 - e_2e_8 + e_3e_5 + e_4e_6,\\
\eta^{\phi_2}_{23} &= e_1e_4 + e_2e_3 + e_5e_8 + e_6e_7, & \eta^{\phi_2}_{24} &= -e_1e_3 + e_2e_4 - e_5e_7 + e_6e_8,\\
\eta^{\phi_2}_{25} &= -e_1e_6 + e_2e_5 - e_3e_8 + e_4e_7, & \eta^{\phi_2}_{26} &= e_1e_5 + e_2e_6 - e_3e_7 - e_4e_8,\\
\eta^{\phi_2}_{27} &= -e_1e_8 + e_2e_7 + e_3e_6 - e_4e_5, & \eta^{\phi_2}_{34} &= e_1e_2 + e_3e_4 + e_5e_6 + e_7e_8,\\
\eta^{\phi_2}_{35} &= -e_1e_7 + e_2e_8 + e_3e_5 - e_4e_6, & \eta^{\phi_2}_{36} &= e_1e_8 + e_2e_7 + e_3e_6 + e_4e_5,\\
\eta^{\phi_2}_{37} &= e_1e_5 - e_2e_6 + e_3e_7 - e_4e_8, & \eta^{\phi_2}_{45} &= -e_1e_8 - e_2e_7 + e_3e_6 + e_4e_5,\\
\eta^{\phi_2}_{46} &= -e_1e_7 + e_2e_8 - e_3e_5 + e_4e_6, & \eta^{\phi_2}_{47} &= e_1e_6 + e_2e_5 + e_3e_8 + e_4e_7,\\
\eta^{\phi_2}_{56} &= -e_1e_2 + e_3e_4 + e_5e_6 - e_7e_8, & \eta^{\phi_2}_{57} &= -e_1e_3 - e_2e_4 + e_5e_7 + e_6e_8,\\
\eta^{\phi_2}_{67} &= -e_1e_4 + e_2e_3 - e_5e_8 + e_6e_7, & &
\end{aligned}$$

and we have another copy

$$\mathrm{span}\left\{\hat\eta^{\phi_2}_{kl} \,|\, 1 \le k < l \le 7\right\} = \mathfrak{spin}(7) \subset \mathfrak{so}(8).$$

The subalgebra

$$\mathrm{span}\left\{\eta^{\phi_2}_{kl} + 2f_{kl} \,|\, 1 \le k < l \le 7\right\} = \widetilde{\mathfrak{spin}}(7) \subset \mathfrak{spin}(8) \oplus \mathfrak{spin}(7)$$

annihilates ϕ_2. Its isotropy group is another copy

$$\widetilde{\mathrm{Spin}}(7) \subset \mathrm{Spin}^7(8).$$

The common annihilator of the twisted pure spinors ϕ_1 and ϕ_2 is generated by the following elements of $\mathfrak{spin}(8) \oplus \mathfrak{spin}(7)$:

$$\begin{aligned}
&e_3e_4 - e_7e_8 - f_1f_2 + f_5f_6, && e_2e_4 - e_6e_8 + f_1f_3 - f_5f_7,\\
&e_2e_3 - e_5e_8 - f_1f_4 + f_6f_7, && e_2e_6 + e_4e_8 - f_1f_5 - f_3f_7,\\
&e_2e_5 + e_3e_8 + f_1f_6 + f_4f_7, && e_2e_3 + e_6e_7 + f_2f_3 + f_6f_7,\\
&e_2e_4 - e_5e_7 + f_2f_4 - f_5f_7, && e_2e_5 + e_4e_7 + f_2f_5 + f_4f_5,\\
&e_2e_6 - e_3e_7 - f_3f_7 + f_2f_6, && e_4e_6 - e_2e_8 - f_3f_5 + f_1f_7,\\
&e_3e_6 + e_2e_7 + f_3f_6 + f_2f_7, && e_2e_7 - e_4e_5 - f_4f_5 + f_2f_7,\\
&e_2e_8 - e_3e_5 - f_1f_7 + f_4f_6, && e_3e_4 + e_5e_6 + f_3f_4 + f_5f_6,
\end{aligned}$$

as can be seen easily by taking appropriate sums of the generators. This elements span a copy of $\mathfrak{g}_2$.

Let us remark that the two copies of $\mathfrak{spin}(7)$ within $\mathfrak{so}(8)$ provided by the sets of 2-forms associated to the spinors ϕ_1 and ϕ_2 are related by triality, and their intersection is a copy of $\mathfrak{g}_2$.

4 Special Riemannian holonomy

In this section, we present the geometrical consequences of the existence of parallel twisted pure spinors. In particular, we establish a correspondence between special Riemannnian holonomies and parallel twisted pure spinors.

4.1 Generic holonomy $\mathrm{SO}(n)$

Proposition 4.1 ([10]). *Every oriented Riemannian manifold admits a spinorially twisted spin structure such that an associated spinor bundle admits a parallel spinor field.*

Indeed, there exists a lift of the horizontal (diagonal) map of the following diagram

$$\begin{array}{ccc} & & \mathrm{Spin}(n)\times_{\mathbb{Z}_2}\mathrm{Spin}(n) \\ & \nearrow & \downarrow \\ \mathrm{SO}(n) & \longrightarrow & \mathrm{SO}(n)\times\mathrm{SO}(n). \end{array}$$

Let $\mathcal{B}$ be the unitary basis of Δ_n described in Section 2 and γ_n be the corresponding real or quaternionic structure of Δ_n. The twisted spinor $\phi_0 \in \Delta_n \otimes \Delta_n$,

$$\phi_0 := \sum_{\psi\in\mathcal{B}} \psi \otimes \gamma_n(\psi)$$

is $\mathrm{SO}(n)$ invariant.

Proposition 4.2 ([10]). *The 2-forms associated to ϕ_0 are multiples of the basic 2-forms $e_p \wedge e_q$ of $\mathfrak{so}(n)$,*

$$\eta^{\phi_0}_{pq} = 2^{[n/2]} e_p \wedge e_q.$$

Note that ϕ_0 is not pure. However, it satisfies the equations

$$e_p e_q \cdot \phi_0 + \kappa^1_{n*}(f_p f_q) \cdot \phi_0 = 0,$$

for $1 \le p < q \le n$.

4.2 Holonomy reduction due to parallel twisted pure spinors

Definition 4.3. Let M be a Spin^r Riemannian manifold and F its (locally defined) auxiliary rank r Riemannian vector bundle.

- A twisted spinor field $\phi \in \Gamma(S(M,F,m))$ is called pure if ϕ_p is pure for every $p \in M$.
- Given a connection θ on the auxiliary bundle $P_{\mathrm{SO}(r)}$, a twisted spinor field $\psi \in \Gamma(S(M,F,m))$ is parallel if

  $$\nabla^\theta_X \psi = 0$$

  for all $X \in \Gamma(TM)$.

Theorem 4.4. *Let M be a Spin^r manifold carrying a twisted pure spinor field $\phi \in \Gamma(S(M,F,m))$ for some $m \in \mathbb{N}$, where $r \ge 3$. Then, in terms of local orthonormal frames $(f_1,\ldots,f_r)$ of F,*

1) *there is a well-defined subbundle $Q \subset \bigwedge^2 T^*M$ locally generated by $\{\eta^\phi_{kl} \,|\, 1 \le k < l \le r\}$;*

2) *there is a well-defined subbundle* $\hat{Q}$ *of* $\mathrm{End}^-(TM)$ *locally generated by* $\{\hat{\eta}^\phi_{kl}\,|\,1\leq k<l\leq r\}$ *whose fibre is isomorphic to* $\mathfrak{spin}(r)$*;*

3) *there is a rank* r *almost even-Clifford Hermitian structure induced by the local maps*

$$\begin{aligned}\left(\mathrm{Cl}^0(F)\right)^2 &\longrightarrow \mathrm{End}^-(TM),\\ f_{ij} &\mapsto \hat{\eta}^\phi_{kl}.\end{aligned}$$

Proof. The proof follows from Proposition 3.11 and Lemma 3.13. ■

Theorem 4.5. *Let* M *be a* Spin^r *Riemannian manifold whose auxiliary bundle* $P_{\mathrm{SO}(r)}$ *is endowed with a connection* θ*. If* M *carries a parallel twisted pure spinor field* $\phi\in\Gamma(S(M,F,m))$ *for some* $m\in\mathbb{N}$*,* $r\geq 3$*, then the manifold* M *admits a rank* r *parallel even-Clifford Hermitian structure and its holonomy algebra is contained in one of the algebras* $\mathfrak{g}$ *of Table 4.*

r (mod 8)	$\mathfrak{g}$
0	$\mathfrak{so}(m_1)\oplus\mathfrak{so}(m_2)\oplus\mathfrak{spin}(r)$
1, 7	$\mathfrak{so}(m)\oplus\mathfrak{spin}(r)$
2, 6	$\mathfrak{u}(m)\oplus\mathfrak{spin}(r)$
3, 5	$\mathfrak{sp}(m)\oplus\mathfrak{spin}(r)$
4	$\mathfrak{sp}(m_1)\oplus\mathfrak{sp}(m_2)\oplus\mathfrak{spin}(r)$

Table 4.

Proof. Suppose $\nabla^\theta\phi=0$. Let $(e_1,\ldots,e_n)$ and $(f_1,\ldots,f_r)$ be local orthonormal frames for TM and F respectively, and $X\in\Gamma(TM)$. Recall that

$$\begin{aligned}\nabla_X e_j &= \omega_{1j}(X)e_1+\cdots+\omega_{nj}(X)e_n,\\ \nabla_X f_j &= \theta_{1j}(X)f_1+\cdots+\theta_{rj}(X)f_r.\end{aligned}$$

On the one hand

$$\begin{aligned}\nabla_X\left(\eta^\phi_{kl}(e_s,e_t)\right) &= \left(\nabla_X\eta^\phi_{kl}\right)(e_s,e_t)+\eta^\phi_{kl}(\nabla_X e_s,e_t)+\eta^\phi_{kl}(e_s,\nabla_X e_t)\\ &= \left(\nabla_X\eta^\phi_{kl}\right)(e_s,e_t)+\sum_{a=1}^n\omega_{as}(X)\eta^\phi_{kl}(e_a,e_t)+\sum_{a=1}^n\omega_{at}(X)\eta^\phi_{kl}(e_s,e_a)\end{aligned}$$

and, on the other,

$$\begin{aligned}\nabla_X\left(\eta^\phi_{kl}(e_s,e_t)\right) &= \nabla_X\left\langle e_se_t\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle\\ &= \left\langle\nabla_X(e_se_t)\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle+\left\langle e_se_t\cdot\nabla_X(\kappa^m_{r*}(f_{kl}))\cdot\phi,\phi\right\rangle\\ &\quad+\left\langle e_se_t\cdot\kappa^m_{r*}(f_{kl})\cdot\nabla^\theta_X\phi,\phi\right\rangle+\left\langle e_se_t\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\nabla^\theta_X\phi\right\rangle\\ &= \left\langle\nabla_X(e_se_t)\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle+\left\langle e_se_t\cdot\nabla_X(\kappa^m_{r*}(f_{kl}))\cdot\phi,\phi\right\rangle\\ &= \left\langle\sum_{a=1}^n\omega_{as}(X)e_ae_t\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle+\left\langle\sum_{a=1}^n\omega_{at}(X)e_se_a\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle\\ &\quad+\left\langle\sum_{a=1}^r\theta_{ak}(X)e_se_t\cdot\kappa^m_{r*}(f_{al})\cdot\phi,\phi\right\rangle+\left\langle\sum_{a=1}^r\theta_{al}(X)e_se_t\cdot\kappa^m_{r*}(f_{ka}))\cdot\phi,\phi\right\rangle\end{aligned}$$

$$= \sum_{a=1}^{n} \omega_{as}(X) \big\langle e_a e_t \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \big\rangle + \sum_{a=1}^{n} \omega_{at}(X) \big\langle e_s e_a \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \big\rangle$$
$$+ \sum_{a=1}^{r} \theta_{ak}(X) \big\langle e_s e_t \cdot \kappa^m_{r*}(f_{al}) \cdot \phi, \phi \big\rangle + \sum_{a=1}^{r} \theta_{al}(X) \big\langle e_s e_t \cdot \kappa^m_{r*}(f_{ka}) \cdot \phi, \phi \big\rangle$$
$$= \sum_{a=1}^{n} \omega_{as}(X) \eta^\phi_{kl}(e_a, e_t) + \sum_{a=1}^{n} \omega_{at}(X) \eta^\phi_{kl}(e_s, e_a)$$
$$+ \sum_{a=1}^{r} \theta_{ak}(X) \eta^\phi_{al}(e_s, e_t) + \sum_{a=1}^{r} \theta_{al}(X) \eta^\phi_{ka}(e_s, e_t).$$

Thus,

$$\big(\nabla_X \big(\eta^\phi_{kl}\big)\big)(e_s, e_t) = \sum_{a=1}^{r} \theta_{ak}(X) \eta^\phi_{al}(e_s, e_t) + \sum_{a=1}^{r} \theta_{al}(X) \eta^\phi_{ka}(e_s, e_t),$$

i.e.,

$$\nabla \eta^\phi_{kl} = \sum_{a=1}^{r} \theta_{ak} \otimes \eta^\phi_{al} + \theta_{al} \otimes \eta^\phi_{ka}.$$

Furthermore, for $X, Y \in \Gamma(TM)$,

$$R^M(X,Y)\big(\hat{\eta}^\phi_{kl}\big) = \sum_{a=1}^{r} \Theta_{ak}(X,Y) \hat{\eta}^\phi_{al} + \Theta_{al}(X,Y) \hat{\eta}^\phi_{ka}, \tag{4.1}$$

which means

$$R^M(X,Y) \in N_{\mathfrak{so}(n)}(\mathfrak{spin}(r)).$$

Such normalizer subalgebras were computed in [2] and the corresponding Lie groups were computed in [1]. ■

4.3 Curvature calculations

In this section we carry out various curvature calculations (as in [27]) which lead us to the generalization of the second Seiberg–Witten equation.

Recall that if ξ and ζ are antisymmetic two forms, and $\hat{\xi}$ and $\hat{\zeta}$ are the corresponding antisymmetric endomorphisms with respect to a positive definite inner product, then

$$\mathbf{tr}\big(\hat{\xi}\hat{\zeta}\big) = -2\xi \bullet \zeta,$$

where $\bullet$ denotes the induced inner product on 2-forms.

If $X, Y, Z, W \in \Gamma(TM)$, let

$$R^M(X,Y,Z,W) = \big\langle R^M(X,Y), Z, W \big\rangle,$$

denote the curvature 4-form of M,

$$\Theta = \sum_{1 \le k < l \le r} \Theta_{kl} \otimes f_{kl}$$

the curvature of θ and

$$\eta^\phi = \sum_{1 \le k < l \le r} \eta^\phi_{kl} \otimes f_{kl},$$

the 2-form with values in $\bigwedge^2 F$ associated to ϕ.

Lemma 4.6. *Let M be a* Spin^r *Riemannian manifold whose auxiliary bundle $P_{\mathrm{SO}(r)}$ is endowed with a connection θ. If M carries a parallel twisted pure spinor field $\phi \in \Gamma(S(M,F,m))$ for some $m \in \mathbb{N}$, $r \geq 3$, then*

$$\Theta = -\frac{4}{\dim(M)} R^M \bullet \eta^\phi, \tag{4.2}$$

where $R^M \bullet \eta^\phi$ denotes the image of $R^M \otimes \eta^\phi$ under the map (contraction of underlined factors)

$$\left(\textstyle\bigwedge^2 T^*M \otimes \underline{\textstyle\bigwedge^2 T^*M}\right) \otimes \left(\underline{\textstyle\bigwedge^2 T^*M} \otimes \textstyle\bigwedge^2 F\right) \xrightarrow{\ \bullet\ } \textstyle\bigwedge^2 T^*M \otimes \textstyle\bigwedge^2 F.$$

Proof. Let $X, Y \in \Gamma(TM)$. Since $r \geq 3$, multiply (4.1) by $\hat{\eta}^\phi_{kp}$ on the left, $p \neq k, l$,

$$\begin{aligned}
&\hat{\eta}^\phi_{kp} R(X,Y) \hat{\eta}^\phi_{kl} - \hat{\eta}^\phi_{pl} R(X,Y) \\
&\quad = \Theta_{kp}(X,Y)\eta^\phi_{kl} + \Theta_{pl}(X,Y)(-\mathrm{Id}_{TM}) + \sum_{a \neq k,l,p} \Theta_{ak}(X,Y)\eta^\phi_{kp}\hat{\eta}^\phi_{al} + \Theta_{al}(X,Y)\hat{\eta}^\phi_{pa},
\end{aligned}$$

where the last summation is absent if $r = 3$. Since

$$\begin{aligned}
&\mathbf{tr}\left(\eta^\phi_{kp}\hat{\eta}^\phi_{lq}\right) = 0, \qquad \text{if} \quad r \neq 4, \\
&\mathbf{tr}\left(\eta^\phi_{kl}\right) = 0,
\end{aligned}$$

it makes sense to take the trace on the right-hand side in order to isolate $\Theta_{pl}(X,Y)$. The left-hand side gives

$$\begin{aligned}
&\mathbf{tr}\left(\hat{\eta}^\phi_{kp} R(X,Y)\hat{\eta}^\phi_{kl} - \hat{\eta}^\phi_{pl} R(X,Y)\right) \\
&\quad = \mathbf{tr}\left(\hat{\eta}^\phi_{kp} R(X,Y)\hat{\eta}^\phi_{kl}\right) - \mathbf{tr}\left(\hat{\eta}^\phi_{pl} R(X,Y)\right) = 2\mathbf{tr}\left(\hat{\eta}^\phi_{lp} R(X,Y)\right),
\end{aligned}$$

and the right-hand side

$$\begin{aligned}
&\mathbf{tr}\left(\Theta_{kp}(X,Y)\hat{\eta}^\phi_{kl} + \Theta_{pl}(X,Y)(-\mathrm{Id}_{TM}) + \sum_{q \neq k,l,p} \Theta_{kq}(X,Y)\eta^\phi_{kp}\hat{\eta}^\phi_{lq} + \Theta_{ql}(X,Y)\hat{\eta}^\phi_{pq}\right) \\
&\quad = -\dim(M)\Theta_{pl}(X,Y).
\end{aligned}$$

Thus

$$2\mathbf{tr}\left(\hat{\eta}^\phi_{pl} R(X,Y)\right) = \dim(M)\Theta_{pl}(X,Y), \tag{4.3}$$

which proves Lemma 4.6. ■

Remark 4.7. (4.2) is, in fact, the prototype of the second Clifford monopole equation, where $-R^M\bullet$ can be substituted by a symmetric endomorphisms of $\bigwedge^2 T^*M$.

Lemma 4.8. *Let M be a* Spin^r *n-dimensional Riemannian manifold whose auxiliary bundle $P_{\mathrm{SO}(r)}$ is endowed with a connection θ. If M carries a parallel twisted pure spinor field $\phi \in \Gamma(S(M,F,m))$ for some $m \in \mathbb{N}$, $r \geq 3$, $r \neq 4$, $n \neq 8$, $n + 4r - 16 \neq 0$, $n + 8r - 16 \neq 0$, then*

$$\hat{\Theta}_{kl} = \frac{\mathrm{R}}{n\left(\frac{n}{4} + 2r - 4\right)} \hat{\eta}^\phi_{kl}$$

for $1 \leq k < l \leq r$, and M is Einstein.

Proof. Let $X, Y \in \Gamma(TM)$. In order to simplify notation and carry out further calculations, let

$$\begin{aligned}\vartheta_{pl}(X,Y) &:= \mathbf{tr}\big(\hat{\eta}^{\phi}_{pl}R(X,Y)\big) = \mathbf{tr}\big(R(X,Y)\hat{\eta}^{\phi}_{pl}\big)\\ &= \sum_i \big\langle R(X,Y)\hat{\eta}^{\phi}_{pl}(e_i), e_i\big\rangle = -\sum_i \big\langle R(X,Y)(e_i), \hat{\eta}^{\phi}_{pl}(e_i)\big\rangle\\ &= \sum_i \big\langle R(X,e_i)\big(\hat{\eta}^{\phi}_{pl}(e_i)\big), Y\big\rangle + \sum_i \big\langle R\big(X,\hat{\eta}^{\phi}_{pl}(e_i)\big)(Y), e_i\big\rangle\\ &= 2\sum_i \big\langle R(X,e_i)\big(\hat{\eta}^{\phi}_{pl}(e_i)\big), Y\big\rangle. \end{aligned} \tag{4.4}$$

Now multiply (4.1) on the left by $\hat{\eta}^{\phi}_{kl}$

$$\begin{aligned}\hat{\eta}^{\phi}_{kl}R(X,Y)\hat{\eta}^{\phi}_{kl} + R(X,Y) &= \hat{\eta}^{\phi}_{kl}\sum_{q\neq k,l}\Theta_{kq}(X,Y)\hat{\eta}^{\phi}_{lq} + \Theta_{ql}(X,Y)\hat{\eta}^{\phi}_{kq}\\ &= -\sum_{q\neq k,l}\Theta_{kq}(X,Y)\hat{\eta}^{\phi}_{kq} + \Theta_{lq}(X,Y)\hat{\eta}^{\phi}_{lq}\\ &= 2\Theta_{kl}(X,Y)\hat{\eta}^{\phi}_{kl} - \sum_q \Theta_{kq}(X,Y)\hat{\eta}^{\phi}_{kq} + \Theta_{lq}(X,Y)\hat{\eta}^{\phi}_{lq}. \end{aligned} \tag{4.5}$$

Consider the sum

$$\begin{aligned}&\sum_{i,j}\big\langle \hat{\eta}^{\phi}_{kl}R(X,e_i)\hat{\eta}^{\phi}_{kl}(e_i), e_j\big\rangle e_j + \sum_{i,j}\langle R(X,e_i)(e_i), e_j\rangle e_j\\ &\quad= 2\sum_{i,j}\big\langle \Theta_{kl}(X,e_i)\hat{\eta}^{\phi}_{kl}(e_i), e_j\big\rangle e_j - \sum_{i,j}\Big\langle \sum_q \Theta_{kq}(X,e_i)\hat{\eta}^{\phi}_{kq}(e_i) + \Theta_{lq}(X,e_i)\hat{\eta}^{\phi}_{lq}(e_i), e_j\Big\rangle e_j.\end{aligned}$$

The left-hand side is

$$\begin{aligned}&\sum_{i,j}\big\langle \hat{\eta}^{\phi}_{kl}R(X,e_i)\hat{\eta}^{\phi}_{kl}(e_i), e_j\big\rangle e_j + \sum_{i,j}\langle R(X,e_i)(e_i), e_j\rangle e_j\\ &\qquad= -\sum_{i,j}\big\langle R(X,e_i)\hat{\eta}^{\phi}_{kl}(e_i), \hat{\eta}^{\phi}_{kl}e_j\big\rangle e_j + \sum_{i,j}\langle R(X,e_i)(e_i), e_j\rangle e_j\\ &\qquad= \hat{\eta}^{\phi}_{kl}\sum_{i,j}\big\langle R(X,e_i)\hat{\eta}^{\phi}_{kl}(e_i), \hat{\eta}^{\phi}_{kl}e_j\big\rangle \hat{\eta}^{\phi}_{kl}(e_j) + \mathrm{Ric}(X)\\ &\qquad= \hat{\eta}^{\phi}_{kl}\sum_j \frac{1}{2}\vartheta_{kl}(X,e'_j)e'_j + \mathrm{Ric}(X) = \frac{1}{2}\hat{\eta}^{\phi}_{kl}\big(\hat{\vartheta}_{kl}(X)\big) + \mathrm{Ric}(X),\end{aligned}$$

and the right-hand side is

$$\begin{aligned}&2\sum_{i,j}\big\langle \Theta_{kl}(X,e_i)\hat{\eta}^{\phi}_{kl}(e_i), e_j\big\rangle e_j - \sum_{i,j}\Big\langle \sum_q \Theta_{kq}(X,e_i)\hat{\eta}^{\phi}_{kq}(e_i) + \Theta_{lq}(X,e_i)\hat{\eta}^{\phi}_{lq}(e_i), e_j\Big\rangle e_j\\ &\quad= -2\sum_j \big\langle \hat{\Theta}_{kl}(X), \hat{\eta}^{\phi}_{kl}(e_j)\big\rangle e_j + \sum_j\sum_q \big\langle \hat{\Theta}_{kq}(X), \hat{\eta}^{\phi}_{kq}(e_j)\big\rangle e_j + \sum_j\sum_q \big\langle \hat{\Theta}_{lq}(X), \hat{\eta}^{\phi}_{lq}(e_j)\big\rangle e_j\\ &\quad= 2\hat{\eta}^{\phi}_{kl}\big(\hat{\Theta}_{kl}(X)\big) - \sum_q \hat{\eta}^{\phi}_{kq}\big(\hat{\Theta}_{kq}(X)\big) - \sum_q \hat{\eta}^{\phi}_{lq}\big(\hat{\Theta}_{lq}(X)\big).\end{aligned}$$

Thus

$$\mathrm{Ric}(X) + \frac{1}{2}\hat{\eta}^{\phi}_{kl}\big(\hat{\vartheta}_{kl}(X)\big) = 2\hat{\eta}^{\phi}_{kl}\big(\hat{\Theta}_{kl}(X)\big) - \sum_q \hat{\eta}^{\phi}_{kq}\big(\hat{\Theta}_{kq}(X)\big) - \sum_q \hat{\eta}^{\phi}_{lq}\big(\hat{\Theta}_{lq}(X)\big).$$

By (4.3) and (4.4)

$$\hat{\vartheta}_{kl}(X)=\frac{n}{2}\hat{\Theta}_{kl}(X),$$

so that

$$\mathrm{Ric}(X)+\frac{n}{4}\hat{\eta}_{kl}^{\phi}(\hat{\Theta}_{kl}(X))=2\hat{\eta}_{kl}^{\phi}\big(\hat{\Theta}_{kl}(X)\big)-\sum_{q}\hat{\eta}_{kq}^{\phi}\big(\hat{\Theta}_{kq}(X)\big)-\sum_{q}\hat{\eta}_{lq}^{\phi}\big(\hat{\Theta}_{lq}(X)\big),$$

i.e.,

$$0=\mathrm{Ric}^{M}+\left(\frac{n}{4}-2\right)\hat{\eta}_{kl}^{\phi}\circ\hat{\Theta}_{kl}+\sum_{q}\hat{\eta}_{kq}^{\phi}\circ\hat{\Theta}_{kq}+\sum_{q}\hat{\eta}_{lq}^{\phi}\circ\hat{\Theta}_{lq}.$$

Let

$$T_k:=\sum_{q}\hat{\eta}_{kq}^{\phi}\circ\hat{\Theta}_{kq}.$$

Now

$$0=\mathrm{Ric}+\left(\frac{n}{4}-2\right)\hat{\eta}_{kl}^{\phi}\circ\hat{\Theta}_{kl}+T_k+T_l. \tag{4.6}$$

Recall that $k\neq l$, and consider

$$0=\sum_{1\leq l\leq r,\,l\neq k}\mathrm{Ric}+\left(\frac{n}{4}-2\right)\sum_{1\leq l\leq r,\,l\neq k}\hat{\eta}_{kl}^{\phi}\circ\hat{\Theta}_{kl}+\sum_{1\leq l\leq r,\,l\neq k}T_k+\sum_{1\leq l\leq r,\,l\neq k}T_l,$$

i.e.,

$$\begin{aligned}0&=(r-1)\mathrm{Ric}+\left(\frac{n}{4}-2\right)T_k+(r-1)T_k+\sum_{l}T_l-T_k\\&=(r-1)\mathrm{Ric}+\left(\frac{n}{4}-2\right)T_k+(r-2)T_k+\sum_{l}T_l\\&=(r-1)\mathrm{Ric}+\left(r+\frac{n}{4}-4\right)T_k+\sum_{l}T_l.\end{aligned}$$

Thus, if $p\neq k$,

$$\begin{aligned}0&=(r-1)\mathrm{Ric}+\left(r+\frac{n}{4}-4\right)T_k+\sum_{l}T_l,\\0&=(r-1)\mathrm{Ric}+\left(r+\frac{n}{4}-4\right)T_p+\sum_{l}T_l,\end{aligned}$$

and, if $(r+n/4-4)\neq 0$, we have

$$T_k=T_p=:\mathbb{T}.$$

Thus,

$$0=(r-1)\mathrm{Ric}+\left(2r+\frac{n}{4}-4\right)\mathbb{T},$$

and if $4-\frac{n}{4}-2r\neq 0$,

$$\mathbb{T}=\frac{r-1}{(4-\frac{n}{4}-2r)}\mathrm{Ric}.$$

Going back to (4.6)

$$0 = \mathrm{Ric} + \left(\frac{n}{4} - 2\right) \hat{\eta}^{\phi}_{kl} \circ \hat{\Theta}_{kl} + T_k + T_l = \mathrm{Ric} + \left(\frac{n}{4} - 2\right) \hat{\eta}^{\phi}_{kl} \circ \hat{\Theta}_{kl} + \frac{2r-2}{\left(4 - \frac{n}{4} - 2r\right)}\mathrm{Ric},$$

i.e.,

$$\left(\frac{n}{4} - 2\right) \hat{\eta}^{\phi}_{kl} \circ \hat{\Theta}_{kl} = -\mathrm{Ric} - \frac{2r-2}{\left(4 - \frac{n}{4} - 2r\right)}\mathrm{Ric} = \frac{\left(\frac{n}{4} - 2\right)}{\left(4 - \frac{n}{4} - 2r\right)}\mathrm{Ric}.$$

If $n \neq 8$,

$$\hat{\eta}^{\phi}_{kl} \circ \hat{\Theta}_{kl} = \frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathrm{Ric}^M. \tag{4.7}$$

Since Ric is symmetric, $\hat{\Theta}_{kl}$ and $\hat{\eta}^{\phi}_{kl}$ commute, and they commute with Ric for all $1 \leq k < l \leq r$. If k, l, p, q are all different,

$$\begin{aligned}
\mathbf{tr}\big(\hat{\Theta}_{pq}\hat{\eta}^{\phi}_{kl}\big) &= -\mathbf{tr}\big(\hat{\Theta}_{pq}\hat{\eta}^{\phi}_{pq}\hat{\eta}^{\phi}_{pq}\hat{\eta}^{\phi}_{sk}\hat{\eta}^{\phi}_{sl}\big) = -\frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathbf{tr}\big(\mathrm{Ric}^M\hat{\eta}^{\phi}_{pq}\hat{\eta}^{\phi}_{sk}\hat{\eta}^{\phi}_{sl}\big) \\
&= -\frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathbf{tr}\big(\mathrm{Ric}^M\hat{\eta}^{\phi}_{sk}\hat{\eta}^{\phi}_{pq}\hat{\eta}^{\phi}_{sl}\big) = -\frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathbf{tr}\big(\hat{\eta}^{\phi}_{sk}\mathrm{Ric}^M\hat{\eta}^{\phi}_{pq}\hat{\eta}^{\phi}_{sl}\big) \\
&= -\frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathbf{tr}\big(\mathrm{Ric}^M\hat{\eta}^{\phi}_{pq}\hat{\eta}^{\phi}_{sl}\hat{\eta}^{\phi}_{sk}\big) = \frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathbf{tr}\big(\mathrm{Ric}^M\hat{\eta}^{\phi}_{pq}\hat{\eta}^{\phi}_{kl}\big) \\
&= -\mathbf{tr}\big(\hat{\Theta}_{pq}\hat{\eta}^{\phi}_{kl}\big),
\end{aligned}$$

so that

$$\mathbf{tr}\big(\hat{\Theta}_{pq}\hat{\eta}^{\phi}_{kl}\big) = 0. \tag{4.8}$$

If p, k, l are all different

$$\begin{aligned}
\mathbf{tr}\big(\hat{\Theta}_{pk}\hat{\eta}^{\phi}_{kl}\big) &= -\mathbf{tr}\big(\hat{\Theta}_{pk}\hat{\eta}^{\phi}_{pk}\hat{\eta}^{\phi}_{pk}\hat{\eta}^{\phi}_{kl}\big) = \frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathbf{tr}\big(\mathrm{Ric}^M\hat{\eta}^{\phi}_{kl}\hat{\eta}^{\phi}_{pk}\big) \\
&= \frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathbf{tr}\big(\hat{\eta}^{\phi}_{kl}\mathrm{Ric}^M\hat{\eta}^{\phi}_{pk}\big) = \frac{1}{\left(4 - \frac{n}{4} - 2r\right)}\mathbf{tr}\big(\mathrm{Ric}^M\hat{\eta}^{\phi}_{pk}\hat{\eta}^{\phi}_{kl}\big) \\
&= -\mathbf{tr}\big(\hat{\Theta}_{pk}\hat{\eta}^{\phi}_{kl}\big),
\end{aligned}$$

i.e.,

$$\mathbf{tr}\big(\hat{\Theta}_{pk}\hat{\eta}^{\phi}_{kl}\big) = 0. \tag{4.9}$$

Now set $X = \hat{\eta}^{\phi}_{st}(e_a)$ and $Y = e_a$ in (4.5) and sum over a

$$\begin{aligned}
&\sum_a \hat{\eta}^{\phi}_{kl} R\big(\hat{\eta}^{\phi}_{st}(e_a), e_a\big)\hat{\eta}^{\phi}_{kl} + R\big(\hat{\eta}^{\phi}_{st}(e_a), e_a\big) \\
&\qquad = 2\sum_a \Theta_{kl}\big(\hat{\eta}^{\phi}_{st}(e_a), e_a\big)\hat{\eta}^{\phi}_{kl} - \sum_a\sum_q \Theta_{kq}\big(\hat{\eta}^{\phi}_{st}(e_a), e_a\big)\hat{\eta}^{\phi}_{kq} + \Theta_{lq}\big(\hat{\eta}^{\phi}_{st}(e_a), e_a\big)\hat{\eta}^{\phi}_{lq}.
\end{aligned}$$

The right-hand side is equal to

$$2\left(\sum_a \big\langle \hat{\Theta}_{kl}\hat{\eta}^{\phi}_{st}(e_a), e_a\big\rangle\right)\hat{\eta}^{\phi}_{kl} - \sum_q \left[\left(\sum_a \big\langle \hat{\Theta}_{kq}\hat{\eta}^{\phi}_{st}(e_a), e_a\big\rangle\right)\hat{\eta}^{\phi}_{kq} + \left(\sum_a \big\langle \hat{\Theta}_{lq}\hat{\eta}^{\phi}_{st}(e_a), e_a\big\rangle\right)\hat{\eta}^{\phi}_{lq}\right]$$

$$= 2\mathbf{tr}\big(\hat\Theta_{kl}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{kl} - \sum_q \big[\mathbf{tr}\big(\hat\Theta_{kq}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{kq} + \mathbf{tr}\big(\hat\Theta_{lq}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{lq}\big].$$

Now let us analyze the terms of the left-hand side by evaluating it on a vector field $Z \in \Gamma(TM)$

$$\begin{aligned}
\sum_a \hat\eta^\phi_{kl} R\big(\hat\eta^\phi_{st}(e_a), e_a\big)\hat\eta^\phi_{kl}(Z) &= \hat\eta^\phi_{kl}\bigg(\sum_a R\big(\hat\eta^\phi_{st}(e_a), e_a\big)\hat\eta^\phi_{kl}(Z)\bigg)\\
&= \hat\eta^\phi_{kl}\bigg(\sum_{a,b}\big\langle R\big(\hat\eta^\phi_{st}(e_a), e_a\big)\hat\eta^\phi_{kl}(Z), e_b\big\rangle e_b\bigg) = -\sum_{a,b}\big\langle R\big(\hat\eta^\phi_{st}(e_a), e_a\big)e_b, \hat\eta^\phi_{kl}(Z)\big\rangle\hat\eta^\phi_{kl}(e_b)\\
&= \sum_{a,b}\big\langle\hat\eta^\phi_{kl}R\big(\hat\eta^\phi_{st}(e_a), e_a\big)e_b, Z\big\rangle\hat\eta^\phi_{kl}(e_b) = -\sum_{a,b}\big\langle\hat\eta^\phi_{kl}R\big(\hat\eta^\phi_{st}(e_a), e_a\big)Z, e_b\big\rangle\hat\eta^\phi_{kl}(e_b)\\
&= \sum_{a,b}\big\langle R\big(\hat\eta^\phi_{st}(e_a), e_a\big)Z, \hat\eta^\phi_{kl}(e_b)\big\rangle\hat\eta^\phi_{kl}(e_b) = \sum_{a,b}\big\langle R\big(\hat\eta^\phi_{st}(e_a), e_a\big)Z, e'_b\big\rangle e'_b\\
&= -\sum_{a,b}\big\langle R\big(e_a, \hat\eta^\phi_{st}(e_a)\big)Z, e'_b\big\rangle e'_b = -\sum_{a,b}\big\langle R(Z, e'_b)e_a, \hat\eta^\phi_{st}(e_a)\big\rangle e'_b\\
&= \sum_{a,b}\langle\hat\eta^\phi_{st}R(Z, e'_b)e_a, e_a\rangle e'_b = \sum_b\bigg(\sum_a\langle\hat\eta^\phi_{st}R(Z, e'_b)e_a, e_a\rangle\bigg)e'_b\\
&= \sum_b \mathbf{tr}\big(\hat\eta^\phi_{st}R(Z, e'_b)\big)e'_b = \sum_b\vartheta_{st}(Z, e'_b)e'_b = \hat\vartheta_{st}(Z),
\end{aligned}$$

i.e.,

$$\sum_a \hat\eta^\phi_{kl}R\big(\hat\eta^\phi_{st}(e_a), e_a\big)\hat\eta^\phi_{kl} = \hat\vartheta_{st},$$

and, as can also be seen from the middle of the previous calculation,

$$\sum_a R\big(\hat\eta^\phi_{st}(e_a), e_a\big) = \hat\vartheta_{st}.$$

Thus, we have

$$2\hat\vartheta_{st} = 2\mathbf{tr}\big(\hat\Theta_{kl}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{kl} - \sum_q \big[\mathbf{tr}\big(\hat\Theta_{kq}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{kq} + \mathbf{tr}\big(\hat\Theta_{lq}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{lq}\big],$$

which, by (4.2) is equivalent to

$$n\hat\Theta_{st} = 2\mathbf{tr}\big(\hat\Theta_{kl}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{kl} - \sum_q \big[\mathbf{tr}\big(\hat\Theta_{kq}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{kq} + \mathbf{tr}\big(\hat\Theta_{lq}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{lq}\big].$$

If $s = k < l \neq t$,

$$n\hat\Theta_{st} = 2\mathbf{tr}\big(\hat\Theta_{sl}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{sl} - \sum_q \big[\mathbf{tr}\big(\hat\Theta_{sq}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{sq} + \mathbf{tr}\big(\hat\Theta_{lq}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{lq}\big].$$

By (4.8) and (4.9),

$$n\hat\Theta_{st} = -\mathbf{tr}\big(\hat\Theta_{st}\hat\eta^\phi_{st}\big)\hat\eta^\phi_{st}.$$

Recalling (4.7)

$$n\hat\Theta_{st} = -\frac{1}{(4 - \frac{n}{4} - 2r)}\mathbf{tr}\big(\mathrm{Ric}^M\big)\hat\eta^\phi_{st} = -\frac{\mathrm{R}}{(4 - \frac{n}{4} - 2r)}\hat\eta^\phi_{st},$$

i.e.,

$$\hat{\Theta}_{st} = \frac{\mathrm{R}}{n\left(\frac{n}{4} + 2r - 4\right)}\hat{\eta}^{\phi}_{st}.$$

Finally, observe that

$$\mathrm{Ric}^M = \left(4 - \frac{n}{4} - 2r\right)\hat{\eta}^{\phi}_{kl}\hat{\Theta}_{kl} = -\frac{\mathrm{R}}{n\left(4 - \frac{n}{4} - 2r\right)}\left(4 - \frac{n}{4} - 2r\right)\hat{\eta}^{\phi}_{kl}\hat{\eta}^{\phi}_{kl} = \frac{\mathrm{R}}{n}\mathrm{Id}_{TM}. \qquad \blacksquare$$

Remark 4.9. The previous lemma implies the identity

$$\Theta = \frac{\mathrm{R}}{n\left(\frac{n}{4} + 2r - 4\right)}\eta^{\phi}, \tag{4.10}$$

which shows that the pair formed by the parallel twisted pure spinor ϕ and the connection θ satisfies, up to a factor, an equation analogous to the second Seiberg–Witten equation in dimension 4.

4.4 Spinorial characterization of special Riemannian holonomies

4.4.1 Kählerian homonomies $\mathrm{U}(n)$ and $\mathrm{SU}(n)$

The Kähler and hyper-Kähler cases have been treated spinorially by various authors [16, 19, 20, 25, 39]. For the sake of completeness, we collect and use some of their ideas to prove the following two corollaries.

Corollary 4.10. *An oriented Riemannian manifold M is Kähler if and only if it admits a* Spin^c *structure endowed with a connection and carrying a parallel (classical) pure spinor field.*

Proof. Let us assume M is a $2m$-dimensional Kähler manifold, J its complex structure, $\bigwedge^{p,q}$ denote the vector bundle of exterior differential forms of type (p,q) and

$$\kappa_M = \bigwedge\nolimits^{m,0} = \det\left(\bigwedge\nolimits^{1,0}\right).$$

By [16], the locally defined Spin bundle decomposes as follows

$$S(TM) = \left(\bigwedge\nolimits^{0,0} \oplus \cdots \oplus \bigwedge\nolimits^{0,m}\right) \otimes \kappa_M^{1/2},$$

so that the anti-canonical Spin^c bundle

$$S(TM) \otimes \kappa_M^{-1/2} = \bigwedge\nolimits^{0,0} \oplus \cdots \oplus \bigwedge\nolimits^{0,m}$$

contains a trivial summand. Thus, the manifold M admits a parallel spinor field $\psi \in \Gamma\left(\bigwedge^{0,0}\right)$ such that

$$(X + \mathrm{i}J(X)) \cdot \psi = 0$$

for all $X \in \Gamma(TM)$ (see [12]).

Conversely, suppose M admits a Spin^c structure carrying a parallel pure spinor field $\psi \in \Gamma\left(S^c(TM)\right)$. If $X \in \Gamma(TM)$, there exists $Y \in \Gamma(TM)$ such that

$$X \cdot \psi = \mathrm{i}Y \cdot \psi.$$

By defining $Y = J(X)$, we see that J is an orthogonal complex structure, and by differentiating

$$\nabla_Z X \cdot \psi = \mathrm{i}\nabla_Z(J(X)) \cdot \psi.$$

Note that the vector $\nabla_Z X$ satisfies

$$\nabla_Z X \cdot \psi = \mathrm{i}J(\nabla_Z X) \cdot \psi,$$

so that

$$\big(\nabla_Z(J(X)) - J(\nabla_Z X)\big) \cdot \psi = 0.$$

Since real tangent vectors do not annihilate spinors,

$$\nabla J = 0.$$ ■

Corollary 4.11. *Let M be a $2m$-dimensional irreducible oriented Riemannian manifold. The manifold M is Calabi–Yau if and only if it admits a* Spin^c *structure endowed with a connection carrying two parallel classical pure spinor fields which are complex-linearly independent at one point.*

Proof. Let us assume M is Calabi–Yau and J is its complex structure. Since M is Spin and κ_M is trivial, we can consider a Spin^c structure with trivial auxiliary complex line bundle $L = \kappa_M$ and flat connection. The Spin^c spinor bundle

$$S(TM) \otimes \kappa_M^{-1/2} = \bigwedge^{0,0} \oplus \cdots \oplus \bigwedge^{0,m}$$

contains two trivial summands generated by parallel spinor fields $\psi_1 \in \Gamma\big(\bigwedge^{0,0}\big)$ and $\psi_2 \in \Gamma\big(\bigwedge^{0,m}\big)$ such that

$$(X + \mathrm{i}J(X)) \cdot \psi_1 = 0 \qquad \text{and} \qquad (X - \mathrm{i}J(X)) \cdot \psi_2 = 0$$

for all $X \in \Gamma(TM)$ (see [12]).

Conversely, suppose M admits a Spin^c bundle carrying two parallel classical pure spinor fields ψ_1 and ψ_2 such that they are complex-linearly independent at one point. We claim that they must be complex-linearly independent everywhere. Suppose there is $z \in \mathbb{C}$ such that $(\psi_1)_y = z(\psi_2)_y$ for some $y \in M$. Since ψ_1 and ψ_2 are parallel, the spinor field $\psi_1 - z\psi_2$ is parallel and its length is constant and equal to zero. Therefore, $\psi_1 = z\psi_2$ everywhere.

Thus, the projective classes $[(\psi_1)_x] \neq [(\psi_2)_x]$ for every $x \in M$. If $X \in \Gamma(TM)$, there exist $Y_1, Y_2 \in \Gamma(TM)$ such that

$$X \cdot \psi_1 = \mathrm{i}Y_1 \cdot \psi_1 \qquad \text{and} \qquad X \cdot \psi_2 = \mathrm{i}Y_2 \cdot \psi_2.$$

By defining

$$Y_1 = J_1(X) \qquad \text{and} \qquad Y_2 = J_2(X),$$

we obtain two parallel complex structures. Since orthogonal complex structures are in one to one correspondence with projective classes of classical Spin^c pure spinors, $J_1 \neq J_2$. If Θ denotes the curvature 2-form of the connection on the auxiliary Spin^c line bundle, by [25]

$$\mathrm{Ric}(X) \cdot \psi_1 = i\hat{\Theta}(X) \cdot \psi_1 \qquad \text{and} \qquad \mathrm{Ric}(X) \cdot \psi_2 = \mathrm{i}\hat{\Theta}(X) \cdot \psi_2,$$

and

$$J_1 \circ \mathrm{Ric}(X) = \hat{\Theta}(X) = J_2 \circ \mathrm{Ric}(X),$$

i.e., J_1 and J_2 coincide in the image of the Ricci tensor. The distribution

$$D = \{X \in TM \mid J_1(X) = J_2(X)\}$$

is parallel and, by irreducibility, it is either equal to TM or trivial. Since $J_1 \neq J_2$, D must be trivial and, therefore, $\mathrm{Ric} = 0$ and θ is flat. ■

4.4.2 Quaternion-Kählerian holonomies $\mathbf{Sp}(n)\mathbf{Sp}(1)$ and $\mathbf{Sp}(n)$

Corollary 4.12. *A Riemannian manifold is quaternion-Kähler if and only if it admits a* Spin^3 *structure endowed with a connection and a twisted spinor bundle carrying a parallel twisted pure spinor field.*

Proof. Let us assume M is quaternion-Kähler so that its orthonormal frame bundle has a parallel reduction to a principal bundle with fiber $\mathrm{Sp}(m)\mathrm{Sp}(1)$. We have the following diagram

$$\begin{array}{ccc} & & \mathrm{Spin}^3(4m) \\ & \nearrow & \downarrow \\ \mathrm{Sp}(m)\mathrm{Sp}(1) & \longrightarrow & \mathrm{SO}(4m)\times\mathrm{SO}(3), \end{array}$$

so that the manifold admits a Spin^3 structure with an induced connection. We can associate a twisted spinor bundle with fibre $\Delta_{4m}\otimes\Delta_3^m$ which contains a trivial $\mathrm{Sp}(m)\mathrm{Sp}(1)$ summand generated by a pure spinor, such as the spinor given in (3.17) in Section 3.4.1.

Conversely, if M admits a Spin^3 structure with a connection and carrying a parallel pure spinor, by Theorem 4.5, we have a parallel quaternion-Kähler structure. ■

Corollary 4.13. *A Riemannian manifold is hyper-Kähler if and only if it admits a* Spin^3 *structure endowed with a connection and a twisted spinor bundle carrying two parallel twisted pure spinor fields complex-lnearly independent at one point.*

Proof. Let us assume M is $4m$-dimensional hyper-Kähler, $m\geq 2$. Its structure group reduces further to $\mathrm{Sp}(m)$. The associated bundle $\Delta_{4m}\otimes\Delta_3^m$ contains the $\widehat{\mathrm{Spin}(3)}$ orbit of the pure spinor in Section 3.4.1, which consists of pure spinors inducing the same quaternionic structure (see Lemma 3.16) and fixed by $\mathrm{Sp}(m)$.

Conversely, suppose M admits a Spin^3 structure with a connection and carrying two parallel pure spinors ψ_1 and ψ_2 complex-linearly independent at one point. Due to the parallelism, they must be complex-linearly independent everywhere. By Theorem 4.5, M is quaternion-Kähler and Einstein [5]. By Lemma 4.8,

$$\frac{\mathrm{R}}{4m(m+2)}\eta_{kl}^{\psi_1}=\Theta_{kl}=\frac{\mathrm{R}}{4m(m+2)}\eta_{kl}^{\psi_2},$$

which also hold when $\dim(M)=8$. If $\mathrm{R}\neq 0$,

$$\eta_{kl}^{\psi_1}=\eta_{kl}^{\psi_2},$$

which means ψ_1 and ψ_2 have the same annihilator $\mathfrak{sp}(m)\oplus\widetilde{\mathfrak{spin}}(3)\subset\mathfrak{spin}(4m)\oplus\mathfrak{spin}(3)$. However, restricted to this subalgebra, the representation $\Delta_{4m}\otimes\Delta_3^{\otimes m}$ has only one trivial 1-dimensional summand and ψ_2 must be a multiple of ψ_1, which is a contradiction. Hence, M is Ricci-flat. ■

4.4.3 Exceptional holonomies $\mathbf{Spin}(7)$ and G_2

Corollary 4.14. *A Riemannian* 8*-dimensional manifold has holonomy contained in* $\mathrm{Spin}(7)$ *if and only if it admits a* Spin^7 *structure endowed with a connection and carrying a parallel pure spinor field.*

Proof. Let us assume M is an 8-dimensional Riemannian manifold with holonomy contained in $\mathrm{Spin}(7)$. Its orthonormal frame bundle has a parallel reduction to a principal bundle with fiber Spin(7). We have the following diagram

$$\begin{array}{ccc} & & \mathrm{Spin}^7(8) \\ & \nearrow & \downarrow \\ \mathrm{Spin}(7) & \longrightarrow & \mathrm{SO}(8)\times\mathrm{SO}(7), \end{array}$$

so that the manifold admits a Spin^7 structure with an induced connection. We can associate a twisted spinor bundle with fibre $\Delta_8 \otimes \Delta_7$ which contains a trivial Spin(7) summand generated by a pure spinor, such as the spinor given in (3.18) in Section 3.4.2.

Conversely, if M admits a Spin^7 structure with a connection and carrying a parallel pure spinor, by Theorem 4.5, it admits a parallel rank 7 even-Clifford structure. ■

Unlike the complex and quaternionic cases, the G_2 holonomy reduction does not arise by the existence of two linearly independent parallel twisted pure spinors belonging to the same $\mathrm{Spin}^7(8)$-orbit. As it could be expected, the holonomy reduction to this exceptional Lie group is due to triality, which in our context is expressed by the interaction of twisted pure spinors whose 2-forms are related by triality.

Corollary 4.15. *A simply connected* 8*-dimensional Riemannian manifold* M *is a product of a flat* 1*-dimensional manifold* $\mathcal{S}$ *and a* 7*-dimensional manifold* N *with holonomy contained in* G_2 *if and only if* M *admits a* Spin^7 *structure with a connection* (*which includes the lift of the Levi-Civita connection*)*, carrying a parallel twisted pure spinor* ϕ_1 *and a parallel spinor* ϕ_2 *such that the vector field*

$$X^{\phi_1,\phi_2} := \sum_{i=1}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle e_i$$

is nonzero at some point of M*.*

Proof. Since $G_2 \subset \mathrm{SO}(7)$, we can embed it as a block $\mathrm{SO}(7) \subset \mathrm{SO}(8)$, in such a way the G_2 leaves invariant the first canonical vector e_1 of $\mathbb{R}^8$. Now, for a 7-dimensional Riemannian manifold N with holonomy contained in G_2, this corresponds to considering the product $\mathcal{S} \times N$ where $\mathcal{S}$ is a flat 1-dimensional manifold. Now that $G_2 \subset \mathrm{SO}(8)$, there are many copies of $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ that contain it. Choose one of such copies. We have the diagram

$$\begin{array}{ccc} & & \mathrm{Spin}^7(8) \\ & \nearrow & \downarrow \\ G_2 \subset \mathrm{Spin}(7) & \longrightarrow & \mathrm{SO}(8)\times\mathrm{SO}(7). \end{array}$$

Now, $\Delta_8 \otimes \Delta_7$ decomposes under the image of such a Spin(7)

$$\Delta_8 \otimes \Delta_7 = \Delta_8^+ \otimes \Delta_7 \oplus \Delta_8^- \otimes \Delta_7 \cong \Delta_7 \otimes \Delta_7 \oplus \Delta_7 \otimes \Delta_7$$

and one of the two summands contains an invariant element which is a pure spinor, say $\phi_1 \in \Delta_8^+ \otimes \Delta_7$. All this translates into a parallel spinor field for $N \times \mathcal{S}$. Since we have a globally defined vector field e_1, we can consider the parallel spinor field $\phi_2 = e_1 \cdot \phi_1 \in \Delta_8^- \otimes \Delta_7$. By Lemma 3.17, the spinor ϕ_2 is also pure and its stabilizer is also a copy of Spin(7), but is not in the same orbit of $\mathrm{Spin}^7(8)$. The corresponding copies of Spin(7) in SO(8) intersect in the original G_2. Note that

$$\begin{aligned} X^{\phi_1,\phi_2} &= \sum_{i=1}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle e_i = \mathrm{Re}\langle e_1 \cdot \phi_1, e_1 \cdot \phi_1\rangle e_1 + \sum_{i=2}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, e_1 \cdot \phi_1\rangle e_i \\ &= \mathrm{Re}\,|\phi_1|^2 e_1 \neq 0. \end{aligned}$$

Conversely, suppose we have a simply connected 8-dimensional Riemannian manifold M with a $\mathrm{Spin}^7(8)$ structure and a connection that carries a parallel twisted pure spinor ϕ_1 and another parallel spinor ϕ_2 such that the vector field

$$X^{\phi_1,\phi_2} = \sum_{i=1}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle e_i$$

is such that $X_p \neq 0$ at some $p \in M$. Note that for any $Z \in \Gamma(TM)$,

$$\begin{aligned}
\nabla_Z X^{\phi_1,\phi_2} &= \nabla_Z \sum_{i=1}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle e_i = \sum_{i=1}^{8} \nabla_Z\big(\mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle e_i\big) \\
&= \sum_{i=1}^{8} \mathrm{Re}\left\langle \nabla_Z e_i \cdot \phi_1, \phi_2\right\rangle e_i + \sum_{i=1}^{8} \mathrm{Re}\left\langle e_i \cdot \nabla^S_Z \phi_1, \phi_2\right\rangle e_i \\
&\quad + \sum_{i=1}^{8} \mathrm{Re}\left\langle e_i \cdot \phi_1, \nabla^S_Z \phi_2\right\rangle e_i + \sum_{i=1}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle \nabla_Z e_i \\
&= \sum_{i=1}^{8} \mathrm{Re}\langle \nabla_Z e_i \cdot \phi_1, \phi_2\rangle e_i + \sum_{i=1}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle \nabla_Z e_i \\
&= \sum_{i=1}^{8} \mathrm{Re}\left\langle \sum_{j=1}^{n} \langle \nabla_Z e_i, e_j\rangle e_j \cdot \phi_1, \phi_2 \right\rangle e_i + \sum_{i=1}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle \sum_{j=1}^{8} \langle \nabla_Z e_i, e_j\rangle e_j \\
&= \sum_{i=1}^{8} \mathrm{Re}\left\langle \sum_{j=1}^{8} \omega_{ji}(Z) e_j \cdot \phi_1, \phi_2 \right\rangle e_i + \sum_{i=1}^{8} \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle \sum_{j=1}^{8} \omega_{ji}(Z) e_j \\
&= \sum_{i=1}^{8}\sum_{j=1}^{8} \omega_{ji}(Z)\, \mathrm{Re}\langle e_j \cdot \phi_1, \phi_2\rangle e_i + \sum_{i=1}^{8}\sum_{j=1}^{8} \omega_{ji}(Z)\, \mathrm{Re}\langle e_i \cdot \phi_1, \phi_2\rangle e_j \\
&= \sum_{i=1}^{8}\sum_{j=1}^{8} \omega_{ji}(Z)\, \mathrm{Re}\langle e_j \cdot \phi_1, \phi_2\rangle e_i + \sum_{j=1}^{8}\sum_{i=1}^{8} \omega_{ij}(Z)\, \mathrm{Re}\langle e_j \cdot \phi_1, \phi_2\rangle e_i = 0,
\end{aligned}$$

i.e., X^{ϕ_1,ϕ_2} is parallel. As a consequence, it has constant length which is non-zero, and therefore M decomposes as a product of a flat 1-dimensional manifold $\mathcal{S}$ and a 7-dimensional Riemannian manifold N. Since M carries a parallel twisted pure spinor, it has holonomy contained in $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. But now that we have proved that $M = \mathcal{S} \times N$, the holonomy of M must reduce to the subgroup

$$\mathrm{Spin}(7) \cap (\{1\} \times \mathrm{SO}(7)) = G_2,$$

i.e., the 7-dimensional Riemannian manifold N has holonomy contained in G_2. ■

5 Clifford monopole equations

Let M be a Spin^r manifold with auxiliary bundle $P_{\mathrm{SO}(r)}$ endowed with a connection θ, F the associated Riemannian rank r vector bundle and $m \in \mathbb{N}$ be such that the twisted Dirac operator

$$\begin{aligned}
\partial\!\!\!/^{\theta}\colon\ &\Gamma(S(M,F,m)) \longrightarrow \Gamma(S(M,F,m)), \\
&\partial\!\!\!/^{\theta}\phi = \sum_{i=1}^{\dim(M)} e_i \cdot \nabla^{\theta}_{e_i}\phi
\end{aligned}$$

is well-defined, where the vectors e_i form a local orthonormal frame of the tangent bundle. The *Clifford monopole equations* are

$$\partial\!\!\!/^{\theta}\phi = 0, \qquad \Theta = E(\eta^{\phi}), \tag{5.1}$$

where

$$\Theta = \sum_{1\le k<l\le r} \Theta_{kl} \otimes f_{kl} \in \Gamma\left(\bigwedge^2 T^*M \otimes \bigwedge^2 F\right)$$

is the curvature of θ,

$$\eta^{\phi} = \sum_{1\le k<l\le r} \eta^{\phi}_{kl} \otimes f_{kl} \in \Gamma\left(\bigwedge^2 T^*M \otimes \bigwedge^2 F\right)$$

is the 2-form with values in $\bigwedge^2 F$ associated to ϕ, and E is a suitable endomorphism of 2-forms. A pair (ϕ, θ) satisfying (5.1) will be called a *Clifford monopole*.

Here, we will show that the Clifford monopole equations restrict to the Seiberg–Witten equations on 4-manifolds, and will also exhibit Clifford monopoles on manifolds with special Riemannian holonomy.

As mentioned in the introduction, preliminary work with A. Quintero indicates the existence of a smooth compact moduli space which, according to the Mathai–Quillen–Atiyah–Jeffrey formalism, will give raise to a topological quantum field theory. Such topological field theory, at least in dimensions 8, might turn out to be a topological twist of an $N = 2$ supersymmetric theory.

5.1 The Clifford monopole equations on 4-manifolds

In this subsection, we will show that the Clifford monopole equations restrict to the Seiberg–Witten monopole equations for appropriate choices of the parameters.

Let us start by recalling that every 4-manifold admits a Spin^c structure [24], i.e., a Spin^2 structure in our notation. Here, we choose $m = 1$ so that $\phi \in \Gamma(S(\Delta_4 \otimes \Delta_2))$. The orthogonal decomposition

$$\Delta_4 \otimes \Delta_2 = \left(\Delta_4^+ \oplus \Delta_4^-\right) \otimes \left(\Delta_2^+ \oplus \Delta_2^-\right) = \Delta_4^+ \otimes \Delta_2^+ \oplus \Delta_4^+ \otimes \Delta_2^- \oplus \Delta_4^- \otimes \Delta_2^+ \oplus \Delta_4^- \otimes \Delta_2^-.$$

implies $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$ where

$$\phi_1 \in \Delta_4^+ \otimes \Delta_2^+, \qquad \phi_2 \in \Delta_4^+ \otimes \Delta_2^-, \qquad \phi_3 \in \Delta_4^- \otimes \Delta_2^+, \qquad \phi_4 \in \Delta_4^- \otimes \Delta_2^-.$$

Thus, we have four Dirac equations

$$\partial\!\!\!/^{\theta}\phi_s = 0, \qquad s = 1, 2, 3, 4.$$

Note that, in this case, $f_1 f_2$ represents a globally defined section of $\bigwedge^2 F$ and acts as multiplication by $\pm\mathrm{i}$ on $\Delta_4 \otimes \Delta_2^{\pm}$ respectively. Now consider

$$\begin{aligned}
\eta^{\phi} &= \sum_{1\le a<b\le 4} \langle e_a e_b \cdot f_{12} \cdot (\phi_1 + \phi_2 + \phi_3 + \phi_4), (\phi_1 + \phi_2 + \phi_3 + \phi_4)\rangle e_a e_b \otimes f_{12} \\
&= \sum_{s,t=1}^{4} \sum_{1\le a<b\le 4} \langle e_a e_b \cdot f_{12} \cdot \phi_s, \phi_t\rangle e_a e_b \otimes f_{12} \\
&= \sum_{s=1}^{4} \sum_{1\le a<b\le 4} \langle e_a e_b \cdot f_{12} \cdot \phi_s, \phi_s\rangle e_a e_b \otimes f_{12}
\end{aligned}$$

$$= \sum_{s=1}^{4}(-1)^{s-1}\mathrm{i}\sum_{1\leq a<b\leq 4}\langle e_ae_b\cdot(\phi_s),\phi_s\rangle e_ae_b\otimes f_{12}$$
$$= \big(\mathrm{i}\tau^{\phi_1}-\mathrm{i}\tau^{\phi_2}+\mathrm{i}\tau^{\phi_3}-\mathrm{i}\tau^{\phi_4}\big)\otimes f_{12},$$

where

$$\tau^{\psi}(X,Y):=\langle X\wedge Y\cdot\psi,\psi\rangle$$

for $\psi\in\Delta_4\otimes\Delta_2$. The 2-forms τ^{ϕ_s} are purely imaginary, and

$$\mathrm{i}\tau^{\phi_1},\mathrm{i}\tau^{\phi_2}\in{\textstyle\bigwedge}^2_+T^*M,\qquad \mathrm{i}\tau^{\phi_3},\mathrm{i}\tau^{\phi_4}\in{\textstyle\bigwedge}^2_-T^*M.$$

Indeed, $\phi_1=(\alpha u_+\otimes u_+ +\beta u_-\otimes u_-)\otimes v_+$ for some $\alpha,\beta\in\mathbb{C}$, so that

$$\mathrm{i}\tau^{\phi_1}=\big(-|\alpha|^2+|\beta|^2\big)(e_1e_2+e_3e_4)-2\,\mathrm{Im}\,\big(\alpha\overline{\beta}\big)(e_1e_3-e_2e_4)$$
$$+2\,\mathrm{Re}\,\big(\alpha\overline{\beta}\big)(e_1e_4+e_2e_3)\in{\textstyle\bigwedge}^2_+T^*M.$$

Similarly for the other 2-forms.

At this point we choose E to be such that $\bigwedge^2_+T^*M$ and $\bigwedge^2_-T^*M$ are invariant subspaces. Since f_{12} trivializes $\bigwedge^2F$, the second Clifford monopole equation

$$\Theta_{12}\otimes f_{12}=E\big(\eta^{\phi}\big)\otimes f_{12}$$

becomes

$$\Theta_{12}=E\big(\eta^{\phi}_{12}\big),$$

which splits as follows

$$\mathrm{i}\Theta^+_{12}=-E\big(\tau^{\phi_1}-\tau^{\phi_2}\big),\qquad \mathrm{i}\Theta^-_{12}=-E\big(\tau^{\phi_3}-\tau^{\phi_4}\big).$$

If we limit ourselves to work with spinor fields in $\Gamma\big(S\big(\Delta_4\otimes\Delta_2^+\big)\big)$, i.e., $\phi_2=0$ and $\phi_4=0$, we have

$$\mathrm{i}\Theta^+_{12}=-E\big(\tau^{\phi_1}\big),\qquad \mathrm{i}\Theta^-_{12}=-E\big(\tau^{\phi_3}\big),$$

and setting $E=\frac{1}{4}\mathrm{Id}_{\bigwedge^2T^*M}$, we have

$$\mathrm{i}\Theta^+_{12}=-\frac{1}{4}\tau^{\phi_1},\qquad \mathrm{i}\Theta^-_{12}=-\frac{1}{4}\tau^{\phi_3}.$$

In 4 dimensions [24], however, if there is a non-zero solution ϕ_1 for

$$\partial\!\!\!/^{\theta}\phi_1=0,$$

then if ϕ_3 is such that

$$\partial\!\!\!/^{\theta}\phi_3=0,$$

it must vanish identically, i.e., $\phi_3\equiv 0$.

Thus, we are left with the Seiberg–Witten equations for pairs (ϕ_1,θ) [12]

$$\partial\!\!\!/^{\theta}\phi_1=0,\qquad \mathrm{i}\Theta=-\frac{1}{4}\tau^{\phi_1}.$$

If, on the other hand, we consider only spinor fields in $\Gamma(S(\Delta_4\otimes\Delta_2^-))$, i.e., $\phi_1=0$ and $\phi_3=0$, and set $E=-\frac{1}{4}\mathrm{Id}_{\bigwedge^2T^*M}$, we end up with the Seiberg–Witten equations (ϕ_2,θ)

$$\partial\!\!\!/^{\theta}\phi_2=0,\qquad \mathrm{i}\Theta=-\frac{1}{4}\tau^{\phi_2}.$$

5.2 8-manifolds with Spin(7) holonomy

Let M be an 8-dimensional manifold with holonomy contained in Spin(7) and θ denote the Levi-Civita connection restricted to the holonomy principal bundle. We have seen that there exists a parallel pure spinor field ϕ_1 which characterizes such holonomy reduction (see (3.18)), where $m = 1$. By (4.2)

$$\Theta = -\frac{1}{4} R^M \bullet \eta^{\phi_1}.$$

Thus (ϕ_1, θ) gives a solution to (5.1) with $E = -\frac{1}{4} R^M \bullet$.

Remark 5.1. Note that spinor can be scaled in order to remove positive constants so that $(\phi_1/2, \theta)$ gives a solution to (5.1) with $E = -R^M \bullet$.

5.3 Quaternion-Kähler manifolds

Let M be a $4m$-dimensional quaternion-Kähler manifold and denote by θ the connection induced by the Levi-Civita connection on the corresponding SO(3)-bundle. We have seen that there exists a parallel pure spinor field ϕ which characterizes the holonomy (see (3.17)). By (4.2) and (4.10),

$$\hat{\Theta} = -\frac{1}{m} R^M \bullet \eta^\phi = \frac{\mathrm{R}}{4m\,(m+2)} \eta^\phi.$$

Thus, we can say that either

- the pair $\left(\phi/\sqrt{m}, \theta\right)$ satisfies (5.1) with $E = -R^M \bullet$;
- or if R is non-negative, the pair $\left(\sqrt{\frac{\mathrm{R}}{4m(m+2)}}\phi, \theta\right)$ satisfies (5.1) with $E = \mathrm{Id}_{\bigwedge^2 T^*M}$;
- or if R is non-positive, the pair $\left(\sqrt{\frac{-\mathrm{R}}{4m(m+2)}}\phi, \theta\right)$ satisfies (5.1) with $E = -\mathrm{Id}_{\bigwedge^2 T^*M}$.

Remark 5.2. The choices $E = \pm\mathrm{Id}_{\bigwedge^2 T^*M}$ are the ones that remind us of the Seiberg–Witten equations in dimension 4.

5.4 Kähler manifolds

Let $(M, \langle\cdot,\cdot\rangle, J)$ be a $2n$-dimensional Kähler manifold with scalar curvature R and canonical Spin^c structure with canonical connection θ. Up to a choice of local frames, the complex structure is in correspondence with the parallel the twisted pure spinor

$$\phi = \underbrace{u_{-1} \otimes \cdots \otimes u_{-1}}_{n \text{ times}} \otimes v_1 \in \Delta_{2n}^- \otimes \Delta_2^+$$

and

$$\hat{\eta}_{12}^\phi = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}.$$

Note that, in this rank, the sections f_{12}, η_{12}^ϕ, $\hat{\eta}_{12}^\phi$ are actually globally defined.

Since

$$\nabla^\theta \phi = 0,$$

we have

$$\nabla\eta_{12}^{\phi}=0 \qquad \text{and} \qquad \big[R^M(X,Y),\hat\eta_{12}^{\phi}\big]=0,$$

where $X,Y\in\Gamma(TM)$. Multiplying by $\hat\eta_{12}^{\phi}$ on the left

$$\hat\eta_{12}^{\phi}R^M(X,Y)\hat\eta_{12}^{\phi}+R^M(X,Y)=0.$$

Setting $Y=e_j$ and summing over i, j,

$$\begin{aligned}
0&=\sum_{i,j}\big\langle\hat\eta_{12}^{\phi}R^M(X,e_i)\hat\eta_{12}^{\phi}(e_i),e_j\big\rangle e_j+\sum_{i,j}\big\langle R^M(X,e_i)(e_i),e_j\big\rangle e_j\\
&=-\sum_{i,j}\big\langle R^M(X,e_i)\hat\eta_{12}^{\phi}(e_i),\hat\eta_{12}^{\phi}e_j\big\rangle e_j+\sum_{i,j}\big\langle R^M(X,e_i)(e_i),e_j\big\rangle e_j\\
&=\hat\eta_{12}^{\phi}\sum_{i,j}\big\langle R^M(X,e_i)\hat\eta_{12}^{\phi}(e_i),\hat\eta_{12}^{\phi}e_j\big\rangle\hat\eta_{12}^{\phi}(e_j)+\mathrm{Ric}(X)\\
&=\hat\eta_{12}^{\phi}\sum_{i,j}\big\langle R(X,e_i)\hat\eta_{12}^{\phi}(e_i),e_j'\big\rangle e_j'+\mathrm{Ric}(X)\\
&=\hat\eta_{12}^{\phi}\big(-R^M\bullet\eta_{12}^{\phi}(X)\big)+\mathrm{Ric}(X),
\end{aligned}$$

i.e.,

$$\mathrm{Ric}=\hat\eta_{12}^{\phi}R^M\bullet\eta_{12}^{\phi},$$

so that

$$R^M\bullet\eta_{12}^{\phi}=-\mathrm{Ric}\circ\eta_{12}^{\phi}.$$

By (2.3),

$$0=R^M(X,Y)\cdot\phi-\Theta_{12}(X,Y)\kappa_2^*(f_{12})\cdot\phi.$$

Proceeding as in [12], set $Y=e_a$, multiply by e_a and sum over a

$$\begin{aligned}
0&=\sum_a e_a\cdot R^M(X,e_a)\cdot\phi-\sum_a\Theta_{12}(X,e_a)e_a\cdot\kappa_{2*}^1(f_{12})\cdot\phi\\
&=-\mathrm{Ric}(X)\cdot\phi-\hat\Theta_{12}(X)\cdot\kappa_{2*}^1(f_{12})\cdot\phi,
\end{aligned}$$

i.e.,

$$\mathrm{Ric}(X)\cdot\phi=-\hat\Theta_{12}(X)\cdot\kappa_{2*}^1(f_{12})\cdot\phi.$$

Now consider, for $i<j$,

$$\begin{aligned}
\mathrm{Ric}_{ji}=\mathrm{Re}\,\big\langle\mathrm{Ric}(e_i)\cdot\phi,e_j\cdot\phi\big\rangle&=-\sum_a\Theta_{12}(e_i,e_a)\,\mathrm{Re}\,\big\langle e_a\cdot\kappa_{2*}^1(f_{12})\cdot\phi,e_j\cdot\phi\big\rangle\\
&=\sum_a\Theta_{12}(e_i,e_a)\,\mathrm{Re}\,\big\langle e_je_a\cdot\kappa_{2*}^1(f_{12})\cdot\phi,\phi\big\rangle=\sum_a\big(\hat\Theta_{12}\big)_{ai}\eta_{12}^{\phi}(e_j,e_a)\\
&=\sum_a\big(\hat\Theta_{12}\big)_{ai}\big(\hat\eta_{12}^{\phi}\big)_{aj}=-\sum_a\big(\hat\eta_{12}^{\phi}\big)_{ja}\big(\hat\Theta_{12}\big)_{ai}=-\big(\hat\eta_{12}^{\phi}\hat\Theta_{12}\big)_{ji},
\end{aligned}$$

i.e.,

$$\mathrm{Ric}=-\hat\eta_{12}^{\phi}\hat\Theta_{12}.$$

Since Ric is symmetric, $\hat{\Theta}_{12}$ and $\hat{\eta}^{\psi}_{12}$ commute and also commute with Ric. Thus,

$$\hat{\Theta}_{12} = \mathrm{Ric} \circ \hat{\eta}^{\phi}_{12}, \qquad \text{i.e.,} \qquad \Theta = -R^M \bullet \eta^{\phi},$$

so that the pair (ϕ, θ) satisfies (5.1) with $E = -R^M \bullet$.

Furthermore, if we assume M is Einstein

$$\Theta = \frac{\mathrm{R}}{2n}\eta^{\phi}.$$

If we assume the scalar curvature R to be constant, we can say that

- if R is non-negative, the pair $\left(\sqrt{\frac{R}{2n}}\phi, \theta\right)$ satisfies (5.1) with $E = \mathrm{Id}_{\bigwedge^2 T^*M}$;
- if R is non-positive, the pair $\left(\sqrt{\frac{-R}{2n}}\phi, \theta\right)$satisfies (5.1) with $E = -\mathrm{Id}_{\bigwedge^2 T^*M}$.

Remark 5.3. The last two statements remind us again of the Seiberg–Witten equations in dimension 4. We could even consider the positive/negative spinor decomposition and the self-dual/anti-self-dual 2-form decomposition to arrive at two sets of equations: the Seiberg–Witten equations and an analogous pair of equations for negative spinors and anti-self-dual forms.

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