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Generalised Chern–Simons Theory and G_2-Instantons over Associative Fibrations

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Abstract. Adjusting conventional Chern–Simons theory to G_2-manifolds, one describes G_2-instantons on bundles over a certain class of 7-dimensional flat tori which fiber non-trivially over T^4, by a pullback argument. Moreover, if $c_2 \neq 0$, any (generic) deformation of the G_2-structure away from such a fibred structure causes all instantons to vanish. A brief investigation in the general context of (conformally compatible) associative fibrations $f : Y^7 \to X^4$ relates G_2-instantons on pullback bundles $f^*E \to Y$ and self-dual connections on the bundle $E \to X$ over the base, a fact which may be of independent interest.

Key words: Chern–Simons; Yang–Mills; G_2-manifolds; associative fibrations

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1 Introduction

This article fits in the context of gauge theory in higher dimensions, following the seminal works of S. Donaldson & R. Thomas, G. Tian and others [4, 16]. The common thread to such generalisations is the presence of a closed differential form on the base manifold Y, inducing an analogous notion of anti-self-dual connections, or *instantons*, on bundles over Y. In the case at hand, G_2-manifolds are 7-dimensional Riemannian manifolds with holonomy in the Lie group G_2, which implies the existence of precisely such a structure. This allows one to make sense of G_2-*instantons* as the energy-minimising gauge classes of connections, solutions to the corresponding Yang–Mills equation.

Heuristically, G_2-instantons are somewhat analogous to flat connections in dimension 3. Given a bundle over a compact 3-manifold, with space of connections $\mathcal{A}$ and gauge group $\mathcal{G}$, the *Chern–Simons functional* is a multi-valued real function on the quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$, with integer periods, whose critical points are precisely the flat connections [3, § 2.5]. Similar theories can be formulated in higher dimensions in the presence of a suitable closed differential form [4, 15]; e.g. on a G_2-manifold (Y, φ), the coassociative 4-form $*\varphi$ allows for the definition of a functional of Chern–Simons type[1]. Its 'gradient', the Chern–Simons 1-form, vanishes precisely at the G_2-instantons, hence it detects the solutions to the Yang–Mills equation. These gauge-theoretic preliminaries are covered in Section 2.

On the other hand, one may understand G_2-manifolds as a particular case of the rich theory of calibrated geometries [6], for which the G_2-structure φ is a calibration 3-form. Then a 3-dimensional submanifold P is said to be *associative* if it is calibrated by φ, i.e., if $\varphi|_P = d\operatorname{Vol}|_P$. The deformation theory of associative submanifolds is known to be obstructed [9], so their occurrence in families, e.g. fibering over a 4-manifold, is nongeneric and somewhat exotic. Nonetheless, we may consider theoretically, at first, the existence of instantons over associative

[1] In fact only the condition $d*\varphi = 0$ is required, so the discussion extends to cases in which the G_2-structure φ is not necessarily torsion-free.

fibrations $f: Y^7 \to X^4$. Given a bundle $E \to X$, a connection $\mathbf{A}$ on its pullback $\mathbf{E}$ is locally of the form

$$\mathbf{A} \overset{\text{loc}}{=} A_t(x) + \sum_{i=1}^{3} \sigma_i(x,t)dt^i,$$

where $\{A_t\}$ is a family of connections on E parametrised by the associative fibers $P_x := f^{-1}(x)$ and $\sigma_i \in \Omega^0(Y, f^*\mathfrak{g}_E)$. In Section 3.1 I prove the following relation between G_2-instantons and families of self-dual connections over the base:

Theorem 1. *Let $f: Y \to X$ define an associative fibration and $\mathbf{E} \to Y$ be the pullback from an indecomposable vector bundle $E \to X$.*

(*i*) *If a connection $\mathbf{A}$ on $\mathbf{E}$ is a G_2-instanton, then $\{A_t\}$ is a family of self-dual connections on E, satisfying*

$$\frac{\partial A_t}{\partial t^i} = d_{A_t}\sigma_i, \qquad i = 1, 2, 3.$$

(*ii*) *If, moreover, the family $A_t \equiv A_{t_0}$ is constant, then $\mathbf{A} = f^*A_{t_0}$ is a pullback.*

NB: We denote henceforth by $\mathcal{M}_+^4$ the moduli space of SD connections on the base and by $\mathcal{M}_\varphi^7$ the moduli space of G_2-instantons relative to G_2-structure φ.

Finally, over the remaining of Section 3, these ideas are applied to a concrete example of certain T^3-fibrations over T^4, topologically equivalent to the 7-torus, which I will call G_2-*torus fibrations* [11]. Deforming the metric (i.e. the lattice) on T^4 induces a change on the fibration map and hence on the G_2-structure, and one can use Chern–Simons formalism to see how this affects the moduli of G_2-instantons:

Theorem 2. *Let $f: \mathbb{T} \to T^4$ be a G_2-torus fibration, $\mathbf{E} \to \mathbb{T}$ be the pullback of an indecomposable vector bundle $E \to T^4$ and φ denote the G_2-structure of $\mathbb{T}$; then*

(*i*) *every SD connection on E lifts to a G_2-instanton on $\mathbf{E}$, i.e.,*

$$f^*\mathcal{M}_+^4 \subset \mathcal{M}_\varphi^7;$$

(*ii*) *if, moreover, $c_2(E) \neq 0$, then any perturbation $\varphi + \phi$ away from the class of fibred structures causes the moduli space of G_2-instantons to vanish, i.e.,*

$$\mathcal{M}_{\varphi+\phi}^7 = \varnothing.$$

The construction of G_2-instantons is a recent and active research area. Indeed Theorem 2 yields nontrivial, albeit nongeneric, examples of G_2-instanton moduli, whenever a complex vector bundle $E \to T^4$ admits SD connections. The interested reader will find other examples in works of Walpuski, Clarke and the author [2, 11, 12, 13, 17]. In the high-energy physics community, solutions to a very similar problem in the context of G_2-structures with torsion have been found eg. for cylinders over nearly-Kähler homogeneous spaces [5] and more generally for cones over nontrivial manifolds admitting real Killing spinors [7].

Finally, a paper just published by Wang [18] makes significant progress towards a Donaldson theory over higher-dimensional foliations, which seems to encompass our G_2-torus fibration as a special, codimension 4 tight foliation, whose leaf space is the smooth 4-manifold X. It is inspiring to speculate whether an invariant of the corresponding foliated moduli space can be explicitly computed for some suitable bundle $\mathbf{E} \to \mathbb{T}$, or indeed if that space coincides with our definition of $\mathcal{M}^7$.

2 Gauge theory over G$_2$-manifolds

I will concisely recall the essentials of gauge theory on G$_2$-manifolds, while referring the interested reader to a more detailed exposition in [12].

Let Y be an oriented smooth 7-manifold; a G$_2$-*structure* is a smooth 3-form $\varphi \in \Omega^3(Y)$ such that, at every point $p \in Y$, one has $\varphi_p = f_p^*(\varphi_0)$ for some frame $f_p : T_pY \to \mathbb{R}^7$ and (adopting the conventions of [14])

$$\varphi_0 = e^{567} + \omega_1 \wedge e^5 + \omega_2 \wedge e^6 + \omega_3 \wedge e^7 \tag{1}$$

with

$$\omega_1 = e^{12} - e^{34}, \qquad \omega_2 = e^{13} - e^{42} \qquad \text{and} \qquad \omega_3 = e^{14} - e^{23}.$$

Moreover, φ determines a Riemannian metric $g(\varphi)$ induced by the pointwise inner-product

$$\langle u, v\rangle e^{1\dots7} := \frac{1}{6}(u \lrcorner \varphi_0) \wedge (v \lrcorner \varphi_0) \wedge \varphi_0, \tag{2}$$

under which $*_\varphi \varphi$ is given pointwise by

$$*\varphi_0 = e^{1234} - \omega_1 \wedge e^{67} - \omega_2 \wedge e^{75} - \omega_3 \wedge e^{56}. \tag{3}$$

Such a pair (Y, φ) is a G$_2$-*manifold* if $d\varphi = 0$ and $d *_\varphi \varphi = 0$.

2.1 The G$_2$-instanton equation

The G$_2$-structure allows for a 7-dimensional analogue of conventional Yang–Mills theory, yielding a notion of (anti-)self-duality for 2-forms. Under the usual identification between 2-forms and matrices, we have $\mathfrak{g}_2 \subset \mathfrak{so}(7) \simeq \Lambda^2$, so we denote $\Lambda^2_{14} := \mathfrak{g}_2$ and Λ^2_7 its orthogonal complement in Λ^2:

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}. \tag{4}$$

It is easy to check that $\Lambda^2_7 = \langle e_1 \lrcorner \varphi_0, \dots, e_7 \lrcorner \varphi_0\rangle$, hence the orthogonal projection onto Λ^2_7 in (4) is given by

$$\begin{aligned} L_{*\varphi_0} :\ & \Lambda^2 \to \Lambda^6, \\ & \eta \mapsto \eta \wedge *\varphi_0 \end{aligned}$$

in the sense that [1, p. 541]

$$L_{*\varphi_0}|_{\Lambda^2_7} :\ \Lambda^2_7 \xrightarrow{\sim} \Lambda^6 \qquad \text{and} \qquad L_{*\varphi_0}|_{\Lambda^2_{14}} = 0.$$

Furthermore, since (4) splits Λ^2 into irreducible representations of G$_2$, a little inspection on generators reveals that $(\Lambda^2)_{\substack{7\\14}}$ is respectively the $\substack{-2\\+1}$-eigenspace of the G$_2$-equivariant linear map

$$\begin{aligned} T_{\varphi_0} :\ & \Lambda^2 \to \Lambda^2, \\ & \eta \mapsto T_{\varphi_0}\eta := *(\eta \wedge \varphi_0). \end{aligned}$$

Consider now a vector bundle $E \to Y$ over a compact G$_2$-manifold (Y, φ); the curvature $F := F_A$ of some connection A decomposes according to the splitting (4):

$$F_A = F_7 \oplus F_{14}, \qquad F_i \in \Omega^2_i(\operatorname{End} E), \qquad i = 7, 14.$$

The L^2-norm of F_A is the *Yang–Mills functional*:

$$\mathrm{YM}(A) := \|F_A\|^2 = \|F_7\|^2 + \|F_{14}\|^2. \tag{5}$$

It is well-known that the values of $\mathrm{YM}(A)$ can be related to a certain characteristic class of the bundle E, given (up to choice of orientation) by

$$\kappa(E) := -\int_Y \mathrm{tr}\left(F_A^2\right) \wedge \varphi.$$

Using the property $d\varphi = 0$, a standard argument of Chern–Weil theory [10] shows that the de Rham class $[\mathrm{tr}(F_A^2)\wedge\varphi]$ is independent of A, thus the integral is indeed a topological invariant. The eigenspace decomposition of T_φ implies (up to a sign)

$$\kappa(E) = -2\|F_7\|^2 + \|F_{14}\|^2,$$

and combining with (5) we get

$$\mathrm{YM}(A) = -\frac{1}{2}\kappa(E) + \frac{3}{2}\|F_{14}\|^2 = \kappa(E) + 3\|F_7\|^2.$$

Hence $\mathrm{YM}(A)$ attains its absolute minimum at a connection whose curvature lies either in Λ^2_7 or in Λ^2_{14}. Moreover, since $\mathrm{YM} \geq 0$, the sign of $\kappa(E)$ obstructs the existence of one type or the other, so we fix $\kappa(E) \geq 0$ and define G_2-*instantons* as connections with $F \in \Lambda^2_{14}$, i.e., such that $\mathrm{YM}(A) = \kappa(E)$. These are precisely the solutions of the G_2-*instanton equation*:

$$F_A \wedge *\varphi = 0 \tag{6a}$$

or, equivalently,

$$F_A - *(F_A \wedge \varphi) = 0. \tag{6b}$$

If instead $\kappa(E) \leq 0$, we may still reverse orientation and consider $F \in \Lambda^2_{14}$, but then the above eigenvalues and energy bounds must be adjusted accordingly, which amounts to a change of the $(-)$ sign in (6b).

2.2 Definition of the Chern–Simons functional ϑ

Gauge theory in higher dimensions can be formulated in terms of the geometric structure of manifolds with exceptional holonomy [4]. In particular, instantons can be characterised as critical points of a Chern–Simons functional, hence zeroes of its gradient 1-form [3]. The explicit case of G_2-manifolds, which we now describe, was first examined in the author's thesis [11].

Let $E \to Y$ be a vector bundle; the space $\mathcal{A}$ is an affine space modelled on $\Omega^1(\mathfrak{g}_E)$ so, fixing a reference connection $A_0 \in \mathcal{A}$,

$$\mathcal{A} = A_0 + \Omega^1(\mathfrak{g}_E)$$

and, accordingly, vectors at $A \in \mathcal{A}$ are 1-forms $a, b, \ldots \in T_A\mathcal{A} \simeq \Omega^1(\mathfrak{g}_E)$ and vector fields are maps $\alpha, \beta, \ldots : \mathcal{A} \to \Omega^1(\mathfrak{g}_E)$. In this notation we define the *Chern–Simons functional* by

$$\vartheta(A) := \tfrac{1}{2}\int_Y \mathrm{tr}\left(d_{A_0}a \wedge a + \frac{2}{3}a \wedge a \wedge a\right) \wedge *\varphi,$$

fixing $\vartheta(A_0) = 0$. This function is obtained by integration of the *Chern–Simons 1-form*

$$\rho(\beta)_A = \rho_A(\beta_A) := \int_Y \mathrm{tr}(F_A \wedge \beta_A) \wedge *\varphi. \tag{7}$$

We find ϑ explicitly by integrating ρ over paths $A(t)=A_0+ta$, from A_0 to any $A=A_0+a$:

$$\begin{aligned}\vartheta(A)-\vartheta(A_0) &= \int_0^1 \rho_{A(t)}(\dot{A}(t))dt = \int_0^1 \left(\int_Y \operatorname{tr}\left(\left(F_{A_0}+td_{A_0}a+t^2 a\wedge a\right)\wedge a\right)\wedge *\varphi\right)dt \\ &= \frac{1}{2}\int_Y \operatorname{tr}\left(d_{A_0}a\wedge a+\frac{2}{3}a\wedge a\wedge a\right)\wedge *\varphi + K,\end{aligned}$$

where $K=K(A_0,a)$ is a constant and vanishes if A_0 is an instanton.

The co-closedness condition $d*\varphi=0$ implies that the 1-form (7) is closed, so the procedure doesn't depend on the path $A(t)$. Indeed, given tangent vectors $a,b\in\Omega^1(\mathfrak{g}_E)$ at A, the leading term in the expansion of ρ,

$$\rho_{A+a}(b)-\rho_A(b)=\int_Y \operatorname{tr}(d_A a\wedge b)\wedge *\varphi + O\big(|b|^2\big),$$

is symmetric by Stokes' theorem:

$$\int_Y \operatorname{tr}(d_A a\wedge b - a\wedge d_A b)\wedge *\varphi = \int_Y d(\operatorname{tr}(b\wedge a)\wedge *\varphi)=0.$$

We conclude that

$$\rho_{A+a}(b)-\rho_A(b)=\rho_{A+b}(a)-\rho_A(a)+O\big(|b|^2\big)$$

and, comparing reciprocal Lie derivatives on parallel vector fields $\alpha\equiv a$, $\beta\equiv b$ near a point A, we have:

$$\begin{aligned}d\rho(\alpha,\beta)_A &= (\mathcal{L}_b\rho)_A(a)-(\mathcal{L}_a\rho)_A(b)=\lim_{h\to 0}\frac{1}{h}\big\{\rho_{A+hb}(a)-\rho_A(a)\big)-\big(\rho_{A+ha}(b)-\rho_A(b)\big)\big\} \\ &= \lim_{h\to 0}\frac{1}{h^2}\underbrace{\big\{\big(\rho_{A+hb}(ha)-\rho_A(ha)\big)-\big(\rho_{A+ha}(hb)-\rho_A(hb)\big)\big\}}_{O\left(|h|^3\right)}=0.\end{aligned}$$

Since $\mathcal{A}$ is contractible, by the Poincaré lemma ρ is the derivative of some function ϑ. Again by Stokes, ρ vanishes along $\mathcal{G}$-orbits $\operatorname{im}(d_A)\simeq T_A\{\mathcal{G}.A\}$. Thus ρ descends to the quotient $\mathcal{B}$ and so does ϑ, locally.

2.3 Periodicity of ϑ

Consider the gauge action of $g\in\mathcal{G}$ and some path $\{A(t)\}_{t\in[0,1]}\subset\mathcal{A}$ connecting an instanton A to $g.A$. The natural projection $p_1: Y\times[0,1]\to Y$ induces a bundle

$$\begin{array}{ccc}\mathbf{E}_g & \xrightarrow{\tilde{p_1}} & E \\ \downarrow & & \downarrow \\ Y\times[0,1] & \xrightarrow{p_1} & Y\end{array}$$

and, using g to identify the fibres $(\mathbf{E}_g)_0 \overset{g}{\simeq} (\mathbf{E}_g)_1$, one may think of $\mathbf{E}_g$ as a bundle over $Y\times S^1$. Moreover, in some local trivialisation, the path $A(t)=A_i(t)dx^i$ gives a connection $\mathbf{A}=\mathbf{A}_0 dt+\mathbf{A}_i dx^i$ on $\mathbf{E}_g$:

$$(\mathbf{A}_0)_{(t,p)}=0, \qquad (\mathbf{A}_i)_{(t,p)}=A_i(t)_p.$$

The corresponding curvature 2-form is $F_{\mathbf{A}}=(F_{\mathbf{A}})_{i0}dx^i\wedge dt+(F_{\mathbf{A}})_{jk}dx^j\wedge dx^k$, where

$$(F_{\mathbf{A}})_{i0}=\dot{A}_i(t), \qquad (F_{\mathbf{A}})_{jk}=(F_A)_{jk}.$$

The periods of ϑ are then of the form

$$\begin{aligned}\vartheta(g.A)-\vartheta(A) &= \int_0^1 \rho_{A(t)}(\dot{A}(t))dt = \int_{Y\times[0,1]} \operatorname{tr}(F_{A(t)}\wedge \dot{A}_i(t)dx^i)\wedge dt\wedge *\varphi \\ &= \int_{Y\times S^1} \operatorname{tr} F_{\mathbf{A}}\wedge F_{\mathbf{A}}\wedge *\varphi = \frac{1}{8\pi^2}\big\langle c_2(\mathbf{E}_g)\smile[*\varphi], Y\times S^1\big\rangle.\end{aligned}$$

The Künneth formula for $Y\times S^1$ gives

$$H^4\big(Y\times S^1,\mathbb{R}\big) = H^4(Y,\mathbb{R})\oplus H^3(Y,\mathbb{R})\otimes \underbrace{H^1\big(S^1,\mathbb{R}\big)}_{\mathbb{Z}}$$

and obviously $H^4(Y)\smile[*\varphi]=0$ so, denoting by $c_2'(\mathbf{E}_g)$ the component lying in $H^3(Y)$ and by $S_g := [\frac{1}{8\pi^2}c_2'(\mathbf{E}_g)]^{PD}$ its normalised Poincaré dual, we are left with

$$\vartheta(g.A)-\vartheta(A) = \langle[*\varphi], S_g\rangle.$$

Consequently, the periods of ϑ lie in the set

$$\left\{\int_{S_g} *\varphi \,\middle|\, S_g\in H_4(Y,\mathbb{R})\right\}.$$

That may seem odd at first, because $*\varphi$ is not, in general, an integral class and so the set of periods is *dense*. However, as long as our interest remains in the study of the moduli space $\mathcal{M} = \operatorname{Crit}(\rho)$ of G_2-instantons, there is not much to worry, for the gradient $\rho = d\vartheta$ is unambiguously defined on $\mathcal{B}$.

3 Instantons over G_2-torus fibrations

Instances of G_2-manifolds fibred by associative submanifolds in the literature are relatively scarce, not least because their deformation theory is zero-index elliptic [9] and therefore any new examples will be somewhat exotic. A few trivial cases include the products $T^7 = T^4\times T^3$ and $K3\times T^3$ and also $CY^3\times S^1$ given a family of curves in the Calabi–Yau [8, § 10.8]. The example I will propose is unique in the sense that the total space is not a Riemannian product.

3.1 Instantons over associative fibrations

We consider pullback bundles over smooth associative fibrations, and relate G_2-instantons to their gauge theory over the base; in particular we do not address the possibility of singular fibres.

Definition 1. A G_2-manifold (Y^7,φ) is called an *associative fibration* over a compact oriented Riemannian four-manifold (X^4,η) if it is the total space of a Riemannian submersion $f: Y\to X$ such that each fibre $P_x := f^{-1}(x)\subset Y$ is a smooth associative submanifold.

Since each fibre P_x is 3-dimensional and orientable, its tangent bundle is differentiably trivial and we may choose global coordinates $t=(t^1,t^2,t^3)$ induced respectively by a global coframe $\{e_5,e_6,e_7\} := \{dt^1,dt^2,dt^3\}$. Thus near each $y\in P_x$ we may complete the triplet into a local orthogonal coframe $\{e_1,\dots,e_7\}$ of T^*Y such that φ_y has the form (1), and the point y is unambiguously described by $(x,t(y))$.

Lemma 1. *Let $f: Y\to X$ define an associative fibration and $\mathbf{E}\to Y$ be the pullback from a vector bundle $E\to X$; then a connection A on E is self-dual if, and only if, f^*A is a* G_2-*instanton on* $\mathbf{E}$.

Proof. Let $F := (F_{f^*A})_y$ be the curvature 2-form at $y \in P_x$; then

$$*_\varphi(F \wedge \varphi) \overset{\text{loc}}{=} *_\varphi\big[F \wedge \big(\varphi|_{P_x} + \omega_1 \wedge e^5 + \omega_2 \wedge e^6 + \omega_3 \wedge e^7\big)\big] = *_\eta F + *_\varphi\big[O(F^-) \wedge f^* d\,\mathrm{Vol}_\eta\,\big],$$

where $O(F^-) := (F_{34} - F_{12})e^5 + (F_{42} - F_{13})e^6 + (F_{23} - F_{14})e^7$ vanishes precisely when A is self-dual, i.e., when $F = *_\eta F$ satisfies the G_2-instanton equation (6b). ■

We are now in position to prove Theorem 1. Let us examine the general form of a G_2-instanton on $\mathbf{E}$. An arbitrary connection $\mathbf{A}$ on $\mathbf{E}$ is locally of the form

$$\mathbf{A}(y) \overset{\text{loc}}{=} A_t(x) + \sum_{i=1}^{3} \sigma_i(x,t) dt^i,$$

where $\{A_t\}_{t \in t(P_x)}$ is a family of connections on E and $\sigma_i \in \Omega^0(Y, f^*\mathfrak{g}_E)$. The curvature of $\mathbf{A}$ is

$$F_{\mathbf{A}} = F_{A_t} + \sum_{i=1}^{3} \left(d_{A_t}\sigma_i - \frac{\partial A_t}{\partial t^i} \right) \wedge dt^i + F_\sigma$$

with

$$F_\sigma := \sum_{i,j=1}^{3} \left(\frac{\partial \sigma_i}{\partial t^j} - \frac{\partial \sigma_j}{\partial t^i} + \frac{1}{2}[\sigma_i, \sigma_j] \right) dt^i \wedge dt^j.$$

Replacing $F_{\mathbf{A}}$ into the G_2-instanton equation (6a) and using the expression (3) of $*\varphi$ in the natural frame $\{e_1, \dots, e_7\}$, we have

$$\left(F_{A_t} + \sum_{i=1}^{3} (d_{A_t}\sigma_i - \frac{\partial A_t}{\partial t^i}) \wedge e^{4+i} + F_\sigma \right) \wedge \big(e^{1234} - \omega_1 \wedge e^{67} - \omega_2 \wedge e^{75} - \omega_3 \wedge e^{56}\big) = 0.$$

Using the following elementary properties

$$\begin{aligned}
&F_{A_t} \wedge e^{1234} = 0, \qquad F_{A_t} \wedge \omega_1 \wedge e^{67} = [(F_{A_t})_{34} - (F_{A_t})_{12}]\big(*e^5\big), \\
&F_{A_t} \wedge \omega_2 \wedge e^{75} = [(F_{A_t})_{42} - (F_{A_t})_{13}]\big(*e^6\big), \qquad F_{A_t} \wedge \omega_3 \wedge e^{56} = [(F_{A_t})_{23} - (F_{A_t})_{14}]\big(*e^7\big), \\
&F_\sigma \wedge e^{4+i} \wedge e^{4+j} = 0, \qquad F_\sigma \wedge e^{1234} = (F_\sigma)_{23}\big(*e^5\big) + (F_\sigma)_{31}\big(*e^6\big) + (F_\sigma)_{12}\big(*e^7\big),
\end{aligned}$$

and the fact that each $d_{A_t}\sigma_i$ and $\frac{\partial A_t}{\partial t^i}$ are locally 1-forms on the base, hence their wedge product with $e^{1234} = d\,\mathrm{Vol}_\eta$ vanishes, the equation simplifies to

$$\sum_{i=1}^{3} \left(d_{A_t}\sigma_i - \frac{\partial A_t}{\partial t^i} \right) \wedge \omega_i = 0 \qquad \text{and} \qquad F_{A_t}^- - Q(F_\sigma) = 0,$$

where Q is the linear map on 2-forms defined by

$$Q\big(dt^i \wedge dt^j\big) = Q\big(e^{4+i} \wedge e^{4+j}\big) := \sum_{k=1}^{3} \epsilon^{ijk}\omega_k.$$

On the other hand, if $\mathbf{A} = A_t + \sum \sigma_i$ is a G_2-instanton, then it minimises the Yang–Mills functional (5). This implies

$$\sum \left\| d_{A_t}\sigma_i - \frac{\partial A_t}{\partial t^i} \right\|^2 + \|F_\sigma\|^2 = 0,$$

since otherwise the pullback component A_t alone would violate the minimum energy:

$$\mathrm{YM}(A_t) = \|F_{A_t}\|^2 < \|F_{\mathbf{A}}\|^2 = \mathrm{YM}(\mathbf{A}).$$

In particular $F_\sigma \equiv 0$ and so every A_t must be SD. Finally, if the family $A_t \equiv A_{t_0}$ is constant, then $d_{A_{t_0}}\sigma_i = 0$ implies $\sigma \equiv 0$, since by assumption $\mathbf{E}$ is indecomposable and therefore does not admit nontrivial parallel sections. This concludes the proof of Theorem 1.

Remark 1. If $\mathcal{M}_+^4$ is discrete, then by continuity the family $\{A_t\}$ is contained in a gauge orbit; if the family is constant, then $\mathbf{A}$ is a pullback.

3.2 G_2-torus fibrations

A 7-torus $T^7 = \mathbb{R}^7/\Lambda$ naturally inherits the G_2-structure φ from $\mathbb{R}^7$. Recall from Section 2.2 that a connection A on some bundle over T^7 is a G_2-instanton if and only if it is a zero of the Chern–Simons 1-form (7):

$$\rho_A(b) = \int_{T^7} \mathrm{tr}(F_A \wedge b) \wedge *\varphi. \tag{8}$$

One asks what is the behaviour of the moduli space of G_2-instantons under perturbations $\varphi \to \varphi + \phi$ of the G_2-structure. More precisely, given suitable assumptions, one asks whether $(\varphi + \phi)$-instantons exist at all once we deform the lattice. As a working example, we consider the following class of flat T^3-fibred 7-tori:

Definition 2. A G_2-*torus fibration structure* is a triplet (η, L, α) in which:

- η is a metric on $\mathbb{R}^4$;
- L is a lattice on the subspace $\Lambda_+^2(\mathbb{R}^4, \eta)$ of η-self-dual 2-forms;
- $\alpha : \mathbb{R}^4 \to \Lambda_+^2(\mathbb{R}^4, \eta)$ is a linear map.

Given the above data, set $V \doteq \mathbb{R}^4 \oplus \Lambda_+^2$ and form the torus $\mathbb{T} = V/\tilde{L}$, with the lattice

$$\tilde{L} \doteq \left\{(\mu, \nu + \alpha\mu) \,|\, \mu \in \mathbb{Z}^4,\, \nu \in L\right\} \subset V.$$

Then $\mathbb{T}$ inherits from V the G_2-structure φ which makes the generators of $\tilde{L}$ orthonormal with respect to the induced inner-product (2). It is straightforward to check that $\mathbb{T}$ is an associative fibration as in Definition 1: denoting by e^5, e^6, e^7 the $(\nu+\alpha\mu)$-orthonormal basis of the fibre Λ_+^2, the flat G_2-structure (1) simplifies to $\varphi|_{\Lambda_+^2} = e^{567} = d\,\mathrm{Vol}_\varphi|_{\Lambda_+^2}$; moreover the lattice $\tilde{L}$ on every tangent subspace normal to the fibre is just the lattice μ from the base, so the corresponding metrics are the same. Although $\mathbb{T}$ fibres over the 4-torus $\mathbb{R}^4/\mu$, the induced metric $g(\varphi)$ is *not*, in general, a Riemannian product.

Suppose the moduli space $\mathcal{M}_+^4$ of self-dual connections on $E \to T^4$ is nonempty; then we have trivial solutions to the G_2-instanton equation on the pullback $\mathbf{E} \to \mathbb{T}$ simply by lifting $\mathcal{M}_+^4$ as in Lemma 1, which proves the first part of Theorem 2:

Corollary 1. *If A is a self-dual connection on $E \to T^4$, then its pullback f^*A by the fibration map $f : \mathbb{T} \to T^4$ is a G_2-instanton on $\mathbf{E}$.*

For future reference, I denote the set of such φ-instantons obtained by lifts from $\mathcal{M}_+^4$ by

$$\widetilde{\mathcal{M}_+^4} := f^*\mathcal{M}_+^4 \subset \mathcal{B}^7. \tag{9}$$

We know from 4-dimensional gauge theory that SD connections on a complex vector bundle $E \to T^4$ correspond to stable holomorphic structures on E, thus in such cases we have examples of G_2-instantons on bundles over $\mathbb{T}$.

3.3 Deformations of $\mathbb{T}$

Working on a bundle $\mathbf{E} \to \mathbb{T}$ with compact structure group over a fixed G_2-torus fibration, let us ponder in generality about the behaviour of instantons under a deformation of the G_2-structure:

$$\varphi \to \varphi + \phi, \qquad *_\varphi\varphi \to *_\varphi\varphi + \xi_\phi, \qquad \xi_\phi := *_{\varphi+\phi}(\varphi+\phi) - *_\varphi\varphi \in \Omega^4(\mathbb{T}).$$

An arbitrary deformation ϕ does not in general preserve the fibred structure of $\mathbb{T}$:

Proposition 1. *A deformation $\xi_\phi \in \Lambda^4(\mathbb{T})$ of the coassociative 4-form $*_\varphi\varphi$ has four orthogonal components, with the following significance:*

$$\Lambda^4\big(\mathbb{R}^4 \oplus \Lambda^2_+\big) = \underbrace{\Lambda^4\big(\mathbb{R}^4\big)}_{\text{(I)}} \oplus \underbrace{\Lambda^3\big(\mathbb{R}^4\big) \otimes \Lambda^1\big(\Lambda^2_+\big)}_{\text{(II)}} \oplus \underbrace{\Lambda^2\big(\mathbb{R}^4\big) \otimes \Lambda^2\big(\Lambda^2_+\big)}_{\text{(III)}} \oplus \underbrace{\Lambda^1\big(\mathbb{R}^4\big) \otimes \Lambda^3\big(\Lambda^2_+\big)}_{\text{(IV)}},$$

(I) *corresponds to a rescaling of the metric η on $\mathbb{R}^4$;*

(II) *redefines the map α;*

(III) *splits as $\mathrm{Hom}(\Lambda^2_+, \Lambda^2_+) \oplus \mathrm{Hom}(\Lambda^2_-, \Lambda^2_+)$, where the first factor modifies the lattice L and the second one affects the conformal class of η;*

(IV) *parametrises deformations transverse to the fibred structures.*

Proof. Let us examine the four cases.

(I) If $\xi_\phi \in \Lambda^4(\mathbb{R}^4) \simeq \mathbb{R}$, then it must be a multiple of $*\varphi|_{\mathbb{R}^4} = e^{1234} = d\,\mathrm{Vol}_\eta$.

(II) Since $\Lambda^3(\mathbb{R}^4) \otimes \Lambda^1(\Lambda^2_+) \simeq \mathbb{R}^4 \otimes (\Lambda^2_+)^* \simeq \mathrm{Hom}(\mathbb{R}^4, \Lambda^2_+)$, such deformations are precisely linear maps $\mathbb{R}^4 \to \Lambda^2_+$.

(III) Clearly $\Lambda^2(\mathbb{R}^4) = \Lambda^2_+ \oplus \Lambda^2_-$ and $\Lambda^2(\Lambda^2_+) \simeq (\Lambda^2_+)^*$, so the product decomposes as

$$\big(\Lambda^2_+ \otimes \big(\Lambda^2_+\big)^*\big) \oplus \big(\Lambda^2_- \otimes \big(\Lambda^2_+\big)^*\big) \simeq \mathrm{Hom}\,\big(\Lambda^2_+, \Lambda^2_+\big) \oplus \mathrm{Hom}\,\big(\Lambda^2_-, \Lambda^2_+\big).$$

Now, on one hand, acting with an endomorphism on Λ^2_+ is equivalent to redefining the triplet $\{e^5, e^6, e^7\}$, hence the lattice $L \subset \Lambda^2_+$. On the other hand, since the orthogonal split $\Lambda^2 = \Lambda^2_- \oplus \Lambda^2_+$ is conformally invariant, a map $\Lambda^2_- \to \Lambda^2_+$ redefines the orthogonal complement of Λ^2_- and hence the conformal class.

(IV) Since $\Lambda^3(\Lambda^2_+) \simeq \mathbb{R}$, this component is just $\Lambda^1(\mathbb{R}^4)$, which is irreducible in the sense that $\mathbb{T}$ has no distinguished subspaces in $\mathbb{R}^4$. Then either every 7-torus is a G_2-fibration, which is obviously false, or these are precisely the deformations away from said structures. ∎

We will now describe what happens to the zeroes of (8) under the corresponding perturbation of the Chern–Simons 1-form:

$$\rho \to \rho_\phi := \rho + r_\phi, \qquad (r_\phi)_A(b) = \int_{\mathbb{T}} \mathrm{tr}(F_A \wedge b) \wedge \xi_\phi.$$

Clearly a φ-instanton A is also a $(\varphi+\phi)$-instanton if and only if $(r_\phi)_A \equiv 0$. There is little reason, however, to expect such a coincidence; as we will see, the topology of the bundle may constrain the existence of instantons under certain – indeed most – deformations.

Denoting henceforth by $\mathcal{A}$ the space of connections over the 7-manifold $\mathbb{T}$, let us briefly digress into the translation action of some vector $v \in \mathbb{T}$ on some $A \in \mathcal{A}$. The first order variation is given by the bundle-valued 1-form

$$(\beta_v)_A := v \lrcorner F_A,$$

which we interpret as a vector in $T_A\mathcal{A}$. Notice first that in the direction β_v the value of the Chern–Simons 1-form is independent of the base-point:

Lemma 2. *The function $\rho(\beta_v) : \mathcal{A} \to \mathbb{R}$ is constant.*

Proof. The computation is straightforward:

$$\begin{aligned}\rho(\beta_v)_{A+ha} &= \int_{\mathbb{T}} \operatorname{tr} F_{A+ha} \wedge v\lrcorner F_{A+ha} \wedge *\varphi = -\frac{1}{2}\int_{\mathbb{T}} \operatorname{tr} F_{A+ha} \wedge F_{A+ha} \wedge (v\lrcorner *\varphi) \\ &= -\frac{1}{2}\int_{\mathbb{T}} (\operatorname{tr} F_A \wedge F_A + d\chi) \wedge (v\lrcorner *\varphi) = -\frac{1}{2}\int_{\mathbb{T}} \operatorname{tr} F_A \wedge F_A \wedge (v\lrcorner *\varphi) = \rho(\beta_v)_A,\end{aligned}$$

where $d\chi$ is the exact differential given by Chern–Weil theory and we use Stokes' theorem and Cartan's identity $d(v\lrcorner *\varphi) = \mathcal{L}_v(*\varphi) = 0$, since φ is constant on the flat torus. ■

Similarly, evaluating r_ϕ on β_v gives

$$r_\phi(\beta_v)_A = \int_{\mathbb{T}} \operatorname{tr}(F_A \wedge (\beta_v)_A) \wedge \xi_\phi = -\tfrac{1}{2}\int_{\mathbb{T}} \operatorname{tr}(F_A \wedge F_A) \wedge (v\lrcorner\xi_\phi) = \langle c_2(E), S_\phi(v)\rangle,$$

where $S_\phi(v) \doteq -\frac{1}{2}[v\lrcorner\xi_\phi]^{PD}$, and this depends only on the topology of E, not on the point A.

Remark 2. Hence we may interpret ϕ as defining a linear functional

$$\begin{aligned} N_\phi :\ & \mathbb{R}^7 \to \mathbb{R}, \\ & v \mapsto \langle c_2(E), S_\phi(v)\rangle, \end{aligned}$$

such that $N_\phi \neq 0$ implies no φ-instanton is still a $(\varphi+\phi)$-instanton. This is, however, a rather weak obstruction, since the map $\phi \mapsto N_\phi$ has kernel of dimension at least 28 and thus, in principle, leaves plenty of possibilities for instantons of perturbed G_2-structures.

Now consider specifically a translation vector on the base $v \in T^4$. Notice that for deformations ϕ of types (I), (II) or (III) the contraction of ξ_ϕ with such v gives $S_\phi(v) = 0$, so ϕ only effectively contributes to the function $\rho(\beta_v)$ when $\xi_\phi \in \Lambda^1(\mathbb{R}^4)$, which means the perturbed torus is no longer a fibred structure (Proposition 1). Moreover, either the bundle E is flat and β_v vanishes identically, or $c_2(E) \neq 0$ and the following holds:

Lemma 3. *If $c_2(E) \neq 0$ and ϕ is of type (IV), then there exists $v \in T^4$ such that $r_\phi(\beta_v)$ is a non-zero constant.*

Proof. Denoting T^3 the typical fibre of f (and setting $\mathrm{Vol}(T^3) = 1$), we may assume

$$\xi_\phi = -2\varepsilon \wedge d\,\mathrm{Vol}_{T^3}$$

for some $0 \neq \varepsilon \in \Lambda^1(T^4)$. One can always choose $v \in T^4$ such that $\varepsilon(v) \neq 0$, and consider $(\beta_v)_A = v\lrcorner F_A$. Then

$$r_\phi(\beta_v)_A = -2\int_{\mathbb{T}} \operatorname{tr}(F_A \wedge v\lrcorner F_A) \wedge \varepsilon \wedge d\,\mathrm{Vol}_{T^3} = -2\int_{T^4} \operatorname{tr}(F_A \wedge v\lrcorner F_A) \wedge \varepsilon = \varepsilon(v) \cdot c_2(E),$$

which is nonzero by assumption. ■

So far we know from Corollary 1 that the set $\mathcal{M}_+^4$ of self-dual connections (modulo gauge) over T^4 lifts to instantons (cf. (9)) of the original G_2-structure φ (i.e. to zeroes of ρ). However, for bundles with non-trivial c_2, this generic case degenerates precisely under deformations of type (IV):

Proposition 2. *Let $\mathbf{E} \to (\mathbb{T}, \varphi)$ be the pullback of a stable $\mathrm{SU}(n)$-bundle E over T^4 with $c_2(E) \neq 0$; then E admits no $(\varphi+\phi)$-instantons, for any perturbation ϕ away from a fibred structure (i.e. of type (IV) in Proposition 1).*

Proof. Fix a lifted φ-instanton $A \in \widetilde{\mathcal{M}^4_+}$; for any $A + ha \in \mathcal{A}$, Lemma 2 gives $\rho_{A+ha}(\beta_v) \equiv \rho_A(\beta_v) = 0$. Taking $v \in T^4$ as in Lemma 3 we have

$$\rho_\phi(\beta_v)_{A+ha} = r_\phi(\beta_v)_{A+ha} + \rho(\beta_v)_{A+ha} = \underbrace{\varepsilon(v) \cdot c_2(E)}_{\neq 0} + \underbrace{\rho(\beta_v)_A}_{0},$$

hence $A + ha$ is not a $(\varphi + \phi)$-instanton. ■

Combining Corollary 1 and Proposition 2 we obtain Theorem 2.

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