Multiple Actions of the Monodromy Matrix in $\mathfrak{gl}(2|1)$-Invariant Integrable Models*

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Abstract. We study $\mathfrak{gl}(2|1)$ symmetric integrable models solvable by the nested algebraic Bethe ansatz. Using explicit formulas for the Bethe vectors we derive the actions of the monodromy matrix entries onto these vectors. We show that the result of these actions is a finite linear combination of Bethe vectors. The obtained formulas open a way for studying scalar products of Bethe vectors.

Key words: algebraic Bethe ansatz; superalgebras; scalar product of Bethe vectors

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1   Introduction

Algebraic Bethe ansatz is one of the most famous applications of the quantum inverse scattering method. It was developed at the end of the 70s of the last century by the Leningrad school [15, 16]. From a mathematical point of view the method consists of the study of the highest weight representations of some Hopf algebra that depends on the model under consideration [12, 22, 23]. This method allows one to find eigenvectors of the transfer matrix (a generating function of the commuting Hamiltonians of the quantum models), leading to a diagonalization of the physical Hamiltonians. The obtained determinant representations for scalar products of the Bethe vectors [29, 45] allow then to calculate the form factors in various integrable systems [25, 27, 30]. Using the form factor representations, many interesting physical results were obtained during the last few years [10, 11, 20, 21, 24, 26, 28, 41, 42, 44].

   The results listed above mostly concerned the models with $\mathfrak{gl}(2)$ symmetry or its q-deformation. An important problem remains open: the case of models based on the $\mathfrak{gl}(N)$ algebras ($N > 2$). These models are quite more involved, therefore they are less studied. The models

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with higher rank symmetries were considered within the framework of a *nested Bethe ansatz* approach [48, 49, 53]. The algebraic version of this method was developed in the works [31, 32, 33, 34]. Later on, new methods to construct Bethe vectors in the models of higher rank symmetries were proposed in [23, 50]. A generalization to the case of superalgebras $\mathfrak{gl}(m|n)$ was given in [8].

Recently, the models with $\mathfrak{gl}(3)$ symmetries were studied more deeply by the nested algebraic Bethe ansatz in the works [3, 4, 5, 7, 37, 38, 39, 51, 52]. There, determinant representations for scalar products of Bethe vectors and form factors of local operators were obtained. Some of these results were generalized to the models with q-deformed $\mathfrak{gl}(3)$ symmetry [35, 36, 46].

In this paper, we calculate multiple actions of the monodromy matrix entries onto Bethe vectors in the models with $\mathfrak{gl}(2|1)$ symmetry. This is a part of a larger program devoted to the study of quantum integrable models based on $\mathfrak{gl}(m|n)$ superalgebras. This program was initiated in [8] and continued in [40], where the Bethe vectors for these models were constructed. The calculation of the monodromy matrix entries multiple actions onto the Bethe vectors is the next step of our investigation. Further steps that include the calculation of the scalar products and form factors of the monodromy matrix entries will be reported elsewhere. They will lead eventually to determinant representations for form factors of local operators in the models of physical interest. In particular, we expect to obtain compact explicit formulas for form factors of local operators in the supersymmetric t-J model [1, 14, 17, 18, 43, 54] and the U-model of strongly correlated electrons [2, 9] since these models are based on the $\mathfrak{gl}(1|2)$ superalgebra. Knowing form factors of local operators, one can then study correlation functions via a summation of their form factor series. We expect to come back on these physical models in further works. We also hope that our results will be of some interest in the context of super-Yang–Mills theories, when studied in the integrable systems framework. Indeed in these theories, the general approach relies on a spin chain based on the $\mathfrak{psl}(2,2|4)$ superalgebra, but there are closed subsectors based on the $\mathfrak{gl}(1|2)$ or $\mathfrak{gl}(2|1)$ superalgebras. We believe that the present results will be a first step to understand these subsectors, and then the entire theory.

The paper is organized as follows. Section 2 is devoted to the definitions and the notation. In Section 3 we present the main results of the paper: multiple actions of the monodromy matrix entries onto Bethe vectors. In Section 4 we consider the action of the transfer matrix onto on-shell Bethe vectors. Sections 5–7 contain proofs of the formulas presented in Section 3. Finally, the appendix gathers some identities for rational functions that we use in our proofs.

2   Definitions

We consider a normalized rational R-matrix

$$R(x, y) = \mathbb{I} + g(x, y)P, \qquad g(x, y) = \frac{c}{x - y}, \tag{2.1}$$

where c is a constant and P is a graded permutation operator [34]. The R-matrix of $\mathfrak{gl}(2|1)$-based models acts in the tensor product $\mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1}$, where $\mathbb{C}^{2|1}$ is the $\mathbb{Z}_2$-graded vector space with the grading $[1] = [2] = 0$, $[3] = 1$. The R-matrix (2.1) satisfies the graded Yang–Baxter equation

$$R_{12}(x, y)R_{13}(x, z)R_{23}(y, z) = R_{23}(y, z)R_{13}(x, z)R_{12}(x, y) \tag{2.2}$$

written in the tensor product of graded spaces $\mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1}$. The subscripts in (2.2) show in which copies of the $\mathbb{C}^{2|1}$ space the R-matrix acts non-trivially. The monodromy matrix $T(u)$ is also graded according to the rule $[T_{ij}(u)] = [i] + [j]$. It satisfies an intertwining relation (RTT relation):

$$R(u, v)\big(T(u) \otimes \mathbb{I}\big)\big(\mathbb{I} \otimes T(v)\big) = \big(\mathbb{I} \otimes T(v)\big)\big(T(u) \otimes \mathbb{I}\big)R(u, v). \tag{2.3}$$

Equation (2.3) holds in the tensor product of graded spaces $\mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1} \otimes \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of a Hamiltonian. It implies commutation relations between the monodromy matrix entries:

$$[T_{ij}(u), T_{kl}(v)\} = (-1)^{[i]([k]+[l])+[k][l]} g(u,v) \big[T_{kj}(v)T_{il}(u) - T_{kj}(u)T_{il}(v) \big], \tag{2.4}$$

where we introduced a graded commutator

$$[T_{ij}(u), T_{kl}(v)\} = T_{ij}(u)T_{kl}(v) - (-1)^{([i]+[j])([k]+[l])} T_{kl}(v)T_{ij}(u).$$

The graded transfer matrix is defined as the supertrace of the monodromy matrix

$$\mathcal{T}(u) = \operatorname{str} T(u) = \sum_{i=1}^{3} (-1)^{[i]} T_{ii}(u).$$

It defines an integrable system, due to the relation $[\mathcal{T}(u), \mathcal{T}(v)] = 0$ which is implied by the relation (2.3).

In order to make our formulas more compact we use several auxiliary functions and conventions on the notation. In addition to the functions $g(x,y)$ we also introduce the functions

$$f(x,y) = 1 + g(x,y) = \frac{x-y+c}{x-y}, \qquad h(x,y) = \frac{f(x,y)}{g(x,y)} = \frac{x-y+c}{c},$$

$$t(x,y) = \frac{g(x,y)}{h(x,y)} = \frac{c^2}{(x-y+c)(x-y)}.$$

The following obvious properties of these functions are useful

$$g(x,y) = -g(y,x), \qquad h(x,y) = \frac{1}{g(x,y-c)}, \qquad f(x-c,y) = \frac{1}{f(y,x)}.$$

Below, we will permanently have to deal with sets of variables which will be denoted by a bar: $\bar{u}$, $\bar{v}$, $\bar{\eta}$ etc. Individual elements of the sets are denoted by subscripts and without a bar: u_k, v_ℓ, η_j etc. The notation $\bar{u} \pm c$ means that all the elements of the set $\bar{u}$ are shifted by $\pm c$: $\bar{u} \pm c = \{u_1 \pm c, \dots, u_n \pm c\}$. As a rule, the number of elements in the sets is not shown explicitly; however we give these cardinalities in special comments to the formulas. Subsets of variables are denoted by roman or other subscripts, easily distinguishable from Latin indices used for individual element: $\bar{u}_{\mathrm{I}}$, $\bar{v}_{\mathrm{II}}$, $\bar{\eta}_{\mathrm{ii}}$, $\bar{\xi}_0$ etc. For example, the notation $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$ means that the set $\bar{u}$ is divided into two disjoint subsets $\bar{u}_{\mathrm{I}}$ and $\bar{u}_{\mathrm{II}}$. We assume that the elements in every subset are ordered in such a way that the sequence of their subscripts is strictly increasing. For the union of two sets into another one we use the notation $\{\bar{v}, \bar{z}\} = \bar{\xi}$. Finally we use a special notation $\bar{u}_j$, $\bar{v}_k$ and so on for the sets $\bar{u} \setminus \{u_j\}$, $\bar{v} \setminus \{v_k\}$ etc.

In order to avoid excessively cumbersome formulas we use shorthand notation for products of functions depending on one or two variables. Namely, whenever such a function depends on a set of variables, this means that we deal with the product of this function with respect to the corresponding set, as follows

$$h(\bar{u}, v) = \prod_{u_j \in \bar{u}} h(u_j, v), \qquad g(x_k, \bar{\xi}_\ell) = \prod_{\substack{\xi_j \in \bar{\xi} \\ \xi_j \neq \xi_\ell}} g(x_k, \xi_j), \qquad f(\bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{I}}) = \prod_{u_j \in \bar{u}_{\mathrm{II}}} \prod_{u_k \in \bar{u}_{\mathrm{I}}} f(u_j, u_k).$$

This notation is also used for the product of commuting operators,

$$T_{ij}(\bar{u}) = \prod_{u_k \in \bar{u}} T_{ij}(u_k), \qquad \text{if} \quad [i] + [j] = 0, \qquad \mod (2). \tag{2.5}$$

One can easily see from the commutation relations (2.4) that in this case $[T_{ij}(u), T_{ij}(v)] = 0$, and hence, the operator product (2.5) is well defined. However, if $[i] + [j] = 1$, then $[T_{ij}(u), T_{ij}(v)] \neq 0$, but we can introduce symmetric operator products

$$\mathbb{T}_{j3}(\bar{v}) = \frac{T_{j3}(v_1) \cdots T_{j3}(v_n)}{\prod\limits_{n \geq \ell > m \geq 1} h(v_\ell, v_m)}, \qquad \mathbb{T}_{3j}(\bar{v}) = \frac{T_{3j}(v_1) \cdots T_{3j}(v_n)}{\prod\limits_{n \geq \ell > m \geq 1} h(v_m, v_\ell)}, \qquad j = 1, 2.$$

In various formulas the Izergin determinant $K_n(\bar{x}|\bar{y})$ appears[1]. It is defined for two sets $\bar{x}$ and $\bar{y}$ with common cardinality $\#\bar{x} = \#\bar{y} = n$,

$$K_n(\bar{x}|\bar{y}) = h(\bar{x}, \bar{y}) \prod_{\ell < m}^{n} g(x_\ell, x_m) g(y_m, y_\ell) \det_n[t(x_i, y_j)]. \tag{2.6}$$

We draw the readers attention that according to the convention on the shorthand notation $h(\bar{x}, \bar{y})$ in (2.6) means the double product of the h-functions over all parameters $\bar{x}$ and $\bar{y}$. It is easy to see from definition (2.6) that $K_1(x|y) = g(x, y)$ and

$$K_n(\bar{x}|\bar{y} + c) = (-1)^n \frac{K_n(\bar{y}|\bar{x})}{f(\bar{y}, \bar{x})}. \tag{2.7}$$

2.1   Bethe vectors

For further calculation we need explicit formulas for $\mathfrak{gl}(2|1)$ Bethe vectors in terms of the monodromy matrix entries $T_{ij}(u)$. We recall that generically Bethe vectors are special polynomials in operators $T_{ij}(u)$ with $i \leq j$ applied to the pseudovacuum vector Ω. This vector possesses the following properties:

$$T_{ii}(u)\Omega = \lambda_i(u)\Omega, \quad i = 1, 2, 3, \qquad T_{ij}(u)\Omega = 0, \quad 3 \geq i > j \geq 1. \tag{2.8}$$

Here $\lambda_i(u)$ are some scalar functions depending on a specific model. Below we will also use the ratios of these functions

$$r_1(u) = \frac{\lambda_1(u)}{\lambda_2(u)}, \qquad r_3(u) = \frac{\lambda_3(u)}{\lambda_2(u)}.$$

We extend the convention on the shorthand notation to the products of the functions λ_i and r_k. For example,

$$\lambda_2(\bar{z}) = \prod_{z_j \in \bar{z}} \lambda_2(z_j), \qquad r_1(\bar{\eta}_\mathbb{I}) = \prod_{\eta_j \in \bar{\eta}_\mathbb{I}} r_1(\eta_j).$$

We denote the Bethe vectors as $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$. They depend on two sets of variables (Bethe parameters) $\bar{u} = \{u_1, \ldots, u_a\}$ and $\bar{v} = \{v_1, \ldots, v_b\}$, where $a, b = 0, 1, \ldots$. Explicit representations for $\mathfrak{gl}(2|1)$ Bethe vectors were obtained in[2] [40].

Definition 2.1. For $\#\bar{u} = a$ and $\#\bar{v} = b$ define a Bethe vector

$$\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \sum g(\bar{v}_\mathbb{I}, \bar{u}_\mathbb{I}) \frac{f(\bar{u}_\mathbb{I}, \bar{u}_\mathbb{II}) g(\bar{v}_\mathbb{II}, \bar{v}_\mathbb{I}) h(\bar{u}_\mathbb{I}, \bar{u}_\mathbb{I})}{\lambda_2(\bar{u})\lambda_2(\bar{v}_\mathbb{II}) f(\bar{v}, \bar{u})} T_{13}(\bar{u}_\mathbb{I}) T_{12}(\bar{u}_\mathbb{II}) \mathbb{T}_{23}(\bar{v}_\mathbb{II})\Omega. \tag{2.9}$$

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[1]Note that by definition this function depends on two sets of variables. Therefore, the convention on shorthand notations for the products is not applicable in this case.

[2]The formulas for the Bethe vectors obtained in [40] differ from (2.9) and (2.10) by a normalization factor $\lambda_2(\bar{v})\lambda_2(\bar{u})f(\bar{v}, \bar{u})$.

Here the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}}\}$ and $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$ with the restriction $\#\bar{u}_{\mathrm{I}} = \#\bar{v}_{\mathrm{I}} = n$, where $n = 0, 1, \ldots, \min(a, b)$. We recall also that we use the shorthand notation for the products of all the functions and the operators in (2.9).

An alternative formula for the Bethe vector is

$$\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \sum K_n(\bar{v}_{\mathrm{I}} | \bar{u}_{\mathrm{I}}) \frac{f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}) g(\bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{I}})}{\lambda_2(\bar{u}_{\mathrm{II}}) \lambda_2(\bar{v}) f(\bar{v}, \bar{u})} \mathbb{T}_{13}(\bar{v}_{\mathrm{I}}) \mathbb{T}_{23}(\bar{v}_{\mathrm{II}}) T_{12}(\bar{u}_{\mathrm{II}}) \Omega, \tag{2.10}$$

where K_n is the Izergin determinant (2.6) and the sum is the same as in (2.9).

A distinctive feature of the Bethe vectors is that under certain conditions on $\bar{u}$ and $\bar{v}$ (Bethe equations), they become eigenvectors of the transfer matrix. In this case we call them *on-shell Bethe vectors*. We will show in Section 4 that the vectors (2.9), (2.10) do possess this property.

It is obvious within the framework of the current approach [23] that the action of any monodromy matrix entry $T_{ij}(u)$ on a Bethe vector produces a linear combination of a *finite* number of Bethe vectors. This follows from the presentations of the monodromy matrix elements and Bethe vectors in terms of the Cartan–Weyl or current generators of the Yangian double $\mathrm{DY}(\mathfrak{gl}(2|1))$ and normal ordering of these generators according to certain cyclic ordering [13, 19]. In summary, we can rewrite the action formulas as normal ordering problem for the current generators and then translate the result of this ordering back into finite sum of the Bethe vectors. However, it is not so obvious if we deal with the explicit representations (2.9), (2.10). Furthermore, in spite of the action of $T_{ij}(z)$ onto $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ formally can be derived via (2.4) and (2.8), actually it is a pretty nontrivial problem.

Fortunately, similarly to the $\mathfrak{gl}(3)$ case [6] the $\mathfrak{gl}(2|1)$ Bethe vectors obey recursion relations over the number of the Bethe parameters [40]. The first recursion has the form

$$T_{12}(z)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(z) f(\bar{v}, z) \mathbb{B}_{a+1,b}(\{\bar{u}; z\}; \bar{v})$$
$$+ \sum_{j=1}^{b} g(z, v_j) g(\bar{v}_j, v_j) T_{13}(z) \mathbb{B}_{a,b-1}(\bar{u}; \bar{v}_j). \tag{2.11}$$

The second recursion reads

$$T_{23}(z)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(z) h(\bar{v}, z) f(z, \bar{u}) \mathbb{B}_{a,b+1}(\bar{u}; \{\bar{v}, z\})$$
$$+ \sum_{j=1}^{a} g(u_j, z) f(u_j, \bar{u}_j) T_{13}(z) \mathbb{B}_{a-1,b}(\bar{u}_j; \bar{v}). \tag{2.12}$$

We recall that in these formulas $\bar{v}_j$ and $\bar{u}_j$ respectively mean $\bar{v} \setminus \{v_j\}$ and $\bar{u} \setminus \{u_j\}$. The shorthand notation for the products of the functions g and f is also used.

Equations (2.11) and (2.12) allow us to built recursively Bethe vectors starting with the simplest cases

$$\mathbb{B}_{a,0}(\bar{u}; \varnothing) = \frac{T_{12}(\bar{u})}{\lambda_2(\bar{u})} \Omega, \qquad \mathbb{B}_{0,b}(\varnothing; \bar{v}) = \frac{\mathbb{T}_{23}(\bar{v})}{\lambda_2(\bar{v})} \Omega. \tag{2.13}$$

One can also easily derive the actions of T_{ij} onto either $\mathbb{B}_{a,0}(\bar{u}; \varnothing)$ or $\mathbb{B}_{0,b}(\varnothing; \bar{v})$, and then, using induction over a or b obtain the action rule in the general case. This will be our main strategy in the derivation of the action formulas.

3 Multiple actions of the operators T_{ij} onto Bethe vectors

The main result of this paper consists of explicit formulas of the multiple actions of the monodromy matrix entries onto Bethe vectors. We show that these actions always reduce to finite linear combinations of Bethe vectors.

Everywhere in this section we assume that $\bar{u}$, $\bar{v}$, and $\bar{z}$ are three sets of generic complex numbers with cardinalities $\#\bar{u} = a$, $\#\bar{v} = b$, and $\#\bar{z} = n$, $a, b, n = 0, 1, \ldots$. We also set $\bar{\eta} = \{\bar{u}, \bar{z}\}$ and $\bar{\xi} = \{\bar{v}, \bar{z}\}$.

3.1 Actions of $T_{ij}(\bar{z})$ with $i < j$

- Multiple action of $T_{13}(z)$:

$$\mathbb{T}_{13}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z})h(\bar{v}, \bar{z})\mathbb{B}_{a+n,b+n}(\bar{\eta}; \bar{\xi}). \tag{3.1}$$

- Multiple action of $T_{12}(z)$:

$$T_{12}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z})h(\bar{\xi}, \bar{z})\sum \frac{g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{z})}\mathbb{B}_{a+n,b}(\bar{\eta}; \bar{\xi}_{\mathrm{II}}). \tag{3.2}$$

Here the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$ with $\#\bar{\xi}_{\mathrm{I}} = n$.

- Multiple action of $T_{23}(z)$:

$$\mathbb{T}_{23}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{z})h(\bar{v}, \bar{z})\sum K_n(\bar{z}|\bar{\eta}_{\mathrm{I}} + c)f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})\mathbb{B}_{a,b+n}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}). \tag{3.3}$$

Here the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$ with $\#\bar{\eta}_{\mathrm{I}} = n$.

3.2 Actions of $T_{ii}(\bar{z})$

In formulas (3.4)–(3.6) the sums are taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$ with $\#\bar{\xi}_{\mathrm{I}} = \#\bar{\eta}_{\mathrm{I}} = n$.

- Multiple action of $T_{11}(z)$:

$$T_{11}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{z})h(\bar{\xi}, \bar{z})$$
$$\times \sum r_1(\bar{\eta}_{\mathrm{I}})\frac{f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{z})f(\bar{\xi}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})}K_n(\bar{\eta}_{\mathrm{I}}|\bar{\xi}_{\mathrm{I}} + c)\mathbb{B}_{a,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{II}}). \tag{3.4}$$

- Multiple action of $T_{22}(z)$:

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{z})h(\bar{\xi}, \bar{z})$$
$$\times \sum \frac{f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{z})}K_n(\bar{z}|\bar{\eta}_{\mathrm{I}} + c)\mathbb{B}_{a,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{II}}). \tag{3.5}$$

- Multiple action of $T_{33}(z)$:

$$T_{33}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z})h(\bar{\xi}, \bar{z})\sum r_3(\bar{\xi}_{\mathrm{I}})\frac{f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})h(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{\eta}_{\mathrm{I}})h(\bar{\eta}_{\mathrm{I}}, \bar{z})f(\bar{\xi}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})}\mathbb{B}_{a,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{II}}). \tag{3.6}$$

3.3 Actions of $T_{ij}(\bar{z})$ with $i > j$

- Multiple action of $T_{21}(z)$:

$$T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z})h(\bar{\xi}, \bar{z})\sum r_1(\bar{\eta}_{\mathrm{I}})\frac{f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{III}})f(\bar{\eta}_{\mathrm{III}}, \bar{\eta}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{z})f(\bar{\xi}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})}$$
$$\times K_n(\bar{z}|\bar{\eta}_{\mathrm{II}} + c)K_n(\bar{\eta}_{\mathrm{I}}|\bar{\xi}_{\mathrm{I}} + c)\mathbb{B}_{a-n,b}(\bar{\eta}_{\mathrm{III}}; \bar{\xi}_{\mathrm{II}}). \tag{3.7}$$

Here the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{III}}\}$ with $\#\bar{\xi}_{\mathrm{I}} = \#\bar{\eta}_{\mathrm{I}} = \#\bar{\eta}_{\mathrm{II}} = n$.

- Multiple action of $T_{32}(z)$:

$$\mathbb{T}_{32}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{\frac{n(n-1)}{2}}\lambda_2(\bar{z})h(\bar{\xi},\bar{z})$$

$$\times \sum r_3(\bar{\xi}_{\mathrm{I}})\frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{III}},\bar{\xi}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{III}},\bar{\xi}_{\mathrm{I}})}{h(\bar{\eta}_{\mathrm{I}},\bar{z})h(\bar{\xi}_{\mathrm{I}},\bar{\eta}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{II}},\bar{z})f(\bar{\xi}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})}h(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{I}})\mathbb{B}_{a,b-n}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{III}}). \qquad (3.8)$$

Here the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}},\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{III}}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}}\}$ with $\#\bar{\xi}_{\mathrm{I}} = \#\bar{\xi}_{\mathrm{II}} = \#\bar{\eta}_{\mathrm{I}} = n$.

- Multiple action of $T_{31}(z)$:

$$\mathbb{T}_{31}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{\frac{n(n+1)}{2}}\lambda_2(\bar{z})h(\bar{\xi},\bar{z})\sum r_3(\bar{\xi}_{\mathrm{I}})r_1(\bar{\eta}_{\mathrm{II}})\frac{g(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{III}},\bar{\xi}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{III}},\bar{\xi}_{\mathrm{I}})}{h(\bar{\eta}_{\mathrm{I}},\bar{z})h(\bar{\xi}_{\mathrm{I}},\bar{\eta}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{II}},\bar{z})}$$

$$\times \frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{III}})f(\bar{\eta}_{\mathrm{III}},\bar{\eta}_{\mathrm{II}})h(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{I}})}{f(\bar{\xi}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})f(\bar{\xi}_{\mathrm{I}},\bar{\eta}_{\mathrm{III}})f(\bar{\xi}_{\mathrm{III}},\bar{\eta}_{\mathrm{II}})}K_n(\bar{\eta}_{\mathrm{II}}|\bar{\xi}_{\mathrm{II}}+c)\mathbb{B}_{a-n,b-n}(\bar{\eta}_{\mathrm{III}};\bar{\xi}_{\mathrm{III}}). \qquad (3.9)$$

Here the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}},\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{III}}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}},\bar{\eta}_{\mathrm{III}}\}$ with $\#\bar{\xi}_{\mathrm{I}} = \#\bar{\xi}_{\mathrm{II}} = \#\bar{\eta}_{\mathrm{I}} = \#\bar{\eta}_{\mathrm{II}} = n$.

The proofs of the multiple action formulas will be given in Sections 5–7.

4 On-shell Bethe vectors

The action formulas (3.1)–(3.9) are valid for generic complex numbers $\bar{z}$, $\bar{u}$, and $\bar{v}$. In this section we consider them for on-shell Bethe vector $\mathbb{B}_{a,b}(\bar{u};\bar{v})$, that is when the parameters $\bar{u}$ and $\bar{v}$ satisfy a system of Bethe equations (see (4.6)).

In order to find explicitly the result of the transfer matrix action onto $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ one should set $n = 1$ in (3.4)–(3.6). Then the subsets $\bar{\eta}_{\mathrm{I}}$ and $\bar{\xi}_{\mathrm{I}}$ consist of one element only. Obviously, there are two essentially different types of partitions of the set $\bar{\eta} = \{z,\bar{u}\}$:

$$\bar{\eta}_{\mathrm{I}} = z, \qquad \bar{\eta}_{\mathrm{II}} = \bar{u}, \qquad\qquad\qquad\qquad\qquad\qquad (4.1)$$

$$\bar{\eta}_{\mathrm{I}} = u_j, \qquad \bar{\eta}_{\mathrm{II}} = \{z,\bar{u}_j\}, \qquad j = 1,\ldots,a. \qquad\qquad (4.2)$$

Similarly, there are two different types of partitions of the set $\bar{\xi} = \{z,\bar{v}\}$:

$$\bar{\xi}_{\mathrm{I}} = z, \qquad \bar{\xi}_{\mathrm{II}} = \bar{v}, \qquad\qquad\qquad\qquad\qquad\qquad (4.3)$$

$$\bar{\xi}_{\mathrm{I}} = v_k, \qquad \bar{\xi}_{\mathrm{II}} = \{z,\bar{v}_k\}, \qquad k = 1,\ldots,b. \qquad\qquad (4.4)$$

Thus, the action of $\mathcal{T}(z)$ onto $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ can be written in the form

$$\mathcal{T}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \tau(z|\bar{u};\bar{v})\mathbb{B}_{a,b}(\bar{u};\bar{v}) + \sum_{j=1}^{a}\Lambda_j\mathbb{B}_{a,b}(\{z,\bar{u}_j\};\bar{v})$$

$$+ \sum_{k=1}^{b}\tilde{\Lambda}_k\mathbb{B}_{a,b}(\bar{u};\{z,\bar{v}_k\}) + \sum_{j=1}^{a}\sum_{k=1}^{b}M_{jk}\mathbb{B}_{a,b}(\{z,\bar{u}_j\};\{z,\bar{v}_k\}),$$

where τ, Λ_j, $\tilde{\Lambda}_k$, and M_{jk} are numerical coefficients. In order to find $\tau(z|\bar{u};\bar{v})$ we substitute the partitions (4.1) and (4.3) into (3.4)–(3.6). We obtain

$$\tau(z|\bar{u},\bar{v}) = \lambda_1(z)f(\bar{u},z) + \lambda_2(z)f(z,\bar{u})f(\bar{v},z) - \lambda_3(z)f(\bar{v},z), \qquad (4.5)$$

where we have used $h(z,z) = 1$ and $K_1(z|z+c) = g(z,z+c) = -1$.

In order to find Λ_j we substitute the partitions (4.2) and (4.3) into (3.4)–(3.6). We find

$$\Lambda_j = \lambda_2(z)h(\bar{v}, z)g(\bar{v}, z)g(z, u_j)\left(r_1(u_j)\frac{f(\bar{u}_j, u_j)}{f(\bar{v}, u_j)} - f(u_j, \bar{u}_j)\right).$$

Similarly, in order to find $\tilde{\Lambda}_k$ we substitute the partitions (4.1) and (4.4) into (3.4)–(3.6). This gives us

$$\tilde{\Lambda}_k = \lambda_2(z)f(z, \bar{u})g(\bar{v}_k, v_k)h(\bar{v}_k, z)g(z, v_k)\left(1 - \frac{r_3(v_k)}{f(v_k, \bar{u})}\right).$$

If $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ is an eigenvector of $\mathcal{T}(z)$, then the coefficients Λ_j and $\tilde{\Lambda}_k$ must vanish for arbitrary z. Setting $\Lambda_j = 0$ for $j = 1, \ldots, a$ and $\tilde{\Lambda}_k = 0$ for $k = 1, \ldots, b$ we arrive at a system of equations

$$r_1(u_j) = \frac{f(u_j, \bar{u}_j)}{f(\bar{u}_j, u_j)}f(\bar{v}, u_j), \qquad j = 1, \ldots, a,$$
$$r_3(v_k) = f(v_k, \bar{u}), \qquad k = 1, \ldots, b. \tag{4.6}$$

Let us check that $M_{jk} = 0$ provided the system (4.6) is fulfilled. Substituting the partitions (4.2) and (4.4) into (3.4)–(3.6) we obtain

$$M_{jk} = \lambda_2(z)h(\bar{v}_k, z)g(\bar{v}_k, v_k)\left(\frac{r_1(u_j)f(\bar{u}_j, u_j)}{f(\bar{v}, u_j)}g(v_k, u_j)g(z, v_k)\right.$$
$$\left. + f(u_j, \bar{u}_j)g(u_j, z)\left[g(z, v_k) + \frac{r_3(v_k)}{f(v_k, \bar{u})}g(v_k, u_j)\right]\right).$$

Substituting here $r_1(u_j)$ and $r_3(v_k)$ from equations (4.6), we immediately find that $M_{jk} = 0$ due to the identity

$$g(v_k, u_j)g(z, v_k) + g(u_j, z)g(z, v_k) + g(u_j, z)g(v_k, u_j) = 0.$$

Thus, the system (4.6) can be treated as the system of Bethe equations for the parameters $\bar{u}$ and $\bar{v}$. If (4.6) holds, then the corresponding Bethe vector $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ is on-shell, i.e., it is an eigenvector of the transfer matrix $\mathcal{T}(z)$. The eigenvalue of this on-shell vector is given by (4.5). At the same time, it is easy to see that the function $\tau(z|\bar{u}, \bar{v})$ has no poles in the points $z = u_j$, $j = 1, \ldots, a$, and $z = v_k$, $k = 1, \ldots, b$ due to the system (4.6).

5   Proofs of multiple actions for T_{ij} with $i < j$

Bethe vectors consist of the elements from the upper triangular part of the monodromy matrix applied to pseudovacuum Ω (2.9), (2.10). Then, it is intuitively clear that actions of the elements T_{ij} with $i < j$ are the simplest. We begin our consideration from the right-upper corner of monodromy matrix and will move along anti-diagonal direction successively proving the action relations.

5.1   Proof for T_{13}

For $n = 1$ equation (3.1) follows directly from the definitions of the Bethe vectors. Let us take, for instance, (2.9) and set there $\bar{u} = \{z, \bar{u}'\}$ and $\bar{v} = \{z, \bar{v}'\}$. Then the product $1/f(\bar{v}, \bar{u})$ vanishes, as it contains $1/f(z, z)$. This zero, however, can be compensated if and only if $z \in \bar{u}_{\mathrm{I}}$ and $z \in \bar{v}_{\mathrm{I}}$. Indeed, in this case the product $g(\bar{v}_{\mathrm{I}}, \bar{u}_{\mathrm{I}})$ contains a singular factor $g(z, z)$. Thus,

we should consider only such partitions, for which $z \in \bar{u}_{\mathrm{I}}$ and $z \in \bar{v}_{\mathrm{I}}$. Therefore we should set: $\bar{u}_{\mathrm{I}} = \{z, \bar{u}'_{\mathrm{I}}\}$ and $\bar{v}_{\mathrm{I}} = \{z, \bar{v}'_{\mathrm{I}}\}$; $\bar{u}_{\mathrm{II}} = \bar{u}'_{\mathrm{II}}$ and $\bar{v}_{\mathrm{II}} = \bar{v}'_{\mathrm{II}}$. Then we obtain

$$\mathbb{B}_{a,b}(\{z, \bar{u}'\}; \{z, \bar{v}'\}) = \sum g(\bar{v}'_{\mathrm{I}}, \bar{u}'_{\mathrm{I}}) \frac{f(\bar{u}'_{\mathrm{I}}, \bar{u}'_{\mathrm{II}}) g(\bar{v}'_{\mathrm{II}}, \bar{v}'_{\mathrm{I}}) h(\bar{u}'_{\mathrm{I}}, \bar{u}'_{\mathrm{I}})}{\lambda_2(\bar{u}') \lambda_2(\bar{v}'_{\mathrm{II}}) f(\bar{v}', \bar{u}')} g(\bar{v}'_{\mathrm{I}}, z) g(z, \bar{u}'_{\mathrm{I}})$$

$$\times \frac{f(z, \bar{u}'_{\mathrm{II}}) g(\bar{v}'_{\mathrm{II}}, z) h(z, \bar{u}'_{\mathrm{I}}) h(\bar{u}'_{\mathrm{I}}, z)}{\lambda_2(z) f(\bar{v}', z) f(z, \bar{u}')} \frac{T_{13}(z)}{h(\bar{u}'_{\mathrm{I}}, z)} \mathbb{T}_{13}(\bar{u}'_{\mathrm{I}}) T_{12}(\bar{u}'_{\mathrm{II}}) \mathbb{T}_{23}(\bar{v}'_{\mathrm{II}}) \Omega.$$

After evident cancellations we arrive at

$$\mathbb{B}_{a,b}(\{z, \bar{u}'\}; \{z, \bar{v}'\}) = \frac{T_{13}(z)}{\lambda_2(z) h(\bar{v}', z)} \mathbb{B}_{a-1,b-1}(\bar{u}'; \bar{v}'),$$

which coincides with (3.1) at $n = 1$. The same result arises from the analysis of equation (2.10).

Now we use induction over n. Assume that (3.1) holds for some $n - 1$. Then

$$\mathbb{T}_{13}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \frac{T_{13}(z_n) \mathbb{T}_{13}(\bar{z}_n)}{h(\bar{z}_n, z_n)} \mathbb{B}_{a,b}(\bar{u}; \bar{v})$$

$$= \lambda_2(\bar{z}_n) \frac{h(\bar{v}, \bar{z}_n)}{h(\bar{z}_n, z_n)} T_{13}(z_n) \mathbb{B}_{a+n-1, b+n-1}(\{\bar{u}, \bar{z}_n\}; \{\bar{v}, \bar{z}_n\})$$

$$= \lambda_2(\bar{z}) h(\bar{v}, \bar{z}) \mathbb{B}_{a+n, b+n}(\{\bar{u}, \bar{z}\}; \{\bar{v}, \bar{z}\}),$$

and thus, (3.1) is proved.

Using (3.1) one can recast recursions (2.11) and (2.12) as follows

$$T_{12}(z) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(z) f(\bar{v}, z) \mathbb{B}_{a+1, b}(\{\bar{u}, z\}; \bar{v})$$

$$+ \lambda_2(z) \sum_{j=1}^{b} g(z, v_j) g(\bar{v}_j, v_j) h(\bar{v}_j, z) \mathbb{B}_{a+1, b}(\{\bar{u}, z\}; \{\bar{v}_j, z\}), \qquad (5.1)$$

and

$$T_{23}(z) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(z) h(\bar{v}, z) \left(f(z, \bar{u}) \mathbb{B}_{a, b+1}(\bar{u}; \{\bar{v}, z\}) \right.$$

$$+ \left. \sum_{j=1}^{a} g(u_j, z) f(u_j, \bar{u}_j) \mathbb{B}_{a, b+1}(\{\bar{u}_j, z\}; \{\bar{v}, z\}) \right).$$

One can easily recognize in these equations the actions (3.2) and (3.3) for $n = 1$. Then one should use induction over n.

5.2   Proof for T_{12}

Assume that (3.2) holds for some $n - 1$. Then

$$T_{12}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = T_{12}(z_n) \lambda_2(\bar{z}_n) h(\bar{\xi}, \bar{z}_n) \sum \frac{g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{z}_n)} \mathbb{B}_{a+n-1, b}(\bar{\eta}; \bar{\xi}_{\mathrm{II}}).$$

Here $\bar{\eta} = \{\bar{z}_n, \bar{u}\}$, $\bar{\xi} = \{\bar{z}_n, \bar{v}\}$, and the sum runs through the partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$ with $\#\bar{\xi}_{\mathrm{I}} = n - 1$. Acting with $T_{12}(z_n)$ we obtain

$$T_{12}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z}) \frac{h(\bar{\xi}, \bar{z}_n)}{h(z_n, \bar{z}_n)} \sum \frac{g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{z}_n)} h(\bar{\xi}_{\mathrm{II}}, z_n) \frac{g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})}{h(\bar{\xi}_{\mathrm{i}}, z_n)} \mathbb{B}_{a+n, b}(\bar{\eta}; \bar{\xi}_{\mathrm{ii}}).$$

Here already $\bar{\eta} = \{\bar{z}, \bar{u}\}$ and $\bar{\xi} = \{\bar{z}, \bar{v}\}$. The sum first is taken over partitions $\{\bar{z}_n, \bar{v}\} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$ with $\#\bar{\xi}_{\mathrm{I}} = n - 1$, and then over partitions $\{z_n, \bar{\xi}_{\mathrm{II}}\} \Rightarrow \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\}$ with $\#\bar{\xi}_{\mathrm{i}} = 1$. One can say that the sum is taken over partitions $\{\bar{z}, \bar{v}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\}$ with restrictions $z_n \notin \bar{\xi}_{\mathrm{I}}$, $\#\bar{\xi}_{\mathrm{I}} = n - 1$, and $\#\bar{\xi}_{\mathrm{i}} = 1$. Presenting $\bar{\xi}_{\mathrm{II}}$ as $\bar{\xi}_{\mathrm{II}} = \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\} \setminus \{z_n\}$ we obtain

$$g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}}) = \frac{g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}) g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{I}})}{g(z_n, \bar{\xi}_{\mathrm{I}})}, \qquad h(\bar{\xi}_{\mathrm{II}}, z_n) = h(\bar{\xi}_{\mathrm{i}}, z_n) h(\bar{\xi}_{\mathrm{ii}}, z_n),$$

and hence,

$$T_{12}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z}) \frac{h(\bar{\xi}, \bar{z}_n)}{h(z_n, \bar{z}_n)} \sum \frac{g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}) g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{I}}) g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})}{g(z_n, \bar{\xi}_{\mathrm{I}}) h(\bar{\xi}_{\mathrm{I}}, \bar{z}_n)} h(\bar{\xi}_{\mathrm{ii}}, z_n) \mathbb{B}_{a+n,b}(\bar{\eta}; \bar{\xi}_{\mathrm{ii}}). \tag{5.2}$$

Observe that the condition $z_n \notin \bar{\xi}_{\mathrm{I}}$ is ensured by the product $g(z_n, \bar{\xi}_{\mathrm{I}})$ in the denominator. Hence, we can say that the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\}$ with the restrictions on the cardinalities of the subsets only. Setting $\{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}\} = \bar{\xi}_0$ we recast (5.2) as follows

$$T_{12}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z}) \frac{h(\bar{\xi}, \bar{z})}{h(z_n, \bar{z}_n)} \sum \frac{g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}) g(z_n, \bar{\xi}_{\mathrm{i}}) h(\bar{\xi}_{\mathrm{i}}, \bar{z}_n)}{g(z_n, \bar{\xi}_0) h(\bar{\xi}_0, \bar{z})} g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_0) \mathbb{B}_{a+n,b}(\bar{\eta}; \bar{\xi}_{\mathrm{ii}}).$$

The sum over partitions $\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}\}$ can be computed via Lemma A.1

$$\sum_{\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}\}} g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}) g(z_n, \bar{\xi}_{\mathrm{i}}) h(\bar{\xi}_{\mathrm{i}}, \bar{z}_n) = g(z_n, \bar{\xi}_0) h(z_n, \bar{z}_n), \tag{5.3}$$

where we took into account that $\#\bar{\xi}_0 = n$. Thus, we arrive at

$$T_{12}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z}) h(\bar{\xi}, \bar{z}) \sum \frac{g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_0)}{h(\bar{\xi}_0, \bar{z})} \mathbb{B}_{a+n,b}(\bar{\eta}; \bar{\xi}_{\mathrm{ii}}),$$

which coincides with (3.2) up to a relabeling of the subsets.

5.3   Proof for T_{23}

Assume that (3.3) holds for some $n - 1$. Let $\#\bar{z} = n$. Then we have

$$\mathbb{T}_{23}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \frac{T_{23}(z_n) \mathbb{T}_{23}(\bar{z}_n)}{h(\bar{z}_n, z_n)} \mathbb{B}_{a,b}(\bar{u}; \bar{v})$$

$$= (-1)^{n-1} \lambda_2(\bar{z}_n) \frac{h(\bar{v}, \bar{z}_n)}{h(\bar{z}_n, z_n)} \sum K_{n-1}(\bar{z}_n | \bar{\eta}_{\mathrm{I}} + c) f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}) T_{23}(z_n) \mathbb{B}_{a,b+n-1}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}).$$

Here $\bar{\eta} = \{\bar{z}_n, \bar{u}\}$, $\bar{\xi} = \{\bar{z}_n, \bar{v}\}$, and the sum runs through the partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$ with $\#\bar{\eta}_{\mathrm{I}} = n - 1$. Acting with $T_{23}(z_n)$ we obtain

$$\mathbb{T}_{23}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{z}) \frac{h(\bar{v}, \bar{z}_n)}{h(\bar{z}_n, z_n)}$$

$$\times \sum K_{n-1}(\bar{z}_n | \bar{\eta}_{\mathrm{I}} + c) f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}) h(\bar{\xi}, z_n) K_1(z_n | \bar{\eta}_{\mathrm{i}} + c) f(\bar{\eta}_{\mathrm{i}}, \bar{\eta}_{\mathrm{ii}}) \mathbb{B}_{a,b+n}(\bar{\eta}_{\mathrm{ii}}; \bar{\xi}).$$

Here already $\bar{\eta} = \{\bar{z}, \bar{u}\}$, $\bar{\xi} = \{\bar{z}, \bar{v}\}$, and the sum is taken first over partitions $\{\bar{z}_n, \bar{u}\} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$ with $\#\bar{\eta}_{\mathrm{I}} = n - 1$, and then over partitions $\{z_n, \bar{\eta}_{\mathrm{II}}\} \Rightarrow \{\bar{\eta}_{\mathrm{i}}, \bar{\eta}_{\mathrm{ii}}\}$ with $\#\bar{\eta}_{\mathrm{i}} = 1$. Substituting here $\bar{\eta}_{\mathrm{II}} = \{\bar{\eta}_{\mathrm{i}}, \bar{\eta}_{\mathrm{ii}}\} \setminus \{z_n\}$ we find

$$\mathbb{T}_{23}(\bar{z}) \mathbb{B}_{a,b}(\bar{u}; \bar{v}) = (-1)^n \lambda_2(\bar{z}) h(\bar{v}, \bar{z})$$

$$\times \sum K_{n-1}(\bar{z}_n | \bar{\eta}_{\mathrm{I}} + c) K_1(z_n | \bar{\eta}_{\mathrm{i}} + c) \frac{f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{i}}) f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{ii}}) f(\bar{\eta}_{\mathrm{i}}, \bar{\eta}_{\mathrm{ii}})}{f(\bar{\eta}_{\mathrm{I}}, z_n)} \mathbb{B}_{a,b+n}(\bar{\eta}_{\mathrm{ii}}; \bar{\xi}).$$

Setting $\bar{\eta}_0 = \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{i}}\}$ and using $K_1(z_n|\bar{\eta}_{\mathrm{i}} + c) = -K_1(\bar{\eta}_{\mathrm{i}}|z_n)/f(\bar{\eta}_{\mathrm{i}}, z_n)$ we obtain

$$\mathbb{T}_{23}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = (-1)^{n-1}\lambda_2(\bar{z})h(\bar{v}, \bar{z})$$
$$\times \sum K_{n-1}(\bar{z}_n|\bar{\eta}_{\mathrm{I}} + c)K_1(\bar{\eta}_{\mathrm{i}}|z_n)f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{i}})\frac{f(\bar{\eta}_0, \bar{\eta}_{\mathrm{ii}})}{f(\bar{\eta}_0, z_n)}\mathbb{B}_{a,b+n}(\bar{\eta}_{\mathrm{ii}}; \bar{\xi}).$$

Now we can compute the sum over partitions $\bar{\eta}_0 \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{i}}\}$ via (A.2)

$$\sum_{\bar{\eta}_0 \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{i}}\}} K_{n-1}(\bar{z}_n|\bar{\eta}_{\mathrm{I}} + c)K_1(\bar{\eta}_{\mathrm{i}}|z_n)f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{i}}) = -f(\bar{\eta}_0, z_n)K_n(\bar{z}|\bar{\eta}_0 + c), \tag{5.4}$$

which gives us

$$\mathbb{T}_{23}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = (-1)^n\lambda_2(\bar{z})h(\bar{v}, \bar{z})\sum K_n(\bar{z}|\bar{\eta}_0 + c)f(\bar{\eta}_0, \bar{\eta}_{\mathrm{ii}})\mathbb{B}_{a,b+n}(\bar{\eta}_{\mathrm{ii}}; \bar{\xi}).$$

This is exactly (3.3) up to the labeling of the subsets.

6 Proof of the multiple action of the operator T_{22}

The proofs for the actions (3.4)–(3.9) are much more involved than the ones considered in the previous section. Fortunately, they all are quite similar. Therefore, we only detail one as a typical example, the other actions being proven in the same manner. We focus on the operator $T_{22}(u)$.

The strategy of the proof is the following. First, we prove equation (3.5) for $a = \#\bar{u} = 0$ and $n = \#\bar{z} = 1$. This can be done either via the standard consideration of the algebraic Bethe ansatz or using induction over $b = \#\bar{v}$. In both cases we use (2.13) and the relation

$$T_{22}(u)T_{23}(v) = f(v, u)T_{23}(v)T_{22}(u) + g(u, v)T_{23}(u)T_{22}(v), \tag{6.1}$$

that follows from (2.4).

The next step of the proof is an induction over a. We assume that (3.5) is valid for $n = 1$ and some a and use recursion (2.11). Hereby, we use some of commutation relations (2.4)

$$T_{22}(u)T_{12}(v) = f(u, v)T_{12}(v)T_{22}(u) + g(v, u)T_{12}(u)T_{22}(v), \tag{6.2}$$
$$[T_{22}(u), T_{13}(v)] = g(u, v)\big(T_{12}(v)T_{23}(u) - T_{12}(u)T_{23}(v)\big). \tag{6.3}$$

Finally, when equation (3.5) is proved for $n = 1$ and arbitrary a and b we use induction over n.

Remark 6.1. We begin the proof with the case $n = 1$, $a = 0$, and arbitrary b. However, one could also begin with the case $n = 1$, $b = 0$, and arbitrary a. For the action of the operator $T_{22}(z)$ this is a matter of choice. For other operators these two starting cases could be essentially different. For instance, one can easily see that $T_{21}(z)\mathbb{B}_{0,b}(\varnothing, \bar{v}) = 0$ for arbitrary b. On the other hand, the action $T_{21}(z)\mathbb{B}_{a,0}(\bar{u}, \varnothing)$ is highly nontrivial, although it is clear that it should coincide with the analogous action in the models with $\mathfrak{gl}(3)$-invariant R-matrix. Obviously, in this case it is better to begin the proof with the vector $\mathbb{B}_{0,b}(\varnothing, \bar{v})$.

6.1 Action of $T_{22}(z)$ at $a = 0$ and $z = 1$

In the particular case $a = 0$ and $n = 1$ equation (3.5) turns into

$$T_{22}(z)\mathbb{B}_{0,b}(\varnothing; \bar{v}) = \lambda_2(z)h(\bar{v}, z)\sum \frac{g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, z)}\mathbb{B}_{0,b}(\varnothing; \bar{\xi}_{\mathrm{II}}). \tag{6.4}$$

The sum is taken over partitions $\{z, \bar{v}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}\}$ with $\#\bar{\xi}_{\rm I} = 1$. We prove this action using the standard scheme of the algebraic Bethe ansatz. The vector $\mathbb{B}_{0,b}(\varnothing; \bar{v})$ is given by the second equation (2.13). Thus, we should move the operator $T_{22}(z)$ to the right through the product of the operators $T_{23}(v_j)$. Using (6.1) we easily find

$$T_{22}(z)\mathbb{B}_{0,b}(\varnothing; \bar{v}) = \Lambda\mathbb{B}_{0,b}(\varnothing; \bar{v}) + \sum_{j=1}^{b} \Lambda_j \mathbb{B}_{0,b}(\varnothing; \{\bar{v}_j, z)),$$

where Λ and Λ_j are some coefficients to be determined. Obviously, in order to obtain the coefficient of $\mathbb{B}_{0,b}(\varnothing; \bar{v})$ one should use only the first term in the r.h.s. of (6.1). From this we immediately find

$$\Lambda = \lambda_2(z) f(\bar{v}, z).$$

Then, due to the symmetry of $\mathbb{T}_{23}(\bar{v})$ over $\bar{v}$ it is enough to find Λ_1 only. Permuting $T_{22}(z)$ with $T_{23}(v_1)$ we should use the second term in the r.h.s. of (6.1). We have

$$T_{22}(z)\frac{\mathbb{T}_{23}(\bar{v})}{\lambda_2(\bar{v})}\Omega = T_{22}(z)\frac{T_{23}(v_1)\mathbb{T}_{23}(\bar{v}_1)}{\lambda_2(\bar{v})h(\bar{v}_1, v_1)}\Omega = g(z, v_1)T_{23}(z)\frac{T_{22}(v_1)\mathbb{T}_{23}(\bar{v}_1)}{\lambda_2(\bar{v})h(\bar{v}_1, v_1)}\Omega + UWT,$$

where UWT means *unwanted terms*, i.e., the terms that cannot give a contribution to the coefficient Λ_1. Now, moving $T_{22}(v_1)$ through the product $\mathbb{T}_{23}(\bar{v})$ we should use only the first term in the r.h.s. of (6.1), which gives us

$$T_{22}(z)\frac{\mathbb{T}_{23}(\bar{v})}{\lambda_2(\bar{v})}\Omega = g(z, v_1)g(\bar{v}_1, v_1)T_{23}(z)\frac{\mathbb{T}_{23}(\bar{v}_1)}{\lambda_2(\bar{v})}\lambda_2(v_1)\Omega + UWT,$$

where we used $g(\bar{v}_1, v_1) = f(\bar{v}_1, v_1)/h(\bar{v}_1, v_1)$. It remains to combine $T_{23}(z)$ and $\mathbb{T}_{23}(\bar{v}_1)$ into $\mathbb{T}_{23}(\{z, \bar{v}_1\})$ and we arrive at

$$T_{22}(z)\frac{\mathbb{T}_{23}(\bar{v})}{\lambda_2(\bar{v})}\Omega = \lambda_2(z)g(z, v_1)g(\bar{v}_1, v_1)h(\bar{v}_1, z)\frac{\mathbb{T}_{23}(\{z, \bar{v}_1\})}{\lambda_2(\bar{v}_1)\lambda_2(z)}\Omega + UWT,$$

leading to

$$\Lambda_1 = \lambda_2(z)g(z, v_1)g(\bar{v}_1, v_1)h(\bar{v}_1, z).$$

Thus, we eventually obtain

$$T_{22}(z)\mathbb{B}_{0,b}(\varnothing; \bar{v}) = \lambda_2(z)f(\bar{v}, z)\mathbb{B}_{0,b}(\varnothing; \bar{v})$$
$$+ \lambda_2(z)\sum_{j=1}^{b} g(z, v_j)g(\bar{v}_j, v_j)h(\bar{v}_j, z)\mathbb{B}_{0,b}(\varnothing; \{\bar{v}_j, z\}). \tag{6.5}$$

It is easy to see that this formula coincides with (6.4). Indeed the first term in (6.5) corresponds to the partition $\bar{\xi}_{\rm I} = z$ and $\bar{\xi}_{\rm II} = \bar{v}$ in (6.4). The other terms arise in the case of the partitions $\bar{\xi}_{\rm I} = v_j$, $j = 1, \ldots, b$, and $\bar{\xi}_{\rm II} = \{z, \bar{v}_j\}$. Thus, action (6.4) is proved.

6.2   Induction over a

For $n = 1$ equation (3.5) takes the form

$$T_{22}(z)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(z)h(\bar{v}, z)\sum \frac{f(\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II})g(\bar{\xi}_{\rm II}, \bar{\xi}_{\rm I})}{h(\bar{\eta}_{\rm I}, z)h(\bar{\xi}_{\rm I}, z)}\mathbb{B}_{a,b}(\bar{\eta}_{\rm II}; \bar{\xi}_{\rm II}). \tag{6.6}$$

The sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\text{I}}, \bar{\xi}_{\text{II}}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}}\}$ with $\#\bar{\xi}_{\text{I}} = \#\bar{\eta}_{\text{I}} = 1$. We assume that (6.6) is valid for some $a \geq 0$ and b arbitrary. Then, due to recursion (2.11) we have

$$T_{22}(z_1)\mathbb{B}_{a+1,b}(\{\bar{u}; z_2\}; \bar{v}) = T_{22}(z_1)\frac{T_{12}(z_2)\mathbb{B}_{a,b}(\bar{u}; \bar{v})}{\lambda_2(z_2)f(\bar{v}, z_2)}$$
$$- T_{22}(z_1)\sum_{j=1}^{b}\frac{g(z_2, v_j)g(\bar{v}_j, v_j)T_{13}(z_2)\mathbb{B}_{a,b-1}(\bar{u}; \bar{v}_j)}{\lambda_2(z_2)f(\bar{v}, z_2)}. \tag{6.7}$$

We see that in order to compute the action of $T_{22}(z_1)$ onto $\mathbb{B}_{a+1,b}(\{z_2, \bar{u}\}; \bar{v})$ we should calculate the successive actions of the operators $T_{22}(z_1)T_{12}(z_2)$ and $T_{22}(z_1)T_{13}(z_2)$. This can be done via (6.2) and (6.3)

$$T_{22}(z_1)T_{12}(z_2)\mathbb{B}_{a,b}(\bar{u}; \bar{v})$$
$$= \big(f(z_1, z_2)T_{12}(z_2)T_{22}(z_1) + g(z_2, z_1)T_{12}(z_1)T_{22}(z_2)\big)\mathbb{B}_{a,b}(\bar{u}; \bar{v}), \tag{6.8}$$
$$T_{22}(z_1)T_{13}(z_2)\mathbb{B}_{a,b-1}(\bar{u}; \bar{v}_j) = T_{13}(z_2)T_{22}(z_1)\mathbb{B}_{a,b-1}(\bar{u}; \bar{v}_j)$$
$$+ g(z_1, z_2)\big(T_{12}(z_2)T_{23}(z_1) - T_{12}(z_1)T_{23}(z_2)\big)\mathbb{B}_{a,b-1}(\bar{u}; \bar{v}_j). \tag{6.9}$$

Thus, we have reduced the action $T_{22}(z_1)\mathbb{B}_{a+1,b}(\{\bar{u}; z_2\}; \bar{v})$ to the calculation of several successive actions. In all of them the operator T_{22} acts either on $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ or on $\mathbb{B}_{a,b-1}(\bar{u}; \bar{v}_j)$, which are known due to the induction assumption. The actions of other operators T_{ij} with $i < j$ are already known for a and b arbitrary.

6.2.1   Successive action of T_{12} and T_{23}

We begin our calculation with the successive action of the operators T_{12} and T_{23}. Using (3.3) we have

$$T_{12}(z_2)T_{23}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(z_1)h(\bar{v}, z_1)\sum\frac{f(\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}})}{h(\bar{\eta}_{\text{I}}, z_1)}T_{12}(z_2)\mathbb{B}_{a,b+1}(\bar{\eta}_{\text{II}}; \bar{\xi}).$$

Here $\bar{\eta} = \{z_1, \bar{u}\}$ and $\bar{\xi} = \{z_1, \bar{v}\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}}\}$ with $\#\bar{\eta}_{\text{I}} = 1$. Then we use (3.2) and find

$$T_{12}(z_2)T_{23}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v})$$
$$= \lambda_2(\bar{z})h(\bar{v}, z_1)h(\bar{\xi}, z_2)\sum\frac{f(\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}})}{h(\bar{\eta}_{\text{I}}, z_1)}\frac{g(\bar{\xi}_{\text{II}}, \bar{\xi}_{\text{I}})}{h(\bar{\xi}_{\text{I}}, z_2)}\mathbb{B}_{a+1,b+1}(\{\bar{\eta}_{\text{II}}, z_2\}; \bar{\xi}_{\text{II}}). \tag{6.10}$$

Here already $\bar{\xi} = \{\bar{z}, \bar{v}\}$, however we still have $\bar{\eta} = \{z_1, \bar{u}\}$. Replacing $\{\bar{\eta}_{\text{II}}, z_2\}$ by $\bar{\eta}_{\text{II}}$ we recast (6.10) in the form

$$T_{12}(z_2)T_{23}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v})$$
$$= \lambda_2(\bar{z})h(\bar{v}, \bar{z})h(z_1, z_2)\sum\frac{f(\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}})g(\bar{\xi}_{\text{II}}, \bar{\xi}_{\text{I}})}{f(\bar{\eta}_{\text{I}}, z_2)h(\bar{\eta}_{\text{I}}, z_1)h(\bar{\xi}_{\text{I}}, z_2)}\mathbb{B}_{a+1,b+1}(\bar{\eta}_{\text{II}}; \bar{\xi}_{\text{II}}).$$

Here $\bar{\eta} = \{\bar{z}, \bar{u}\}$, and the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}}\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_{\text{I}}, \bar{\xi}_{\text{II}}\}$ with $\#\bar{\eta}_{\text{I}} = \#\bar{\xi}_{\text{I}} = 1$. Note that the condition $z_2 \notin \bar{\eta}_{\text{I}}$ is ensured automatically. Indeed, if $z_2 \in \bar{\eta}_{\text{I}}$, then $1/f(\bar{\eta}_{\text{I}}, z_2) = 0$.

Replacing here $z_1 \leftrightarrow z_2$ we obtain

$$T_{12}(z_1)T_{23}(z_2)\mathbb{B}_{a,b}(\bar{u}; \bar{v})$$
$$= \lambda_2(\bar{z})h(\bar{v}, \bar{z})h(z_2, z_1)\sum\frac{f(\bar{\eta}_{\text{I}}, \bar{\eta}_{\text{II}})g(\bar{\xi}_{\text{II}}, \bar{\xi}_{\text{I}})}{f(\bar{\eta}_{\text{I}}, z_1)h(\bar{\eta}_{\text{I}}, z_2)h(\bar{\xi}_{\text{I}}, z_1)}\mathbb{B}_{a+1,b+1}(\bar{\eta}_{\text{II}}; \bar{\xi}_{\text{II}}).$$

Thus, we find

$$
g(z_1, z_2)\big(T_{12}(z_2)T_{23}(z_1) - T_{12}(z_1)T_{23}(z_2)\big)\mathbb{B}_{a,b}(\bar{u}; \bar{v})
$$
$$
= \lambda_2(\bar{z})h(\bar{v}, \bar{z}) \sum \frac{f(\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II})g(\bar{\xi}_\mathrm{II}, \bar{\xi}_\mathrm{I})}{h(\bar{\eta}_\mathrm{I}, \bar{z})} \mathbb{B}_{a+1,b+1}(\bar{\eta}_\mathrm{II}; \bar{\xi}_\mathrm{II})
$$
$$
\times \left\{ \frac{f(z_1, z_2)}{g(\bar{\eta}_\mathrm{I}, z_2)h(\bar{\xi}_\mathrm{I}, z_2)} + \frac{f(z_2, z_1)}{g(\bar{\eta}_\mathrm{I}, z_1)h(\bar{\xi}_\mathrm{I}, z_1)} \right\}. \tag{6.11}
$$

6.2.2 Successive action of T_{13} and T_{22}

Combining the actions (6.6) and (3.1) we obtain

$$
T_{13}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v})
$$
$$
= \lambda_2(\bar{z})h(\bar{v}, z_1) \sum \frac{f(\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II})g(\bar{\xi}_\mathrm{II}, \bar{\xi}_\mathrm{I})h(\bar{\xi}_\mathrm{II}, z_2)}{h(\bar{\eta}_\mathrm{I}, z_1)h(\bar{\xi}_\mathrm{I}, z_1)} \mathbb{B}_{a+1,b+1}(\{\bar{\eta}_\mathrm{II}, z_2\}; \{\bar{\xi}_\mathrm{II}, z_2\}).
$$

Here $\bar{\eta} = \{\bar{u}, z_1\}$ and $\bar{\xi} = \{\bar{v}, z_1\}$. Replacing $\{\bar{\eta}_\mathrm{II}, z_2\}$ with $\bar{\eta}_\mathrm{II}$ and $\{\bar{\xi}_\mathrm{II}, z_2\}$ with $\bar{\xi}_\mathrm{II}$ we arrive at

$$
T_{13}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z})h(\bar{v}, \bar{z})h(z_1, z_2) \sum \frac{f(\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II})g(\bar{\xi}_\mathrm{II}, \bar{\xi}_\mathrm{I})\mathbb{B}_{a+1,b+1}(\bar{\eta}_\mathrm{II}; \bar{\xi}_\mathrm{II})}{f(\bar{\eta}_\mathrm{I}, z_2)g(z_2, \bar{\xi}_\mathrm{I})h(\bar{\eta}_\mathrm{I}, z_1)h(\bar{\xi}_\mathrm{I}, \bar{z})}. \tag{6.12}
$$

Here already $\bar{\eta} = \{\bar{u}, \bar{z}\}$ and $\bar{\xi} = \{\bar{v}, \bar{z}\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II}\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_\mathrm{I}, \bar{\xi}_\mathrm{II}\}$ with $\#\bar{\eta}_\mathrm{I} = \#\bar{\xi}_\mathrm{I} = 1$.

Now we are able to calculate the successive action $T_{22}(z_1)T_{13}(z_2)$ on $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$. Indeed, due to (6.9) this successive action is given by a combination of (6.11) and (6.12). A straightforward calculation leads us to the following representation:

$$
T_{22}(z_1)T_{13}(z_2)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z})h(\bar{v}, \bar{z})h(z_2, z_1) \sum \frac{f(\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II})g(\bar{\xi}_\mathrm{II}, \bar{\xi}_\mathrm{I})}{h(\bar{\eta}_\mathrm{I}, z_1)h(\bar{\xi}_\mathrm{I}, z_1)} \mathbb{B}_{a+1,b+1}(\bar{\eta}_\mathrm{II}; \bar{\xi}_\mathrm{II}).
$$

Remark 6.2. Taking into account (3.1) we conclude that if the action (6.6) is valid on the vector $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$, then it is also valid on vectors of the special type $\mathbb{B}_{a+1,b+1}(\bar{u}'; \bar{v}')$, if $\bar{u}' \cap \bar{v}' \neq \varnothing$.

6.2.3 Successive action of T_{12} and T_{22}

Using (6.6) we obtain

$$
T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(z_1)h(\bar{v}, z_1) \sum \frac{f(\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II})g(\bar{\xi}_\mathrm{II}, \bar{\xi}_\mathrm{I})}{h(\bar{\eta}_\mathrm{I}, z_1)h(\bar{\xi}_\mathrm{I}, z_1)} T_{12}(z_2)\mathbb{B}_{a,b}(\bar{\eta}_\mathrm{II}; \bar{\xi}_\mathrm{II}).
$$

Here $\bar{\eta} = \{\bar{u}, z_1\}$ and $\bar{\xi} = \{\bar{v}, z_1\}$. Applying (3.2) to this formula we find

$$
T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v})
$$
$$
= \lambda_2(\bar{z})h(\bar{v}, z_1) \sum \frac{f(\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II})g(\bar{\xi}_\mathrm{II}, \bar{\xi}_\mathrm{I})}{h(\bar{\eta}_\mathrm{I}, z_1)h(\bar{\xi}_\mathrm{I}, z_1)} h(\bar{\xi}_\mathrm{II}, z_2) \frac{g(\bar{\xi}_\mathrm{ii}, \bar{\xi}_\mathrm{i})}{h(\bar{\xi}_\mathrm{i}, z_2)} \mathbb{B}_{a+1,b}(\{\bar{\eta}_\mathrm{II}, z_2\}; \bar{\xi}_\mathrm{ii}).
$$

Here we first have partitions $\bar{\eta} = \{\bar{u}, z_1\} \Rightarrow \{\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II}\}$ and $\bar{\xi} = \{\bar{v}, z_1\} \Rightarrow \{\bar{\xi}_\mathrm{I}, \bar{\xi}_\mathrm{II}\}$. Then we combine $\bar{\xi}_\mathrm{II}$ with z_2 and divide this set into new subsets $\{\bar{\xi}_\mathrm{II}, z_2\} \Rightarrow \{\bar{\xi}_\mathrm{i}, \bar{\xi}_\mathrm{ii}\}$. The restrictions are: $\#\bar{\xi}_\mathrm{i} = \#\bar{\xi}_\mathrm{I} = \#\bar{\eta}_\mathrm{I} = 1$, $z_2 \notin \bar{\eta}_\mathrm{I}$, and $z_2 \notin \bar{\xi}_\mathrm{I}$. As we already did before, we replace $\{\bar{\eta}_\mathrm{II}, z_2\}$ with $\bar{\eta}_\mathrm{II}$ and use $\bar{\xi}_\mathrm{II} = \{\bar{\xi}_\mathrm{i}, \bar{\xi}_\mathrm{ii}\} \setminus \{z_2\}$. Then

$$
T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \lambda_2(\bar{z})h(\bar{v}, \bar{z})h(z_1, z_2)
$$
$$
\times \sum \frac{f(\bar{\eta}_\mathrm{I}, \bar{\eta}_\mathrm{II})g(\bar{\xi}_\mathrm{i}, \bar{\xi}_\mathrm{I})g(\bar{\xi}_\mathrm{ii}, \bar{\xi}_\mathrm{I})g(\bar{\xi}_\mathrm{ii}, \bar{\xi}_\mathrm{i})}{f(\bar{\eta}_\mathrm{I}, z_2)h(\bar{\eta}_\mathrm{I}, z_1)h(\bar{\xi}_\mathrm{I}, z_1)g(z_2, \bar{\xi}_\mathrm{I})h(\bar{\xi}_\mathrm{I}, z_2)h(\bar{\xi}_\mathrm{i}, z_2)} \mathbb{B}_{a+1,b}(\bar{\eta}_\mathrm{II}; \bar{\xi}_\mathrm{ii}). \tag{6.13}
$$

Setting $\bar{\xi}_0 = \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}\}$ we recast (6.13) as follows

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})h(\bar{v},\bar{z})h(z_1,z_2)$$
$$\times \sum \frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{i}},\bar{\xi}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{ii}},\bar{\xi}_0)}{g(\bar{\eta}_{\mathrm{I}},z_2)h(\bar{\eta}_{\mathrm{I}},\bar{z})h(\bar{\xi}_{\mathrm{I}},z_1)g(z_2,\bar{\xi}_{\mathrm{I}})h(\bar{\xi}_0,z_2)}\mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{ii}}).$$

The sum over partitions of the set $\bar{\xi} = \{z_1, z_2, \bar{v}\}$ is organized as follows: first, we have partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_0\}$; second we divide $\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}\}$. The latter sum consists of two terms and can be computed straightforwardly. This leads us to

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})h(\bar{v},\bar{z})h(\bar{z},\bar{z}) \sum \frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{ii}},\bar{\xi}_0)}{g(\bar{\eta}_{\mathrm{I}},z_2)h(\bar{\eta}_{\mathrm{I}},\bar{z})h(\bar{\xi}_0,\bar{z})}\mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{ii}}),$$

and relabeling $\bar{\xi}_0 \to \bar{\xi}_{\mathrm{I}}$, $\bar{\xi}_{\mathrm{ii}} \to \bar{\xi}_{\mathrm{II}}$ we finally obtain

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar{u};\bar{v})$$
$$= \lambda_2(\bar{z})h(\bar{v},\bar{z})h(\bar{z},\bar{z}) \sum \frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})}{g(\bar{\eta}_{\mathrm{I}},z_2)h(\bar{\eta}_{\mathrm{I}},\bar{z})h(\bar{\xi}_{\mathrm{I}},\bar{z})}\mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{II}}). \tag{6.14}$$

Here the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$, $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$. The cardinalities of the subsets are $\#\bar{\eta}_{\mathrm{I}} = 1$, $\#\bar{\xi}_{\mathrm{I}} = 2$.

6.2.4   Successive action of T_{22} and T_{12}

Using (6.8) and (6.14) we are able to calculate the action of $T_{22}(z_1)T_{12}(z_2)$ onto $\mathbb{B}_{a,b}(\bar{u};\bar{v})$. It is clear that for this we should take the following combination: equation (6.14) multiplied with $f(z_1,z_2)$ and the same equation with $z_1 \leftrightarrow z_2$ multiplied with $g(z_2,z_1)$. This straightforward calculation gives

$$T_{22}(z_1)T_{12}(z_2)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})h(\bar{v},\bar{z})h(\bar{z},\bar{z}) \sum \frac{f(\bar{\eta}_{\mathrm{I}},\bar{\eta}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})}{h(\bar{\eta}_{\mathrm{I}},z_1)h(\bar{\xi}_{\mathrm{I}},\bar{z})}\mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}};\bar{\xi}_{\mathrm{II}}). \tag{6.15}$$

Here the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$, $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$. The cardinalities of the subsets are $\#\bar{\eta}_{\mathrm{I}} = 1$, $\#\bar{\xi}_{\mathrm{I}} = 2$.

6.3   Recursion formula

Now everything is ready for the use of recursion (6.7). Due to (5.1) we can write it as follows

$$\mathbb{B}_{a+1,b}(\{\bar{u},z_2\};\bar{v}) = \frac{1}{\lambda_2(z_2)f(\bar{v},z_2)}\big(T_{12}(z_2)\mathbb{B}_{a,b}(\bar{u};\bar{v}) - \Psi\big), \tag{6.16}$$

where

$$\Psi = \lambda_2(z_2)h(\bar{v},z_2) \sum_{z_2 \notin \bar{\xi}_{\mathrm{I}}} \frac{g(\bar{\xi}_{\mathrm{II}},\bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}},z_2)}\mathbb{B}_{a+1,b}(\bar{\eta};\bar{\xi}_{\mathrm{II}}). \tag{6.17}$$

Here $\bar{\eta} = \{z_2, \bar{u}\}$ and $\bar{\xi} = \{z_2, \bar{v}\}$, and we used $h(z_2, z_2) = 1$. The sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$ with $\#\bar{\xi}_{\mathrm{I}} = 1$. One more restriction $z_2 \notin \bar{\xi}_{\mathrm{I}}$ is shown explicitly by the subscript of the sum.

   Recall that we assume that the action of $T_{22}(z_1)$ on the vectors $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ is given by (6.6) at some value of $a \geq 0$ and arbitrary b. All the vectors in the linear combination (6.17) have

the form $\mathbb{B}_{a+1,b}(\{\bar{u}, z_2\}; \{\bar{v}_j, z_2\})$, that is $\{\bar{u}, z_2\} \cap \{\bar{v}_j, z_2\} \neq \varnothing$. Hence, taking into account Remark 6.2, the action of $T_{22}(z_1)$ on these vectors is known and it is given by (6.6):

$$T_{22}(z_1)\Psi = \lambda_2(\bar{z})h(\bar{v}, z_2) \sum_{z_2 \notin \bar{\xi}_{\mathrm{I}}} \frac{g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, z_2)} h(\bar{\xi}_{\mathrm{II}}, z_1) \frac{f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})}{h(\bar{\eta}_{\mathrm{I}}, z_1)h(\bar{\xi}_{\mathrm{i}}, z_1)} \mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{ii}}). \tag{6.18}$$

In this formula $\bar{\eta} = \{\bar{z}, \bar{u}\}$ and $\bar{\xi} = \{\bar{z}, \bar{v}\}$. At the first step we have partitions $\{z_2, \bar{v}\} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$. Then we obtain additional partitions $\{\bar{z}, \bar{v}\} \Rightarrow \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$. Hereby $\#\bar{\eta}_{\mathrm{I}} = \#\bar{\xi}_{\mathrm{I}} = \#\bar{\xi}_{\mathrm{i}} = 1$. Thus, one can say that the set $\bar{\xi} = \{\bar{z}, \bar{v}\}$ is divided into subsets $\{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\}$ with the restrictions $z_1 \notin \bar{\xi}_{\mathrm{I}}$ and $z_2 \notin \bar{\xi}_{\mathrm{I}}$. Substituting $\bar{\xi}_{\mathrm{II}} = \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\} \setminus \{z_1\}$ into (6.18) we obtain

$$T_{22}(z_1)\Psi = \lambda_2(\bar{z})h(\bar{v}, z_2) \sum_{z_2 \notin \bar{\xi}_{\mathrm{I}}} \frac{g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})h(\bar{\xi}_{\mathrm{ii}}, z_1)}{g(z_1, \bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}}, z_2)h(\bar{\eta}_{\mathrm{I}}, z_1)} \mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{ii}}). \tag{6.19}$$

Observe that the restriction $z_1 \notin \bar{\xi}_{\mathrm{I}}$ holds automatically due to the factor $g(z_1, \bar{\xi}_{\mathrm{I}})^{-1}$. In order to get rid of the restriction $z_2 \notin \bar{\xi}_{\mathrm{I}}$ we present $T_{22}(z_1)\Psi$ as a difference of two terms. The first term is just the sum over partitions in (6.19), where no restrictions on the partitions of the set $\bar{\xi}$ are imposed. In the second term we simply set $\bar{\xi}_{\mathrm{I}} = z_2$. Thus,

$$T_{22}(z_1)\Psi = \Psi' - \Psi'',$$

where

$$\Psi' = \lambda_2(\bar{z})h(\bar{v}, z_2) \sum \frac{g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})h(\bar{\xi}_{\mathrm{ii}}, z_1)}{g(z_1, \bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}}, z_2)h(\bar{\eta}_{\mathrm{I}}, z_1)} \mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{ii}}), \tag{6.20}$$

and

$$\Psi'' = \lambda_2(\bar{z})h(\bar{v}, z_2) \sum \frac{g(\bar{\xi}_{\mathrm{ii}}, z_2)g(\bar{\xi}_{\mathrm{i}}, z_2)g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})h(\bar{\xi}_{\mathrm{ii}}, z_1)}{g(z_1, z_2)h(\bar{\eta}_{\mathrm{I}}, z_1)} \mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{ii}}). \tag{6.21}$$

In (6.21) we have $\{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\} = \{\bar{v}, z_1\}$, therefore

$$\Psi'' = \lambda_2(\bar{z})h(\bar{v}, z_1)f(\bar{v}, z_2) \sum \frac{g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})}{h(\bar{\xi}_{\mathrm{i}}, z_1)h(\bar{\eta}_{\mathrm{I}}, z_1)} \mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{ii}}). \tag{6.22}$$

In (6.20) we can take the sum over partitions into subsets $\bar{\xi}_{\mathrm{i}}$ and $\bar{\xi}_{\mathrm{I}}$, because it consists of two terms only

$$\frac{g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}})}{g(z_1, \bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{I}}, z_2)} + \frac{g(\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}})}{g(z_1, \bar{\xi}_{\mathrm{i}})h(\bar{\xi}_{\mathrm{i}}, z_2)} = \frac{h(z_1, z_2)}{h(\bar{\xi}_0, z_2)},$$

where $\bar{\xi}_0 = \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}\}$. Thus,

$$\Psi' = \lambda_2(\bar{z})h(\bar{v}, z_2)h(z_1, z_2) \sum \frac{g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_0)f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})h(\bar{\xi}_{\mathrm{ii}}, z_1)}{h(\bar{\xi}_0, z_2)h(\bar{\eta}_{\mathrm{I}}, z_1)} \mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{ii}}), \tag{6.23}$$

and extracting the product $h(\bar{\xi}, z_1)$ we recast (6.23) as follows

$$\Psi' = \lambda_2(\bar{z})h(\bar{v}, \bar{z})h(\bar{z}, \bar{z}) \sum \frac{g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_0)f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}})}{h(\bar{\xi}_0, \bar{z})h(\bar{\eta}_{\mathrm{I}}, z_1)} \mathbb{B}_{a+1,b}(\bar{\eta}_{\mathrm{II}}; \bar{\xi}_{\mathrm{ii}}).$$

Here the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}\}$, $\bar{\xi} \Rightarrow \{\bar{\xi}_0, \bar{\xi}_{\mathrm{ii}}\}$ with $\#\bar{\eta}_{\mathrm{I}} = 1$, $\#\bar{\xi}_0 = 2$. Comparing this expression with (6.15) we see that

$$\Psi' = T_{22}(z_1)T_{12}(z_2)\mathbb{B}_{a,b}(\bar{u}; \bar{v}).$$

Thus, we find from the recursion (6.16)

$$T_{22}(z_1)\mathbb{B}_{a+1,b}(\{\bar{u},z_2\};\bar{v}) = \frac{T_{22}(z_1)T_{12}(z_2)\mathbb{B}_{a,b}(\bar{u};\bar{v}) - \Psi' + \Psi''}{\lambda_2(z_2)f(\bar{v},z_2)} = \frac{\Psi''}{\lambda_2(z_2)f(\bar{v},z_2)}.$$

Substituting (6.22) in this expression, we arrive at

$$T_{22}(z_1)\mathbb{B}_{a+1,b}(\{\bar{u},z_2\};\bar{v}) = \lambda_2(z_1)h(\bar{v},z_1)\sum \frac{g(\bar{\xi}_{\rm I\hspace{-1pt}I},\bar{\xi}_{\rm I})f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm I\hspace{-1pt}I})}{h(\bar{\xi}_{\rm I},z_1)h(\bar{\eta}_{\rm I},z_1)}\mathbb{B}_{a+1,b}(\bar{\eta}_{\rm I\hspace{-1pt}I};\bar{\xi}_{\rm I\hspace{-1pt}I}),$$

where we have relabeled $\bar{\xi}_{\rm i} \to \bar{\xi}_{\rm I}$ and $\bar{\xi}_{\rm ii} \to \bar{\xi}_{\rm I\hspace{-1pt}I}$. Thus, the induction step is completed.

6.4   Induction over n

Actually, the induction over n for the action of $T_{22}(\bar{z})$ is a combination of the corresponding proofs for the actions of $T_{12}(\bar{z})$ and $T_{23}(\bar{z})$. Assume that (3.5) is valid for some $n-1$. Then

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{n-1}\lambda_2(\bar{z}_n)h(\bar{\xi},\bar{z}_n)$$
$$\times \sum \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm I\hspace{-1pt}I})g(\bar{\xi}_{\rm I\hspace{-1pt}I},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}_n)}K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)T_{22}(z_n)\mathbb{B}_{a,b}(\bar{\eta}_{\rm I\hspace{-1pt}I};\bar{\xi}_{\rm I\hspace{-1pt}I}).$$

Here the sum is taken over partitions $\{\bar{z}_n,\bar{v}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm I\hspace{-1pt}I}\}$ and $\{\bar{z}_n,\bar{u}\} = \bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I},\bar{\eta}_{\rm I\hspace{-1pt}I}\}$ with $\#\bar{\xi}_{\rm I} = \#\bar{\eta}_{\rm I} = n-1$. Acting with $T_{22}(z_n)$ onto $\mathbb{B}_{a,b}(\bar{\eta}_{\rm I\hspace{-1pt}I};\bar{\xi}_{\rm I\hspace{-1pt}I})$ we obtain

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^n\lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z}_n)}{h(z_n,\bar{z}_n)}\sum \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm I\hspace{-1pt}I})g(\bar{\xi}_{\rm I\hspace{-1pt}I},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}_n)}K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)$$
$$\times h(\bar{\xi}_{\rm I\hspace{-1pt}I},z_n)\frac{f(\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})}{h(\bar{\xi}_{\rm i},z_n)}K_1(z_n|\bar{\eta}_{\rm i}+c)\mathbb{B}_{a,b}(\bar{\eta}_{\rm ii};\bar{\xi}_{\rm ii}).$$

Here already $\bar{\xi} = \{\bar{z},\bar{v}\}$ and $\bar{\eta} = \{\bar{z},\bar{u}\}$, and we have additional partitions $\{\bar{\xi}_{\rm I\hspace{-1pt}I},z_n\} \Rightarrow \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ and $\{\bar{\eta}_{\rm I\hspace{-1pt}I},z_n\} \Rightarrow \{\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\}$ with $\#\bar{\xi}_{\rm i} = \#\bar{\eta}_{\rm i} = 1$. Thus, we can say that we have the sum over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I},\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\}$ with restrictions $z_n \notin \bar{\eta}_{\rm I}$ and $z_n \notin \bar{\xi}_{\rm I}$.

Substituting $\bar{\xi}_{\rm I\hspace{-1pt}I} = \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\} \setminus \{z_n\}$, $\bar{\eta}_{\rm I\hspace{-1pt}I} = \{\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\} \setminus \{z_n\}$ and denoting $\bar{\xi}_0 = \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm I}\}$, $\bar{\eta}_0 = \{\bar{\eta}_{\rm i},\bar{\eta}_{\rm I}\}$ we obtain

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^n\lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z})}{h(z_n,\bar{z}_n)}\sum \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm i})f(\bar{\eta}_0,\bar{\eta}_{\rm ii})g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_0)}{f(\bar{\eta}_{\rm I},z_n)g(z_n,\bar{\xi}_{\rm I})h(\bar{\xi}_{\rm I},\bar{z}_n)h(\bar{\xi}_0,z_n)}$$
$$\times K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)K_1(z_n|\bar{\eta}_{\rm i}+c)\mathbb{B}_{a,b}(\bar{\eta}_{\rm ii};\bar{\xi}_{\rm ii}). \tag{6.24}$$

Observe that the restrictions $z_n \notin \bar{\eta}_{\rm I}$ and $z_n \notin \bar{\xi}_{\rm I}$ hold automatically due to the factors $f(\bar{\eta}_{\rm I},z_n)$ and $g(z_n,\bar{\xi}_{\rm I})$ in the denominator of (6.24). Using $K_1(z_n|\bar{\eta}_{\rm i}+c) = -K_1(\bar{\eta}_{\rm i}|z_n)/f(\bar{\eta}_{\rm i},z_n)$ we recast (6.24) in the form

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{n-1}\lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z})}{h(z_n,\bar{z}_n)}\sum \frac{f(\bar{\eta}_0,\bar{\eta}_{\rm ii})g(\bar{\xi}_{\rm ii},\bar{\xi}_0)}{f(\bar{\eta}_0,z_n)g(z_n,\bar{\xi}_0)h(\bar{\xi}_0,\bar{z})}$$
$$\times \left\{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(z_n,\bar{\xi}_{\rm I})h(\bar{\xi}_{\rm I},\bar{z}_n)\right\}\left\{K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)K_1(\bar{\eta}_{\rm i}|z_n)f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm i})\right\}\mathbb{B}_{a,b}(\bar{\eta}_{\rm ii};\bar{\xi}_{\rm ii}).$$

The sums over partitions $\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm I}\}$ and $\bar{\eta}_0 \Rightarrow \{\bar{\eta}_{\rm i},\bar{\eta}_{\rm I}\}$ (see the terms in braces) were already computed (see (5.3) and (5.4)). Thus, we arrive at

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^n\lambda_2(\bar{z})h(\bar{\xi},\bar{z})\sum \frac{f(\bar{\eta}_0,\bar{\eta}_{\rm ii})g(\bar{\xi}_{\rm ii},\bar{\xi}_0)}{h(\bar{\xi}_0,\bar{z})}K_n(\bar{z}|\bar{\eta}_0+c)\mathbb{B}_{a,b}(\bar{\eta}_{\rm ii};\bar{\xi}_{\rm ii}),$$

which ends the proof.

7   Induction over n for the actions of $T_{ij}(\bar{z})$ with $i > j$

The action formulas for all other elements of the monodromy matrix can be proved exactly in the same manner. However, it is clear that the technical difficulty of the proofs increases when moving from the right top corner of the monodromy matrix to the left bottom corner. It is due to the form of the recursion formulas and the commutation relations (2.4). For example, we have seen that for the derivation of the action of $T_{22}(\bar{z})$ one should know the actions of $T_{12}(\bar{z})$ and $T_{23}(\bar{z})$ onto Bethe vectors. The latest are relatively simple. However, one can easily convince oneself that to get the action of $T_{ij}(\bar{z})$ with $i > j$, it is necessary to know the actions of the diagonal elements $T_{ii}(\bar{z})$, which are more involved. Therefore, we omit the detailed proofs of the multiple actions of the operators $T_{11}(\bar{z})$ (3.4), $T_{33}(\bar{z})$ (3.6), and the operators $T_{ij}(\bar{z})$ from the lower-triangular part of the monodromy matrix (3.7)–(3.9). However, as an illustration of the method, we prove the multiple action of the operator $T_{21}(\bar{z})$ assuming that the action of a single operator $T_{21}(z)$ is known.

As previously, the proof goes by induction over $n = \#\bar{z}$. We assume that the action (3.7) holds for some $n - 1$. Then acting successively with $T_{21}(\bar{z}_n)$ and $T_{21}(z_n)$ on $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ we obtain

$$
\begin{aligned}
T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = {} & \lambda_2(\bar{z})\frac{h(\bar{\xi}, \bar{z}_n)}{h(z_n, \bar{z}_n)} \sum r_1(\bar{\eta}_{\mathrm{I}}) \frac{f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{III}})f(\bar{\eta}_{\mathrm{III}}, \bar{\eta}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{II}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{z}_n)f(\bar{\xi}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})} \\
& \times K_{n-1}(\bar{z}_n|\bar{\eta}_{\mathrm{II}} + c)K_{n-1}(\bar{\eta}_{\mathrm{I}}|\bar{\xi}_{\mathrm{I}} + c)h(\bar{\xi}_{\mathrm{II}}, z_n)r_1(\bar{\eta}_{\mathrm{i}}) \frac{f(\bar{\eta}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{i}})f(\bar{\eta}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{iii}})f(\bar{\eta}_{\mathrm{iii}}, \bar{\eta}_{\mathrm{i}})g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})}{h(\bar{\xi}_{\mathrm{i}}, z_n)f(\bar{\xi}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{i}})} \\
& \times K_1(z_n|\bar{\eta}_{\mathrm{ii}} + c)K_1(\bar{\eta}_{\mathrm{i}}|\bar{\xi}_{\mathrm{i}} + c)\mathbb{B}_{a-n,b}(\bar{\eta}_{\mathrm{iii}}; \bar{\xi}_{\mathrm{ii}}).
\end{aligned}
\tag{7.1}
$$

Here $\#\bar{\eta}_{\mathrm{i}} = \#\bar{\eta}_{\mathrm{ii}} = \#\bar{\xi}_{\mathrm{i}} = 1$ and $\#\bar{\eta}_{\mathrm{I}} = \#\bar{\eta}_{\mathrm{II}} = \#\bar{\xi}_{\mathrm{I}} = n - 1$. Originally we have partitions $\{\bar{z}_n, \bar{u}\} = \bar{\eta} \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{III}}\}$ and $\{\bar{z}_n, \bar{v}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{II}}\}$. Then we have additional partitions $\{z_n, \bar{\eta}_{\mathrm{III}}\} \Rightarrow \{\bar{\eta}_{\mathrm{i}}, \bar{\eta}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{iii}}\}$ and $\{z_n, \bar{\xi}_{\mathrm{II}}\} \Rightarrow \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\}$. Thus, in equation (7.1) we have $\{\bar{z}, \bar{u}\} = \bar{\eta}$ and $\{\bar{z}, \bar{v}\} = \bar{\xi}$. Setting there $\bar{\eta}_{\mathrm{III}} = \{\bar{\eta}_{\mathrm{i}}, \bar{\eta}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{iii}}\} \setminus \{z_n\}$ and $\bar{\xi}_{\mathrm{II}} = \{\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{ii}}\} \setminus \{z_n\}$ we arrive at

$$
\begin{aligned}
T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = {} & \lambda_2(\bar{z})\frac{h(\bar{\xi}, \bar{z})}{h(z_n, \bar{z}_n)} \sum r_1(\bar{\eta}_{\mathrm{I}})r_1(\bar{\eta}_{\mathrm{i}}) \frac{f(\bar{\eta}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{i}})f(\bar{\eta}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{iii}})f(\bar{\eta}_{\mathrm{iii}}, \bar{\eta}_{\mathrm{i}})g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{i}})}{h(\bar{\xi}_{\mathrm{i}}, z_n)f(\bar{\xi}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{i}})} \\
& \times \frac{f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{ii}})f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{iii}})f(\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{iii}}, \bar{\eta}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{I}})g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, z_n)h(\bar{\xi}_{\mathrm{I}}, \bar{z}_n)f(\bar{\xi}_{\mathrm{I}}, \bar{\eta}_{\mathrm{I}})f(\bar{\xi}_{\mathrm{ii}}, \bar{\eta}_{\mathrm{I}})f(\bar{\eta}_{\mathrm{II}}, z_n)g(z_n, \bar{\xi}_{\mathrm{I}})} \\
& \times K_{n-1}(\bar{z}_n|\bar{\eta}_{\mathrm{II}} + c)K_1(z_n|\bar{\eta}_{\mathrm{ii}} + c)K_{n-1}(\bar{\eta}_{\mathrm{I}}|\bar{\xi}_{\mathrm{I}} + c)K_1(\bar{\eta}_{\mathrm{i}}|\bar{\xi}_{\mathrm{i}} + c)\mathbb{B}_{a-n,b}(\bar{\eta}_{\mathrm{iii}}; \bar{\xi}_{\mathrm{ii}}).
\end{aligned}
\tag{7.2}
$$

Now we set $\{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{i}}\} = \bar{\eta}_0$, $\{\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{ii}}\} = \bar{\eta}_{0'}$, and $\{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}\} = \bar{\xi}_0$. We also transform $K_1(z_n|\bar{\eta}_{\mathrm{ii}} + c) = -K_1(\bar{\eta}_{\mathrm{ii}}|z_n)/f(\bar{\eta}_{\mathrm{ii}}, z_n)$ and $K_1(\bar{\eta}_{\mathrm{i}}|\bar{\xi}_{\mathrm{i}} + c) = -K_1(\bar{\xi}_{\mathrm{i}}|\bar{\eta}_{\mathrm{i}})/f(\bar{\xi}_{\mathrm{i}}, \bar{\eta}_{\mathrm{i}})$. Then (7.2) takes the form

$$
\begin{aligned}
T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = {} & \lambda_2(\bar{z})\frac{h(\bar{\xi}, \bar{z})}{h(z_n, \bar{z}_n)} \sum \frac{r_1(\bar{\eta}_0)f(\bar{\eta}_{0'}, \bar{\eta}_0)f(\bar{\eta}_{0'}, \bar{\eta}_{\mathrm{iii}})f(\bar{\eta}_{\mathrm{iii}}, \bar{\eta}_0)g(\bar{\xi}_{\mathrm{ii}}, \bar{\xi}_0)g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}})}{h(\bar{\xi}_{\mathrm{I}}, \bar{z})f(\bar{\xi}_{\mathrm{I}}, \bar{\eta}_0)f(\bar{\eta}_{0'}, z_n)f(\bar{\xi}_{\mathrm{ii}}, \bar{\eta}_0)g(z_n, \bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{i}}, z_n)} \\
& \times \left\{ K_{n-1}(\bar{z}_n|\bar{\eta}_{\mathrm{II}} + c)K_1(\bar{\eta}_{\mathrm{ii}}|z_n)f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{ii}}) \right\}\left\{ K_{n-1}(\bar{\eta}_{\mathrm{I}}|\bar{\xi}_{\mathrm{I}} + c)K_1(\bar{\xi}_{\mathrm{i}}|\bar{\eta}_{\mathrm{i}})f(\bar{\eta}_{\mathrm{i}}, \bar{\eta}_{\mathrm{I}}) \right\} \\
& \times \mathbb{B}_{a-n,b}(\bar{\eta}_{\mathrm{iii}}; \bar{\xi}_{\mathrm{ii}}).
\end{aligned}
\tag{7.3}
$$

The sums over partitions in braces can be computed via (A.2):

$$
\sum_{\bar{\eta}_{0'} \Rightarrow \{\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{ii}}\}} K_{n-1}(\bar{z}_n|\bar{\eta}_{\mathrm{II}} + c)K_1(\bar{\eta}_{\mathrm{ii}}|z_n)f(\bar{\eta}_{\mathrm{II}}, \bar{\eta}_{\mathrm{ii}}) = -f(\bar{\eta}_{0'}, z_n)K_n(\bar{z}|\bar{\eta}_{0'} + c),
$$

and

$$
\sum_{\bar{\eta}_0 \Rightarrow \{\bar{\eta}_{\mathrm{I}}, \bar{\eta}_{\mathrm{i}}\}} K_{n-1}(\bar{\eta}_{\mathrm{I}}|\bar{\xi}_{\mathrm{I}} + c)K_1(\bar{\xi}_{\mathrm{i}}|\bar{\eta}_{\mathrm{i}})f(\bar{\eta}_{\mathrm{i}}, \bar{\eta}_{\mathrm{I}}) = (-1)^{n-1}\frac{K_n(\bar{\xi}_0|\bar{\eta}_0)}{f(\bar{\xi}_{\mathrm{I}}, \bar{\eta}_0)}.
$$

Substituting this into (7.3) we arrive at

$$T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z})}{h(z_n,\bar{z}_n)}\sum \frac{r_1(\bar{\eta}_0)f(\bar{\eta}_{0'},\bar{\eta}_0)f(\bar{\eta}_{0'},\bar{\eta}_{\mathrm{iii}})f(\bar{\eta}_{\mathrm{iii}},\bar{\eta}_0)g(\bar{\xi}_{\mathrm{ii}},\bar{\xi}_0)}{h(\bar{\xi}_0,\bar{z})f(\bar{\xi}_{\mathrm{ii}},\bar{\eta}_0)g(z_n,\bar{\xi}_0)}$$
$$\times K_n(\bar{z}|\bar{\eta}_{0'}+c)K_n(\bar{\eta}_0|\bar{\xi}_0+c)\big\{g(\bar{\xi}_{\mathrm{i}},\bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{i}},\bar{z}_n)g(z_n,\bar{\xi}_{\mathrm{i}})\big\}\mathbb{B}_{a-n,b}(\bar{\eta}_{\mathrm{iii}};\bar{\xi}_{\mathrm{ii}}),$$

where we again replaced $K_n(\bar{\xi}_0|\bar{\eta}_0)$ by $K_n(\bar{\eta}_0|\bar{\xi}_0+c)$ via (2.7). The sum over partitions in braces can be calculated via Lemma A.1

$$\sum_{\bar{\xi}_0\Rightarrow\{\bar{\xi}_{\mathrm{I}},\bar{\xi}_{\mathrm{i}}\}} g(\bar{\xi}_{\mathrm{i}},\bar{\xi}_{\mathrm{I}})h(\bar{\xi}_{\mathrm{i}},\bar{z}_n)g(z_n,\bar{\xi}_{\mathrm{i}}) = h(z_n,\bar{z}_n)g(z_n,\bar{\xi}_0).$$

Thus, we finally obtain

$$T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})h(\bar{\xi},\bar{z})\sum \frac{r_1(\bar{\eta}_0)f(\bar{\eta}_{0'},\bar{\eta}_0)f(\bar{\eta}_{0'},\bar{\eta}_{\mathrm{iii}})f(\bar{\eta}_{\mathrm{iii}},\bar{\eta}_0)g(\bar{\xi}_{\mathrm{ii}},\bar{\xi}_0)}{h(\bar{\xi}_0,\bar{z})f(\bar{\xi}_{\mathrm{ii}},\bar{\eta}_0)}$$
$$\times K_n(\bar{z}|\bar{\eta}_{0'}+c)K_n(\bar{\eta}_0|\bar{\xi}_0+c)\mathbb{B}_{a-n,b}(\bar{\eta}_{\mathrm{iii}};\bar{\xi}_{\mathrm{ii}}),$$

which coincides with the original formula up to the labeling of the subsets.

8   Conclusion

In this paper we obtained compact expressions for the multiple actions of the operators T_{ij} onto the Bethe vectors in the models with $\mathfrak{gl}(2|1)$ symmetry. Formally, our method can be extended to higher rank algebras as well, although the technical complexity of the calculations increases significantly with the rank and can be rapidly untractable.

Comparing the actions of T_{ij} onto $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ with the analogous actions in the $\mathfrak{gl}(3)$-based models [6] one can observe that they are quite similar. The main difference is that some of the Izergin determinants in the $\mathfrak{gl}(3)$ actions are replaced with the Cauchy determinants in their $\mathfrak{gl}(2|1)$ analogs. This is the consequence of $\mathbb{Z}_2$ grading that 'trivializes' some of the relations and helps to go further in the calculation. It is well possible that there exist some general formulas, which valid for both cases. It would be important to find such formulas, especially for the models with graded algebras of higher rank.

The obtained formulas allow one to tackle the problem of scalar products of the Bethe vectors. For this we can use explicit expressions for the dual Bethe vectors $\mathbb{C}_{a,b}(\bar{u},\bar{v})$ obtained in [40]. Actually, they can be obtained by a transposition[3] of (2.9), (2.10), for instance,

$$\mathbb{C}_{a,b}(\bar{u},\bar{v}) = (-1)^{b(b-1)/2}\sum g(\bar{v}_{\mathrm{I}},\bar{u}_{\mathrm{I}})\frac{f(\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}})g(\bar{v}_{\mathrm{II}},\bar{v}_{\mathrm{I}})h(\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{I}})}{\lambda_2(\bar{v}_{\mathrm{II}})\lambda_2(\bar{u})f(\bar{v},\bar{u})}\Omega^\dagger\mathbb{T}_{32}(\bar{v}_{\mathrm{II}})T_{21}(\bar{u}_{\mathrm{II}})\mathbb{T}_{31}(\bar{u}_{\mathrm{I}}).$$

Having this representation we reduce the evaluation of the scalar product to the calculation of successive multiple actions of the operators T_{31}, T_{21}, and T_{32} onto Bethe vectors. This will be the subject of our further publication.

A   Identities for rational functions

Lemma A.1. *Let $\bar{w}$, $\bar{u}$ and $\bar{v}$ be sets of complex variables with $\#\bar{u} = m_1$, $\#\bar{v} = m_2$, and $\#\bar{w} = m_1 + m_2$, where m_1 and m_2 are fixed arbitrary integers. Then*

$$\sum g(\bar{w}_{\mathrm{I}},\bar{u})g(\bar{w}_{\mathrm{II}},\bar{v})g(\bar{w}_{\mathrm{II}},\bar{w}_{\mathrm{I}}) = \frac{g(\bar{w},\bar{u})g(\bar{w},\bar{v})}{g(\bar{u},\bar{v})}. \tag{A.1}$$

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[3]More precisely, one should use an antimorphism of the algebra $\mathfrak{gl}(2|1)$ relating T_{ij} and T_{ji} (see [40] for details).

The sum is taken with respect to all partitions of the set $\bar{w}$ into subsets $\bar{w}_{\mathrm{I}}$ and $\bar{w}_{\mathrm{II}}$ with $\#\bar{w}_{\mathrm{I}} = m_1$ and $\#\bar{w}_{\mathrm{II}} = m_2$.

The proof of this lemma is given in [47]. Let us show how Lemma A.1 works. In equation (5.3) we have a sum

$$I = \sum_{\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}\}} g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}) g(z_n, \bar{\xi}_{\mathrm{i}}) h(\bar{\xi}_{\mathrm{i}}, \bar{z}_n),$$

where $\#\bar{\xi}_{\mathrm{i}} = 1$ and $\#\bar{\xi}_{\mathrm{I}} = n - 1$. First, we reduce this sum to the form (A.1) using $h(u, v) = 1/g(u, v - c)$. We have

$$I = -h(\bar{\xi}_0, \bar{z}_n) \sum_{\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}\}} g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}) \frac{g(\bar{\xi}_{\mathrm{i}}, z_n)}{h(\bar{\xi}_{\mathrm{I}}, \bar{z}_n)} = -h(\bar{\xi}_0, \bar{z}_n) \sum_{\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\mathrm{I}}, \bar{\xi}_{\mathrm{i}}\}} g(\bar{\xi}_{\mathrm{i}}, \bar{\xi}_{\mathrm{I}}) g(\bar{\xi}_{\mathrm{i}}, z_n) g(\bar{\xi}_{\mathrm{I}}, \bar{z}_n - c).$$

Now we can directly apply (A.1), and we arrive at

$$I = -h(\bar{\xi}_0, \bar{z}_n) \frac{g(\bar{\xi}_0, z_n) g(\bar{\xi}_0, \bar{z}_n - c)}{g(\bar{z}_n - c, z_n)} = g(z_n, \bar{\xi}_0) h(z_n, \bar{z}_n).$$

Lemma A.2. *Let $\bar{w}$, $\bar{u}$ and $\bar{v}$ be sets of complex variables with $\#\bar{u} = m_1$, $\#\bar{v} = m_2$, and $\#\bar{w} = m_1 + m_2$. Then*

$$\sum K_{m_1}(\bar{w}_{\mathrm{I}} | \bar{u}) K_{m_2}(\bar{v} | \bar{w}_{\mathrm{II}}) f(\bar{w}_{\mathrm{II}}, \bar{w}_{\mathrm{I}}) = (-1)^{m_1} f(\bar{w}, \bar{u}) K_{m_1 + m_2}(\{\bar{u} - c, \bar{v}\} | \bar{w}). \tag{A.2}$$

The sum is taken with respect to all partitions of the set $\bar{w}$ into subsets $\bar{w}_{\mathrm{I}}$ and $\bar{w}_{\mathrm{II}}$ with $\#\bar{w}_{\mathrm{I}} = m_1$ and $\#\bar{w}_{\mathrm{II}} = m_2$.

The proof of this lemma is given in [5].

Acknowledgements

References

[1] Anderson P.W., Singular forward scattering in the $2D$ Hubbard model and a renormalized Bethe ansatz ground state, *Phys. Rev. Lett.* **65** (1990), 2306–2308.

[2] Bedürftig G., Frahm H., Thermodynamics of an integrable model for electrons with correlated hopping, *J. Phys. A: Math. Gen.* **28** (1995), 4453–4468, cond-mat/9504103.

[3] Belliard S., Pakuliak S., Ragoucy E., Universal Bethe ansatz and scalar products of Bethe vectors, *SIGMA* **6** (2010), 094, 22 pages, arXiv:1012.1455.

[4] Belliard S., Pakuliak S., Ragoucy E., Slavnov N.A., The algebraic Bethe ansatz for scalar products in SU(3)-invariant integrable models, *J. Stat. Mech. Theory Exp.* **2012** (2012), P10017, 25 pages, arXiv:1207.0956.

[5] Belliard S., Pakuliak S., Ragoucy E., Slavnov N.A., Highest coefficient of scalar products in SU(3)-invariant integrable models, *J. Stat. Mech. Theory Exp.* **2012** (2012), P09003, 17 pages, arXiv:1206.4931.

[6] Belliard S., Pakuliak S., Ragoucy E., Slavnov N.A., Bethe vectors of GL(3)-invariant integrable models, *J. Stat. Mech. Theory Exp.* **2013** (2013), P02020, 24 pages, arXiv:1210.0768.

[7] Belliard S., Pakuliak S., Ragoucy E., Slavnov N.A., Form factors in SU(3)-invariant integrable models, *J. Stat. Mech. Theory Exp.* **2013** (2013), P04033, 16 pages, arXiv:1211.3968.

[8] Belliard S., Ragoucy E., The nested Bethe ansatz for 'all' closed spin chains, *J. Phys. A: Math. Theor.* **41** (2008), 295202, 33 pages, arXiv:0804.2822.

[9] Bracken A.J., Gould M.D., Links J.R., Zhang Y.-Z., A new supersymmetric and exactly solvable model of correlated electrons, *Phys. Rev. Lett.* **74** (1995), 2768–2771, cond-mat/9410026.

[10] Caux J.-S., Calabrese P., Slavnov N.A., One-particle dynamical correlations in the one-dimensional Bose gas, *J. Stat. Mech. Theory Exp.* **2007** (2007), P01008, 20 pages, cond-mat/0611321.

[11] Caux J.-S., Maillet J.M., Computation of dynamical correlation functions of Heisenberg chains in a magnetic field, *Phys. Rev. Lett.* **95** (2005), 077201, 3 pages, cond-mat/0502365.

[12] Drinfel'd V.G., Hopf algebras and the quantum Yang–Baxter equation, *Soviet Math. Dokl.* **32** (1985), 254–258.

[13] Enriquez B., Khoroshkin S., Pakuliak S., Weight functions and Drinfeld currents, *Comm. Math. Phys.* **276** (2007), 691–725, math.QA/0610398.

[14] Essler F.H.L., Korepin V.E., Higher conservation laws and algebraic Bethe Ansätze for the supersymmetric *t-J* model, *Phys. Rev. B* **46** (1992), 9147–9162.

[15] Faddeev L.D., Sklyanin E.K., Takhtajan L.A., Quantum inverse problem. I, *Theoret. Math. Phys.* **40** (1979), 688–706.

[16] Faddeev L.D., Takhtajan L.A., Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model, *J. Sov. Math.* **24** (1984), 241–267.

[17] Foerster A., Karowski M., Algebraic properties of the Bethe ansatz for an spl(2, 1)-supersymmetric *t-J* model, *Nuclear Phys. B* **396** (1993), 611–638.

[18] Förster D., Staggered spin and statistics in the supersymmetric *t-J* model, *Phys. Rev. Lett.* **63** (1989), 2140–2143.

[19] Frappat L., Khoroshkin S., Pakuliak S., Ragoucy É., Bethe ansatz for the universal weight function, *Ann. Henri Poincaré* **10** (2009), 513–548, arXiv:0810.3135.

[20] Göhmann F., Klümper A., Seel A., Integral representations for correlation functions of the XXZ chain at finite temperature, *J. Phys. A: Math. Gen.* **37** (2004), 7625–7651, hep-th/0405089.

[21] Göhmann F., Klümper A., Seel A., Integral representation of the density matrix of the XXZ chain at finite temperatures, *J. Phys. A: Math. Gen.* **38** (2005), 1833–1841, cond-mat/0412062.

[22] Jimbo M., A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation, *Lett. Math. Phys.* **10** (1985), 63–69.

[23] Khoroshkin S., Pakuliak S., A computation of universal weight function for quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$, *J. Math. Kyoto Univ.* **48** (2008), 277–321, arXiv:0711.2819.

[24] Kitanine N., Kozlowski K., Maillet J.M., Slavnov N.A., Terras V., Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions, *J. Stat. Mech. Theory Exp.* **2009** (2009), P04003, 66 pages, arXiv:0808.0227.

[25] Kitanine N., Kozlowski K., Maillet J.M., Slavnov N.A., Terras V., A form factor approach to the asymptotic behavior of correlation functions, *J. Stat. Mech. Theory Exp.* **2011** (2011), P12010, 27 pages, arXiv:1110.0803.

[26] Kitanine N., Maillet J.M., Slavnov N.A., Terras V., Spin-spin correlation functions of the XXZ-$\frac{1}{2}$ Heisenberg chain in a magnetic field, *Nuclear Phys. B* **641** (2002), 487–518, hep-th/0201045.

[27] Kitanine N., Maillet J.M., Terras V., Form factors of the XXZ Heisenberg spin-$\frac{1}{2}$ finite chain, *Nuclear Phys. B* **554** (1999), 647–678, math-ph/9807020.

[28] Kitanine N., Maillet J.M., Terras V., Correlation functions of the XXZ Heisenberg spin-$\frac{1}{2}$ chain in a magnetic field, *Nuclear Phys. B* **567** (2000), 554–582, math-ph/9907019.

[29] Korepin V.E., Calculation of norms of Bethe wave functions, *Comm. Math. Phys.* **86** (1982), 391–418.

[30] Korepin V.E., Izergin A.G., Essler F.H.L., Uglov D.B., Correlation function of the spin-$\frac{1}{2}$ XXX antiferromagnet, *Phys. Lett. A* **190** (1994), 182–184, cond-mat/9403066.

[31] Kulish P.P., Classical and quantum inverse problem method and generalized Bethe ansatz, *Phys. D* **3** (1981), 246–257.

[32] Kulish P.P., Reshetikhin N.Yu., Diagonalisation of GL(N) invariant transfer matrices and quantum N-wave system (Lee model), *J. Phys. A: Math. Gen.* **16** (1983), L591–L596.

[33] Kulish P.P., Reshetikhin N.Yu., On GL_3-invariant solutions of the Yang–Baxter equation and associated quantum systems, *J. Sov. Math.* **34** (1986), 1948–1971.

[34] Kulish P.P., Sklyanin E.K., Solutions of the Yang–Baxter equation, *J. Sov. Math.* **19** (1982), 1596–1620.

[35] Pakuliak S., Ragoucy E., Slavnov N.A., Scalar products in models with the GL(3) trigonometric R-matrix: general case, *Theoret. Math. Phys.* **180** (2014), 795–814, arXiv:1401.4355.

[36] Pakuliak S., Ragoucy E., Slavnov N.A., Scalar products in models with a GL(3) trigonometric R-matrix: highest coefficient, *Theoret. Math. Phys.* **178** (2014), 314–335, arXiv:1311.3500.

[37] Pakuliak S., Ragoucy E., Slavnov N.A., Form factors of local operators in a one-dimensional two-component Bose gas, *J. Phys. A: Math. Theor.* **48** (2015), 435001, 21 pages, arXiv:1503.00546.

[38] Pakuliak S., Ragoucy E., Slavnov N.A., GL(3)-based quantum integrable composite models. II. Form factors of local operators, *SIGMA* **11** (2015), 064, 18 pages, arXiv:1502.01966.

[39] Pakuliak S., Ragoucy E., Slavnov N.A., Zero modes method and form factors in quantum integrable models, *Nuclear Phys. B* **893** (2015), 459–481, arXiv:1412.6037.

[40] Pakuliak S., Ragoucy E., Slavnov N.A., Bethe vectors for models based on the super-Yangian $Y(\mathfrak{gl}(m|n))$, arXiv:1604.02311.

[41] Pereira R.G., Sirker J., Caux J.-S., Hagemans R., Maillet J.M., White S.R., Affleck I., The dynamical spin structure factor for the anisotropic spin-1/2 Heisenberg chain, *Phys. Rev. Lett.* **96** (2006), 257202, 4 pages, cond-mat/0603681.

[42] Pereira R.G., Sirker J., Caux J.-S., Hagemans R., Maillet J.M., White S.R., Affleck I., Dynamical structure factor at small q for the XXZ spin-1/2 chain, *J. Stat. Mech. Theory Exp.* **2007** (2007), P08022, 64 pages, arXiv:0706.4327.

[43] Schlottmann P., Integrable narrow-band model with possible relevance to heavy fermion systems, *Phys. Rev. B* **36** (1987), 5177–5185.

[44] Seel A., Bhattacharyya T., Göhmann F., Klümper A., A note on the spin-$\frac{1}{2}$ XXZ chain concerning its relation to the Bose gas, *J. Stat. Mech. Theory Exp.* **2007** (2007), P08030, 10 pages, arXiv:0705.3569.

[45] Slavnov N.A., Calculation of scalar products of wave functions and form-factors in the framework of the algebraic Bethe ansatz, *Theoret. Math. Phys.* **79** (1989), 502–508.

[46] Slavnov N.A., Scalar products in GL(3)-based models with trigonometric R-matrix. Determinant representation, *J. Stat. Mech. Theory Exp.* **2015** (2015), P03019, 25 pages, arXiv:1501.06253.

[47] Slavnov N.A., Multiple commutation relations in the models with $\mathfrak{gl}(2|1)$ symmetry, arXiv:1604.05343.

[48] Sutherland B., Further results for the many-body problem in one dimension, *Phys. Rev. Lett.* **20** (1968), 98–100.

[49] Sutherland B., A general model for multicomponent quantum systems, *Phys. Rev. B* **12** (1975), 3795–3805.

[50] Tarasov V., Varchenko A., Combinatorial formulae for nested Bethe vectors, *SIGMA* **9** (2013), 048, 28 pages, math.QA/0702277.

[51] Wheeler M., Multiple integral formulae for the scalar product of on-shell and off-shell Bethe vectors in SU(3)-invariant models, *Nuclear Phys. B* **875** (2013), 186–212, arXiv:1306.0552.

[52] Wheeler M., Scalar products in generalized models with SU(3)-symmetry, *Comm. Math. Phys.* **327** (2014), 737–777, arXiv:1204.2089.

[53] Yang C.N., Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.* **19** (1967), 1312–1315.

[54] Zhang F.C., Rice T.M., Effective Hamiltonian for the superconducting Cu oxides, *Phys. Rev. B* **37** (1988), 3759–3761.