

Realization of $U_q(\mathfrak{sp}_{2n})$ within the Differential Algebra on Quantum Symplectic Space

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Abstract. We realize the Hopf algebra $U_q(\mathfrak{sp}_{2n})$ as an algebra of quantum differential operators on the quantum symplectic space $\mathcal{X}(f_s; \mathbb{R})$ and prove that $\mathcal{X}(f_s; \mathbb{R})$ is a $U_q(\mathfrak{sp}_{2n})$ -module algebra whose irreducible summands are just its homogeneous subspaces. We give a coherence realization for all the positive root vectors under the actions of Lusztig's braid automorphisms of $U_q(\mathfrak{sp}_{2n})$.

Key words: quantum symplectic group; quantum symplectic space; quantum differential operators; differential calculus; module algebra

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1 Introduction

Quantum analogues of differential forms and differential operators on quantum groups or Hopf algebras or quantum spaces have been studied extensively since the end of 1980s (see [4, 7, 10, 15, 17, 23, 24], etc. and references therein). As a main theme of noncommutative (differential) geometry, the general theory of bicovariant differential calculus on quantum groups or Hopf algebras was established in [24]. Woronowicz's axiomatic description of bicovariant bimodules (namely, Hopf bimodules in Hopf algebra theory) is not only used to construct/classify the first order differential calculi (FODC) on Hopf algebras, but also leads to the appearance of Woronowicz's braiding [24, Proposition 3.1] (also see [22, Theorem 6.3]). Actually, the defining condition of Yetter–Drinfeld module appeared implicitly in Woronowicz's work a bit earlier than [20, 25] (see [24, formula (2.39)]), as was witnessed by Schauenburg in [22, Corollaries 6.4 and 6.5] proving that the category of Woronowicz's bicovariant bimodules is categorically equivalent to the category of Yetter–Drinfeld modules, while the latter has currently served as an important working framework for classifying the finite-dimensional pointed Hopf algebras. The coupled pair consists of a quantum group and its corresponding quantum space on which it coacts, both of which in the pair were intimately interrelated [21]. On the other hand, the covariant differential calculus on the quantum space \mathbb{C}_q^n was built by Wess–Zumino [23] so as to extend the covariant coaction of the quantum group $GL_q(n)$ to quantum derivatives. Along the way, many pioneering works appeared by Ogievetsky et al. [17, 18, 19], etc.

Recall that for any bialgebra \mathcal{A} , by a quantum space for \mathcal{A} we mean a right \mathcal{A} -comodule algebra \mathcal{X} . Here, we let \mathcal{A} denote a certain Hopf quotient of the FRT bialgebra $\mathcal{A}(\mathbb{R})$, which is related with a standard R -matrix R of the $ABCD$ series (cf. [10, 21]), and we set $\mathcal{X} := \mathcal{X}_r(f_s; \mathbb{R})$

(adopting the notation in the book [10]). For the definition of polynomials f_s in types $ABCD$, we refer to [10, Definitions 4, 8, 12 in Sections 9.2 and 9.3]. Roughly speaking, viewing $U_q(\mathfrak{g})$ as the Hopf dual object of quantum group G_q in types $ABCD$, one sees that the aforementioned quantum space \mathcal{X} is a left $U_q(\mathfrak{g})$ -module algebra. As a benefit of the viewpoint, this allows one to do the crossed product construction to enlarge the quantum enveloping algebra $U_q(\mathfrak{g})$ into a quantum enveloping parabolic subalgebra of the same type but with a higher rank. This actually contributes an evidence to support Majid's conjecture [14] on the rank-inductive construction of $U_q(\mathfrak{g})$'s via his double-bosonization procedure (see also a recent work [5] for confirming Majid's claim in the classical cases).

For types B and D , under the assumption that q is not a root of unity, Fiore [2] used the standard R -matrix for the quantum group $SO_q(N)$ ($N = 2n + 1$ or $2n$) to define some quantum differential operators on the quantum Euclidean space \mathbb{R}_q^N . Then he realized $U_{q^{-1}}(\mathfrak{so}_N)$ within the differential algebra $\text{Diff}(\mathbb{R}_q^N)$ such that \mathbb{R}_q^N is a left $U_{q^{-1}}(\mathfrak{so}_N)$ -module algebra, and further developed the corresponding quantum Euclidean geometry in his subsequent works. There were many works [17, 18, 19], prior to [2], using quantum differential operators to describe the $GL_q(n)$ and $SO_q(n)$, q -Lorentz algebra, and q -deformed Poincaré algebra, etc.

For type A , there appeared several special discussions in rank 1 case, see [9, 16, 23], etc. To our interest, for arbitrary rank, different from [17] and [2], the second author [6] introduced the notion of quantum divided power algebra $\mathcal{A}_q(n)$ for q both generic and root of unity. He defined q -derivatives over $\mathcal{A}_q(n)$ and realized the U -module algebra structure of $\mathcal{A}_q(n)$ for $U = U_q(\mathfrak{sl}_n)$, $\mathfrak{u}_q(\mathfrak{sl}_n)$. A coherence realization of all the positive root vectors in terms of the quantum differential operators was provided (in the modified q -Weyl algebra $\mathcal{W}_q(2n)$) which are compatible with the actions of Lusztig's braid automorphisms [13]. Especially, this discussion of q -derivatives resulted in the definition of the quantum universal enveloping algebras of abelian Lie algebras for the first time, and even the new Hopf algebra structure so-called the n -rank Taft algebra (see [7, 11]) in root of unity case. Based on the realization in [6], Gu and Hu [3] gave further explicit results of the module structures on the quantum Grassmann algebra defined over the quantum divided power algebra, the quantum de Rham complexes and their cohomological modules, as well as the descriptions of the Loewy filtrations of a class of interesting indecomposable modules for Lusztig's small quantum group $\mathfrak{u}_q(\mathfrak{sl}_n)$.

For type C , it seems lack of corresponding discussions over the quantum symplectic space in the literature. Here we consider the quantum enveloping algebra $U_q(\mathfrak{sp}_{2n})$ with $n \geq 2$ and its corresponding quantum symplectic space $\mathcal{X}(f_s; \mathbb{R})$. We assume that q is not a root of unity. We define the q -analogues $\partial_i := \partial_q / \partial x_i$ of the classical partial derivatives and introduce left- and right-multiplication operators x_{i_L} and x_{i_R} as in [9]. Our discussion also does not use the R -matrix as a tool as in [2]. We consider the subalgebra U_q^{2n} generated by some quantum differential operators in the quantum differential algebra $\text{Diff}(\mathcal{X}(f_s; \mathbb{R}))$ (we call it the *modified* q -Weyl algebra of type C , distinctive from the ordinary one, since it contains some extra automorphisms as group-likes inside). Furthermore, we check the Serre relations of U_q^{2n} and show $\mathcal{X}(f_s; \mathbb{R})$ is a $U_q(\mathfrak{sp}_{2n})$ -module algebra. At last, we show that the positive root vectors of $U_q(\mathfrak{sp}_{2n})$ defined by Lusztig's braid automorphisms in [13] can be realized precisely by means of the quantum differential operators defined in Section 5.

The paper is organized as follows. Section 2 gives the definition of the quantum symplectic space $\mathcal{X}(f_s; \mathbb{R})$ and derives some useful formulas. In Section 3, we define the quantum differential operators on $\mathcal{X}(f_s; \mathbb{R})$ and a subalgebra U_q^{2n} of $\text{Diff}(\mathcal{X}(f_s; \mathbb{R}))$. We prove that the generators of U_q^{2n} satisfy the Serre relations which implies that U_q^{2n} is a quotient algebra of $U_q(\mathfrak{sp}_{2n})$. We show that $\mathcal{X}(f_s; \mathbb{R})$ is a $U_q(\mathfrak{sp}_{2n})$ -module algebra whose irreducible summands are just its homogeneous subspaces. In Section 4, we provide inductive formulas to calculate all the positive root vectors under the actions of Lusztig's braid automorphisms of $U_q(\mathfrak{sp}_{2n})$ from simple root vectors. In Section 5, we give a coherence realization for all the positive root vectors of $U_q(\mathfrak{sp}_{2n})$.

For simplicity, we write \mathcal{X} for $\mathcal{X}(f_s; \mathbb{R})$. Let \mathbb{N}_0 (resp. \mathbb{N}) be the set of nonnegative (resp. positive) integers, \mathbb{R} denote the set of real numbers, \mathbf{k} the underlying field of characteristic 0. Assume that q is invertible in \mathbf{k} and is not a root of unity. Let $n \geq 2$ be a positive integer. Set $I = \{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$ and $I^+ = \{1, \dots, n\}$.

2 Preliminaries

2.1. Recall that the q -number $[m]_q$ for $m \in \mathbb{Z}$ is defined by $[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$. Note that $[0]_q = 0$. For $m \in \mathbb{N}$, the q -factorial is defined by setting $[m]_q! := [1]_q [2]_q \cdots [m]_q$ and $[0]_q! := 1$. The q -binomial coefficients are defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_q := \frac{[m]_q [m-1]_q \cdots [m-k+1]_q}{[1]_q [2]_q \cdots [k]_q}$$

for $m, k \in \mathbb{Z}$ with $k > 0$, and $\begin{bmatrix} m \\ 0 \end{bmatrix}_q := 1$. So if $k > m \geq 0$, then $\begin{bmatrix} m \\ k \end{bmatrix}_q = 0$. Set $[A, B]_v = AB - vBA$ for $v \in \mathbf{k}$. When $v = 1$, $[\cdot, \cdot]_1$ is the commutator $[\cdot, \cdot]$. The following three lemmas can be checked directly and will be used many times in Sections 4 and 5.

Lemma 2.1. For $u, v \in \mathbf{k}$ and $u \neq 0$, if $AB = uBA$, then

$$\begin{aligned} [A, BC]_v &= uB[A, C]_{v/u}, \\ [A, CB]_v &= [A, C]_{v/u}B, \\ [CA, B]_v &= u[C, B]_{v/u}A, \\ [AC, B]_v &= A[C, B]_{v/u}. \end{aligned} \tag{2.1}$$

Lemma 2.2. For $u, v, w \in \mathbf{k}$ and $u \neq 0$, if $AC = uCA$, then

$$\begin{aligned} [[A, B]_v, C]_w &= [A, [B, C]_{w/u}]_{uv}, \\ [[B, A]_v, C]_w &= u[[B, C]_{w/u}, A]_{v/u}. \end{aligned} \tag{2.2}$$

Lemma 2.3. We have

$$\begin{aligned} [A, B]_q &= -q[B, A]_{q^{-1}}, \\ [AB, C]_{q^2} &= A[B, C]_q + q[A, C]_q B, \end{aligned} \tag{2.3}$$

$$[[A, B]_q, C]_q = [A, [B, C]_{q^2}] + [[A, C]_q, B]_q. \tag{2.4}$$

2.2. Recall that the simple roots of \mathfrak{sp}_{2n} are $\alpha_1 = 2\epsilon_1$ and $\alpha_i = \epsilon_i - \epsilon_{i-1}$ for $2 \leq i \leq n$, where $\epsilon_i = (\delta_{1i}, \dots, \delta_{ni})$ and $\epsilon_1, \dots, \epsilon_n$ form a canonical basis of \mathbb{R}^n . Note that here α_1 is chosen to be longer than other simple roots. Let Δ^+ be the set of positive roots of \mathfrak{sp}_{2n} , then

$$\Delta^+ = \{2\epsilon_i, \pm\epsilon_l + \epsilon_k \mid 1 \leq i \leq n, 1 \leq l < k \leq n\}.$$

2.3. Recall that the quantum universal enveloping algebra $U_q(\mathfrak{sp}_{2n})$ generated by $\{E_i, F_i, K_i, K_i^{-1}, i \in I^+\}$ has the defining relations as follows:

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{2.5}$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \tag{2.6}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{2.7}$$

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{q_i} E_i^t E_j E_i^{1-a_{ij}-t} = 0, \quad i \neq j, \quad (2.8)$$

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{q_i} F_i^t F_j F_i^{1-a_{ij}-t} = 0, \quad i \neq j, \quad (2.9)$$

where $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$, $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$, and the Cartan matrix (a_{ij}) of \mathfrak{sp}_{2n} in our indices is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Note that $q_1 = q^2$, $q_i = q$ for $1 < i \leq n$. The relations (2.8) and (2.9) are usually called the *Serre relations*.

The algebra $U_q(\mathfrak{sp}_{2n})$ is a Hopf algebra equipped with coproduct Δ , counit ε and antipode S defined by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad (2.10)$$

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad (2.11)$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i^{\pm 1}) = 1, \quad (2.12)$$

$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i^{\pm 1}) = K_i^{\mp 1},$$

for $i \in I^+$.

2.4. Set $\lambda = q - q^{-1}$. By [10, Proposition 16 in Section 9.3.4], the quantum symplectic space \mathcal{X} is the algebra with generators x_i , $i \in I$, and defining relations:

$$x_j x_i = q x_i x_j, \quad i < j, \quad -i \neq j, \quad (2.13)$$

$$x_i x_{-i} = q^2 x_{-i} x_i + q^2 \lambda \Omega_{i+1}, \quad i \in I^+, \quad (2.14)$$

where $\Omega_i := \sum_{i \leq j \leq n} q^{j-i} x_{-j} x_j$ for $i \in I^+$, and \mathcal{X} is a vector space with basis

$$\{x_{-n}^{a_{-n}} \cdots x_n^{a_n} \mid a_{-n}, \dots, a_n \in \mathbb{N}_0\}.$$

By definition, for $1 \leq i \leq n-1$, we have

$$\Omega_i = x_{-i} x_i + q \Omega_{i+1}. \quad (2.15)$$

From relations (2.13) and (2.14), we can obtain the following identities:

$$\Omega_i x_k = \begin{cases} q^2 x_k \Omega_i, & -n \leq k \leq -i, \\ x_k \Omega_i, & -i < k < i, \\ q^{-2} x_k \Omega_i, & i \leq k \leq n, \end{cases} \quad (2.16)$$

and

$$\Omega_i \Omega_j = \Omega_j \Omega_i, \quad i, j \in I^+.$$

Set $x^a := x_{-n}^{a_{-n}} \cdots x_n^{a_n}$ and $a := (a_{-n}, a_{1-n}, \dots, a_n)$, where $a_{-n}, \dots, a_n \in \mathbb{N}_0$. We call the monomial $x_{i_1}^{a_{i_1}} \cdots x_{i_m}^{a_{i_m}}$ whose subscripts are placed in an increasing order a *normal monomial*. Write $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^{2n}$ with 1 in the i -position and 0 elsewhere. Then $a = \sum_{i \in I} a_i \varepsilon_i$. Set $|a| = \sum_{i \in I} a_i$. Thus $\mathcal{X} = \bigoplus_m \mathcal{X}^m$ is an \mathbb{N}_0 -graded algebra with $\mathcal{X}^m = \text{Span}_{\mathbf{k}}\{x^a \mid |a| = m\}$.

By induction and using relations (2.13)–(2.16), we get

$$\begin{aligned} x_i x_{-i}^m &= q^{2m} x_{-i}^m x_i + q^{m+1} \lambda[m]_q \Omega_{i+1} x_{-i}^{m-1}, \\ x_i^m x_{-i} &= q^{2m} x_{-i} x_i^m + q^{m+1} \lambda[m]_q \Omega_{i+1} x_i^{m-1}, \end{aligned}$$

for $i \in I^+$ and $m \in \mathbb{N}_0$. Hence, for $i \in I^+$, we have

$$x_{-i} x^a = \left(\prod_{j=-n}^{-i-1} q^{a_j} \right) x^{a+\varepsilon_{-i}}, \quad x^a x_i = \left(\prod_{j=i+1}^n q^{a_j} \right) x^{a+\varepsilon_i}, \quad (2.17)$$

$$x_i x^a = \left(\prod_{j=-n}^{i-1} q^{a_j} \right) q^{a_{-i}} x^{a+\varepsilon_i} + \left(\prod_{j=-n}^{-i-1} q^{-a_j} \right) q^{a_{-i}+1} \lambda[a_{-i}]_q \Omega_{i+1} x^{a-\varepsilon_{-i}}, \quad (2.18)$$

$$x^a x_{-i} = \left(\prod_{j=1-i}^n q^{a_j} \right) q^{a_i} x^{a+\varepsilon_{-i}} + \left(\prod_{k=i}^n q^{a_k} \right) \left(\prod_{j=-n}^{-i-1} q^{-2a_j} \right) q \lambda[a_i]_q \Omega_{i+1} x^{a-\varepsilon_i}. \quad (2.19)$$

The following lemma will be used later.

Lemma 2.4. *For $i \in I^+$, we have*

$$\Omega_i x^a = \left(\prod_{l=-n}^{-i} q^{2a_l} \right) \left(\sum_{j=i}^n q^{j-i+(a_{1-j}+\cdots+a_{j-1})} x^{a+\varepsilon_{-j}+\varepsilon_j} \right). \quad (2.20)$$

Proof. We prove this lemma by induction on i from n to 1. From (2.13) and (2.16), we have

$$\begin{aligned} \Omega_n x^a &= q^{2a_{-n}} x_{-n}^{a_{-n}} \Omega_n x_{1-n}^{a_{1-n}} \cdots x_n^{a_n} = q^{2a_{-n}} x_{-n}^{a_{-n}+1} x_n x_{1-n}^{a_{1-n}} \cdots x_n^{a_n} \\ &= q^{2a_{-n}} q^{a_{1-n}+\cdots+a_{n-1}} x^{a+\varepsilon_{-n}+\varepsilon_n}. \end{aligned}$$

So the formula (2.20) holds for $i = n$. Suppose (2.20) holds for $i > 1$. Then from (2.13), (2.15) and (2.16), we obtain

$$\begin{aligned} \Omega_{i-1} x^a &= \left(\prod_{l=-n}^{1-i} q^{2a_l} \right) x_{-n}^{a_{-n}} \cdots x_{1-i}^{a_{1-i}} \Omega_{i-1} x_{2-i}^{a_{2-i}} \cdots x_n^{a_n} \\ &= \left(\prod_{l=-n}^{1-i} q^{2a_l} \right) x_{-n}^{a_{-n}} \cdots x_{1-i}^{a_{1-i}} (x_{1-i} x_{i-1} + q \Omega_i) x_{2-i}^{a_{2-i}} \cdots x_n^{a_n} \\ &= \left(\prod_{l=-n}^{1-i} q^{2a_l} \right) q^{a_{2-i}+\cdots+a_{i-2}} x^{a+\varepsilon_{1-i}+\varepsilon_{i-1}} + q^{1+2a_{1-i}} \Omega_i x^a. \end{aligned}$$

The induction hypothesis completes the proof. ■

3 Quantum differential operators on $\mathcal{X}(f_s; \mathbf{R})$

3.1. We define some quantum analogs of differential operators on \mathcal{X} .

Definition 3.1. For any normal monomial x^a and $i \in I$, set

$$\begin{aligned}\partial_i \cdot x^a &:= [a_i]_q x^{a-\varepsilon_i}, \\ x_{i_L} \cdot x^a &:= x_i x^a, \\ x_{i_R} \cdot x^a &:= x^a x_i, \\ \mu_i \cdot x^a &:= q^{a_i} x^a, \\ \mu_i^{-1} \cdot x^a &:= q^{-a_i} x^a.\end{aligned}$$

Let $\text{Diff}(\mathcal{X})$ be the unital algebra of quantum differential operators on \mathcal{X} generated by ∂_i , x_{i_L} , x_{i_R} , μ_i and μ_i^{-1} with $i \in I$. This algebra can be described precisely as the smash product of a quantum group \mathfrak{D}_q and the symplectic space \mathcal{X} , where the associative algebra \mathfrak{D}_q generated by ∂_i 's ($i \in I$) as well as μ_i 's ($i \in I$), acting on \mathcal{X} , is a Hopf algebra. For a detailed treatment for type A case, one can refer to [6], where the quantum differential operators algebra is the (modified) quantum Weyl algebra (of type A). Since we only use the actions of these quantum differential operators on \mathcal{X} , we omit the explicit presentation of $\text{Diff}(\mathcal{X})$.

Since $\mu_k \mu_l = \mu_l \mu_k$, we write

$$\tau_i := \prod_{j=i}^n \mu_j \quad \text{and} \quad \tau_{-i} := \prod_{j=-n}^{-i} \mu_j$$

for $i \in I^+$. Now we define a subalgebra of $\text{Diff}(\mathcal{X})$.

Definition 3.2. For $i \in I^+$ with $i \geq 2$, set

$$\begin{aligned}e_1 &:= [2]_q^{-1} q^{-1} \mu_1^{-1} (\tau_{-2}^{-1} x_{-1_L} + q^2 \tau_2^{-1} x_{-1_R}) \partial_1, \\ f_1 &:= [2]_q^{-1} q^{-1} \mu_{-1}^{-1} (\tau_2^{-1} x_{1_R} + q^2 \tau_{-2}^{-1} x_{1_L}) \partial_{-1}, \\ k_1 &:= \mu_{-1}^2 \mu_1^{-2}, \\ e_i &:= \mu_{i-1} \mu_i^{-1} \tau_{-i-1}^{-1} x_{-i_L} \partial_{1-i} - \tau_i^{-1} x_{i-1_R} \partial_i, \\ f_i &:= -\mu_{i-1} \mu_{-i}^{-1} \tau_{i+1}^{-1} x_{i_R} \partial_{i-1} + \tau_{-i}^{-1} x_{1-i_L} \partial_{-i}, \\ k_i &:= \mu_{-i} \mu_{1-i}^{-1} \mu_{i-1} \mu_i^{-1}.\end{aligned}$$

Let U_q^{2n} be the subalgebra of $\text{Diff}(\mathcal{X})$ generated by $\{e_i, f_i, k_i, k_i^{-1} \mid i \in I^+\}$.

Applying the operators defined in Definition 3.2 to any normal monomial x^a , and using Definition 3.1 and (2.17)–(2.19), we get

$$e_1 \cdot x^a = [a_1]_{q^2} x^{a+\varepsilon_{-1}-\varepsilon_1} + \lambda \begin{bmatrix} a_1 \\ 2 \end{bmatrix}_q q^{2-2(a_{-n}+\dots+a_{-2})} \Omega_2 x^{a-2\varepsilon_1}, \quad (3.1)$$

$$f_1 \cdot x^a = [a_{-1}]_{q^2} x^{a-\varepsilon_{-1}+\varepsilon_1} + \lambda \begin{bmatrix} a_{-1} \\ 2 \end{bmatrix}_q q^{2-2(a_{-n}+\dots+a_{-2})} \Omega_2 x^{a-2\varepsilon_{-1}}, \quad (3.2)$$

$$k_1 \cdot x^a = q^{2(a_{-1}-a_1)} x^a, \quad (3.3)$$

$$e_i \cdot x^a = q^{a_{i-1}-a_i} [a_{1-i}]_q x^{a+\varepsilon_{-i}-\varepsilon_{1-i}} - [a_i]_q x^{a+\varepsilon_{i-1}-\varepsilon_i}, \quad (3.4)$$

$$f_i \cdot x^a = [a_{-i}]_q x^{a-\varepsilon_{-i}+\varepsilon_{1-i}} - q^{a_{1-i}-a_{-i}} [a_{i-1}]_q x^{a-\varepsilon_{i-1}+\varepsilon_i}, \quad (3.5)$$

$$k_i \cdot x^a = q^{a_{-i}-a_{1-i}+a_{i-1}-a_i} x^a, \quad (3.6)$$

for $1 < i \leq n$.

The following two lemmas will be used later.

Lemma 3.3. For $i, j \in I^+$, we have

$$e_i.\Omega_j = \begin{cases} 0, & i \neq j, \\ -x_{-j}x_{j-1}, & i = j > 1, \\ x_{-1}^2, & i = j = 1, \end{cases} \quad \text{and} \quad f_i.\Omega_j = \begin{cases} 0, & i \neq j, \\ x_{1-j}x_j, & i = j > 1, \\ x_1^2, & i = j = 1. \end{cases}$$

Proof. It follows immediately from (3.1) and (3.5). ■

Lemma 3.4. For any two normal monomials x^a, x^b and $i \in I^+$ we have

$$\begin{aligned} k_i^{\pm 1}.(x^a x^b) &= (k_i^{\pm 1}.x^a)(k_i^{\pm 1}.x^b), \\ e_i.(x^a x^b) &= (e_i.x^a)(k_i.x^b) + x^a(e_i.x^b), \\ f_i.(x^a x^b) &= (f_i.x^a)x^b + (k_i^{-1}.x^a)(f_i.x^b). \end{aligned}$$

Proof. We prove this lemma by induction on $|a|$. For $|a| = 1$, write $x^a = x_j$, $j \in I$. The assertion of this lemma for $|a| = 1$ can be derived from the relations (2.17)–(2.20), (3.1)–(3.6) and Lemma 3.3 directly. We omit this straightforward and lengthy verification. Suppose that the lemma holds for any normal monomial x^a with $|a| = m$. Let x^c be a normal monomial with $|c| = m + 1$. We can write $x^c = x_j x^a$, where $|a| = m$ and j is the smallest index in (c_{-n}, \dots, c_n) such that $c_j \neq 0$. Since $x^a x^b$ can be written as a linear combination of normal monomials, by the induction hypothesis, we get

$$k_i.(x^c x^b) = k_i.(x_j x^a x^b) = (k_i.x_j)(k_i.(x^a x^b)) = (k_i.x_j)(k_i.x^a)(k_i.x^b) = (k_i.x^c)(k_i.x^b).$$

Then

$$\begin{aligned} e_i.(x^c x^b) &= e_i.(x_j x^a x^b) = (e_i.x_j)(k_i.(x^a x^b)) + x_j(e_i.(x^a x^b)) \\ &= (e_i.x_j)(k_i.x^a)(k_i.x^b) + x_j(e_i.x^a)(k_i.x^b) + x_j x^a(e_i.x^b) \\ &= (e_i.(x_j x^a))(k_i.x^b) + x^c(e_i.x^b) = (e_i.x^c)(k_i.x^b) + x^c(e_i.x^b). \end{aligned}$$

Other relations can be proved similarly. ■

The following lemma can be easily checked by definition.

Lemma 3.5. For any $m \in \mathbb{Z}$ we have

$$[m+1]_q = q[m]_q + q^{-m} = q^{-1}[m]_q + q^m, \quad (3.7)$$

$$\begin{bmatrix} m+1 \\ 2 \end{bmatrix}_q - \begin{bmatrix} m \\ 2 \end{bmatrix}_q = [m]_{q^2}, \quad (3.8)$$

$$\begin{bmatrix} m+1 \\ 2 \end{bmatrix}_q - q^2 \begin{bmatrix} m \\ 2 \end{bmatrix}_q = q^{1-m}[m]_q. \quad (3.9)$$

Now we state one of our main theorems.

Theorem 3.6. The generators e_i, f_i, k_i, k_i^{-1} , $i \in I^+$, of U_q^{2n} satisfy the relations (2.5)–(2.9) after replacing E_i, F_i, K_i, K_i^{-1} by e_i, f_i, k_i, k_i^{-1} , respectively. Hence, there is a unique surjective algebra homomorphism $\Psi: U_q(\mathfrak{sp}_{2n}) \rightarrow U_q^{2n}$ mapping E_i, F_i, K_i, K_i^{-1} to e_i, f_i, k_i, k_i^{-1} , respectively.

Proof. The relations (2.5) are clear. Using (3.1)–(3.6), the relations (2.6) can be easily checked. For (2.7), we only prove the case $i = j = 1$, the others can be checked similarly. For any normal monomial x^a , using (3.1)–(3.3) and Lemmas 3.3 and 3.4, we get

$$\begin{aligned}
e_1 f_1 . x^a &= [a_{-1}]_{q^2} e_1 . x^{a-\varepsilon_{-1}+\varepsilon_1} + \lambda \begin{bmatrix} a_{-1} \\ 2 \end{bmatrix}_q q^{2-2(a_{-n}+\dots+a_{-2})} e_1 . (\Omega_2 x^{a-2\varepsilon_{-1}}) \\
&= [a_{-1}]_{q^2} e_1 . x^{a-\varepsilon_{-1}+\varepsilon_1} + \lambda \begin{bmatrix} a_{-1} \\ 2 \end{bmatrix}_q q^{2-2(a_{-n}+\dots+a_{-2})} \Omega_2 (e_1 . x^{a-2\varepsilon_{-1}}) \\
&= [a_{-1}]_{q^2} [a_1 + 1]_{q^2} x^a \\
&\quad + \lambda \left(\begin{bmatrix} a_1 + 1 \\ 2 \end{bmatrix}_q [a_{-1}]_{q^2} + \begin{bmatrix} a_{-1} \\ 2 \end{bmatrix}_q [a_1]_{q^2} \right) q^{2-2(a_{-n}+\dots+a_{-2})} \Omega_2 x^{a-\varepsilon_{-1}-\varepsilon_1} \\
&\quad + \lambda^2 \begin{bmatrix} a_{-1} \\ 2 \end{bmatrix}_q \begin{bmatrix} a_1 \\ 2 \end{bmatrix}_q q^{4-4(a_{-n}+\dots+a_{-2})} \Omega_2 \Omega_2 x^{a-2\varepsilon_{-1}-2\varepsilon_1}
\end{aligned}$$

and

$$\begin{aligned}
f_1 e_1 . x^a &= [a_1]_{q^2} f_1 . x^{a+\varepsilon_{-1}-\varepsilon_1} + \lambda \begin{bmatrix} a_1 \\ 2 \end{bmatrix}_q q^{2-2(a_{-n}+\dots+a_{-2})} f_1 . (\Omega_2 x^{a-2\varepsilon_1}) \\
&= [a_1]_{q^2} f_1 . x^{a+\varepsilon_{-1}-\varepsilon_1} + \lambda \begin{bmatrix} a_1 \\ 2 \end{bmatrix}_q q^{2-2(a_{-n}+\dots+a_{-2})} (k_1^{-1} . \Omega_2) (f_1 . x^{a-2\varepsilon_1}) \\
&= [a_1]_{q^2} [a_{-1} + 1]_{q^2} x^a \\
&\quad + \lambda \left(\begin{bmatrix} a_{-1} + 1 \\ 2 \end{bmatrix}_q [a_1]_{q^2} + \begin{bmatrix} a_1 \\ 2 \end{bmatrix}_q [a_{-1}]_{q^2} \right) q^{2-2(a_{-n}+\dots+a_{-2})} \Omega_2 x^{a-\varepsilon_{-1}-\varepsilon_1} \\
&\quad + \lambda^2 \begin{bmatrix} a_{-1} \\ 2 \end{bmatrix}_q \begin{bmatrix} a_1 \\ 2 \end{bmatrix}_q q^{4-4(a_{-n}+\dots+a_{-2})} \Omega_2 \Omega_2 x^{a-2\varepsilon_{-1}-2\varepsilon_1}.
\end{aligned}$$

Using (3.7) and (3.8), we obtain

$$\begin{aligned}
[e_1, f_1] . x^a &= ([a_{-1}]_{q^2} [a_1 + 1]_{q^2} - [a_1]_{q^2} [a_{-1} + 1]_{q^2}) x^a \\
&= ([a_{-1}]_{q^2} q^{2a_1} - [a_1]_{q^2} q^{2a_{-1}}) x^a = [a_{-1} - a_1]_{q^2} x^a.
\end{aligned}$$

Since $q_1 = q^2$,

$$\frac{k_1 - k_1^{-1}}{q_1 - q_1^{-1}} . x^a = \frac{q^{2(a_{-1}-a_1)} - q^{-2(a_{-1}-a_1)}}{q^2 - q^{-2}} x^a = [a_{-1} - a_1]_{q^2} x^a.$$

Hence

$$[e_1, f_1] = \frac{k_1 - k_1^{-1}}{q_1 - q_1^{-1}}.$$

Consider the first Serre relation (2.8). For the case $i = 1$ and $j = 2$, we need to prove

$$e_1^2 e_2 - [2]_{q^2} e_1 e_2 e_1 + e_2 e_1^2 = 0.$$

Set $e_{1,2} := [e_1, e_2]_{q^2}$. It is equivalent to show

$$[e_1, e_{1,2}]_{q^{-2}} = 0. \tag{3.10}$$

By (3.1), (3.4), (3.6) and Lemmas 3.3–3.5, we get

$$\begin{aligned}
e_{1,2} . x^a &= -[a_1]_{q^2} q^{2+a_{-1}-a_2} x^{a+\varepsilon_{-2}-\varepsilon_1} - [a_2]_{q^2} q^{-2a_1} x^{a+\varepsilon_{-1}-\varepsilon_2} \\
&\quad - \lambda [a_1]_{q^2} [a_2]_{q^2} q^{3-2(a_{-n}+\dots+a_{-2})-a_1} \Omega_2 x^{a-\varepsilon_1-\varepsilon_2}.
\end{aligned}$$

From Lemmas 3.3 and 3.4 and the relation $k_2 e_1 k_2^{-1} = q^{-2} e_1$ which has been proved before, it is easy to show

$$e_{1,2} \cdot (\Omega_2 x^a) = \Omega_2(e_{1,2} \cdot x^a) + (e_1 e_2 \cdot \Omega_2)(k_1 k_2 \cdot x^a).$$

Using the above two formulas and the identity

$$q[a_1]_q [a_1 - 1]_{q^2} - [a_1]_{q^2} [a_1 - 1]_q - \lambda \begin{bmatrix} a_1 \\ 2 \end{bmatrix}_q q^{1-a_1} = 0,$$

which is easy to check, we can verify $[e_1, e_{1,2}]_{q^{-2}} \cdot x^a = 0$ by direct computation. So the relation (3.10) holds.

Consider the first Serre relation (2.8) for $i = 2, j = 1$. We need to prove

$$e_2^3 e_1 - [3]_q e_2^2 e_1 e_2 + [3]_q e_2 e_1 e_2^2 - e_1 e_2^3 = 0.$$

It is equivalent to show

$$[e_2, [e_{1,2}, e_2]]_{q^2} = 0. \tag{3.11}$$

In order to verify the first Serre relation (2.8) for $j = i \pm 1$ and $i, j > 1$, i.e.,

$$e_i^2 e_{i \pm 1} - [2]_q e_i e_{i \pm 1} e_i + e_{i \pm 1} e_i^2 = 0,$$

it is equivalent to show

$$[e_i, [e_i, e_{i \pm 1}]]_{q^{-1}} = 0. \tag{3.12}$$

It is not hard to check (3.11) and (3.12) after applying their left hand sides to x^a . For $|i - j| > 2$, the first Serre relation is $[e_i, e_j] = 0$, which is obvious.

The second Serre relation can be verified similarly.

This completes the proof. ■

Due to the above theorem, we can realize the elements of the quantum group $U_q(\mathfrak{sp}_{2n})$ as certain q -differential operators on \mathcal{X} . In other words, \mathcal{X} is a left $U_q(\mathfrak{sp}_{2n})$ -module.

Let $(H, m, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. Recall that an algebra A is called a left H -module algebra if A is a left H -module, and the multiplication map and the unit map of A are left H -module homomorphisms, that is,

$$h \cdot 1_A = \varepsilon(h) 1_A, \tag{3.13}$$

$$h \cdot (a' a'') = \sum (h_{(1)} \cdot a') (h_{(2)} \cdot a''), \tag{3.14}$$

for any $h \in H, a', a'' \in A$, where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

Theorem 3.7. *The algebra \mathcal{X} is a left $U_q(\mathfrak{sp}_{2n})$ -module algebra.*

Proof. It is sufficient to check (3.13) and (3.14) on the generators of $U_q(\mathfrak{sp}_{2n})$, since ε and Δ are algebra homomorphisms. The relations (2.12) and (3.1)–(3.6) imply (3.13). Lemma 3.4, (2.10) and (2.11) imply (3.14). ■

3.2. We consider the decomposition of \mathcal{X} into a direct sum of irreducible $U_q(\mathfrak{sp}_{2n})$ -submodules. Recall that $\mathcal{X} = \bigoplus_{m \in \mathbb{N}_0} \mathcal{X}^m$, where \mathcal{X}^m is the subspace of homogeneous elements of degree m .

Proposition 3.8. *The vector space \mathcal{X}^m is a finite-dimensional irreducible $U_q(\mathfrak{sp}_{2n})$ -module with highest weight vector x_{-n}^m and highest weight $m\varepsilon_n$.*

Proof. The assertion follows at once from the facts that the symmetric powers of the vector representation of \mathfrak{sp}_{2n} are irreducible and the theory of finite-dimensional representations of $U_q(\mathfrak{g})$ is very similar to that of \mathfrak{g} when q is not a root of unity (see [12]), especially, they have the same character formulas for the irreducible modules. ■

4 Positive root vectors of $U_q(\mathfrak{sp}_{2n})$

We are going to list all positive root vectors of $U_q(\mathfrak{sp}_{2n})$ in U_q^{2n} . We first recall some notions.

Let m_{ij} be equal to 2, 3, 4 when $a_{ij}a_{ji}$ is equal to 0, 1, 2, respectively, where a_{ij} are the entries of the Cartan matrix of \mathfrak{sp}_{2n} . The braid group \mathcal{B} associated with \mathfrak{sp}_{2n} is the group generated by elements s_1, \dots, s_n subject to the relations

$$s_i s_j s_i s_j \cdots = s_j s_i s_j s_i \cdots, \quad i \neq j,$$

where there are m_{ij} s 's on each side. Lusztig introduced actions of braid groups on $U_q(\mathfrak{g})$ in [12, 13]. The following two propositions can be found in many books, for example [8, 10, 13], etc.

Proposition 4.1. *To every $i, i \in I^+$, there corresponds an algebra automorphism T_i of $U_q(\mathfrak{sp}_{2n})$ which acts on the generators K_j, E_j, F_j as*

$$\begin{aligned} T_i(K_j) &= K_j K_i^{-a_{ij}}, & T_i(E_i) &= -F_i K_i^{-1}, & T_i(F_i) &= -K_i E_i, \\ T_i(E_j) &= \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r E_i^{(-a_{ij}-r)} E_j E_i^{(r)}, & \text{for } i &\neq j, \\ T_i(F_j) &= \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{-r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}, & \text{for } i &\neq j, \end{aligned}$$

where $E_i^{(r)} = E_i^r / [r]_{q_i}!$ and $F_i^{(r)} = F_i^r / [r]_{q_i}!$. The mapping $s_i \mapsto T_i$ determines a homomorphism of the braid group \mathcal{B} into the group of algebra automorphisms of $U_q(\mathfrak{sp}_{2n})$.

The operators T_i defined by Proposition 4.1 are Lusztig's $T_{i,-1}''$ [13, Section 37.1.3].

Proposition 4.2. *The operators T_i satisfy the following relations.*

1. For $i, j \in I^+$ with $|i - j| > 1$, we have

$$T_i(E_j) = E_j, \quad T_i T_j = T_j T_i.$$

2. For $2 \leq i, j \leq n$ with $|i - j| = 1$, we have

$$T_i(E_j) = [E_i, E_j]_q, \quad T_i T_j(E_i) = E_j, \quad T_i T_j T_i = T_j T_i T_j.$$

3. For $1 \leq i \neq j \leq 2$, we have

$$\begin{aligned} T_1(E_2) &= [E_1, E_2]_{q^2}, & [2]_q T_2(E_1) &= [E_2, [E_2, E_1]_{q^2}], \\ T_1 T_2 T_1(E_2) &= E_2, & T_2 T_1 T_2(E_1) &= E_1, & T_1 T_2 T_1 T_2 &= T_2 T_1 T_2 T_1. \end{aligned}$$

The Weyl group W of \mathfrak{sp}_{2n} generated by reflections w_1, \dots, w_n (corresponding to the simple roots of \mathfrak{sp}_{2n}) has the longest element w_0 whose reduced expression is

$$w_0 = \gamma_1 \cdots \gamma_n,$$

where $\gamma_i = w_i w_{i-1} \cdots w_1 \cdots w_{i-1} w_i$ (cf. [1]). Write $w_0 = w_{i_1} w_{i_2} \cdots w_{i_N}$ for this reduced expression. Then

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = w_{i_1}(\alpha_{i_2}), \quad \dots, \quad \beta_N = w_{i_1} w_{i_2} \cdots w_{i_{N-1}}(\alpha_{i_N})$$

exhaust all positive roots of \mathfrak{sp}_{2n} .

Definition 4.3. The elements

$$E_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(E_{i_r}), \quad 1 \leq r \leq N,$$

are called *positive root vectors* of $U_q(\mathfrak{sp}_{2n})$ corresponding to the roots β_r 's.

Set

$$\alpha_{i,i} = 2\epsilon_i \quad \text{and} \quad \alpha_{\pm l,k} = \pm\epsilon_l + \epsilon_k$$

for $1 \leq i \leq n$ and $1 \leq l < k \leq n$. We can list all positive roots in the ordering according to the above reduced expression for the longest element w_0 as follows

$$\begin{aligned} &\alpha_{1,1}, \\ &\alpha_{1,2}, \alpha_{2,2}, \alpha_{-1,2}, \\ &\alpha_{2,3}, \alpha_{1,3}, \alpha_{3,3}, \alpha_{-1,3}, \alpha_{-2,3}, \\ &\cdots, \\ &\alpha_{n-1,n}, \alpha_{n-2,n}, \dots, \alpha_{1,n}, \alpha_{n,n}, \alpha_{-1,n}, \alpha_{-2,n}, \dots, \alpha_{1-n,n}. \end{aligned}$$

Write

$$E_{i,i} = E_{\alpha_{i,i}} \quad \text{and} \quad E_{\pm l,k} = E_{\alpha_{\pm l,k}}.$$

It is clear that $E_{1,1} = E_1$ and we will check in Corollary 4.5 that $E_{\alpha_i} = E_i$ for all $1 < i \leq n$. Set

$$T_{\gamma_i} = T_i T_{i-1} \cdots T_1 \cdots T_{i-1} T_i, \quad 1 \leq i \leq n.$$

By Definition 4.3, all the positive root vectors of $U_q(\mathfrak{sp}_{2n})$ associated to the above ordering of Δ^+ are as follows, for any $1 < j \leq n$,

$$E_{j-1,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}}(E_j), \tag{4.1}$$

$$E_{i,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_{i+2}(E_{i+1}), \quad \text{for } 1 \leq i < j-1, \tag{4.2}$$

$$E_{j,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_2(E_1), \tag{4.3}$$

$$E_{-i,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j \cdots T_1 \cdots T_i(E_{i+1}), \quad \text{for } 1 \leq i < j. \tag{4.4}$$

Lemma 4.4. For $1 < i \leq j \leq n$, we have

$$T_1([E_2, E_1]_{q^2}) = E_2, \tag{4.5}$$

$$T_{\gamma_i} T_{i+1}(E_i) = [E_i, E_{i+1}]_q, \tag{4.6}$$

$$T_{\gamma_{j-1}} T_j T_{\gamma_{j-1}}(E_j) = E_j, \tag{4.7}$$

$$T_{\gamma_j} T_{j+1} T_j \cdots T_i = (T_j T_{j+1})(T_{j-1} T_j) \cdots (T_i T_{i+1}) T_{\gamma_{i-1}}(T_i T_{i+1} \cdots T_{j+1}). \tag{4.8}$$

Proof. It is easy to see from (2.6) and (2.7) that

$$[[E_1, E_2]_{q^2}, K_1^{-1}]_{q^2} = 0 \quad \text{and} \quad [E_2, F_1] = 0,$$

then by Proposition 4.1 and relations (2.1) and (2.2), we get

$$\begin{aligned} T_1([E_2, E_1]_{q^2}) &= [T_1(E_2), T_1(E_1)]_{q^2} = [[E_1, E_2]_{q^2}, -F_1 K_1^{-1}]_{q^2} \\ &= -[[E_1, E_2]_{q^2}, F_1] K_1^{-1} = -[[E_1, F_1], E_2]_{q^2} K_1^{-1} \\ &= -\left[\frac{K_1 - K_1^{-1}}{q^2 - q^{-2}}, E_2 \right]_{q^2} K_1^{-1} = E_2. \end{aligned}$$

The relation (4.6) is clear, since for $i > 1$ we have

$$T_{\gamma_i} T_{i+1}(E_i) = T_i T_{\gamma_{i-1}} T_i T_{i+1}(E_i) = T_i T_{\gamma_{i-1}}(E_{i+1}) = T_i(E_{i+1}) = [E_i, E_{i+1}]_q.$$

We use induction on j to prove (4.7). For $j = 2$, this is obvious by Proposition 4.2(3). Now suppose that (4.7) holds for some j with $2 < j < n$. Then Proposition 4.2(2) and induction yield

$$\begin{aligned} T_{\gamma_j} T_{j+1} T_{\gamma_j}(E_{j+1}) &= T_j T_{\gamma_{j-1}} (T_j T_{j+1} T_j) T_{\gamma_{j-1}} T_j(E_{j+1}) \\ &= T_j T_{\gamma_{j-1}} (T_{j+1} T_j T_{j+1}) T_{\gamma_{j-1}} T_j(E_{j+1}) \\ &= T_j T_{j+1} T_{\gamma_{j-1}} T_j T_{\gamma_{j-1}} (T_{j+1} T_j(E_{j+1})) \\ &= T_j T_{j+1} (T_{\gamma_{j-1}} T_j T_{\gamma_{j-1}}(E_j)) = T_j T_{j+1}(E_j) = E_{j+1}. \end{aligned}$$

To prove (4.8), we use induction on $j - i$. For $j - i = 0$, we have

$$T_{\gamma_j} T_{j+1} T_j = T_j T_{\gamma_{j-1}} T_j T_{j+1} T_j = T_j T_{\gamma_{j-1}} T_{j+1} T_j T_{j+1} = T_j T_{j+1} T_{\gamma_{j-1}} T_j T_{j+1}.$$

Suppose that (4.8) holds for some $j - i - 1 > 0$. Then by induction, we get

$$\begin{aligned} T_{\gamma_j} T_{j+1} T_j \cdots T_{i+1} T_i &= (T_j T_{j+1})(T_{j-1} T_j) \cdots (T_{i+1} T_{i+2}) T_{\gamma_i} (T_{i+1} T_{i+2} \cdots T_{j+1}) T_i \\ &= (T_j T_{j+1})(T_{j-1} T_j) \cdots (T_{i+1} T_{i+2}) (T_{\gamma_i} T_{i+1} T_i) T_{i+2} \cdots T_{j+1} \\ &= (T_j T_{j+1})(T_{j-1} T_j) \cdots (T_{i+1} T_{i+2}) (T_i T_{i+1}) T_{\gamma_{i-1}} (T_i T_{i+1} T_{i+2} \cdots T_{j+1}). \end{aligned}$$

So (4.8) holds. ■

Using (4.7) and Proposition 4.2(1), we get the following corollary easily.

Corollary 4.5. *For any $1 < j \leq n$, we have*

$$T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j \cdots T_1 \cdots T_{j-1}(E_j) = E_j,$$

that is, $E_{1-j,j} = E_{\alpha_j} = E_j$.

Proposition 4.6. *The positive root vectors of $U_q(\mathfrak{sp}_{2n})$ have the following commutation relations:*

$$E_{1,2} = [E_1, E_2]_q, \tag{4.9}$$

$$E_{-i,j} = [E_{-i,j-1}, E_j]_q, \quad 3 \leq i+2 \leq j \leq n, \tag{4.10}$$

$$E_{i,j} = [E_{i,j-1}, E_j]_q, \quad 3 \leq i+2 \leq j \leq n, \tag{4.11}$$

$$E_{j-1,j} = [E_{j-1}, E_{j-2,j}]_q, \quad 3 \leq j \leq n, \tag{4.12}$$

$$E_{j,j} = [2]_q^{-1} [E_{1,j}, E_{-1,j}], \quad 2 \leq j \leq n. \tag{4.13}$$

Proof. Relation (4.9) is clear. For $i \geq 1$, by Proposition 4.2 and the identity (4.8), we obtain that

$$\begin{aligned} T_{\gamma_{i+1}} T_{i+2} T_{i+1} T_{\gamma_i}(E_{i+1}) &= T_{i+1} T_{i+2} T_{\gamma_i} T_{i+1} T_{i+2} T_{\gamma_i}(E_{i+1}) = T_{i+1} T_{i+2} T_{\gamma_i} T_{i+1} T_{\gamma_i}([E_{i+2}, E_{i+1}]_q) \\ &= [T_{i+1} T_{\gamma_i} T_{i+2} T_{i+1}(E_{i+2}), T_{i+1} T_{i+2} T_{\gamma_i} T_{i+1} T_{\gamma_i}(E_{i+1})]_q \\ &= [T_{i+1} T_{\gamma_i} T_{i+2} T_{i+1}(E_{i+2}), T_{i+1} T_{i+2}(E_{i+1})]_q = [T_{i+1} T_{\gamma_i}(E_{i+1}), E_{i+2}]_q. \end{aligned}$$

So Proposition 4.2, (4.4), (4.8) and the above formula show that for $1 \leq i < j - 1$

$$E_{-i,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j \cdots T_1 \cdots T_i(E_{i+1})$$

$$\begin{aligned}
 &= (T_{\gamma_1} \cdots T_{\gamma_{j-2}})(T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_{i+1}) T_{\gamma_i}(E_{i+1}) \\
 &= (T_{\gamma_1} \cdots T_{\gamma_{j-2}})(T_{j-1} T_j) \cdots (T_{i+2} T_{i+3})(T_{i+1} T_{i+2}) T_{\gamma_i}(T_{i+1} \cdots T_{j-1} T_j) T_{\gamma_i}(E_{i+1}) \\
 &= (T_{\gamma_1} \cdots T_{\gamma_{j-2}})(T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) T_{i+1} T_{\gamma_i} T_{i+2} T_{i+1} T_{i+2} T_{\gamma_i}(E_{i+1}) \\
 &= (T_{\gamma_1} \cdots T_{\gamma_{j-2}})(T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) T_{i+1} T_{\gamma_i} T_{i+1} T_{i+2} T_{i+1} T_{\gamma_i}(E_{i+1}) \\
 &= (T_{\gamma_1} \cdots T_{\gamma_{j-2}})(T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) T_{\gamma_{i+1}} T_{i+2} T_{i+1} T_{\gamma_i}(E_{i+1}) \\
 &= (T_{\gamma_1} \cdots T_{\gamma_{j-2}})(T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) [T_{i+1} T_{\gamma_i}(E_{i+1}), E_{i+2}]_q \\
 &= (T_{\gamma_1} \cdots T_{\gamma_{j-2}}) [T_{j-1} \cdots T_{i+2} T_{i+1} T_{\gamma_i}(E_{i+1}), (T_{j-1} T_j) \cdots (T_{i+2} T_{i+3})(E_{i+2})]_q \\
 &= [T_{\gamma_1} \cdots T_{\gamma_{j-2}} T_{j-1} \cdots T_{i+2} T_{i+1} T_{\gamma_i}(E_{i+1}), E_j]_q = [E_{-i, j-1}, E_j]_q.
 \end{aligned}$$

Hence the relation (4.10) holds.

For $j = i + 2$, the relations (4.2) and (4.6) yield

$$E_{i, i+2} = T_{\gamma_1} \cdots T_{\gamma_{i+1}} T_{i+2}(E_{i+1}) = T_{\gamma_1} \cdots T_{\gamma_i}([E_{i+1}, E_{i+2}]_q) = [E_{i, i+1}, E_{i+2}]_q.$$

For $j > i + 2$, by (4.2), (4.6) and (4.8), we have

$$\begin{aligned}
 E_{i, j} &= T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_{i+2}(E_{i+1}) \\
 &= T_{\gamma_1} \cdots T_{\gamma_{j-2}}(T_{j-1} T_j)(T_{j-2} T_{j-1}) \cdots (T_{i+2} T_{i+3}) T_{\gamma_{i+1}} T_{i+2} \cdots T_{j-1} T_j(E_{i+1}) \\
 &= T_{\gamma_1} \cdots T_{\gamma_{j-2}}(T_{j-1} T_j)(T_{j-2} T_{j-1}) \cdots (T_{i+2} T_{i+3}) T_{\gamma_{i+1}} T_{i+2}(E_{i+1}) \\
 &= T_{\gamma_1} \cdots T_{\gamma_{j-2}}(T_{j-1} T_j)(T_{j-2} T_{j-1}) \cdots (T_{i+2} T_{i+3})([E_{i+1}, E_{i+2}]_q) \\
 &= [T_{\gamma_1} \cdots T_{\gamma_{j-2}} T_{j-1} T_{j-2} \cdots T_{i+2}(E_{i+1}), E_j]_q = [E_{i, j-1}, E_j]_q.
 \end{aligned}$$

So the relation (4.11) holds.

For $j \geq 3$, using the relations (4.1) (4.4) and (4.11), we have

$$\begin{aligned}
 E_{j-1, j} &= T_{\gamma_1} \cdots T_{\gamma_{j-1}}(E_j) = T_{\gamma_1} \cdots T_{\gamma_{j-2}} T_{j-1} T_{\gamma_{j-2}} T_{j-1}(E_j) \\
 &= T_{\gamma_1} \cdots T_{\gamma_{j-2}} T_{j-1} T_{\gamma_{j-2}}([E_{j-1}, E_j]_q) \\
 &= [T_{\gamma_1} \cdots T_{\gamma_{j-2}} T_{j-1} T_{\gamma_{j-2}}(E_{j-1}), T_{\gamma_1} \cdots T_{\gamma_{j-2}} T_{j-1}(E_j)]_q \\
 &= [E_{2-j, j-1}, T_{\gamma_1} \cdots T_{\gamma_{j-2}}([E_{j-1}, E_j]_q)]_q = [E_{j-1}, [T_{\gamma_1} \cdots T_{\gamma_{j-2}}(E_{j-1}), E_j]_q]_q \\
 &= [E_{j-1}, [E_{j-2, j-1}, E_j]_q]_q = [E_{j-1}, E_{j-2, j}]_q.
 \end{aligned}$$

That is, the relation (4.12) holds.

It is easy to check that $T_1 T_{\gamma_i} = T_{\gamma_i} T_1$ for any $i \in I^+$, so for $j \geq 2$, using (4.2)–(4.5) and Proposition 4.2(3), we obtain

$$\begin{aligned}
 E_{j, j} &= T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_2(E_1) \\
 &= [2]_q^{-1} T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_3([E_2, [E_2, E_1]_{q^2}]) \\
 &= [2]_q^{-1} [T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_3(E_2), T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_3([E_2, E_1]_{q^2})] \\
 &= [2]_q^{-1} [E_{1, j}, T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_3([E_2, E_1]_{q^2})] \\
 &= [2]_q^{-1} [E_{1, j}, T_{\gamma_2} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_3 T_1([E_2, E_1]_{q^2})] \\
 &= [2]_q^{-1} [E_{1, j}, T_{\gamma_2} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_3(E_2)] \\
 &= [2]_q^{-1} [E_{1, j}, T_{\gamma_2} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_3 T_1 T_2 T_1(E_2)] \\
 &= [2]_q^{-1} [E_{1, j}, T_1 T_{\gamma_2} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_3 T_2 T_1(E_2)] = [2]_q^{-1} [E_{1, j}, E_{-1, j}].
 \end{aligned}$$

This proves (4.13). ■

Remark 4.7. By Proposition 4.6, we can perform a double induction first on i then on j with $1 \leq i \leq j \leq n$ to obtain all the positive root vectors $E_{\pm i, j}$ from simple root vectors.

5 Realization of positive root vectors of $U_q(\mathfrak{sp}_{2n})$

In order to realize all the positive root vectors of $U_q(\mathfrak{sp}_{2n})$ directly and concisely as certain operators in $\text{Diff}(\mathcal{X})$, we introduce some new operators.

Definition 5.1. For $i \in I^+$, set

$$\begin{aligned}\Lambda_0 &= \tau_{n+1} = \tau_{-n-1} := 1, & \Lambda_{-i} &:= \prod_{j=-i}^{-1} \mu_j, & \Lambda_i &:= \prod_{j=1}^i \mu_j, \\ \mathfrak{D}_{-i} &:= \mu_i \tau_{-i-1}^{-1} \partial_{-i}, & \mathfrak{D}_i &:= \tau_1^{-1} \Lambda_{i-1}^{-1} \partial_i, \\ \mathfrak{X}_{-i_L} &:= \mu_i^{-1} \mu_{-i} x_{-i_L}, & \mathfrak{X}_{i_R} &:= \Lambda_i^2 x_{i_R},\end{aligned}$$

and

$$\begin{aligned}\Phi_0 &:= 0, & \Psi_{n+1} &:= 0, \\ \Phi_i &:= \sum_{j=1}^i q^{j-i} \Lambda_{j-1}^2 \mathfrak{D}_{-j} \mathfrak{D}_j, & \Psi_i &:= \tau_{-i}^2 \sum_{j=i}^n q^{j-i} \tau_{-j}^{-2} \mathfrak{X}_{-j_L} \mathfrak{X}_{j_R}, \\ \mathfrak{X}_{-i_R} &:= q^i \Lambda_{1-i}^2 (\mu_i^2 \mathfrak{X}_{-i_L} + \lambda \mu_{-i}^2 \Psi_{i+1} \mathfrak{D}_i).\end{aligned}$$

Then we get

$$\Psi_i = \mathfrak{X}_{-i_L} \mathfrak{X}_{i_R} + q \mu_{-i}^2 \Psi_{i+1}, \quad (5.1)$$

$$\Phi_i = \Lambda_{i-1}^2 \mathfrak{D}_{-i} \mathfrak{D}_i + q^{-1} \Phi_{i-1}. \quad (5.2)$$

The commutation relations in the following three lemmas will be used frequently in this section.

Lemma 5.2.

1. For $k, l \in I$ and $i \in I^+$, we have

$$\mathfrak{D}_k \mu_l = q^{\delta_{kl}} \mu_l \mathfrak{D}_k, \quad \mathfrak{X}_{i_R} \mu_k = q^{-\delta_{i,k}} \mu_k \mathfrak{X}_{i_R}, \quad \mathfrak{X}_{-i_L} \mu_k = q^{-\delta_{-i,k}} \mu_k \mathfrak{X}_{-i_L}.$$

2. For $i, j \in I^+$ with $i < j$, we have

$$\begin{aligned}[\mathfrak{D}_j, \mathfrak{D}_i]_q &= [\mathfrak{D}_{-i}, \mathfrak{D}_{-j}]_q = 0, \\ [\mathfrak{X}_{j_R}, \mathfrak{X}_{i_R}]_q &= [\mathfrak{X}_{-i_L}, \mathfrak{X}_{-j_L}]_q = 0, \\ [\mathfrak{X}_{i_R}, \mathfrak{D}_j]_q &= [\mathfrak{X}_{-j_L}, \mathfrak{D}_{-i}]_q = 0, \\ [\mathfrak{D}_i, \mathfrak{X}_{j_R}]_q &= [\mathfrak{D}_{-j}, \mathfrak{X}_{-i_L}]_q = 0.\end{aligned}$$

3. For $i, j \in I^+$ with $i \neq j$, we have

$$\begin{aligned}[\mathfrak{D}_i, \mathfrak{D}_{-j}] &= [\mathfrak{X}_{-i_L}, \mathfrak{X}_{j_R}] = 0, \\ [\mathfrak{D}_i, \mathfrak{X}_{-j_L}] &= [\mathfrak{D}_{-i}, \mathfrak{X}_{j_R}] = 0, \\ [\mathfrak{D}_i, \mathfrak{D}_{-i}]_q &= [\mathfrak{X}_{i_R}, \mathfrak{X}_{-i_L}]_q = 0, \\ [\mathfrak{X}_{-i_L}, \mathfrak{D}_i]_q &= [\mathfrak{D}_{-i}, \mathfrak{X}_{i_R}]_q = 0.\end{aligned}$$

4. For $i \in I^+$, we have

$$\begin{aligned}\mathfrak{D}_i \mathfrak{X}_{i_R} &= q\lambda^{-1}(q^2\mu_i^2 - 1), & \mathfrak{X}_{i_R} \mathfrak{D}_i &= q\lambda^{-1}(\mu_i^2 - 1), \\ \mathfrak{D}_{-i} \mathfrak{X}_{-i_L} &= \lambda^{-1}(q^2\mu_{-i}^2 - 1), & \mathfrak{X}_{-i_L} \mathfrak{D}_{-i} &= \lambda^{-1}(\mu_{-i}^2 - 1).\end{aligned}$$

Then

$$\begin{aligned}[\mathfrak{D}_i, \mathfrak{X}_{i_R}] &= q^2\mu_i^2, & [\mathfrak{D}_i, \mathfrak{X}_{i_R}]_{q^2} &= q^2, & [\mathfrak{X}_{i_R}, \mathfrak{D}_i]_{q^{-2}} &= -1, \\ [\mathfrak{D}_{-i}, \mathfrak{X}_{-i_L}] &= q\mu_{-i}^2, & [\mathfrak{D}_{-i}, \mathfrak{X}_{-i_L}]_{q^2} &= q, & [\mathfrak{X}_{-i_L}, \mathfrak{D}_{-i}]_{q^{-2}} &= -q^{-1}.\end{aligned}$$

Proof. Applying both sides of each identity to any normal monomial x^a , using (2.17) and Definitions 3.1 and 5.1, we can obtain these commutation relations. \blacksquare

By Definition 5.1, Lemmas 2.1 and 5.2 and (5.2), it is easy to check the following lemma.

Lemma 5.3. *The operators Φ_i and Ψ_i satisfy the following commutation relations.*

1. For $i \in I^+$ and $t, k, l \in I$ with $|t| < i$ and $|k| > i$, we have

$$\begin{aligned}[\Psi_i, \mu_t] &= [\Phi_i, \mu_k] = 0, \\ [\Psi_i, \mu_k]_{q^{-1}} &= [\Phi_i, \mu_t]_q = 0, \\ [\Psi_i, \mu_{\pm i}]_{q^{-1}} &= [\Phi_i, \mu_{\pm i}]_q = 0.\end{aligned}$$

2. For $i, j \in I^+$ with $i < j$, we have

$$\begin{aligned}[\mathfrak{D}_i, \Psi_j]_q &= [\mathfrak{D}_{-i}, \Psi_j]_{q^{-1}} = 0, \\ [\Psi_j, \mathfrak{X}_{-i_L}]_{q^{-1}} &= [\Psi_j, \mathfrak{X}_{i_R}]_q = 0, \\ [\Phi_i, \mathfrak{X}_{-j_L}]_{q^{-1}} &= [\Phi_i, \mathfrak{X}_{j_R}]_q = 0.\end{aligned}$$

3. For $i, j \in I^+$ with $i \leq j$, we have

$$\begin{aligned}[\Phi_i, \mathfrak{D}_j]_{q^{-1}} &= [\Phi_i, \mathfrak{D}_{-j}]_q = 0, \\ [\Psi_i, \mathfrak{X}_{j_R} \mathfrak{D}_{j+1}] &= -q^{j+2-i} \tau_{-i}^2 \tau_{-j-1}^{-2} \mathfrak{X}_{-j-1_L} \mathfrak{X}_{j_R}, \\ [\Psi_i, \mathfrak{X}_{-j-1_L} \mathfrak{D}_{-j}] &= -q^{j+1-i} \tau_{-i}^2 \tau_{-j-1}^{-2} \mathfrak{X}_{-j-1_L} \mathfrak{X}_{j_R},\end{aligned}$$

so

$$[\Psi_i, \mathfrak{X}_{j_R} \mathfrak{D}_{j+1}] = q[\Psi_i, \mathfrak{X}_{-j-1_L} \mathfrak{D}_{-j}]. \quad (5.3)$$

4. For $i \in I^+$ we have

$$\begin{aligned}[\mathfrak{D}_i, \Psi_i]_q &= q\mathfrak{X}_{-i_L}, \\ [\Psi_i, \mathfrak{X}_{-i_L}]_q &= 0, \\ [\Phi_i, \mathfrak{X}_{i_R}]_q &= q^2\Lambda_i^2\mathfrak{D}_{-i}, \\ [\Phi_i, \mathfrak{X}_{-i_L}]_{q^{-1}} &= \Lambda_{i-1}^2\mu_{-i}^2\mathfrak{D}_i, \\ [\Phi_i, \mathfrak{X}_{-i_L}]_q &= \Lambda_{i-1}^2\mathfrak{D}_i - \lambda q^{-1}\mathfrak{X}_{-i_L}\Phi_{i-1}.\end{aligned} \quad (5.4)$$

5. For $i, k \in I^+$, we have

$$[\mathfrak{D}_k, [\mathfrak{D}_k, \Psi_i]_q]_{q^{-1}} = 0. \quad (5.5)$$

From now on, Lemma 2.1 is frequently used without extra explanation.

Lemma 5.4. *The operators \mathfrak{X}_{-i_R} for $i \in I^+$ satisfy the following commutation relations.*

1. For $i \in I^+$, $k, l \in I$ with $|k| < i$ and $|l| \neq i$, we have

$$[\mathfrak{X}_{-i_R}, \mu_k] = [\mathfrak{X}_{-i_R}, \mu_l \mu_l^{-1}] = [\mathfrak{X}_{-i_R}, \mu_i \mu_i^{-1}]_q = 0. \quad (5.6)$$

2. For $i \in I^+$, we have

$$[\mathfrak{D}_i, \mathfrak{X}_{-i_R}]_q = [\mathfrak{X}_{-i_R}, \mathfrak{X}_{-i_L}] = 0, \quad (5.7)$$

$$[\mathfrak{X}_{-i_R}, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q = q^2 \mathfrak{X}_{-i-1_R} - q^{i+3} \Lambda_{-i}^2 \mu_i^2 \mathfrak{X}_{-i-1_L}, \quad (5.8)$$

$$[\mathfrak{X}_{-i_R}, \mathfrak{X}_{-i-1_L} \mathfrak{D}_{-i}]_q = -q^{i+2} \Lambda_{-i}^2 \mu_i^2 \mathfrak{X}_{-i-1_L}. \quad (5.9)$$

3. For $i, j \in I^+$ with $i < j$, we have

$$[\mathfrak{X}_{i_R}, \mathfrak{X}_{-j_R}] = [\mathfrak{X}_{-j_R}, \mathfrak{X}_{-i_L}]_q = 0, \quad [\mathfrak{D}_i, \mathfrak{X}_{-j_R}] = [\mathfrak{D}_{-i}, \mathfrak{X}_{-j_R}]_q = 0. \quad (5.10)$$

Proof. Using Lemmas 5.2 and 5.3, we can prove this lemma directly. We only show (5.8) for example. For $i \in I^+$, by definition of \mathfrak{X}_{-i_R} , Lemma 5.2 and (5.1), we have

$$\begin{aligned} [\mathfrak{X}_{-i_R}, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q &= [q^i \Lambda_{1-i}^2 (\mu_i^2 \mathfrak{X}_{-i_L} + \lambda \mu_{-i}^2 \Psi_{i+1} \mathfrak{D}_i), \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q \\ &= q^i \Lambda_{1-i}^2 [\mu_i^2 \mathfrak{X}_{-i_L}, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q + q^i \lambda \Lambda_{1-i}^2 [\mu_{-i}^2 \Psi_{i+1} \mathfrak{D}_i, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q \\ &= q^i \Lambda_{1-i}^2 \mu_i^2 [\mathfrak{X}_{-i_L}, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_{q^{-1}} + q^i \lambda \Lambda_{1-i}^2 \mu_{-i}^2 [\Psi_{i+1} \mathfrak{D}_i, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q \\ &= 0 + q^i \lambda \Lambda_{-i}^2 [(\mathfrak{X}_{-i-1_L} \mathfrak{X}_{i+1_R} + q \mu_{-i-1}^2 \Psi_{i+2}) \mathfrak{D}_i, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q \\ &= q^i \lambda \Lambda_{-i}^2 \mathfrak{X}_{-i-1_L} (\mathfrak{X}_{i+1_R} \mathfrak{D}_i \mathfrak{X}_{i_R} \mathfrak{D}_{i+1} - \mathfrak{X}_{i_R} \mathfrak{D}_{i+1} \mathfrak{X}_{i+1_R} \mathfrak{D}_i) \\ &\quad + q^{i+1} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} [\mathfrak{D}_i, \mathfrak{X}_{i_R}]_{q^2} \mathfrak{D}_{i+1} \\ &= q^i \lambda \Lambda_{-i}^2 \mathfrak{X}_{-i-1_L} (\mathfrak{D}_i \mathfrak{X}_{i_R} \mathfrak{X}_{i+1_R} \mathfrak{D}_{i+1} - \mathfrak{X}_{i_R} \mathfrak{D}_i \mathfrak{D}_{i+1} \mathfrak{X}_{i+1_R}) \\ &\quad + q^{i+3} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} \mathfrak{D}_{i+1} \\ &= q^i \lambda \Lambda_{-i}^2 \mathfrak{X}_{-i-1_L} ([\mathfrak{D}_i, \mathfrak{X}_{i_R}] \mathfrak{X}_{i+1_R} \mathfrak{D}_{i+1} - \mathfrak{X}_{i_R} \mathfrak{D}_i [\mathfrak{D}_{i+1}, \mathfrak{X}_{i+1_R}]) \\ &\quad + q^{i+3} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} \mathfrak{D}_{i+1} \\ &= q^{i+3} \Lambda_{-i}^2 \mathfrak{X}_{-i-1_L} (\mu_i^2 (\mu_{i+1}^2 - 1) - (\mu_i^2 - 1) \mu_{i+1}^2) + q^{i+3} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} \mathfrak{D}_{i+1} \\ &= q^{i+3} \Lambda_{-i}^2 (\mu_{i+1}^2 - \mu_i^2) \mathfrak{X}_{-i-1_L} + q^{i+3} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} \mathfrak{D}_{i+1} \\ &= q^2 \mathfrak{X}_{-i-1_R} - q^{i+3} \Lambda_{-i}^2 \mu_i^2 \mathfrak{X}_{-i-1_L}, \end{aligned}$$

that is, (5.8) holds. ■

Corollary 5.5. *For $i, j \in I^+$ with $i \leq j$, we have*

$$[\mathfrak{X}_{-j_R}, [\mathfrak{D}_j, \Psi_i]_q] = 0. \quad (5.11)$$

Proof. For $1 \leq i \leq j \leq n$, Lemma 5.2 yields

$$\begin{aligned} [\mathfrak{D}_j, \Psi_i]_q &= \left[\mathfrak{D}_j, \tau_{-i}^2 \sum_{k=i}^n q^{k-i} \tau_{-k}^{-2} \mathfrak{X}_{-k_L} \mathfrak{X}_{k_R} \right]_q = \tau_{-i}^2 \sum_{k=i}^n q^{k-i} \tau_{-k}^{-2} [\mathfrak{D}_j, \mathfrak{X}_{-k_L} \mathfrak{X}_{k_R}]_q \\ &= -\lambda \tau_{-i}^2 \sum_{k=i}^{j-1} q^{k-i} \tau_{-k}^{-2} \mathfrak{X}_{-k_L} \mathfrak{X}_{k_R} \mathfrak{D}_j + q^{j-i-1} \tau_{-i}^2 \tau_{-j}^{-2} \mathfrak{X}_{-j_L} [\mathfrak{D}_j, \mathfrak{X}_{j_R}]_{q^2} \end{aligned}$$

$$= -\lambda \sum_{k=i}^{j-1} q^{k-i} \tau_{-i}^2 \tau_{-k}^{-2} \mathfrak{X}_{-kL} \mathfrak{X}_{kR} \mathfrak{D}_j + q^{j-i+1} \tau_{-i}^2 \tau_{-j}^{-2} \mathfrak{X}_{-jL},$$

then by (5.6), (5.7) and (5.10), we have

$$\begin{aligned} [\mathfrak{X}_{-jR}, [\mathfrak{D}_j, \Psi_i]_q] &= -\lambda \sum_{k=i}^{j-1} q^{k-i} \tau_{-i}^2 \tau_{-k}^{-2} [\mathfrak{X}_{-jR}, \mathfrak{X}_{-kL} \mathfrak{X}_{kR} \mathfrak{D}_j] + q^{j-i+1} \tau_{-i}^2 \tau_{-j}^{-2} [\mathfrak{X}_{-jR}, \mathfrak{X}_{-jL}] \\ &= -\lambda \sum_{k=i}^{j-1} q^{k-i} \tau_{-i}^2 \tau_{-k}^{-2} [\mathfrak{X}_{-jR}, \mathfrak{X}_{-kL}]_q \mathfrak{X}_{kR} \mathfrak{D}_j = 0, \end{aligned}$$

that is, (5.11) holds. ■

We are now in the position to realize all the positive root vectors $E_{\pm i, j}$ of $U_q(\mathfrak{sp}_{2n})$ as $e_{\pm i, j}$ in $\text{Diff}(\mathcal{X})$.

Definition 5.6. For $i, j \in I^+$ with $i < j$, set

$$\begin{aligned} e_{-i, j} &:= (-1)^{i+j} q^{-2} (\mathfrak{X}_{iR} \mathfrak{D}_j - [\mathfrak{D}_j, \Psi_{i+1}]_q \mathfrak{D}_{-i}), \\ e_{i, i} &:= [2]_q^{-1} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-iR} \mathfrak{D}_i + q^{-2} [\mathfrak{D}_i, \Psi_1]_q \mathfrak{D}_i), \\ e_{i, j} &:= (-1)^{j+1} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-iL} \mathfrak{D}_j + q^{i-1} \mathfrak{X}_{-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q). \end{aligned}$$

The next lemma says that the operators which realize the simple root vectors of $U_q(\mathfrak{sp}_{2n})$ defined in Definition 5.6 coincide with those defined in Definition 3.2.

Lemma 5.7. $e_{1,1} = e_1$ and $e_{1-i, i} = e_i$ for $1 < i \leq n$.

Proof. From (2.20), it is easy to show that

$$\Omega_i x^a = \tau_1^{-1} \tau_{-1} \tau_{-i}^2 \sum_{j=i}^n q^{j-i-2} \tau_{-j}^{-2} \mathfrak{X}_{-jL} \mathfrak{X}_{jR} \cdot x^a$$

for any normal monomial x^a . Then it yields from (2.19) that

$$\begin{aligned} x_{-iR} &= \tau_1 \tau_{-i}^{-1} \mu_i^2 \Lambda_{1-i} \mathfrak{X}_{-iL} + \lambda \tau_1 \tau_{-1} \sum_{j=i+1}^n q^{j-i-1} \tau_{-j}^{-2} \mathfrak{X}_{-jL} \mathfrak{X}_{jR} \mathfrak{D}_i \\ &= \tau_1 \tau_{-i}^{-1} \mu_i^2 \Lambda_{1-i} \mathfrak{X}_{-iL} + \lambda \tau_1 \Lambda_{-i} \tau_{-i-1}^{-1} \Psi_{i+1} \mathfrak{D}_i. \end{aligned}$$

Write e_i in terms of the new operators defined in Definition 5.1. By (5.4), we get

$$\begin{aligned} e_1 &= [2]_q^{-1} q^{-1} \mu_1^{-1} (\tau_{-2}^{-1} x_{-1L} + q^2 \tau_2^{-1} x_{-1R}) \partial_1 \\ &= [2]_q^{-1} \tau_{-1}^{-1} (q^{-1} \mathfrak{X}_{-1L} + q \mu_1^2 \mathfrak{X}_{-1L} + q \lambda \mu_{-1}^2 \Psi_2 \mathfrak{D}_1) \tau_1 \mathfrak{D}_1 \\ &= [2]_q^{-1} \tau_1 \tau_{-1}^{-1} (q^{-1} \mathfrak{X}_{-1L} + \mathfrak{X}_{-1R}) \mathfrak{D}_1 = [2]_q^{-1} \tau_1 \tau_{-1}^{-1} (q^{-2} [\mathfrak{D}_1, \Psi_1]_q + \mathfrak{X}_{-1R}) \mathfrak{D}_1 = e_{1,1}, \end{aligned}$$

and for $i > 1$

$$\begin{aligned} e_i &= \mu_{i-1} \mu_i^{-1} \tau_{-i-1}^{-1} x_{-iL} \partial_{1-i} - \tau_i^{-1} x_{i-1R} \partial_i = q^{-1} \mathfrak{X}_{-iL} \mathfrak{D}_{1-i} - q^{-2} \mathfrak{X}_{i-1R} \mathfrak{D}_i \\ &= q^{-2} ([\mathfrak{D}_i, \Psi_i]_q \mathfrak{D}_{1-i} - \mathfrak{X}_{i-1R} \mathfrak{D}_i) = e_{1-i, i}. \end{aligned}$$

This completes the proof. ■

Proposition 5.8. The commutation relations (4.9)–(4.13) remain valid if E is replaced by e .

Proof. (1) To prove $e_{1,2} = [e_1, e_2]_{q^2}$, we compute the following four brackets first. By (2.3), (5.1), (5.8), (5.9) and Lemma 5.2, we get

$$\begin{aligned}
[\mathfrak{X}_{-1R}\mathfrak{D}_1, \mathfrak{X}_{1R}\mathfrak{D}_2]_{q^2} &= \mathfrak{X}_{-1R}[\mathfrak{D}_1, \mathfrak{X}_{1R}\mathfrak{D}_2]_q + q[\mathfrak{X}_{-1R}, \mathfrak{X}_{1R}\mathfrak{D}_2]_q\mathfrak{D}_1 \\
&= \mathfrak{X}_{-1R}[\mathfrak{D}_1, \mathfrak{X}_{1R}]_{q^2}\mathfrak{D}_2 + q(q^2\mathfrak{X}_{-2R} - q^4\Lambda_{-1}^2\mu_1^2\mathfrak{X}_{-2L})\mathfrak{D}_1 \\
&= q^2\mathfrak{X}_{-1R}\mathfrak{D}_2 + q^3\mathfrak{X}_{-2R}\mathfrak{D}_1 - q^5\mu_{-1}^2\mu_1^2\mathfrak{X}_{-2L}\mathfrak{D}_1 \\
&= q^3(\mu_1^2\mathfrak{X}_{-1L} + \lambda\mu_{-1}^2\Psi_2\mathfrak{D}_1)\mathfrak{D}_2 + q^3\mathfrak{X}_{-2R}\mathfrak{D}_1 - q^5\mu_{-1}^2\mu_1^2\mathfrak{X}_{-2L}\mathfrak{D}_1 \\
&= q^3\mu_1^2\mathfrak{X}_{-1L}\mathfrak{D}_2 + q^2\lambda\mu_{-1}^2(\mathfrak{X}_{-2L}\mathfrak{X}_{2R} + q\mu_{-2}^2\Psi_3)\mathfrak{D}_2\mathfrak{D}_1 \\
&\quad + q^3\mathfrak{X}_{-2R}\mathfrak{D}_1 - q^5\mu_{-1}^2\mu_1^2\mathfrak{X}_{-2L}\mathfrak{D}_1 \\
&= q^3\mu_1^2\mathfrak{X}_{-1L}\mathfrak{D}_2 + q^3\mu_{-1}^2(\mu_2^2 - 1)\mathfrak{X}_{-2L}\mathfrak{D}_1 + q^3\lambda\mu_{-1}^2\mu_{-2}^2\Psi_3\mathfrak{D}_2\mathfrak{D}_1 \\
&\quad + q^3\mathfrak{X}_{-2R}\mathfrak{D}_1 - q^5\mu_{-1}^2\mu_1^2\mathfrak{X}_{-2L}\mathfrak{D}_1 \tag{4.9'a} \\
&= q^3\mu_1^2\mathfrak{X}_{-1L}\mathfrak{D}_2 - q^3\mu_{-1}^2\mathfrak{X}_{-2L}\mathfrak{D}_1 - q^5\mu_{-1}^2\mu_1^2\mathfrak{X}_{-2L}\mathfrak{D}_1 + q^2[2]_q\mathfrak{X}_{-2R}\mathfrak{D}_1,
\end{aligned}$$

$$\begin{aligned}
[q^{-1}\mathfrak{X}_{-1L}\mathfrak{D}_1, \mathfrak{X}_{1R}\mathfrak{D}_2]_{q^2} &= q^{-1}\mathfrak{X}_{-1L}[\mathfrak{D}_1, \mathfrak{X}_{1R}]_{q^4}\mathfrak{D}_2 = \lambda^{-1}\mathfrak{X}_{-1L}((q^2\mu_1^2 - 1) - q^4(\mu_1^2 - 1))\mathfrak{D}_2 \\
&= q\mathfrak{X}_{-1L}(q^2 + 1 - q^2\mu_1^2)\mathfrak{D}_2 = q(q[2]_q - q^2\mu_1^2)\mathfrak{X}_{-1L}\mathfrak{D}_2 \\
&= q^2[2]_q\mathfrak{X}_{-1L}\mathfrak{D}_2 - q^3\mu_1^2\mathfrak{X}_{-1L}\mathfrak{D}_2, \tag{4.9'b}
\end{aligned}$$

$$\begin{aligned}
[\mathfrak{X}_{-1R}\mathfrak{D}_1, -q\mathfrak{X}_{-2L}\mathfrak{D}_{-1}]_{q^2} &= -q\mathfrak{X}_{-1R}[\mathfrak{D}_1, \mathfrak{X}_{-2L}\mathfrak{D}_{-1}]_q - q^2[\mathfrak{X}_{-1R}, \mathfrak{X}_{-2L}\mathfrak{D}_{-1}]_q\mathfrak{D}_1 \\
&= -q\mathfrak{X}_{-1R}\mathfrak{X}_{-2L}[\mathfrak{D}_1, \mathfrak{D}_{-1}]_q + q^5\mu_{-1}^2\mu_1^2\mathfrak{X}_{-2L}\mathfrak{D}_1 \\
&= q^5\mu_{-1}^2\mu_1^2\mathfrak{X}_{-2L}\mathfrak{D}_1, \tag{4.9'c}
\end{aligned}$$

$$\begin{aligned}
[q^{-1}\mathfrak{X}_{-1L}\mathfrak{D}_1, -q\mathfrak{X}_{-2L}\mathfrak{D}_{-1}]_{q^2} &= -q[\mathfrak{X}_{-1L}, \mathfrak{X}_{-2L}\mathfrak{D}_{-1}]_q\mathfrak{D}_1 = -q^2\mathfrak{X}_{-2L}[\mathfrak{X}_{-1L}, \mathfrak{D}_{-1}]\mathfrak{D}_1 \\
&= q^3\mathfrak{X}_{-2L}\mu_{-1}^2\mathfrak{D}_1 = q^3\mu_{-1}^2\mathfrak{X}_{-2L}\mathfrak{D}_1. \tag{4.9'd}
\end{aligned}$$

It is easy to see that

$$[e_1, e_2]_{q^2} = [e_{1,1}, e_{-1,2}]_{q^2} = -q^{-2}[2]_q^{-1}\tau_1\tau_{-1}^{-1}[\mathfrak{X}_{-1R}\mathfrak{D}_1 + q^{-1}\mathfrak{X}_{-1L}\mathfrak{D}_1, \mathfrak{X}_{1R}\mathfrak{D}_2 - q\mathfrak{X}_{-2L}\mathfrak{D}_{-1}]_{q^2}.$$

So from (4.9'a)–(4.9'd), we obtain

$$[e_1, e_2]_{q^2} = -\tau_1\tau_{-1}^{-1}(\mathfrak{X}_{-1L}\mathfrak{D}_2 + \mathfrak{X}_{-2R}\mathfrak{D}_1) = e_{1,2}.$$

(2) To prove $e_{-i,j} = [e_{-i,j-1}, e_j]_q$ for $3 \leq i+2 \leq j \leq n$, we compute the following four brackets first. By (2.4), (5.3) and Lemma 5.2, for $3 \leq i+2 \leq j \leq n$, we get

$$\begin{aligned}
[\mathfrak{X}_{iR}\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]_q &= \mathfrak{X}_{iR}[\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]_q \\
&= \mathfrak{X}_{iR}[\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R}]_{q^2}\mathfrak{D}_j = q^2\mathfrak{X}_{iR}\mathfrak{D}_j, \tag{4.10'a}
\end{aligned}$$

$$\begin{aligned}
[[\mathfrak{D}_{j-1}, \Psi_{i+1}]_q\mathfrak{D}_{-i}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]_q &= [[\mathfrak{D}_{j-1}, \Psi_{i+1}]_q, \mathfrak{X}_{j-1R}\mathfrak{D}_j]_q\mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]]_{q^2}\mathfrak{D}_{-i} + [[\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]_q, \Psi_{i+1}]_q\mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]]_{q^2}\mathfrak{D}_{-i} + [[\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R}]_{q^2}\mathfrak{D}_j, \Psi_{i+1}]_q\mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]]_{q^2}\mathfrak{D}_{-i} + q^2[\mathfrak{D}_j, \Psi_{i+1}]_q\mathfrak{D}_{-i}, \tag{4.10'b}
\end{aligned}$$

$$[\mathfrak{X}_{iR}\mathfrak{D}_{j-1}, q\mathfrak{X}_{-jL}\mathfrak{D}_{1-j}]_q = q[\mathfrak{X}_{iR}\mathfrak{D}_{j-1}, \mathfrak{X}_{-jL}]\mathfrak{D}_{1-j} = 0, \tag{4.10'c}$$

$$\begin{aligned}
[[\mathfrak{D}_{j-1}, \Psi_{i+1}]_q\mathfrak{D}_{-i}, q\mathfrak{X}_{-jL}\mathfrak{D}_{1-j}]_q &= [[\mathfrak{D}_{j-1}, \Psi_{i+1}]_q, q\mathfrak{X}_{-jL}\mathfrak{D}_{1-j}]_q\mathfrak{D}_{-i} \\
&= q[\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{-jL}\mathfrak{D}_{1-j}]]_{q^2}\mathfrak{D}_{-i} + q[[\mathfrak{D}_{j-1}, \mathfrak{X}_{-jL}\mathfrak{D}_{1-j}]_q, \Psi_{i+1}]_q\mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]]_{q^2}\mathfrak{D}_{-i} + q[[\mathfrak{D}_{j-1}, \mathfrak{X}_{-jL}]\mathfrak{D}_{1-j}, \Psi_{i+1}]_q\mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1R}\mathfrak{D}_j]]_{q^2}\mathfrak{D}_{-i}. \tag{4.10'd}
\end{aligned}$$

From (4.10'a)–(4.10'd), it is easy to see that

$$\begin{aligned} [e_{-i,j-1}, e_j]_q &= [e_{-i,j-1}, e_{1-j,j}]_q \\ &= (-1)^{i+j} q^{-4} [\mathfrak{X}_{iR} \mathfrak{D}_{j-1} - [\mathfrak{D}_{j-1}, \Psi_{i+1}]_q \mathfrak{D}_{-i}, \mathfrak{X}_{j-1R} \mathfrak{D}_j - q \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q \\ &= (-1)^{i+j} q^{-2} (\mathfrak{X}_{iR} \mathfrak{D}_j - [\mathfrak{D}_j, \Psi_{i+1}]_q \mathfrak{D}_{-i}) = e_{-i,j}. \end{aligned}$$

(3) To prove $e_{i,j} = [e_{i,j-1}, e_j]_q$ for $3 \leq i+2 \leq j \leq n$, we compute the following four brackets first. By (2.2), (5.8), (5.9), Lemmas 5.2 and 5.3, for $3 \leq i+2 \leq j \leq n$, we get

$$[\mathfrak{X}_{-iL} \mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R} \mathfrak{D}_j]_q = \mathfrak{X}_{-iL} [\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R}]_{q^2} \mathfrak{D}_j = q^2 \mathfrak{X}_{-iL} \mathfrak{D}_j, \quad (4.11'a)$$

$$\begin{aligned} [q^{i-1} \mathfrak{X}_{1-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q, \mathfrak{X}_{j-1R} \mathfrak{D}_j]_q &= q^{i-1} [\mathfrak{X}_{1-jR}, \mathfrak{X}_{j-1R} \mathfrak{D}_j]_q [\Phi_i, \mathfrak{X}_{-iL}]_q \\ &= q^{i-1} (q^2 \mathfrak{X}_{-jR} - q^{j+2} \Lambda_{1-j}^2 \mu_{j-1}^2 \mathfrak{X}_{-jL}) [\Phi_i, \mathfrak{X}_{-iL}]_q \\ &= q^{i+1} \mathfrak{X}_{-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q - q^{i+j+1} \Lambda_{1-j}^2 \mu_{j-1}^2 \mathfrak{X}_{-jL} [\Phi_i, \mathfrak{X}_{-iL}]_q, \end{aligned} \quad (4.11'b)$$

$$[\mathfrak{X}_{-iL} \mathfrak{D}_{j-1}, -q \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q = -q \mathfrak{X}_{-iL} \mathfrak{X}_{-jL} [\mathfrak{D}_{j-1}, \mathfrak{D}_{1-j}]_q = 0, \quad (4.11'c)$$

$$\begin{aligned} [q^{i-1} \mathfrak{X}_{1-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q, -q \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q &= -q^i [\mathfrak{X}_{1-jR}, \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q [\Phi_i, \mathfrak{X}_{-iL}]_q \\ &= q^{i+j+1} \Lambda_{1-j}^2 \mu_{j-1}^2 \mathfrak{X}_{-jL} [\Phi_i, \mathfrak{X}_{-iL}]_q. \end{aligned} \quad (4.11'd)$$

From (4.11'a)–(4.11'd), it is easy to see that

$$\begin{aligned} [e_{i,j-1}, e_j]_q &= (-1)^{j+1} q^{-2} \tau_1 \tau_{-1}^{-1} [\mathfrak{X}_{-iL} \mathfrak{D}_{j-1} + q^{i-1} \mathfrak{X}_{1-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q, \mathfrak{X}_{j-1R} \mathfrak{D}_j - q \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q \\ &= (-1)^{j+1} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-iL} \mathfrak{D}_j + q^{i-1} \mathfrak{X}_{-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q) = e_{i,j}. \end{aligned}$$

(4) To prove $e_{j-1,j} = [e_{j-1}, e_{j-2,j}]_q$ for $3 \leq j \leq n$, we compute the following four brackets first. By (2.2) and Lemmas 5.2–5.4, for $3 \leq j \leq n$, we get

$$\begin{aligned} [\mathfrak{X}_{j-2R} \mathfrak{D}_{j-1}, \mathfrak{X}_{2-jL} \mathfrak{D}_j]_q &= q^{-1} [\mathfrak{X}_{j-2R}, \mathfrak{X}_{2-jL} \mathfrak{D}_j]_{q^2} \mathfrak{D}_{j-1} \\ &= q^{-1} [\mathfrak{X}_{j-2R}, \mathfrak{X}_{2-jL}]_q \mathfrak{D}_j \mathfrak{D}_{j-1} = 0, \end{aligned} \quad (4.12'a)$$

$$\begin{aligned} q^{j-3} [\mathfrak{X}_{j-2R} \mathfrak{D}_{j-1}, \mathfrak{X}_{-jR} [\Phi_{j-2}, \mathfrak{X}_{2-jL}]_q]_q &= q^{j-2} \mathfrak{X}_{-jR} [\mathfrak{X}_{j-2R}, [\Phi_{j-2}, \mathfrak{X}_{2-jL}]_q] \mathfrak{D}_{j-1} \\ &= -q^{j-3} \mathfrak{X}_{-jR} [[\Phi_{j-2}, \mathfrak{X}_{j-2R}]_q, \mathfrak{X}_{2-jL}]_{q^2} \mathfrak{D}_{j-1} = -q^{j-1} \mathfrak{X}_{-jR} [\Lambda_{j-2}^2 \mathfrak{D}_{2-j}, \mathfrak{X}_{2-jL}]_{q^2} \mathfrak{D}_{j-1} \\ &= -q^{j-1} \Lambda_{j-2}^2 \mathfrak{X}_{-jR} [\mathfrak{D}_{2-j}, \mathfrak{X}_{2-jL}]_{q^2} \mathfrak{D}_{j-1} = -q^j \Lambda_{j-2}^2 \mathfrak{X}_{-jR} \mathfrak{D}_{j-1}, \end{aligned} \quad (4.12'b)$$

$$\begin{aligned} -q [\mathfrak{X}_{1-jL} \mathfrak{D}_{2-j}, \mathfrak{X}_{2-jL} \mathfrak{D}_j]_q &= -q [\mathfrak{X}_{1-jL} \mathfrak{D}_{2-j}, \mathfrak{X}_{2-jL}]_q \mathfrak{D}_j \\ &= -q \mathfrak{X}_{1-jL} [\mathfrak{D}_{2-j}, \mathfrak{X}_{2-jL}]_{q^2} \mathfrak{D}_j = -q^2 \mathfrak{X}_{1-jL} \mathfrak{D}_j, \end{aligned} \quad (4.12'c)$$

$$\begin{aligned} -q^{j-2} [\mathfrak{X}_{1-jL} \mathfrak{D}_{2-j}, \mathfrak{X}_{-jR} [\Phi_{j-2}, \mathfrak{X}_{2-jL}]_q]_q &= -q^{j-2} \mathfrak{X}_{-jR} [\mathfrak{X}_{1-jL} \mathfrak{D}_{2-j}, [\Phi_{j-2}, \mathfrak{X}_{2-jL}]_q]_q \\ &= -q^{j-2} \mathfrak{X}_{-jR} \mathfrak{X}_{1-jL} [\mathfrak{D}_{2-j}, [\Phi_{j-2}, \mathfrak{X}_{2-jL}]_q]_q \\ &= q^{j-1} \mathfrak{X}_{-jR} \mathfrak{X}_{1-jL} [\Phi_{j-2}, [\mathfrak{X}_{2-jL}, \mathfrak{D}_{2-j}]_{q^{-2}}]_{q^2} = q^{j-1} \mathfrak{X}_{-jR} \mathfrak{X}_{1-jL} [\Phi_{j-2}, -q^{-1}]_{q^2} \\ &= q^{j-1} \lambda \mathfrak{X}_{-jR} \mathfrak{X}_{1-jL} \Phi_{j-2}. \end{aligned} \quad (4.12'd)$$

From (4.12'a)–(4.12'd), it is easy to see that

$$\begin{aligned} [e_{j-1}, e_{j-2,j}]_q &= (-1)^j q^{-2} \tau_1 \tau_{-1}^{-1} [\mathfrak{X}_{j-2R} \mathfrak{D}_{j-1} - q \mathfrak{X}_{1-jL} \mathfrak{D}_{2-j}, \mathfrak{X}_{2-jL} \mathfrak{D}_j + q^{j-3} \mathfrak{X}_{-jR} [\Phi_{j-2}, \mathfrak{X}_{2-jL}]_q]_q \\ &= (-1)^{j+1} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{1-jL} \mathfrak{D}_j + q^{j-2} \mathfrak{X}_{-jR} (\Lambda_{j-2}^2 \mathfrak{D}_{j-1} - \lambda q^{-1} \mathfrak{X}_{1-jL} \Phi_{j-2})) \\ &= (-1)^{j+1} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{1-jL} \mathfrak{D}_j + q^{j-2} \mathfrak{X}_{-jR} [\Phi_{j-1}, \mathfrak{X}_{1-jL}]_q) = e_{j-1,j}. \end{aligned}$$

(5) To prove $e_{j,j} = [2]_q^{-1}[e_{1,j}, e_{-1,j}]$ for $2 \leq j \leq n$, we compute the following four brackets first. By Lemma 5.2, (5.1), (5.5) and (5.11), for $2 \leq j \leq n$, we get

$$[\mathfrak{X}_{-1L} \mathfrak{D}_j, \mathfrak{X}_{1R} \mathfrak{D}_j] = q^{-1}[\mathfrak{X}_{-1L}, \mathfrak{X}_{1R}]_q \mathfrak{D}_j^2 = -\lambda \mathfrak{X}_{-1L} \mathfrak{X}_{1R} \mathfrak{D}_j^2, \quad (4.13'a)$$

$$\begin{aligned} -[\mathfrak{X}_{-1L} \mathfrak{D}_j, [\mathfrak{D}_j, \Psi_2]_q \mathfrak{D}_{-1}] &= -q^{-1}[\mathfrak{X}_{-1L}, [\mathfrak{D}_j, \Psi_2]_q \mathfrak{D}_{-1}]_q \mathfrak{D}_j = -[\mathfrak{D}_j, \Psi_2]_q [\mathfrak{X}_{-1L}, \mathfrak{D}_{-1}] \mathfrak{D}_j \\ &= q\mu_{-1}^2 [\mathfrak{D}_j, \Psi_2]_q \mathfrak{D}_j = q\mu_{-1}^2 [\mathfrak{D}_j, q^{-1}\mu_{-1}^{-2}(\Psi_1 - \mathfrak{X}_{-1L} \mathfrak{X}_{1R})]_q \mathfrak{D}_j \\ &= [\mathfrak{D}_j, \Psi_1]_q \mathfrak{D}_j + \lambda \mathfrak{X}_{-1L} \mathfrak{X}_{1R} \mathfrak{D}_j^2, \end{aligned} \quad (4.13'b)$$

$$[\mathfrak{X}_{-jR} \mathfrak{D}_1, \mathfrak{X}_{1R} \mathfrak{D}_j] = \mathfrak{X}_{-jR} [\mathfrak{D}_1, \mathfrak{X}_{1R} \mathfrak{D}_j]_q = \mathfrak{X}_{-jR} [\mathfrak{D}_1, \mathfrak{X}_{1R}]_{q^2} \mathfrak{D}_j = q^2 \mathfrak{X}_{-jR} \mathfrak{D}_j, \quad (4.13'c)$$

$$\begin{aligned} -[\mathfrak{X}_{-jR} \mathfrak{D}_1, [\mathfrak{D}_j, \Psi_2]_q \mathfrak{D}_{-1}] &= -q[\mathfrak{X}_{-jR}, [\mathfrak{D}_j, \Psi_2]_q \mathfrak{D}_{-1}]_{q^{-1}} \mathfrak{D}_1 \\ &= -q[\mathfrak{X}_{-jR}, [\mathfrak{D}_j, \Psi_2]_q] \mathfrak{D}_{-1} \mathfrak{D}_1 = 0. \end{aligned} \quad (4.13'd)$$

It is easy to see that

$$\begin{aligned} [e_{1,j}, e_{-1,j}] &= (-1)^{i+j} q^{-2} [\tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-1L} \mathfrak{D}_j + \mathfrak{X}_{-jR} \mathfrak{D}_1), \mathfrak{X}_{1R} \mathfrak{D}_j - [\mathfrak{D}_j, \Psi_2]_q \mathfrak{D}_{-1}] \\ &= (-1)^{i+j} q^{-2} \tau_1 \tau_{-1}^{-1} [\mathfrak{X}_{-1L} \mathfrak{D}_j + \mathfrak{X}_{-jR} \mathfrak{D}_1, \mathfrak{X}_{1R} \mathfrak{D}_j - [\mathfrak{D}_j, \Psi_2]_q \mathfrak{D}_{-1}]. \end{aligned}$$

So from (4.13'a)–(4.13'd), we get

$$[e_{1,j}, e_{-1,j}] = (-1)^{i+j} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-jR} \mathfrak{D}_j + q^{-2} [\mathfrak{D}_j, \Psi_1]_q \mathfrak{D}_j) = [2]_q e_{j,j}.$$

We complete the proof. ■

Hence, we can obtain the operators $e_{\pm i,j}$ from e_i by the same inductive formulas that we used to get $E_{\pm i,j}$ from E_i . In other words, all the positive root vectors $E_{\pm i,j}$ of $U_q(\mathfrak{sp}_{2n})$ can be realized by the operators $e_{\pm i,j}$ in the subalgebra U_q^{2n} of $\text{Diff}(\mathcal{X})$.

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