Commuting Differential Operators of Rank 3 Associated to a Curve of Genus 2

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Abstract. In this paper, we construct some examples of commuting differential operators L_1 and L_2 with rational coefficients of rank 3 corresponding to a curve of genus 2.

Key words: commuting differential operators; rank 3; genus 2

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1 Introduction

The study of the commutation equation

$$[L_1, L_2] = 0$$

of two scalar differential operators

$$L_1 = \frac{d^n}{dx^n} + \sum_{i=0}^{n-1} f_i(x)\frac{d^i}{dx^i} \qquad \text{and} \qquad L_2 = \frac{d^m}{dx^m} + \sum_{j=0}^{m-1} g_j(x)\frac{d^j}{dx^j}, \qquad n < m,$$

is one of the classical problems of the theory of ordinary differential equations.

Burchnall and Chaundy in [1, 2, 3] have shown that *"each pair of commuting operators L_1 and L_2 is connected by a nontrivial polynomial algebraic relation $Q(L_1, L_2) = 0$"*. The equation $Q(z, w) = 0$ determines a smooth compact algebraic curve Ξ of finite genus g. For a generic point $P \in \Xi$, there exist common eigenfunctions $\psi(x, P)$ on Ξ such that $L_1\psi = \lambda\psi$ and $L_2\psi = \mu\psi$. The dimension l of the space of these functions corresponding to $P \in \Xi$ is called the *rank* of the commuting pair (L_1, L_2). For simplicity, in this paper we denote "the commuting differential operators of rank l corresponding to a curve of genus g" by *"(l, g)-operators"*.

Burchnall and Chaundy also made significant progress in solving the commutation equation for relatively prime orders m and n. In this case, the rank l equals to 1. The study of this case was completed by Krichever [11, 12], who also obtained explicit formulas of the function ψ and the coefficients of L_1 and L_2 in terms of the Riemann Θ-function. Let us remark that there are several papers related to this case, for instance [5, 6, 23, 25, 28, 29].

But for high rank case i.e. $l > 1$, it is much more complicated. In [10], the problem of classifying (l, g)-operators was solved by reducing the computation of the coefficients to a Riemann problem. In [13, 14] I.M. Krichever and S.P. Novikov developed a method of deforming the Tyurin parameters on the moduli space of framed holomorphic bundles over algebraic curves. By using this method, in certain cases the Riemann problem can be avoided and they found all $(2,1)$-operators. Let us remark that J. Dixmier in [4] also discovered an example of $(2,1)$-operators with polynomial coefficients. Furthermore, P.G. Grinevich found the condition of $(2,1)$-operators with rational coefficients [7]. S.P. Novikov and P.G. Grinevich [24] clarified the spectral data related to formally self-adjoint $(2,1)$-operators. In [21] O.I. Mokhov obtained all $(3,1)$-operators. A.E. Mironov in [17, 19] introduced a σ-invariance to simplify the Krichever–Novikov system [14] and constructed some examples of $(2,2)$-operators, $(2,4)$-operators with polynomial coefficients and also in [18, 20] formally self-adjoint $(2,g)$-operators and $(3,g)$-operators. Recently, an interesting paper is due to O.I. Mokhov in [22] who constructed examples of $(2k,g)$-operators and $(3k,g)$-operators with polynomial coefficients for arbitrary genus g. For more related results, please see [8, 9, 13, 15, 16, 25, 26, 27] and references therein.

The aim of this paper is to construct examples of commuting differential operators L_1 and L_2 with rational coefficients of rank 3 corresponding to a curve of genus 2, which is different from those in [22].

2 The commuting operators of rank 3 and genus 2

In this section we want to construct (3,2)-operators. The first step is to use a σ-invariance, due to A.E. Mironov [17], to simplify the Krichever–Novikov system (2). The second step is to solve the simplified system by making a crucial hypothesis

$$\gamma_1 = \gamma, \qquad \gamma_2 = a\gamma, \qquad \gamma_3 = \bar{a}\gamma, \qquad a = \frac{-1+\sqrt{3}\mathbf{i}}{2}.$$

The last step is to construct the commuting differential operators L_1 and L_2.

2.1 The general principle

Let Γ be a curve of genus 2 defined in $\mathbb{C}^2$ by the equation

$$w^2 = z^6 + c_5 z^5 + c_4 z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0.$$

On the curve Γ, there is a holomorphic involution

$$\sigma: \ \Gamma \to \Gamma \qquad \text{by} \quad \sigma(z,w) = (z,-w),$$

which has six fixed ramification points. It induces an action on the space of function by $(\sigma f)(x,P) = f(x,\sigma(P))$. Let us take $q = (0,\sqrt{c_0}) \in \Gamma$. For a generic point $P \in \Gamma$ there exist common eigenfunctions $\psi_j(x,P)$, $j = 0,1,2$ with an essential singularity at q, of the operators L_1 and L_2. Without loss of generality, we assume that $\psi_j(x,P)$ are normalized by

$$\frac{d^i}{dx^i}\psi_j(x_0,P) = \delta_{ij},$$

where x_0 is a fixed point. Notice that on $\Gamma - \{q\}$, $\psi_j(x,P)$ are meromorphic and have six simple poles at $P_1,\dots,P_6$ independent of x. Let us consider the Wronskian matrix

$$\vec{\Psi}(x,P;x_0) = \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 \\ \psi_0' & \psi_1' & \psi_2' \\ \psi_0'' & \psi_1'' & \psi_2'' \end{pmatrix}$$

of the vector-valued function $\vec{\Psi}(x,P;x_0)$, and

$$\vec{\Psi}_x \vec{\Psi}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \chi_0 & \chi_1 & \chi_2 \end{pmatrix}, \tag{1}$$

where $\chi_j = \chi_j(x, P)$ are independent of x_0 and meromorphic functions on Γ with six poles at $P_1(x), \dots, P_6(x)$ coinciding with the poles of $\psi_j(x, P)$ at $x = x_0$. In a neighborhood of q, the functions $\chi_j(x, P)$ have the form

$$\begin{aligned}&\chi_0(x, P) = k + w_0(x) + O\big(k^{-1}\big), \qquad \chi_1(x, P) = w_1(x) + O\big(k^{-1}\big),\\ &\chi_2(x, P) = O\big(k^{-1}\big),\end{aligned} \tag{2}$$

where k^{-1} is a local parameter near q. The expansion of χ_j in a neighborhood of the pole $P_i(x)$ has the form

$$\chi_j(x, P) = -\frac{\gamma_i'(x)\alpha_{ij}(x)}{k - \gamma_i(x)} + d_{ij}(x) + O(k - \gamma_i(x)), \qquad \alpha_{i2} = 1, \tag{3}$$

where $k - \gamma_i(x)$ is a local parameter near $P_i(x)$ for $1 \le i \le 6$ and $0 \le j \le 2$.

Lemma 2.1 ([11])**.** *The parameters $\gamma_i(x)$, $\alpha_{ij}(x)$ and $d_{ij}(x)$, $1 \le i \le 6$, $0 \le j \le 2$ satisfy the system*

$$\begin{aligned}&\mathrm{Eq}[i, 0] := \alpha_{i0}(x)\alpha_{i1}(x) + \alpha_{i0}(x)d_{i2}(x) - \alpha_{i0}'(x) - d_{i0}(x) = 0,\\ &\mathrm{Eq}[i, 1] := \alpha_{i1}(x)^2 - \alpha_{i0}(x) + \alpha_{i1}(x)d_{i2}(x) - \alpha_{i1}'(x) - d_{i1}(x) = 0.\end{aligned} \tag{4}$$

2.2 Explicit forms of $\chi_j(x, P)$

In this subsection, we discuss explicit forms of $\chi_j(x, P)$ corresponding to the curve Γ defined by $w^2 = 1 + c_3z^3 + c_4z^4 + z^6$. In order to do this, we assume that

$$\sigma\chi_2(x, P) = \chi_2(x, P), \qquad \sigma P_s(x) = P_{s+3}(x), \qquad s = 1, 2, 3, \tag{5}$$

and

$$\gamma_1 = \gamma, \qquad \gamma_2 = a\gamma, \qquad \gamma_3 = \bar{a}\gamma, \qquad a = \frac{-1 + \sqrt{3}\,\mathbf{i}}{2}. \tag{6}$$

Theorem 2.2. *Let γ be a solution of*

$$1 + c_3\gamma^3 + \gamma^6 - 6(-3)^{\frac14}c_4^{\frac14}\gamma'^{\frac32} = 0, \tag{7}$$

then functions χ_0, χ_1, χ_2 are given by the formulas

$$\begin{aligned}&\chi_2(x, P) = -\sum_{s=1}^{3}\frac{\gamma_s'}{z - \gamma_s} - \sum_{s=1}^{3}\frac{\gamma_s'}{\gamma_s} = \frac{3z^3\gamma'}{\gamma^4 - z^3\gamma},\\ &\chi_1(x, P) = \tau_1 - \sum_{s=1}^{3}\frac{G_s\gamma_s'}{z - \gamma_s} + \frac{w(z)h_1}{2(z - \gamma_1)(z - \gamma_2)(z - \gamma_3)},\\ &\chi_0(x, P) = \frac{\tau_0}{2} + \frac{1}{2z} - \sum_{s=1}^{3}\frac{H_s\gamma_s'}{z - \gamma_s} - \frac{w(z)(\gamma_1\gamma_2\gamma_3 + zh_0)}{2z(z - \gamma_1)(z - \gamma_2)(z - \gamma_3)},\end{aligned} \tag{8}$$

with G_s, H_s, τ_0, τ_1 defined in (9)–(14).

Proof. By using the σ-invariance of $\chi_2(x, P)$, we know

$$\gamma_s(x) = \gamma_{s+3}(x), \qquad d_{s2}(x) = d_{s+3,2}(x), \qquad s = 1, 2, 3.$$

According to the properties of $\chi_j(x,P)$ in (2), (3) and (5), we could assume that the functions $\chi_j(x,P)$ are of the form in (8) with unknown functions $G_s = G_s(x)$, $H_s = H_s(x)$, $\tau_r = \tau_r(x)$ and $h_r = h_r(x)$ for $s = 1,2,3$ and $r = 0,1$.

Substituting (6) into (8), we have

$$\chi_2(x,P) = \frac{3z^3\gamma'}{\gamma^4 - z^3\gamma},$$

which yields that

$$d_{i2} = -\frac{2\gamma'}{\gamma}, \qquad i = 1,\dots,6.$$

For simplicity we use the following notations

$$a_1 = 1, \qquad a_2 = a, \qquad a_3 = \bar{a},$$
$$a_{s+3} = a_s, \qquad G_{s+3} = G_s, \qquad H_{s+3} = H_s, \qquad s = 1,2,3.$$

It follows from (3) that

$$\alpha_{s0} = H_s + \frac{w(a_s\gamma)h_0}{6\gamma^2\gamma'} + \frac{a_s^2 w(a_s\gamma)}{6\gamma'}, \qquad \alpha_{s1} = G_s - \frac{w(a_s\gamma)h_1}{6\gamma^2\gamma'},$$

$$d_{s0} = \frac{\tau_0}{2} + \frac{a_s^2}{2\gamma} + \frac{\gamma' \sum_{m=1}^{2}(1 - a_s^2 a_{s+m})H_{s+m}}{3\gamma}$$
$$+ \frac{(h_0 + 2(a_s\gamma)^2)w(a_s\gamma) - (h_0 + (a_s\gamma)^2)a_s\gamma w'(a_s\gamma)}{6\gamma^3},$$
$$d_{s1} = \tau_1 - \frac{G_{s+1}\gamma'}{(a_s - a_{s+1})\gamma} - \frac{G_{s+2}\gamma'}{(a_s - a_{s+2})\gamma} + \frac{(a_s\gamma w'(a_s\gamma) - w(a_s\gamma))h_1}{6\gamma^3},$$
$$\alpha_{s+3,r} = \sigma\alpha_{sr}, \qquad d_{s+3,1} = \sigma d_{s1}, \qquad r = 0,1, \qquad s = 1,2,3.$$

By substituting α_{ij} and d_{ij} into (4), we get twelve equations

$$\mathrm{Eq}[i,0] = 0, \qquad \mathrm{Eq}[i,1] = 0, \qquad i = 1,\dots,6.$$

We now try to solve these equations. Firstly, it follows from

$$\mathrm{Eq}[s+3,1] - \mathrm{Eq}[s,1] = 0, \qquad s = 1,2,3$$

that

$$G_s = \frac{h_1' - h_0 - (a_s\gamma)^2}{2h_1} + \frac{\gamma'}{2\gamma} - \frac{\gamma''}{2\gamma'}, \qquad s = 1,2,3. \tag{9}$$

By using (9) and $\mathrm{Eq}[s+3,0] - \mathrm{Eq}[s,0] = 0$, we get

$$H_s = \frac{(h_0 + (a_s\gamma)^2)h_1' - 2h_0' - 7a_s^2\gamma\gamma'}{2h_1} - \frac{(h_0 + (a_s\gamma)^2)^2}{2h_1^2} \tag{10}$$
$$- \frac{h_0\gamma'}{2h_1\gamma} + \frac{(h_0 + (a_s\gamma)^2)\gamma''}{2h_1\gamma'}, \qquad s = 1,2,3. \tag{11}$$

Furthermore, by solving

$$\mathrm{Eq}[s+3,1] + \mathrm{Eq}[s,1] = 0, \qquad \mathrm{Eq}[s+3,0] + \mathrm{Eq}[s,0] = 0,$$

we have

$$\begin{aligned}\mathrm{Neq}[s,1] := -\tau_1 &+ \frac{h_0^2 + 6h_0(a_s\gamma)^2 + 6a_s\gamma^4 - 6h_0h_1' - 6(a_s\gamma)^2h_1' + 3h_1'^2}{4h_1^2}\\ &+ \frac{3h_0' - h_1'' + 9a_s^2\gamma\gamma'}{2h_1} + \frac{3h_0\gamma' - 2h_1'\gamma'}{2h_1\gamma} + \frac{\gamma''}{2\gamma} + \frac{\gamma'''}{2\gamma'}\\ &- \frac{3\gamma'^2}{4\gamma^2} - \frac{\gamma''^2}{4\gamma'^2} - \frac{h_1\gamma''}{2h_1\gamma'} + \frac{h_1^2w(a_s\gamma)}{36\gamma^4\gamma'^2} = 0, \qquad s = 1,2,3,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Neq}[s,0] := -\tau_0 &- \frac{a_s^2}{\gamma} + \frac{4h_0'' + 16a_s^2\gamma\gamma' + 21(a_s\gamma)^2}{2h_1}\\ &+ \frac{(3h_0' + 9a_s\gamma\gamma' - h_1'')(h_0 + (a_s\gamma)^2) - 4h_0'h_1' - 13a_s^2\gamma\gamma'h_1'}{h_1^2}\\ &+ \frac{(h_0 + (a_s\gamma)^2)^3 - 6h_1'(h_0 + (a_s\gamma)^2)^2 + 5h_1'^2(h_0 + (a_s\gamma)^2)}{2h_1^3}\\ &+ \frac{6h_0'\gamma' - h_0\gamma''}{h_1\gamma} + \frac{(3h_0^2 - 4h_0h_1')\gamma'}{h_1^2\gamma} - \frac{(h_0 + (a_s\gamma)^2)\gamma'''}{h_1\gamma'}\\ &+ \frac{(h_0 + (a_s\gamma)^2)h_1'\gamma''}{h_1^2\gamma'} + \frac{(h_0 + (a_s\gamma)^2)\gamma''^2}{2h_1\gamma'^2} + \frac{3h_0\gamma'^2}{2h_1\gamma^2}\\ &- \frac{(h_0 + (a_s\gamma)^2)h_1w(a_s\gamma)}{18\gamma^4\gamma'^2}, \qquad s = 1,2,3.\end{aligned}$$

Let us remark that we have reduced twelve equations to six equations

$$\mathrm{Neq}[s,0] = 0, \qquad \mathrm{Neq}[s,1] = 0, \qquad s = 1,2,3,$$

with four unknown functions τ_1, τ_0, h_1 and h_0.

Let us take

$$h_1 = i(-3)^{\frac{3}{4}}c_4^{-\frac{1}{4}}\gamma\sqrt{\gamma'}, \qquad h_0 = \frac{i(-3)^{\frac{3}{4}}(\gamma\gamma'' - 4\gamma'^2)}{2c_4^{\frac{1}{4}}\sqrt{\gamma'}}. \tag{12}$$

From $\mathrm{Neq}[1,1] = 0$, we get

$$\tau_1 = \frac{4\gamma'^2 - 9\gamma\gamma''}{2\gamma^2} + \frac{4\gamma'\gamma''' - 3\gamma''^2}{4\gamma'^2} + \frac{i(\gamma^3 - 1)^2}{4\sqrt{3c_4}\gamma^2\gamma'}. \tag{13}$$

By using (13), we conclude that $\mathrm{Neq}[2,1] = 0$ and $\mathrm{Neq}[3,1] = 0$ always hold true.

From the equation $\mathrm{Neq}[1,0] = 0$, we obtain

$$\begin{aligned}\tau_0 = &\frac{i(\gamma^3 - 1)^2}{\sqrt{3c_4}\gamma^3} - \frac{1}{\gamma} - \frac{i(\gamma^3 - 1)^2\gamma''}{4\sqrt{3c_4}\gamma^2\gamma'^2} - \frac{2i(-3)^{\frac{3}{4}}c_4^{\frac{3}{4}}\gamma^3}{27\gamma'^{\frac{3}{2}}} - \frac{i(-3)^{\frac{3}{4}}(\gamma^3 - 1)^2}{18c_4^{\frac{1}{4}}\gamma\gamma'^{\frac{3}{2}}}\\ &- \frac{3\gamma'''}{\gamma} + \frac{10\gamma'\gamma''}{\gamma^2} - \frac{4\gamma'^3}{\gamma^3} + \frac{\gamma^{(4)}}{\gamma'} - \frac{5\gamma''\gamma'''}{2\gamma'^2} + \frac{3\gamma''^3}{2\gamma'^3} - \frac{3\gamma''^2}{\gamma\gamma'}.\end{aligned} \tag{14}$$

By using (14), both $\mathrm{Neq}[2,0] = 0$ and $\mathrm{Neq}[3,0] = 0$ reduce to the same equation

$$1 + c_3\gamma^3 + \gamma^6 - 6(-3)^{\frac{1}{4}}c_4^{\frac{1}{4}}\gamma'^{\frac{3}{2}} = 0,$$

which is exactly the equation (7). Thus we complete the proof of the theorem. ■

Generally, solutions of (7) are not useful for us to construct $(3,2)$-operators with "good" coefficients. But when we choose $c_3 = 2$ or -2, there are rational solutions. In what follows let us suppose

$$c_3 = -2, \qquad c_4 = -\frac{\epsilon^4}{3888}, \qquad \epsilon < 0.$$

The equation (7) is rewritten as

$$1 - 2\gamma^3 + \gamma^6 + \epsilon\gamma'^{\frac{3}{2}} = 0. \tag{15}$$

It is easy to check that when $(x+s_0)^3 + \epsilon^2 > 0$,

$$\gamma = \frac{x+s_0}{((x+s_0)^3+\epsilon^2)^{\frac{1}{3}}}, \qquad s_0 \in \mathbb{C}$$

is a solution of (15). Without loss of generality, we set $s_0 = 0$. In this case we would like to write $\gamma = \gamma(x;\epsilon)$. As a corollary of Theorem 2.2, we have

Corollary 2.3. *Let* $\gamma(x;\epsilon) = \frac{x}{(x^3+\epsilon^2)^{\frac{1}{3}}}$ *be a solution of* (15). *Then we have*

$$\begin{aligned}
\chi_0(x,P) &= \frac{1}{2z} - \frac{x^3(\epsilon^2+x^3)}{5832} + \frac{10(z^3-1)}{\kappa} + \frac{\epsilon^2x^3z}{216\kappa} \\
&\quad - \frac{108w(z)+\epsilon^2z^2}{6\kappa} - \frac{x^3w(z)}{2\kappa z} + \frac{16\epsilon^2z^3}{\kappa x^3}, \\
\chi_1(x,P) &= \frac{132\epsilon^2z^3 - x^3[204-204z^3+108w(z)+\epsilon^2z^2]}{12x^2\kappa}, \qquad \chi_2(x,P) = -\frac{3\epsilon^2z^3}{x\kappa},
\end{aligned} \tag{16}$$

where $\kappa = (\epsilon^2+x^3)z^3 - x^3$ *and* $w(z) = \sqrt{1-2z^3-\frac{\epsilon^2}{3888}z^4+z^6}$.

By using (16), let us expand $\chi_j(x,P)$ in a neighborhood of $z=0$

$$\begin{aligned}
\chi_0(x,P) &= \frac{1}{z} + \zeta_1 - \frac{\epsilon^2}{216}z + \frac{2\epsilon^2}{3x^2}z^2 + O\big(z^3\big), \\
\chi_1(x,P) &= \zeta_2 + \frac{\epsilon^2}{12x^2}z^2 + O\big(z^3\big), \qquad \chi_2(x,P) = \frac{3\epsilon^2}{x^4} + O\big(z^4\big),
\end{aligned}$$

where

$$\zeta_1 = \frac{28}{x^2} - \frac{\epsilon^2x^3+x^6}{5832} \qquad \text{and} \qquad \zeta_2 = \frac{26}{x^2}. \tag{17}$$

2.3 Commuting differential operators of rank 3

Let Γ be a smooth curve of genus 2 defined by the equation

$$w^2 = 1 - 2z^3 - \frac{\epsilon^4}{3888}z^4 + z^6 \tag{18}$$

on the (z,w)-plane.

Theorem 2.4. *The operator* L_1 *corresponding to the meromorphic function*

$$\lambda = \frac{1+w(z)}{2z^3} - \frac{1}{2}$$

on Γ *with the unique pole at* $q = (0, 1)$ *and* $L_1\psi = \lambda\psi$ *has the form*

$$L_1 = \frac{d^9}{dx^9} + \sum_{n=0}^{7} f_n \frac{d^n}{dx^n}, \tag{19}$$

where

$$\begin{aligned} f_0 &= \frac{152}{243} - \frac{58240}{x^9} - \frac{55\epsilon^2}{243x^3} - \frac{37\epsilon^4 x^3}{11337408} + \frac{115\epsilon^2 x^6}{11337408} + \frac{37x^9}{1417176} \\ &\quad + \frac{\epsilon^6 x^9}{198359290368} + \frac{\epsilon^4 x^{12}}{66119763456} + \frac{\epsilon^2 x^{15}}{66119763456} + \frac{x^{18}}{198359290368}, \\ f_1 &= \frac{58240}{x^8} + \frac{55\epsilon^2}{243x^2} - \frac{152x}{243} + \frac{5\epsilon^4 x^4}{5668704} + \frac{2\epsilon^2 x^7}{177147} + \frac{17x^{10}}{1417176}, \\ f_2 &= -\frac{43200}{x^7} + \frac{26\ \epsilon^2}{243x} - \frac{73x^2}{243} + \frac{\epsilon^4 x^5}{1259712} + \frac{\epsilon^2 x^8}{419904} + \frac{x^{11}}{629856}, \\ f_3 &= -\frac{143\epsilon^2}{1944} + \frac{19120}{x^6} + \frac{79x^3}{486} + \frac{\epsilon^4 x^6}{11337408} + \frac{\epsilon^2 x^9}{5668704} + \frac{x^{12}}{11337408}, \\ f_4 &= -\frac{4800}{x^5} - \frac{2\epsilon^2 x}{243} + \frac{16x^4}{243}, \qquad f_5 = -\frac{24}{x^4} + \frac{\epsilon^2 x^2}{216} + \frac{x^5}{108}, \\ f_6 &= \frac{384}{x^3} + \frac{\epsilon^2 x^3}{1944} + \frac{x^6}{1944}, \qquad f_7 = -\frac{78}{x^2}. \end{aligned} \tag{20}$$

Proof. By using (1), we have

$$\psi_j'''(x,P) = \chi_2(x,P)\psi_j''(x,P) + \chi_1(x,P)\psi_j'(x,P) + \chi_0(x,P)\psi_j(x,P). \tag{21}$$

It follows from (21) that the equation $L_1\psi_j = \lambda(z)\psi_j$ can be rewritten as

$$Q_0(x,z)\psi_j(x,P) + Q_1(x,z)\psi_j'(x,P) + Q_2(x,z)\psi_j''(x,P) = \lambda(z)\psi_j. \tag{22}$$

According to the independence of $\chi_0(x,P)$, $\chi_1(x,P)$ and $\chi_2(x,P)$ at $x = x_0$, we conclude that the system (22) is equivalent to three equations

$$Q_0(x,z) = \lambda(z), \qquad Q_1(x,z) = 0, \qquad Q_2(x,z) = 0.$$

By expanding $Q_j(x,z)$ at $z = 0$, we have

$$0 = Q_j(x,z) - \delta_j^0\lambda(z) = Q_{j,-2}\frac{1}{z^2} + Q_{j,-1}\frac{1}{z} + Q_{j0} + O(z).$$

Then by solving $Q_{j,-s} = 0$ for $s, j = 0, 1, 2$, we get the coefficients of L_1 given by

$$\begin{aligned} f_0 &= -1 - \zeta_1^3 - \frac{4\epsilon^2 + 3}{2x^3} - \zeta_2\zeta_1'\zeta_2' - -\zeta_2^2\zeta_1'' + 6\zeta_1'\zeta_1'' + 3\zeta_1''\zeta_2'' + 3\zeta_2'\zeta_1''' \\ &\quad + \zeta_1\left(-\frac{\epsilon^2}{72} - 3\zeta_2\zeta_1' + 3\zeta_1'''\right) + \zeta_1'\zeta_2''' + 2\zeta_2\zeta_1^{(4)} - \zeta_1^{(6)}, \\ f_1 &= -\frac{1}{4x^2} + 6\zeta_1'^2 + 9\zeta_1\zeta_1'' + 12\zeta_2'\zeta_1'' + 9\zeta_1'\zeta_2'' + 3\zeta_2''^2 - \zeta_2^2(3\zeta_1' + \zeta_2'') + 3\zeta_1\zeta_2''' \\ &\quad + 4\zeta_2'\zeta_2''' + \zeta_2\left(-\frac{\epsilon^2}{72} - 3\zeta_1^2 - 3\zeta_1\zeta_2' - \zeta_2'^2 + 9\zeta_1''' + 2\zeta_2^{(4)}\right) - 6\zeta_1^{(5)} - \zeta_2^{(6)}, \\ f_2 &= 3\big[-\zeta_2^2\zeta_2' + 5\zeta_1'\zeta_2' + 5\zeta_2'\zeta_2'' - \zeta_1\zeta_2^2 + 3\zeta_1(\zeta_1' + \zeta_2'') + \zeta_2(5\zeta_1'' + 3\zeta_2''') - 5\zeta_1^{(4)} - 2\zeta_2^{(5)}\big], \\ f_3 &= \frac{\epsilon^2}{72} + 3\zeta_1^2 - \zeta_2^3 + 9\zeta_1\zeta_2' + 9\zeta_2'^2 + 3\zeta_2(4\zeta_1' + 5\zeta_2'') - 21\zeta_1''' - 15\zeta_2^{(4)}, \end{aligned}$$

$$f_4 = 15\zeta_2\zeta_2' - \zeta_2(2(-3\zeta_1 - 9\zeta_2') + 21\zeta_2') - 18\zeta_1'' - 21\zeta_2''',$$
$$f_5 = 3\zeta_2^2 - 9\zeta_1' - 18\zeta_2'', \qquad f_6 = -3\zeta_1 - 9\zeta_2', \qquad f_7 = -3\zeta_2.$$

By substituting ζ_1 and ζ_2 in (17) into the above formula, we obtain explicit expressions of f_j in (20). ∎

Next we want to look for a 12^{th}-order differential operator

$$L_2 = \frac{d^{12}}{dx^{12}} + \sum_{m=0}^{10} g_m \frac{d^m}{dx^m}, \tag{23}$$

such that $[L_1, L_2] = 0$. Let us sketch out our ideas and omit tedious computations. The commutation equation $[L_1, L_2] = 0$ is written as

$$0 = \left[\frac{d^9}{dx^9} + \sum_{n=0}^{7} f_n \frac{d^n}{dx^n}, \frac{d^{12}}{dx^{12}} + \sum_{m=0}^{10} g_m \frac{d^m}{dx^m}\right] = \sum_{k=0}^{18} W_k(f,g)\frac{d^k}{dx^k}, \tag{24}$$

which yields that

$$W_k(f,g) = 0, \qquad k = 0, \ldots, 18.$$

By using eleven equations $W_k(f,g) = 0$, $k = 8, \ldots, 18$, we could obtain explicit forms of $g_m = h_m(x; \rho_0, \ldots, \rho_{10-m}) + \rho_{11-m}$ with integral constants ρ_{11-m}. The last eight equations will determine some integral constants. For simplicity, we take all arbitrary parameters to be zero, and then obtain all coefficients g_j as follows

$$\begin{aligned}
g_0 = {} & \frac{45660160}{x^{12}} - \frac{4928\epsilon^2}{729x^6} - \frac{20048}{729x^3} - \frac{605\epsilon^2 x^3}{708588} + \frac{4553x^6}{708588} + \frac{79\epsilon^6 x^6}{99179645184} \\
& + \frac{269\epsilon^4 x^9}{16529940864} + \frac{683\epsilon^2 x^{12}}{16529940864} + \frac{\epsilon^8 x^{12}}{1156831381426176} + \frac{661x^{15}}{24794911296} \\
& + \frac{\epsilon^6 x^{15}}{289207845356544} + \frac{\epsilon^4 x^{18}}{192805230237696} + \frac{\epsilon^2 x^{21}}{289207845356544} + \frac{x^{24}}{1156831381426176}, \\
g_1 = {} & -\frac{45660160}{x^{11}} + \frac{4928\epsilon^2}{729x^5} + \frac{20048}{729x^2} - \frac{203\epsilon^4 x}{2834352} + \frac{1691\epsilon^2 x^4}{2834352} + \frac{7111x^7}{708588} \\
& + \frac{55\epsilon^6 x^7}{49589822592} + \frac{127\epsilon^4 x^{10}}{16529940864} + \frac{217\epsilon^2 x^{13}}{16529940864} + \frac{325x^{16}}{49589822592}, \\
g_2 = {} & \frac{27758080}{x^{10}} - \frac{182\epsilon^2}{27x^4} + \frac{296}{9x} - \frac{413\epsilon^4 x^2}{5668704} + \frac{4339\epsilon^2 x^5}{2834352} + \frac{6595x^8}{1417176} \\
& + \frac{\epsilon^6 x^8}{3673320192} + \frac{\epsilon^4 x^{11}}{918330048} + \frac{5\epsilon^2 x^{14}}{3673320192} + \frac{x^{17}}{1836660096}, \\
g_3 = {} & -\frac{5992}{729} - \frac{11567360}{x^9} + \frac{1028\epsilon^2}{729x^3} + \frac{25\epsilon^4 x^3}{1417176} + \frac{457\epsilon^2 x^6}{708588} + \frac{1393x^9}{1417176} \\
& + \frac{\epsilon^6 x^9}{49589822592} + \frac{\epsilon^4 x^{12}}{16529940864} + \frac{\epsilon^2 x^{15}}{16529940864} + \frac{x^{18}}{49589822592}, \\
g_4 = {} & \frac{3395840}{x^8} + \frac{271\epsilon^2}{243x^2} - \frac{2834x}{243} + \frac{193\epsilon^4 x^4}{11337408} + \frac{317\epsilon^2 x^7}{2834352} + \frac{307x^{10}}{2834352}, \\
g_5 = {} & -\frac{693504}{x^7} - \frac{13\epsilon^2}{243x} + \frac{221x^2}{243} + \frac{\epsilon^4 x^5}{314928} + \frac{\epsilon^2 x^8}{104976} + \frac{x^{11}}{157464}, \\
g_6 = {} & -\frac{167\epsilon^2}{972} + \frac{86464}{x^6} + \frac{316x^3}{243} + \frac{\epsilon^4 x^6}{5668704} + \frac{\epsilon^2 x^9}{2834352} + \frac{x^{12}}{5668704},
\end{aligned}$$

$$g_7 = -\frac{672}{x^5} + \frac{\epsilon^2 x}{486} + \frac{109x^4}{486}, \qquad g_8 = -\frac{2856}{x^4} + \frac{\epsilon^2 x^2}{108} + \frac{x^5}{54},$$
$$g_9 = \frac{824}{x^3} + \frac{\epsilon^2 x^3}{1458} + \frac{x^6}{1458}, \qquad g_{10} = -\frac{104}{x^2}. \tag{25}$$

Remark 2.5. By analogy with the process of getting f_j in (20), we could obtain the above g_j in (25) by choosing another meromorphic function with a unique pole of order 4 at $z = 0$ on Γ

$$\mu(z) = \frac{1 + w(z)}{2z^4} - \frac{1}{2z}.$$

Remark 2.6. One could find another operator L_3 of order 15 from $[L_1, L_3] = 0$. Furthermore as in [17], the commutative ring of differential operators generated by L_1, L_2 and L_3 is isomorphic to the ring of meromorphic functions on Γ with the pole at $q = (0, 1)$.

2.4 The corresponding Burchnall–Chaundy curve

According to the Burchnall–Chaundy's correspondence in [1, 2, 3], for each pair of commuting operators L_1 and L_2 there is a Burchnall–Chaundy curve defined by a minimal nontrivial polynomial $Q(z, w) = 0$ such that $Q(L_1, L_2) = 0$ (or $Q(L_2, L_1) = 0$). Obviously, the above curve Γ defined by (18) is not the Burchnall–Chaundy curve for L_1 and L_2 given in (19) and (24). Actually the corresponding Burchnall–Chaundy curve $\tilde{\Gamma}$ is given by

$$w^3 - \frac{\epsilon^4}{15552} w^2 = z^4 + z^3,$$

that is to say,

$$L_2^3 - \frac{\epsilon^4}{15552} L_2^2 = L_1^4 + L_1^3.$$

The curve $\tilde{\Gamma}$ has a cuspidal singularity at $(0, 0)$. The operators L_1 and L_2 correspond to those meromorphic functions on Γ

$$\lambda = \frac{1 + w(z)}{2z^3} - \frac{1}{2}, \qquad \mu = \frac{1 + w(z)}{2z^4} - \frac{1}{2z}$$

defining a birational equivalence

$$\pi : \Gamma \to \tilde{\Gamma}, \qquad \pi(z, w) = (\lambda, \mu).$$

The inverse image of the cuspidal point is the point $\sigma(q)$, where $q = (0, 1) \in \Gamma$. In order to make π to be a morphism, we must complement $\tilde{\Gamma}$ at infinity by a cuspidal point of the type $(3, 4)$, then its inverse image is the point q.

3 Concluding remarks

In summary by using a σ-invariance to simplify the Krichever–Novikov system, we have constructed a pair of commuting differential operators L_1 in (19) and L_2 in (23) of rank 3 with rational coefficients corresponding to the singular curve $\tilde{\Gamma}$, which is birationally equivalent to the smooth curve Γ of genus 2.

Let us remark that all of coefficients of L_1 and L_2 are polynomials with respect to the parameter ϵ. So if we take

$$\mathcal{L}_1 = \lim_{\epsilon \to 0} L_1, \qquad \mathcal{L}_2 = \lim_{\epsilon \to 0} L_2,$$

then

$$[\mathcal{L}_1, \mathcal{L}_2] = 0, \qquad \mathcal{L}_2^3 = \mathcal{L}_1^4 + \mathcal{L}_1^3.$$

More precisely, we have

$$\mathcal{L}_1 = \mathcal{L}^3 - 1, \qquad \mathcal{L}_2 = \mathcal{L}^4 - \mathcal{L},$$

where

$$\mathcal{L} = \frac{d^3}{dx^3} - \frac{26}{x^2}\frac{d}{dx} - \frac{28}{x^3} + \frac{x^6}{5832}.$$

So, when $\epsilon = 0$ this is a trivial example.

How about the case $\epsilon \neq 0$? Let us comment that in this case, by a direct verification there is not such kind of $\mathcal{L}$ of order 3 commuting with L_1 and L_2. Furthermore, according to the result in [29], any rank one operator with rational coefficients whose second highest coefficient is zero has the property that the limit as x goes to ∞ of the coefficients is zero. So, for example, the absence of a $\frac{d^{11}}{dx^{11}}$ term in L_2 and the x^6 in the coefficient of its $\frac{d^9}{dx^9}$ term which means that L_2 is not a rank 1 operator.

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