# An Introduction to the $q$-Laguerre–Hahn Orthogonal $q$-Polynomials*

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**Abstract.** Orthogonal $q$-polynomials associated with $q$-Laguerre–Hahn form will be studied as a generalization of the $q$-semiclassical forms via a suitable $q$-difference equation. The concept of class and a criterion to determinate it will be given. The $q$-Riccati equation satisfied by the corresponding formal Stieltjes series is obtained. Also, the structure relation is established. Some illustrative examples are highlighted.

*Key words:* orthogonal $q$-polynomials; $q$-Laguerre–Hahn form; $q$-difference operator; $q$-difference equation; $q$-Riccati equation

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## 1 Introduction and preliminary results

The concept of the usual Laguerre–Hahn polynomials were extensively studied by several authors [1, 2, 4, 6, 8, 9, 10, 15, 18]. They constitute a very remarkable family of orthogonal polynomials taking consideration of most of the monic orthogonal polynomials sequences (MOPS) found in literature. In particular, semiclassical orthogonal polynomials are Laguerre–Hahn MOPS [15, 20]. The Laguerre–Hahn set of form (linear functional) is invariant under the standard perturbations of forms [2, 9, 18, 20]. It is well known that a usual Laguerre–Hahn polynomial satisfies a fourth order differential equation with polynomials coefficients but the converse remains not proved until now [20]. Discrete Laguerre–Hahn polynomials were studied in [13]. These families are already extensions of discrete semiclassical polynomials [19]. In literature, analysis and characterization of the $q$-Laguerre–Hahn orthogonal $q$-polynomials have not been yet presented in a unified way. However, several authors have studied the fourth order $q$-difference equation related to some examples of $q$-Laguerre–Hahn orthogonal $q$-polynomials such as the co-recursive and the $r$th associated of $q$-classical polynomials [11, 12]. More generally, the fourth order difference equation of Laguerre–Hahn orthogonal on special non-uniform lattices polynomials was established in [4]. For other relevant works in the domain of orthogonal $q$-polynomials and $q$-difference equation theory see [3, 21] and [5].

So the aim of this contribution is to establish a basic theory of $q$-Laguerre–Hahn orthogonal $q$-polynomials. We give some characterization theorems for this case such as the structure relation and the $q$-Riccati equation. We extend the concept of the class of the usual Laguerre–Hahn forms to the $q$-Laguerre–Hahn case. Moreover, we show that some standard transformation and perturbation carried out on the $q$-Laguerre–Hahn forms lead to new $q$-Laguerre–Hahn forms; the class of the resulting forms is analyzed and some examples are treated.

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We denote by $\mathcal{P}$ the vector space of the polynomials with coefficients in $\mathbb{C}$ and by $\mathcal{P}'$ its dual space whose elements are forms. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$ , $n \geq 0$ the moments of $u$. A linear operator $T : \mathcal{P} \longrightarrow \mathcal{P}$ has a transpose ${}^tT : \mathcal{P}' \longrightarrow \mathcal{P}'$ defined by

$$\langle {}^tTu, f\rangle = \langle u, Tf\rangle, \qquad u \in \mathcal{P}', \qquad f \in \mathcal{P}.$$

For instance, for any form $u$, any polynomial $g$ and any $(a, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, we let $H_q u$, $gu$, $h_a u$, $Du$, $(x - c)^{-1}u$ and $\delta_c$, be the forms defined as usually [20] and [16] for the results related to the operator $H_q$

$$\begin{aligned}&\langle H_q u, f\rangle := -\langle u, H_q f\rangle, \qquad \langle gu, f\rangle := \langle u, gf\rangle, \qquad \langle h_a u, f\rangle := \langle u, h_a f\rangle,\\ &\langle Du, f\rangle := -\langle u, f'\rangle, \qquad \langle (x - c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle, \qquad \langle \delta_c, f\rangle := f(c),\end{aligned}$$

where for all $f \in \mathcal{P}$ and $q \in \widetilde{\mathbb{C}} := \left\{z \in \mathbb{C},\ z \neq 0,\ z^n \neq 1,\ n \geq 1\right\}$ [16]

$$(H_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x}, \qquad (h_a f)(x) = f(ax), \qquad (\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}.$$

In particular, this yields to

$$(H_q u)_n = -[n]_q(u)_{n-1}, \qquad n \geq 0,$$

where $(u)_{-1} = 0$ and $[n]_q := \frac{q^n-1}{q-1}$, $n \geq 0$ [15]. It is obvious that when $q \to 1$, we meet again the derivative $D$.

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product $uf$ is the polynomial [20]

$$(uf)(x) := \langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta}\rangle = \sum_{i=0}^{n}\left(\sum_{j=i}^{n}(u)_{j-i}\, f_j\right)x^i,$$

where $f(x) = \sum\limits_{i=0}^{n} f_i x^i$. This allows us to define the Cauchy's product of two forms:

$$\langle uv, f\rangle := \langle u, vf\rangle, \qquad f \in \mathcal{P}.$$

The product defined as before is commutative [20]. Particularly, the inverse $u^{-1}$ of $u$ if there exists is defined by $uu^{-1} = \delta_0$.

The Stieltjes formal series of $u \in \mathcal{P}'$ is defined by

$$S(u)(z) := -\sum_{n\geq 0}\frac{(u)_n}{z^{n+1}}.$$

A form $u$ is said to be regular whenever there is a sequence of monic polynomials $\{P_n\}_{n\geq 0}$, $\deg P_n = n$, $n \geq 0$ such that $\langle u, P_n P_m\rangle = r_n\delta_{n,m}$ with $r_n \neq 0$ for any $n, m \geq 0$. In this case, $\{P_n\}_{n\geq 0}$ is called a monic orthogonal polynomials sequence MOPS and it is characterized by the following three-term recurrence relation (Favard's theorem)

$$\begin{aligned}&P_0(x) = 1, \qquad P_1(x) = x - \beta_0,\\ &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \qquad n \geq 0,\end{aligned} \tag{1.1}$$

where $\beta_n = \frac{\langle u, xP_n^2\rangle}{r_n} \in \mathbb{C}$, $\gamma_{n+1} = \frac{r_{n+1}}{r_n} \in \mathbb{C} \setminus \{0\}$, $n \geq 0$.

The shifted MOPS $\{\widehat{P}_n := a^{-n}(h_a P_n)\}_{n\geq 0}$ is then orthogonal with respect to $\widehat{u} = h_{a^{-1}}u$ and satisfies (1.1) with [20]

$$\widehat{\beta}_n = \frac{\beta_n}{a}, \qquad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \qquad n \geq 0.$$

Moreover, the form $u$ is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any form will be normalized.

The form $u$ is said to be positive definite if and only if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$ for all $n \geq 0$. When $u$ is regular, $\{P_n\}_{n\geq 0}$ is a symmetrical MOPS if and only if $\beta_n = 0$, $n \geq 0$ or equivalently $(u)_{2n+1} = 0$, $n \geq 0$.

Given a regular form $u$ and the corresponding MOPS $\{P_n\}_{n\geq 0}$, we define the associated sequence of the first kind $\{P_n^{(1)}\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ by [20, equations (2.8) and (2.9)]

$$P_n{}^{(1)}(x) = \Big\langle u, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi}\Big\rangle = (u\theta_0 P_{n+1})(x), \qquad n \geq 0.$$

The following well known results (see [16, 17, 20]) will be needed in the sequel.

**Lemma 1.** *Let $u \in \mathcal{P}'$. $u$ is regular if and only if $\Delta_n(u) \neq 0$, $n \geq 0$ where*

$$\Delta_n(u) := \det\big((u)_{\mu+\nu}\big)_{\mu,\nu=0}^{n}, \qquad n \geq 0$$

*are the Hankel determinants.*

**Lemma 2.** *For $f, g \in \mathcal{P}$, $u, v \in \mathcal{P}'$, $(a, b, c) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}^2$, and $n \geq 1$, we have*

$$(x - c)\big((x - c)^{-1}u\big) = u, \qquad (x - c)^{-1}((x - c)u) = u - (u)_0\delta_c, \tag{1.2}$$

$$(u\theta_0 f)(x) = a_n x^{n-1}(u)_0 + \textit{lower order terms}, \qquad f(x) = \sum_{k=0}^{n} a_k x^k, \tag{1.3}$$

$$u\theta_0(fg) = g(u\theta_0 f) + (fu)\theta_0 g, \tag{1.4}$$

$$u\theta_0(fP_{k+1}) = fP_k^{(1)}, \qquad k + 1 \geq \deg f, \tag{1.5}$$

$$\theta_b - \theta_c = (b - c)\theta_b \circ \theta_c, \qquad \theta_b \circ \theta_c = \theta_c \circ \theta_b, \tag{1.6}$$

$$h_a(gu) = (h_{a^{-1}}g)(h_a u), \qquad h_a(uv) = (h_a u)(h_a v), \qquad h_a\big(x^{-1}u\big) = ax^{-1}h_a u, \tag{1.7}$$

$$h_{q^{-1}} \circ H_q = H_{q^{-1}}, \qquad H_q \circ h_{q^{-1}} = q^{-1}H_{q^{-1}}, \qquad \textit{in} \quad \mathcal{P}, \tag{1.8}$$

$$h_{q^{-1}} \circ H_q = q^{-1}H_{q^{-1}}, \qquad H_q \circ h_{q^{-1}} = H_{q^{-1}}, \qquad \textit{in} \quad \mathcal{P}', \tag{1.9}$$

$$H_q(fg)(x) = (h_q f)(x)(H_q g)(x) + g(x)(H_q f)(x), \tag{1.10}$$

$$H_q(gu) = (h_{q^{-1}}g)H_q u + q^{-1}(H_{q^{-1}}g)u, \tag{1.11}$$

$$H_{q^{-1}}(u\theta_0 f)(x) = q(H_q u)\theta_0(h_{q^{-1}}f)(x) + (u\theta_0 H_{q^{-1}}f)(x), \tag{1.12}$$

$$S(fu)(z) = f(z)S(u)(z) + (u\theta_0 f)(z), \tag{1.13}$$

$$S(uv)(z) = -zS(u)(z)S(v)(z), \tag{1.14}$$

$$S(x^{-n}u)(z) = z^{-n}S(u)(z), \qquad S(u^{-1})(z) = z^{-2}(S(u)(z))^{-1}, \tag{1.15}$$

$$S(H_q u)(z) = q^{-1}(H_{q^{-1}}(S(u)))(z), \qquad (h_{q^{-1}}S(u))(z) = qS(h_q u)(z). \tag{1.16}$$

**Definition 1.** A form $u$ is called $q$-Laguerre–Hahn when it is regular and satisfies the $q$-difference equation

$$H_q(\Phi u) + \Psi u + B\big(x^{-1}u(h_q u)\big) = 0, \tag{1.17}$$

where $\Phi$, $\Psi$, $B$ are polynomials, with $\Phi$ monic. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called $q$-Laguerre–Hahn MOPS.

**Remark 1.** When $B = 0$ and the form $u$ is regular then $u$ is $q$-semiclassical [17]. When $u$ is regular and not $q$-semiclassical then $u$ is called a strict $q$-Laguerre–Hahn form.

**Lemma 3.** *Let $u$ be a regular form. If $u$ is a strict $q$-Laguerre–Hahn form satisfying (1.17) and there exist two polynomials $\Delta$ and $\Omega$ such that*

$$\Delta u + \Omega\big(x^{-1}u(h_q u)\big) = 0 \tag{1.18}$$

*then $\Delta = \Omega = 0$.*

**Proof.** The operation $\Delta \times (1.17) - B \times (1.18)$ gives

$$\Omega H_q(\Phi u) + (\Omega\Psi - \Delta B)u = 0.$$

According to (1.9) and (1.11), the above equation becomes

$$H_q((h_q\Omega)\Phi u) + (\Omega\Psi - (H_q\Omega)\Phi - \Delta B)u = 0.$$

Then $\Delta = \Omega = 0$ because the form $u$ is regular and not $q$-semiclassical. ■

**Lemma 4.** *Consider the sequence $\{\widehat{P}_n\}_{n\geq 0}$ obtained by shifting $P_n$, i.e. $\widehat{P}_n(x) = a^{-n}P_n(ax)$, $n \geq 0$, $a \neq 0$. When $u$ satisfies (1.17), then $\widehat{u} = h_{a^{-1}}u$ fulfills the $q$-difference equation*

$$H_q(\widehat{\Phi}\widehat{u}) + \widehat{\Psi}\widehat{u} + \widehat{B}\big(x^{-1}\widehat{u}(h_q\widehat{u})\big) = 0,$$

*where* $\widehat{\Phi}(x) = a^{-\deg\Phi}\Phi(ax)$, $\widehat{\Psi}(x) = a^{1-\deg\Phi}\Psi(ax)$, $\widehat{B}(x) = a^{-\deg\Phi}B(ax)$.

**Proof.** With $u = h_a\widehat{u}$, we have $\Psi u = \Psi(h_a\widehat{u}) = h_a\big((h_a\Psi)\widehat{u}\big)$ from (1.7). Further,

$$H_q(\Phi u) = H_q\big(\Phi(h_a\widehat{u})\big) = H_q\big(h_a\big((h_a\Phi)\widehat{u}\big)\big) = a^{-1}h_a\big(H_q\big((h_a\Phi)\widehat{u}\big)\big)$$

from (1.7) and (1.9).

Moreover, by virtue of (1.7) an other time we get

$$B\big(x^{-1}u(h_q u)\big) = B\big(x^{-1}(h_a\widehat{u})(h_{aq}\widehat{u})\big) = B\big(x^{-1}h_a\big(\widehat{u}h_q\widehat{u}\big)\big) = a^{-1}h_a\big((h_aB)\big(x^{-1}\widehat{u}(h_q\widehat{u})\big)\big).$$

Equation (1.17) becomes

$$h_a\big(H_q\big(\Phi(ax)\widehat{u}\big) + a\Psi(ax)\widehat{u} + B(ax)\big(x^{-1}\widehat{u}(h_q\widehat{u})\big)\big) = 0.$$

Hence the desired result. ■

## 2 Class of a $q$-Laguerre–Hahn form

It is obvious that a $q$-Laguerre–Hahn form satisfies an infinite number of $q$-difference equations type (1.17). Indeed, multiplying (1.17) by a polynomial $\chi$ and taking into account (1.7), (1.11) we obtain

$$H_q\big((h_q\chi)\Phi u\big) + \big\{\chi\Psi - \Phi(H_q\chi)\big\}u + (\chi B)\big(x^{-1}u(h_q u)\big) = 0. \tag{2.1}$$

Put $t = \deg\Phi$, $p = \deg\Psi$, $r = \deg B$ with $d = \max(t, r)$ and $s = \max(p-1, d-2)$. Thus, there exists $u \to \hbar(u) \subset \mathbb{N}\cup\{-1\}$ from the set of $q$-Laguerre–Hahn forms into the subsets of $\mathbb{N}\cup\{-1\}$.

**Definition 2.** The minimum element of $\hbar(u)$ will be called the class of $u$. When $u$ is of class $s$, the sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to $u$ is said to be of class $s$.

**Proposition 1.** *The number $s$ is an integer positive or zero. In other words, if $p = 0$, then $d \geq 2$ or if $0 \leq d \leq 1$, then necessarily $p \geq 1$.*

**Proof.** Let us show that in case $s = -1$, the form $u$ is not regular, which is a contradiction. Indeed, when $s = -1$, we have

$$\Phi(x) = c_1 x + c_0, \qquad \Psi(x) = a_0, \qquad B(x) = b_1 x + b_0$$

with $c_1 = 1$ or $c_1 = 0$ and $c_0 = 1$, and where $a_0 \neq 0$.

The condition $\langle H_q(\Phi u) + \Psi u + B\big(x^{-1}u(h_q u)\big), x^n\rangle = 0$, $0 \leq n \leq 4$ gives successively

$$\begin{aligned}
&a_0 + b_1 = 0, \\
&(qb_1 - c_1)(u)_1 + b_0 - c_0 = 0, && (2.2)\\
&\big(q^2 b_1 - (1+q)c_1\big)\big((u)_2 - (u)_1^2\big) = 0, && (2.3)\\
&\big(q^3 b_1 - \big(1+q+q^2\big)c_1\big)(u)_3 + \big\{\big(1+q^2\big)b_0 + q(1+q)b_1(u)_1 - \big(1+q+q^2\big)c_0\big\}(u)_2 \\
&\qquad + qb_0(u)_1^2 = 0, && (2.4)\\
&\big(q^4 b_1 - (1+q)\big(1+q^2\big)c_1\big)(u)_4 + \big\{\big(1+q^3\big)b_0 + q\big(1+q^2\big)b_1(u)_1 - (1+q)\big(1+q^2\big)c_0\big\}(u)_3 \\
&\qquad + q^2 b_1 (u)_2^2 + q(1+q)b_0(u)_1(u)_2 = 0. && (2.5)
\end{aligned}$$

Suppose $q^2 b_1 - (1+q)c_1 \neq 0$. From (2.3)

$$\Delta_1 = \begin{vmatrix} 1 & (u)_1 \\ (u)_1 & (u)_2 \end{vmatrix} = 0.$$

Contradiction.

Suppose $q^2 b_1 = (1+q)c_1 = 0$ implies $b_1 = 0 = c_1$ implies (2.2) $b_0 = c_0 = 1$. Thus (2.4) $(u)_2 - (u)_1^2 = 0$, hence $\Delta_1 = 0$. Contradiction.

Suppose $q^2 b_1 = (1+q)c_1 \neq 0$ with $c_1 = 1$. From (2.2) and (2.4), (2.5), we have

$$\begin{aligned}
&(u)_1 = q(c_0 - b_0), \\
&(u)_3 = q(c_0 - 2b_0)(u)_2 + q^3 b_0 (c_0 - b_0)^2, \qquad (2.6)\\
&(u)_4 = (u)_2^2 + q^2 b_0^2 (u)_2 - q^4 b_0^2 (c_0 - b_0)^2.
\end{aligned}$$

On the other hand, let us consider the Hankel determinant

$$\Delta_2 = \begin{vmatrix} 1 & (u)_1 & (u)_2 \\ (u)_1 & (u)_2 & (u)_3 \\ (u)_2 & (u)_3 & (u)_4 \end{vmatrix}.$$

With (2.6), we get $\Delta_2 = 0$. Contradiction. ∎

**Proposition 2.** *Let $u$ be a strict $q$-Laguerre–Hahn form satisfying*

$$H_q(\Phi_1 u) + \Psi_1 u + B_1\big(x^{-1}uh_q u\big) = 0, \tag{2.7}$$

*and*

$$H_q(\Phi_2 u) + \Psi_2 u + B_2(x^{-1}uh_q u) = 0, \tag{2.8}$$

*where $\Phi_1$, $\Psi_1$, $B_1$, $\Phi_2$, $\Psi_2$, $B_2$ are polynomials, $\Phi_1$, $\Phi_2$ monic and $\deg \Phi_i = t_i$, $\deg \Psi_i = p_i$, $\deg B_i = r_i$, $d_i = \max(t_i, r_i)$, $s_i = \max(p_i - 1, d_i - 2)$ for $i \in \{1, 2\}$. Let $\Phi = \gcd(\Phi_1, \Phi_2)$. Then, there exist two polynomials $\Psi$ and $B$ such that*

$$H_q(\Phi u) + \Psi u + B\big(x^{-1}uh_q u\big) = 0, \tag{2.9}$$

*with*

$$s = \max(p-1, d-2) = s_1 - t_1 + t = s_2 - t_2 + t, \tag{2.10}$$

*where* $t = \deg\Phi$, $p = \deg\Psi$, $r = \deg B$ *and* $d = \max(t, r)$.

**Proof.** With $\Phi = \gcd(\Phi_1, \Phi_2)$, there exist two co-prime polynomials $\widetilde{\Phi}_1$, $\widetilde{\Phi}_2$ such that

$$\Phi_1 = \Phi\widetilde{\Phi}_1, \qquad \Phi_2 = \Phi\widetilde{\Phi}_2. \tag{2.11}$$

Taking into account (1.11) equations (2.7), (2.8) become for $i \in \{1, 2\}$

$$\big(h_{q^{-1}}\widetilde{\Phi}_i\big)H_q(\Phi u) + \big\{\Psi_i + q^{-1}H_{q^{-1}}\widetilde{\Phi}_i\big\}u + B_i\big(x^{-1}uh_q u\big) = 0. \tag{2.12}$$

The operation $(h_{q^{-1}}\widetilde{\Phi}_2) \times (2.12_{i=1}) - (h_{q^{-1}}\widetilde{\Phi}_1) \times (2.12_{i=2})$ gives

$$\begin{aligned}&\big\{\big(h_{q^{-1}}\widetilde{\Phi}_2\big)\big(\Psi_1 + q^{-1}\Phi\big(H_{q^{-1}}\widetilde{\Phi}_1\big)\big) - \big(h_{q^{-1}}\widetilde{\Phi}_1\big)\big(\Psi_2 + q^{-1}\Phi\big(H_{q^{-1}}\widetilde{\Phi}_2\big)\big)\big\}u \\ &\quad + \big\{\big(h_{q^{-1}}\widetilde{\Phi}_2\big)B_1 - \big(h_{q^{-1}}\widetilde{\Phi}_1\big)B_2\big\}\big(x^{-1}uh_q u\big) = 0.\end{aligned}$$

From the fact that $u$ is a strict $q$-Laguerre–Hahn form and by virtue of Lemma 3 we get

$$\begin{aligned}&\big(h_{q^{-1}}\widetilde{\Phi}_1\big)\big(\Psi_2 + q^{-1}\Phi\big(H_{q^{-1}}\widetilde{\Phi}_2\big)\big) = \big(h_{q^{-1}}\widetilde{\Phi}_2\big)\big(\Psi_1 + q^{-1}\Phi\big(H_{q^{-1}}\widetilde{\Phi}_1\big)\big), \\ &\big(h_{q^{-1}}\widetilde{\Phi}_1\big)B_2 = \big(h_{q^{-1}}\widetilde{\Phi}_2\big)B_1.\end{aligned}$$

Thus, there exist two polynomials $\Psi$ and $B$ such that

$$\begin{aligned}&\Psi_1 + q^{-1}\Phi\big(H_{q^{-1}}\widetilde{\Phi}_1\big) = \big(h_{q^{-1}}\widetilde{\Phi}_1\big)\Psi, \qquad \Psi_2 + q^{-1}\Phi\big(H_{q^{-1}}\widetilde{\Phi}_2\big) = \big(h_{q^{-1}}\widetilde{\Phi}_2\big)\Psi, \\ &B_1 = (h_{q^{-1}}\widetilde{\Phi}_1)B, \qquad B_2 = (h_{q^{-1}}\widetilde{\Phi}_2)B.\end{aligned} \tag{2.13}$$

Then, formulas (2.7), (2.8) become

$$\big(h_{q^{-1}}\widetilde{\Phi}_i\big)\big\{H_q(\Phi u) + \Psi u + B\big(x^{-1}uh_q u\big)\big\} = 0, \qquad i \in \{1, 2\}. \tag{2.14}$$

But the polynomials $h_{q^{-1}}\widetilde{\Phi}_1$ and $h_{q^{-1}}\widetilde{\Phi}_2$ are also co-prime. Using the Bezout identity, there exist two polynomials $A_1$ and $A_2$ such that

$$A_1\big(h_{q^{-1}}\widetilde{\Phi}_1\big) + A_2\big(h_{q^{-1}}\widetilde{\Phi}_2\big) = 1.$$

Consequently, the operation $A_1 \times (2.14_{i=1}) + A_2 \times (2.14_{i=2})$ leads to (2.9). With (2.11) and (2.13) it is easy to prove (2.10). ■

**Proposition 3.** *For any $q$-Laguerre–Hahn form $u$, the triplet* $(\Phi, \Psi, B)$ ($\Phi$ *monic*) *which realizes the minimum of* $\hbar(u)$ *is unique.*

**Proof.** If $s_1 = s_2$ in (2.9), (2.10) and $s_1 = s_2 = s = \min\hbar(u)$, then $t_1 = t = t_2$. Consequently, $\Phi_1 = \Phi = \Phi_2$, $B_1 = B = B_2$ and $\Psi_1 = \Psi = \Psi_2$. ■

Then, it's necessary to give a criterion which allows us to simplify the class. For this, let us recall the following lemma:

**Lemma 5.** *Consider $u$ a regular form, $\Phi$, $\Psi$ and $B$ three polynomials, $\Phi$ monic. For any zero $c$ of $\Phi$, denoting*

$$\begin{aligned}
&\Phi(x) = (x-c)\Phi_c(x),\\
&q\Psi(x) + \Phi_c(x) = (x-cq)\Psi_{cq}(x) + r_{cq},\\
&qB(x) = (x-cq)B_{cq}(x) + b_{cq}.
\end{aligned}\tag{2.15}$$

*The following statements are equivalent:*

$$\begin{aligned}
&H_q(\Phi u) + \Psi u + B\big(x^{-1}uh_qu\big) = 0,\\
&H_q(\Phi_c u) + \Psi_{cq}u + B_{cq}\big(x^{-1}uh_qu\big) + r_{cq}(x-cq)^{-1}u + b_{cq}(x-cq)^{-1}\big(x^{-1}uh_qu\big)\\
&\qquad - \big\{\langle u, \Psi_{cq}\rangle + \langle x^{-1}uh_qu, B_{cq}\rangle\big\}\delta_{cq} = 0.
\end{aligned}\tag{2.16}$$

**Proof.** The proof is obtained straightforwardly by using the relations in (1.2) and in (2.1). ■

**Proposition 4.** *A regular form $u$ $q$-Laguerre–Hahn satisfying (1.17) is of class $s$ if and only if*

$$\begin{aligned}
&\prod_{c\in\mathcal{Z}_\Phi}\Big\{|q(h_q\Psi)(c) + (H_q\Phi)(c)| + |q(h_qB)(c)|\\
&\qquad + \big|\langle u, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi) + q(h_qu(\theta_0\circ\theta_{cq}B))\rangle\big|\Big\} > 0,
\end{aligned}\tag{2.17}$$

*where $\mathcal{Z}_\Phi$ is the set of roots of $\Phi$.*

**Proof.** Let $c$ be a root of $\Phi$: $\Phi(x) = (x-c)\Phi_c(x)$. On account of (2.15) we have

$$\begin{aligned}
&r_{cq} = q\Psi(cq) + \Phi_c(cq) = q(h_q\Psi)(c) + (H_q\Phi)(c), \qquad b_{cq} = qB(cq) = q(h_qB)(c),\\
&\Psi_{cq}(x) = q(\theta_{cq}\Psi)(x) + (\theta_{cq}\Phi_c)(x) = q(\theta_{cq}\Psi)(x) + (\theta_{cq}\circ\theta_c\Phi)(x),\\
&B_{cq}(x) = q(\theta_{cq}B)(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle u, \Psi_{cq}\rangle + \langle x^{-1}uh_qu, B_{cq}\rangle &= \langle u, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi)\rangle + \langle uh_qu, q\theta_0\circ\theta_{cq}B\rangle\\
&= \langle u, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi)\rangle + \langle u, q(h_qu(\theta_0\circ\theta_{cq}B))\rangle\\
&= \langle u, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi) + q(h_qu(\theta_0\circ\theta_{cq}B))\rangle.
\end{aligned}$$

The condition (2.17) is necessary. Let us suppose that $c$ fulfils the conditions

$$r_{cq} = 0, \qquad b_{cq} = 0, \qquad \langle u, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi) + q(h_qu(\theta_0\circ\theta_{cq}B))\rangle = 0.$$

Then on account of Lemma 5 (2.16) becomes

$$H_q(\Phi_c u) + \Psi_{cq}u + B_{cq}\big(x^{-1}uh_qu\big) = 0$$

with $s_c = \max(\max(\deg\Phi_c, \deg B_{cq}) - 2, \deg\Psi_c - 1) < s$, what contradicts with $s := \min\hbar(u)$.

The condition (2.17) is sufficient. Let us suppose $u$ to be of class $\widetilde{s} < s$. There exist three polynomials $\widetilde{\Phi}$ (monic) $\deg\widetilde{\Phi} = \widetilde{t}$, $\widetilde{\Psi}$, $\deg\widetilde{\Phi} = \widetilde{p}$, $\widetilde{B}$, $\deg\widetilde{B} = \widetilde{r}$ such that

$$H_q(\widetilde{\Phi}u) + \widetilde{\Psi}u + \widetilde{B}(x^{-1}uh_qu) = 0$$

with $\widetilde{s} = \max(\widetilde{d}-2, \widetilde{p}-1)$ where $\widetilde{d} := \max(\widetilde{t}, \widetilde{r})$. By Proposition 2, it exists a polynomial $\chi$ such that

$$\Phi = \chi\,\widetilde{\Phi}, \qquad \Psi = (h_{q^{-1}}\chi)\widetilde{\Psi} - q^{-1}(H_{q^{-1}}\chi)\widetilde{\Phi}, \qquad B = (h_{q^{-1}}\chi)\widetilde{B}.$$

Since $\widetilde{s} < s$ hence $\deg \chi \geq 1$. Let $c$ be a zero of $\chi$ : $\chi(x) = (x-c)\chi_c(x)$. On account of (1.10) we have

$$q\Psi(x) + \Phi_c(x) = (x - cq)\big\{(h_{q^{-1}}\chi_c)(x)\widetilde{\Psi}(x) - q^{-1}(H_{q^{-1}}\chi_c)(x)\widetilde{\Phi}(x)\big\}.$$

Thus $r_{cq} = 0$ and $b_{cq} = 0$. Moreover, with (1.8) we have

$$\begin{aligned}
&\big\langle u, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi) + q\big(h_q u(\theta_0\circ\theta_{cq}B)\big)\big\rangle \\
&\quad = \big\langle u, (h_{q^{-1}}\chi_c)\widetilde{\Psi} - q^{-1}(H_{q^{-1}}\chi_c)\widetilde{\Phi} + (h_q u)\theta_0((h_{q^{-1}}\chi_c)\widetilde{B})\big\rangle \\
&\quad = \big\langle u, (h_{q^{-1}}\chi_c)\widetilde{\Psi} - (H_q\circ h_{q^{-1}}\chi_c)\widetilde{\Phi} + (h_q u)\theta_0((h_{q^{-1}}\chi_c)\widetilde{B})\big\rangle \\
&\quad = \langle \widetilde{\Psi}u, h_{q^{-1}}\chi_c\rangle + \langle H_q(\widetilde{\Phi}u), h_{q^{-1}}\chi_c\rangle + \langle \widetilde{B}\big(x^{-1}uh_q u\big), h_{q^{-1}}\chi_c\rangle \\
&\quad = \big\langle H_q(\widetilde{\Phi}u) + \widetilde{\Psi}u + \widetilde{B}\big(x^{-1}uh_q u\big), h_{q^{-1}}\chi_c\big\rangle = 0.
\end{aligned}$$

This is contradictory with (2.17). Consequently, $\widetilde{s} = s$, $\widetilde{\Phi} = \Phi$, $\widetilde{\Psi} = \Psi$ and $\widetilde{B} = B$. ■

**Remark 2.** When $q \longrightarrow 1$ we recover again the criterion which allows us to simplify a usual Laguerre–Hahn form [6].

**Remark 3.** When $B = 0$ and $s = 0$, the form $u$ is usually called $q$-classical [16]. When $B = 0$ and $s = 1$, the symmetrical $q$-semiclassical orthogonal $q$-polynomials of class one are exhaustively described in [14].

**Proposition 5.** *Let $u$ be a symmetrical $q$-Laguerre–Hahn form of class $s$ satisfying (1.17). The following statements hold*

(*i*) *If $s$ is odd, then the polynomials $\Phi$ and $B$ are odd and $\Psi$ is even.*

(*ii*) *If $s$ is even, then the polynomials $\Phi$ and $B$ are even and $\Psi$ is odd.*

**Proof.** Writing

$$\Phi(x) = \Phi^{\mathrm{e}}\big(x^2\big) + x\Phi^{\mathrm{o}}\big(x^2\big), \qquad \Psi(x) = \Psi^{\mathrm{e}}\big(x^2\big) + x\Psi^{\mathrm{o}}\big(x^2\big), \qquad B(x) = B^{\mathrm{e}}\big(x^2\big) + xB^{\mathrm{o}}\big(x^2\big),$$

then (1.17) becomes

$$\begin{aligned}
&H_q\big(\Phi^{\mathrm{e}}\big(x^2\big)u\big) + x\Psi^{\mathrm{o}}\big(x^2\big)u + B^{\mathrm{e}}\big(x^2\big)\big(x^{-1}uh_q u\big) \\
&\qquad + H_q\big(x\Phi^{\mathrm{o}}\big(x^2\big)u\big) + \Psi^{\mathrm{e}}\big(x^2\big)u + xB^{\mathrm{o}}\big(x^2\big)\big(x^{-1}uh_q u\big) = 0.
\end{aligned}$$

Denoting

$$\begin{aligned}
&w^{\mathrm{e}} = H_q\big(\Phi^{\mathrm{e}}\big(x^2\big)u\big) + x\Psi^{\mathrm{o}}\big(x^2\big)u + B^{\mathrm{e}}\big(x^2\big)\big(x^{-1}uh_q u\big), \\
&w^{\mathrm{o}} = H_q\big(x\Phi^{\mathrm{o}}\big(x^2\big)u\big) + \Psi^{\mathrm{e}}\big(x^2\big)u + xB^{\mathrm{o}}\big(x^2\big)\big(x^{-1}uh_q u\big).
\end{aligned} \tag{2.18}$$

Then,

$$w^{\mathrm{o}} + w^{\mathrm{e}} = 0. \tag{2.19}$$

From (2.19) we get

$$(w^{\mathrm{o}})_n = -(w^{\mathrm{e}})_n, \qquad n \geq 0. \tag{2.20}$$

From definitions in (2.18) and (2.20) we can write for $n \geq 0$

$$(w^{\mathrm{e}})_{2n} = \langle u, x^{2n+1}\Psi^{\mathrm{o}}\big(x^2\big) - [2n]_q x^{2n-1}\Phi^{\mathrm{e}}\big(x^2\big)\rangle + \langle uh_q u, x^{2n-1}B^{\mathrm{e}}\big(x^2\big)\rangle,$$

$$(w^{\mathrm{o}})_{2n+1} = \langle u, x^{2n+1}\Psi^{\mathrm{e}}(x^2) - [2n+1]_q x^{2n+1}\Phi^{\mathrm{o}}(x^2)\rangle + \langle uh_q u, x^{2n+1}B^{\mathrm{o}}(x^2)\rangle. \tag{2.21}$$

Now, with the fact that $u$ is a symmetrical form then $uh_qu$ is also a symmetrical form. Indeed,

$$\begin{aligned}(uh_qu)_{2n+1} &= \sum_{k=0}^{2n+1}(h_qu)_k(u)_{2n+1-k} = \sum_{k=0}^{2n+1} q^k(u)_k(u)_{2n+1-k} \\ &= \sum_{k=0}^{n} q^{2k}(u)_{2k}(u)_{2(n-k)+1} + \sum_{k=0}^{n} q^{2k+1}(u)_{2k+1}(u)_{2(n-k)} = 0, \qquad n \geq 0.\end{aligned}$$

Thus (2.21) gives

$$(w^{\mathrm{o}})_{2n+1} = 0 = (w^{\mathrm{e}})_{2n}, \qquad n \geq 0. \tag{2.22}$$

On account of (2.19) and (2.22) we deduce $w^{\mathrm{o}} = w^{\mathrm{e}} = 0$. Consequently $u$ satisfies two $q$-difference equations

$$H_q\big(\Phi^{\mathrm{e}}(x^2)u\big) + x\Psi^{\mathrm{o}}(x^2)u + B^{\mathrm{e}}(x^2)\big(x^{-1}uh_qu\big) = 0, \tag{2.23}$$

and

$$H_q\big(x\Phi^{\mathrm{o}}(x^2)u\big) + \Psi^{\mathrm{e}}(x^2)u + xB^{\mathrm{o}}(x^2)\big(x^{-1}uh_qu\big) = 0. \tag{2.24}$$

$(i)$ If $s = 2k+1$, with $s = \max(d-2, p-1)$ we get $d \leq 2k+3$, $p \leq 2k+2$ then $\deg(x\Psi^{\mathrm{o}}(x^2)) \leq 2k+1$, $\deg(\Phi^{\mathrm{e}}(x^2)) \leq 2k+2$ and $\deg(B^{\mathrm{e}}(x^2)) \leq 2k+2$. So, in accordance with (2.23), we obtain the contradiction $s = 2k+1 \leq 2k$. Necessary $\Phi^{\mathrm{e}} = B^{\mathrm{e}} = \Psi^{\mathrm{o}} = 0$.

$(ii)$ If $s = 2k$, with $s = \max(d-2, p-1)$ we get $d \leq 2k+2$, $p \leq 2k+1$ then $\deg(\Psi^{\mathrm{e}}(x^2)) \leq 2k$, $\deg(x\Phi^{\mathrm{o}}(x^2)) \leq 2k+1$ and $\deg(xB^{\mathrm{o}}(x^2)) \leq 2k+1$. So, in accordance with (2.24), we obtain the contradiction $s = 2k \leq 2k-1$. Necessary $\Phi^{\mathrm{o}} = B^{\mathrm{o}} = \Psi^{\mathrm{e}} = 0$. Hence the desired result. ■

## 3 Different characterizations of $q$-Laguerre–Hahn forms

One of the most important characterizations of the $q$-Laguerre–Hahn forms is given in terms of a non homogeneous second order $q$-difference equation so called $q$-Riccati equation fulfilled by its formal Stieltjes series. See also [6, 8, 10, 15] for the usual case and [13] for the discrete one.

**Proposition 6.** *Let $u$ be a regular form. The following statement are equivalents:*

(a) *$u$ belongs to the $q$-Laguerre–Hahn class, satisfying (1.17).*

(b) *The Stieljes formal series $S(u)$ satisfies the $q$-Riccati equation*

$$(h_{q^{-1}}\Phi)(z)H_{q^{-1}}(S(u))(z) = B(z)S(u)(z)(h_{q^{-1}}S(u))(z) + C(z)S(u)(z) + D(z), \tag{3.1}$$

*where $\Phi$ and $B$ are polynomials defined in (1.17) and*

$$\begin{aligned}&C(z) = -(H_{q^{-1}}\Phi)(z) - q\Psi(z), \\ &D(z) = -\big\{H_{q^{-1}}(u\theta_0\Phi)(z) + q(u\theta_0\Psi)(z) + q(uh_qu)\big(\theta_0^2B\big)(z)\big\}.\end{aligned} \tag{3.2}$$

**Proof.** $(a) \Rightarrow (b)$. Suppose that $(a)$ is satisfied, then there exist three polynomials $\Phi$ (monic), $\Psi$ and $B$ such that $H_q(\Phi u)+\Psi u+B(x^{-1}uh_qu) = 0$. From (1.11) the above $q$-difference equation becomes

$$(h_{q^{-1}}\Phi)(H_qu) + \big\{\Psi + q^{-1}(H_{q^{-1}}\Phi)\big\}u + B\big(x^{-1}uh_qu\big) = 0.$$

From definition of $S(u)$ and the linearity of $S$ we obtain

$$S\big((h_{q^{-1}}\Phi)(H_q u)\big)(z) + S(\Psi u)(z) + q^{-1}S\big((H_{q^{-1}}\Phi)u\big)(z) + S\big(B(x^{-1}uh_q u)\big)(z) = 0. \tag{3.3}$$

Moreover,

$$\begin{aligned}
S(\Psi u)(z) &\overset{\text{by (1.13)}}{=} \Psi(z)S(u)(z) + (u\theta_0\Psi)(z),\\
q^{-1}S((H_{q^{-1}}\Phi)u)(z) &\overset{\text{by (1.13)}}{=} q^{-1}(H_{q^{-1}}\Phi)(z)S(u)(z) + q^{-1}(u\theta_0(H_{q^{-1}}\Phi))(z),\\
S\big((h_{q^{-1}}\Phi)(H_q u)\big)(z) &\overset{\text{by (1.13)}}{=} (h_{q^{-1}}\Phi)(z)S(H_q u)(z) + \big((H_q u)\theta_0(h_{q^{-1}}\Phi)\big)(z)\\
&\overset{\text{by (1.16)}}{=} q^{-1}(h_{q^{-1}}\Phi)(z)H_{q^{-1}}(S(u))(z) + \big((H_q u)\theta_0(h_{q^{-1}}\Phi)\big)(z),\\
S\big(B(x^{-1}uh_q u)\big)(z) &\overset{\text{by (1.13)}}{=} B(z)S\big(x^{-1}uh_q u)\big)(z) + \big((x^{-1}uh_q u)\theta_0 B\big)(z)\\
&\overset{\text{by (1.15)}}{=} z^{-1}B(z)S\big(uh_q u)\big)(z) + \big((uh_q u)\theta_0^2 B\big)(z)\\
&\overset{\text{by (1.14)}}{=} -B(z)S(u)(z)S(h_q u)(z) + \big((uh_q u)\theta_0^2 B\big)(z)\\
&\overset{\text{by (1.16)}}{=} -q^{-1}B(z)S(u)(z)(h_{q^{-1}}S(u))(z) + \big((uh_q u)\theta_0^2 B\big)(z),
\end{aligned}$$

and

$$(u\theta_0(H_{q^{-1}}\Phi))(z) + q\big((H_q u)\theta_0(h_{q^{-1}}\Phi)\big)(z) \overset{\text{by (1.12)}}{=} H_{q^{-1}}(u\theta_0\Phi)(z).$$

(3.3) becomes

$$\begin{aligned}
(h_{q^{-1}}\Phi)(z)H_{q^{-1}}(S(u))(z) &= B(z)S(u)(z)(h_{q^{-1}}S(u))(z) - (H_{q^{-1}}\Phi + q\Psi)(z)S(u)(z)\\
&\quad - \big\{H_{q^{-1}}(u\theta_0\Phi) + qu\theta_0\Psi + q(uh_q u)\theta_0^2 B\big\}(z).
\end{aligned}$$

The previous relation gives (3.1) with (3.2).

$(b) \Rightarrow (a)$. Let $u \in \mathcal{P}'$ regular with its formal Stieltjes series $S(u)$ satisfying (3.1). Likewise as in the previous implication, formula (3.1) leads to

$$\begin{aligned}
&S\big\{H_q(\Phi u) - q^{-1}(C + H_{q^{-1}}\Phi)u + B\big(x^{-1}uh_q u\big)\big\}\\
&\qquad = q^{-1}D - q^{-1}u\theta_0 C + \big((uh_q u)\theta_0^2 B\big) + ((H_q u)\theta_0(h_{q^{-1}}\Phi)),
\end{aligned}$$

which implies

$$\begin{aligned}
&S\big\{H_q(\Phi u) - q^{-1}(C + H_{q^{-1}}\Phi)u + B\big(x^{-1}uh_q u\big)\big\} = 0,\\
&D(z) = (u\theta_0 C)(z) - q\big((uh_q u)\big(\theta_0^2 B\big)\big)(z) - q((H_q u)\theta_0(h_{q^{-1}}\Phi))(z).
\end{aligned}$$

According to (3.2) and (1.12) we deduce that

$$H_q(\Phi u) + \Psi u + B\big(x^{-1}uh_q u\big) = 0,$$

with

$$\Psi = -q^{-1}(C + H_{q^{-1}}\Phi). \tag{3.4}$$

■

We are going to give the criterion which allows us to simplify the class of $q$-Laguerre–Hahn form in terms of the coefficients corresponding to the previous characterization.

**Proposition 7.** *A regular form $u$ $q$-Laguerre–Hahn satisfying (3.1) is of class $s$ if and only if*

$$\prod_{c\in Z_\Phi}\big\{|B(cq)|+|C(cq)|+|D(cq)|\big\}>0, \tag{3.5}$$

*where $Z_\Phi$ is the set of roots of $\Phi$ with*

$$s=\max\big(\deg B-2,\deg C-1,\deg D\big). \tag{3.6}$$

**Proof.** By comparing (2.17) and (3.5), it is enough to prove the following equalities

$$|C(cq)|=\big|q(h_q\Psi)(c)+(H_q\Phi)(c)\big|,$$
$$|D(cq)|=\big|\big\langle u,q(\theta_{cq}\Psi)+(\theta_{cq}\circ\theta_c\Phi)+q\big(h_qu(\theta_0\circ\theta_{cq}B)\big)\big\rangle\big|.$$

Indeed, on account of (3.2), the definition of the polynomial $uf$, the definition of the product form $uv$ and (1.8) we have

$$C(cq)=-(H_{q^{-1}}\Phi)(cq)-q\Psi(cq)=-(H_q\Phi)(c)-q(h_q\Psi)(c),$$

and

$$\begin{aligned}D(cq)&=-\big\{H_{q^{-1}}(u\theta_0\Phi)(cq)+q(u\theta_0\Psi)(cq)+q(uh_qu)({\theta_0}^2B)(cq)\big\}\\&=-\big\{H_q(u\theta_0\Phi)(c)+\langle u,q\theta_{cq}\Psi\rangle+\langle uh_qu,q\theta_0\circ\theta_{cq}B\rangle\big\}\\&=-\big\{H_q(u\theta_0\Phi)(c)+\big\langle u,q\theta_{cq}\Psi+q\big(h_qu(\theta_0\circ\theta_{cq}B)\big)\big\rangle\big\}.\end{aligned}$$

Moreover,

$$H_q(u\theta_0\Phi)(c)\overset{\text{by (1.6)}}{=}\frac{(u\theta_0\Phi)(cq)-(u\theta_0\Phi)(c)}{(q-1)c}=\Big\langle u,\frac{\theta_{cq}\Phi-\theta_c\Phi}{cq-c}\Big\rangle=\big\langle u,\theta_{cq}\circ\theta_c\Phi\big\rangle.$$

Thus (2.17) is equivalent to (3.5). To prove (3.6), according to the definition of the class we may write

$$s=\max\big(\deg B-2,\deg\Phi-2,\deg\Psi-1\big). \tag{3.7}$$

• If $\deg\Psi\neq\max\big(\deg B-1,\deg\Phi-1\big)$, on account of (3.2) and (3.7) we get the following implications

$$\deg B\le\deg\Phi\Rightarrow\begin{cases}\deg C=s+1,\\ \deg D\le s\end{cases}\Rightarrow\max\big(\deg B-2,\deg C-1,\deg D\big)=s,$$
$$\deg B>\deg\Phi\Rightarrow\begin{cases}\deg C\le s+1,\\ \deg D=s\end{cases}\Rightarrow\max\big(\deg B-2,\deg C-1,\deg D\big)=s.$$

• If $\deg\Psi=\max\big(\deg B-1,\deg\Phi-1\big)$ and $\deg B>\deg\Phi$ then $s+1=\deg\Psi=\deg B-1>\deg\Phi-1$. Consequently, $\max\big(\deg B-2,\deg C-1,\deg D\big)=s$.

• If $\deg\Psi=\max\big(\deg B-1,\deg\Phi-1\big)$ and $\deg B=\deg\Phi$ then $\deg\Psi=\deg B-1=\deg\Phi-1$ which implies $\deg B-2=s$, $\deg C-1\le s$, $\deg D\le s$. Therefore $\max\big(\deg B-2,\deg C-1,\deg D\big)=s$.

• If $\deg\Psi=\max\big(\deg B-1,\deg\Phi-1\big)$ and $\deg B<\deg\Phi$ then $\deg\Psi=\deg\Phi-1$ and $s=\deg\Psi-1$. Writing $\Phi(x)=x^{p+1}+\text{lower order terms}$, $\Psi(x)=a_px^p+\cdots+a_0$, by virtue of (3.2) and (1.3), it is worth noting that $C(z)=-\big([p+1]_{q^{-1}}+qa_p\big)z^{p-1}+\text{lower order terms}$ and $D(z)=-\big([p]_{q^{-1}}+qa_p\big)z^{p-1}+\text{lower order terms}$ with $[p+1]_{q^{-1}}\neq[p]_{q^{-1}}$ assuming either $\deg C=s$ or $\deg D=s$. Thus, $\max\big(\deg B-2,\deg C-1,\deg D\big)=s$.

Hence the desired result (3.6). ■

An other important characterization of the $q$-Laguerre–Hahn forms is the structure relation. See also [6, 15] for the usual case and [13] for the discrete one.

**Proposition 8.** *Let $u$ be a regular form and $\{P_n\}_{n\geq 0}$ be its MOPS. The following statements are equivalent:*

*(i) $u$ is a $q$-Laguerre–Hahn form satisfying (1.17).*

*(ii) There exist an integer $s \geq 0$, two polynomials $\Phi$ (monic), $B$ with $t = \deg\Phi \leq s+2$, $r = \deg B \leq s+2$ and a sequence of complex numbers $\{\lambda_{n,\nu}\}_{n,\nu\geq 0}$ such that*

$$\Phi(x)(H_qP_{n+1})(x) - h_q(BP_n^{(1)})(x) = \sum_{\nu=n-s}^{n+d} \lambda_{n,\nu}P_\nu(x), \qquad n > s, \quad \lambda_{n,n-s} \neq 0, \tag{3.8}$$

*where $d = \max(t,r)$ and $\{P_n^{(1)}\}_{n\geq 0}$ be the associated sequence of the first kind for the sequence $\{P_n\}_{n\geq 0}$.*

**Proof.** $(i) \Rightarrow (ii)$. Beginning with the expression $\Phi(x)(H_qP_{n+1})(x) - h_q\big(BP_n^{(1)}\big)(x)$ which is a polynomial of degree at most $n+d$. Then, there exists a sequence of complex numbers $\{\lambda_{n,\nu}\}_{n\geq 0,\, 0\leq\nu\leq n+d}$ such that

$$\Phi(x)(H_qP_{n+1})(x) - (h_qB)(x)\big(h_qP_n^{(1)}\big)(x) = \sum_{\nu=0}^{n+d} \lambda_{n,\nu}P_\nu(x), \qquad n \geq 0. \tag{3.9}$$

Multiplying both sides of (3.9) by $P_m$, $0 \leq m \leq n+d$ and applying $u$ we get

$$\langle u, \Phi P_m(H_qP_{n+1})\rangle - \langle h_qu, B(h_{q^{-1}}P_m)(u\theta_0P_{n+1})\rangle = \lambda_{n,m}\langle u, P_m^2\rangle, \\ n \geq 0, \qquad 0 \leq m \leq n+d. \tag{3.10}$$

On the other hand, applying $H_q(\Phi u) + \Psi u + B(x^{-1}uh_qu) = 0$ to $P_{n+1}(h_{q^{-1}}P_m)$, on account of the definitions, (1.10) and (1.8) we obtain

$$\begin{aligned} 0 &= \langle H_q(\Phi u) + \Psi u + B\big(x^{-1}uh_qu\big), P_{n+1}(h_{q^{-1}}P_m)\rangle \\ &= \big\langle u, \Psi P_{n+1}(h_{q^{-1}}P_m) - \Phi H_q\big(P_{n+1}(h_{q^{-1}}P_m)\big)\big\rangle + \big\langle h_qu, u\theta_0(BP_{n+1}(h_{q^{-1}}P_m))\big\rangle \\ &= \langle u, \big\{\Psi(h_{q^{-1}}P_m) - q^{-1}\Phi(H_{q^{-1}}P_m)\big\}P_{n+1} - \Phi P_m(H_qP_{n+1})\rangle \\ &\quad + \langle h_qu, u\theta_0(BP_{n+1}(h_{q^{-1}}P_m))\rangle. \end{aligned}$$

Thus, for $n \geq 0$, $0 \leq m \leq n+d$

$$\begin{aligned} \langle u, \Phi P_m(H_qP_{n+1})\rangle &= \big\langle u, \big\{\Psi(h_{q^{-1}}P_m) - q^{-1}\Phi(H_{q^{-1}}P_m)\big\}P_{n+1}\big\rangle \\ &\quad + \langle h_qu, u\theta_0(BP_{n+1}(h_{q^{-1}}P_m))\rangle. \end{aligned} \tag{3.11}$$

Using (3.10), (3.11) to eliminate $\langle u, \Phi P_m(H_qP_{n+1})\rangle$ we get for $n \geq 0$, $0 \leq m \leq n+d$

$$\begin{aligned} &\big\langle u, \big\{\Psi(h_{q^{-1}}P_m) - q^{-1}\Phi(H_{q^{-1}}P_m)\big\}P_{n+1}\big\rangle \\ &\quad + \big\langle h_qu, u\theta_0(BP_{n+1}(h_{q^{-1}}P_m)) - (h_{q^{-1}}P_m)B(u\theta_0P_{n+1})\big\rangle = \lambda_{n,m}\langle u, P_m^2\rangle. \end{aligned} \tag{3.12}$$

Moreover, by virtue of (1.5) we have $B(u\theta_0P_{n+1}) = u\theta_0(BP_{n+1})$, $n > s$. Therefore, taking into account (1.4) and definitions, (3.12) yields for $n > s$, $0 \leq m \leq n+d$

$$\big\langle u, \big\{\Psi(h_{q^{-1}}P_m) - q^{-1}\Phi(H_{q^{-1}}P_m) + B((h_qu)\theta_0(h_{q^{-1}}P_m))\big\}P_{n+1}\big\rangle = \lambda_{n,m}\langle u, P_m^2\rangle$$

with

$$\deg\big\{\Psi(h_{q^{-1}}P_m) - q^{-1}\Phi(H_{q^{-1}}P_m) + B((h_q u)\theta_0(h_{q^{-1}}P_m))\big\} \leq m+s+1.$$

Consequently, the orthogonality of $\{P_n\}_{n\geq 0}$ with respect to $u$ gives

$$\lambda_{n,m} = 0, \qquad 0 \leq m \leq n-s-1, \quad n \geq s+1, \qquad \lambda_{n,n-s} \neq 0.$$

Hence the desired result (3.8).

$(ii) \Rightarrow (i)$. Let $v$ be the form defined by

$$v := H_q(\Phi u) + B\big(x^{-1}uh_q u\big) + \left(\sum_{i=0}^{s+1} a_i x^i\right) u$$

with $a_i \in \mathbb{C}$, $0 \leq i \leq s+1$. From definitions and the hypothesis of $(ii)$ we may write successively

$$\begin{aligned}
\langle v, P_{n+1}\rangle &= \big\langle H_q(\Phi u) + B\big(x^{-1}uh_q u\big), P_{n+1}\big\rangle + \langle u, P_{n+1}\sum_{i=0}^{s+1} a_i x^i\rangle \\
&= -\big\langle u, \Phi(H_q P_{n+1}) - (h_q u)\theta_0(BP_{n+1})\big\rangle + \langle u, P_{n+1}\sum_{i=0}^{s+1} a_i x^i\rangle \\
&= -\big\langle u, \sum_{\nu=n-s}^{n+d} \lambda_{n,\nu} P_\nu\big\rangle + \langle u, P_{n+1}\sum_{i=0}^{s+1} a_i x^i\rangle \\
&= -\sum_{\nu=n-s}^{n+d} \lambda_{n,\nu}\langle u, P_\nu\rangle + \sum_{i=0}^{s+1} a_i\langle u, x^i P_{n+1}\rangle, \qquad n > s.
\end{aligned}$$

From assumption of orthogonality of $\{P_n\}_{n\geq 0}$ with respect to $u$ we get

$$\langle v, P_n\rangle = 0, \qquad n \geq s+2.$$

In order to get $\langle v, P_n\rangle = 0$, for any $n \geq 0$, we shall choose $a_i$ with $i = 0, 1, \dots, s+1$, such that $\langle v, P_i\rangle = 0$, for $i = 0, 1, \dots, s+1$. These coefficients $a_i$ are determined in a unique way. Thus, we have deduced the existence of polynomial $\Psi(x) = \sum_{i=0}^{s+1} a_i x^i$ such that $\langle v, P_n\rangle = 0$, for any $n \geq 0$. This leads to $H_q(\Phi u) + \Psi u + B(x^{-1}uh_q u) = 0$ and the point $(i)$ is then proved. ∎

# 4 Applications

## 4.1 The co-recursive of a $q$-Laguerre–Hahn form

Let $\mu$ be a complex number, $u$ a regular form and $\{P_n\}_{n\geq 0}$ be its corresponding MOPS satisfying (1.1). We define the co-recursive $\big\{P_n^{[\mu]}\big\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ as the family of monic polynomials satisfying the following three-term recurrence relation [20, Definition 4.2]

$$\begin{aligned}
&P_0^{[\mu]}(x) = 1, \qquad P_1^{[\mu]}(x) = x - \beta_0 - \mu, \\
&P_{n+2}^{[\mu]}(x) = (x - \beta_{n+1})P_{n+1}^{[\mu]}(x) - \gamma_{n+1}P_n^{[\mu]}(x), \qquad n \geq 0.
\end{aligned}$$

Denoting by $u^{[\mu]}$ its corresponding regular form. It is well known that [20, equation (4.14)]

$$u^{[\mu]} = u\big(\delta - \mu x^{-1}u\big)^{-1}.$$

**Proposition 9.** *If $u$ is a $q$-Laguerre–Hahn form of class $s$, then $u^{[\mu]}$ is a $q$-Laguerre–Hahn form of the same class $s$.*

**Proof.** The relation linking $S(u)$ and $S(u^{[\mu]})$ is [20, equation (4.15)] $S(u^{[\mu]}) = \frac{S(u)}{1+\mu S(u)}$ or equivalently

$$S(u) = \frac{S(u^{[\mu]})}{1 - \mu S(u^{[\mu]})}. \tag{4.1}$$

From definitions and by virtue of (4.1) we have

$$h_{q^{-1}} S(u) = \frac{h_{q^{-1}} S(u^{[\mu]})}{1 - \mu h_{q^{-1}} S(u^{[\mu]})}$$

and

$$\begin{aligned}(H_{q^{-1}} S(u))(z) &= \frac{\frac{(h_{q^{-1}} S(u^{[\mu]}))(z)}{1-\mu(h_{q^{-1}} S(u^{[\mu]}))(z)} - \frac{S(u^{[\mu]})(z)}{1-\mu S(u^{[\mu]})(z)}}{(q^{-1}-1)z} \\ &= \frac{(H_{q^{-1}} S(u^{[\mu]}))(z)}{\big(1 - \mu(h_{q^{-1}} S(u^{[\mu]}))(z)\big)\big(1 - \mu S(u^{[\mu]})(z)\big)}.\end{aligned}$$

Replacing the above results in (3.1) the $q$-Riccati equation becomes

$$\begin{aligned}&(h_{q^{-1}}\Phi)\frac{H_{q^{-1}} S(u^{[\mu]})}{\big(1 - \mu h_{q^{-1}} S(u^{[\mu]})\big)\big(1 - \mu S(u^{[\mu]})\big)} \\ &\qquad = B\frac{S(u^{[\mu]})}{1 - \mu S(u^{[\mu]})}\frac{h_{q^{-1}} S(u^{[\mu]})}{1 - \mu h_{q^{-1}} S(u^{[\mu]})} + C\frac{S(u^{[\mu]})}{1 - \mu S(u^{[\mu]})} + D.\end{aligned}$$

Equivalently

$$\begin{aligned}(h_{q^{-1}}\Phi) H_{q^{-1}} S(u^{[\mu]}) &= B S(u^{[\mu]}) h_{q^{-1}} S(u^{[\mu]}) + C S(u^{[\mu]})\big(1 - \mu h_{q^{-1}} S(u^{[\mu]})\big) \\ &\quad + D\big(1 - \mu h_{q^{-1}} S(u^{[\mu]})\big)\big(1 - \mu S(u^{[\mu]})\big).\end{aligned}$$

Therefore the $q$-Riccati equation satisfied by $S(u^{[\mu]})$

$$(h_{q^{-1}}\Phi^{[\mu]}) H_{q^{-1}} S(u^{[\mu]}) = B^{[\mu]} S(u^{[\mu]}) h_{q^{-1}} S(u^{[\mu]}) + C^{[\mu]} S(u^{[\mu]}) + D^{[\mu]}, \tag{4.2}$$

where

$$\begin{aligned}&K\Phi^{[\mu]}(x) = \Phi(x) + \mu(1-q)x(h_q D)(x), && KB^{[\mu]}(x) = B(x) - \mu C(x) + \mu^2 D(x), \\ &KC^{[\mu]}(x) = C(x) - 2\mu D(x), && KD^{[\mu]}(x) = D(x),\end{aligned} \tag{4.3}$$

the non zero constant $K$ is chosen such that the polynomial $\Phi^{[\mu]}$ is monic. $u^{[\mu]}$ is then a $q$-Laguerre–Hahn form.

On account of (3.2), (3.4) and (4.3) we get

$$K\Psi^{[\mu]} = \Psi + \mu\big(q^{-1}D + h_q D\big). \tag{4.4}$$

As a consequence, the regular form $u^{[\mu]}$ fulfils the following $q$-difference equation

$$H_q\big(\Phi^{[\mu]} u^{[\mu]}\big) + \Psi^{[\mu]} u^{[\mu]} + B^{[\mu]}\big(x^{-1} u^{[\mu]} h_q u^{[\mu]}\big) = 0. \tag{4.5}$$

We suppose that the $q$-Riccati equation (3.1) of $u$ is irreducible of class $s$. With respect to the class, we use the result (3.5) of Proposition 7 and get for every zero $c$ of $\Phi^{[\mu]}$:

- If $D(cq) \neq 0$, then $D^{[\mu]}(cq) = K^{-1}D(cq) \neq 0$ and equation (4.2) is not reducible.
- We suppose that $D(cq) = 0$. From the fact that $\Phi^{[\mu]}(c) = 0$, the first relation in (4.3) leads to $\Phi(c) = 0$ and the third equality in (4.3) gives $C^{[\mu]}(cq) = K^{-1}C(cq)$.

If $C(cq) \neq 0$, then the equation (4.2) is still not reducible. If $C(cq) = 0 = D(cq)$, then $B^{[\mu]}(cq) = K^{-1}B(cq) \neq 0$ since $u$ is of class $s$. We conclude that

$$\left|B^{[\mu]}(cq)\right| + \left|C^{[\mu]}(cq)\right| + \left|D^{[\mu]}(cq)\right| > 0.$$

Consequently, the class $s^{[\mu]}$ of $u^{[\mu]}$ is given by $s^{[\mu]} = \max\big(\deg B^{[\mu]} - 2, \deg C^{[\mu]} - 1, \deg D^{[\mu]}\big)$. Accordingly to the last equality in (4.3) and (3.6) we get $s^{[\mu]} = \max\big(\deg B^{[\mu]} - 2, \deg C^{[\mu]} - 1, \deg D\big)$. A discussion on the degree leads to $s^{[\mu]} = s$. ■

**Example 1.** Let $u$ be a $q$-classical form satisfying the $q$-analog of the distributional equation of Pearson type

$$H_q(\phi u) + \psi u = 0, \tag{4.6}$$

where $\phi$ is a monic polynomial of degree at most two and $\psi$ a polynomial of degree one, the co-recursive $u^{[\mu]}$ of $u$ is a $q$-Laguerre–Hahn form of class zero. $u^{[\mu]}$ and the Stieltjes function $S(u^{[\mu]})$ satisfy, respectively, the $q$-difference equation (4.5) and the $q$-Riccati equation (4.2) where on account of (4.3), (4.4)

$$\begin{aligned}
&K\Phi^{[\mu]}(x) = \frac{\phi''(0)}{2}x^2 + \left\{\phi'(0) + \mu(q-1)\left(\frac{\phi''(0)}{2} + q\psi'(0)\right)\right\}x + \phi(0),\\
&K\Psi^{[\mu]}(x) = \psi'(0)x + \psi(0) - \mu\big(q^{-1}+1\big)\left(\frac{\phi''(0)}{2} + q\psi'(0)\right),\\
&KB^{[\mu]}(x) = \mu\left\{\left(\big(q^{-1}+1\big)\frac{\phi''(0)}{2} + q\psi'(0)\right)x + \phi'(0) + q\psi(0) - \left(\frac{\phi''(0)}{2} + q\psi'(0)\right)\mu\right\},\\
&KC^{[\mu]}(x) = -\left(q\psi'(0) + (q^{-1}+1)\frac{\phi''(0)}{2}\right)x - \phi'(0) - q\psi(0) + 2\mu\left(\frac{\phi''(0)}{2} + q\psi'(0)\right),\\
&KD^{[\mu]}(x) = -\frac{\phi''(0)}{2} - q\psi'(0).
\end{aligned}$$

### 4.2 The associated of a $q$-Laguerre–Hahn form

Let $u$ be a regular form and $\{P_n\}_{n\geq 0}$ its corresponding MOPS satisfying (1.1). The associated sequence of the first kind $\big\{P_n^{(1)}\big\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ satisfies the following three-term recurrence relation [20]

$$\begin{aligned}
&P_0^{(1)}(x) = 1, \qquad P_1^{(1)}(x) = x - \beta_1,\\
&P_{n+2}^{(1)}(x) = (x - \beta_{n+2})P_{n+1}^{(1)}(x) - \gamma_{n+2}P_n^{(1)}(x), \qquad n \geq 0.
\end{aligned}$$

Denoting by $u^{(1)}$ its corresponding regular form.

**Proposition 10.** *If $u$ is a $q$-Laguerre–Hahn form of class $s$, then $u^{(1)}$ is a $q$-Laguerre–Hahn form of the same class $s$.*

**Proof.** We assume that the formal Stieltjes function $S(u)$ of $u$ satisfies (3.1). The relationship between $S(u^{(1)})$ and $S(u)$ is [20, equation (4.7)]

$$\gamma_1 S(u^{(1)})(z) = -\frac{1}{S(u)(z)} - (z - \beta_0).$$

Consequently,

$$S(u)(z) = -\frac{1}{\gamma_1 S(u^{(1)})(z) + (z-\beta_0)}. \tag{4.7}$$

From definitions and by virtue of (4.7) we have

$$h_{q^{-1}}(S(u))(z) = -\frac{1}{\gamma_1 h_{q^{-1}}(S(u^{(1)}))(z) + q^{-1}z - \beta_0}$$

and

$$H_{q^{-1}}(S(u))(z) = \frac{\gamma_1 H_{q^{-1}}(S(u^{(1)}))(z) + 1}{\big(\gamma_1 h_{q^{-1}}(S(u^{(1)}))(z) + q^{-1}z - \beta_0\big)\big(\gamma_1 S(u^{(1)})(z) + z - \beta_0\big)}.$$

Substituting in (3.1) the $q$-Riccati equation becomes

$$\begin{aligned}(h_{q^{-1}}\Phi)(z)&\frac{\gamma_1 H_{q^{-1}}(S(u^{(1)}))(z) + 1}{\big(\gamma_1 h_{q^{-1}}(S(u^{(1)}))(z) + q^{-1}z - \beta_0\big)\big(\gamma_1 S(u^{(1)})(z) + z - \beta_0\big)}\\ &= \frac{B(z)}{\big(\gamma_1 h_{q^{-1}}(S(u^{(1)}))(z) + q^{-1}z - \beta_0\big)\big(\gamma_1 S(u^{(1)})(z) + z - \beta_0\big)}\\ &\quad - \frac{C(z)}{\big(\gamma_1 S(u^{(1)})(z) + z - \beta_0\big)} + D(z).\end{aligned}$$

Equivalently

$$\begin{aligned}\gamma_1&\big\{(h_{q^{-1}}\Phi)(z) + \big(q^{-1}-1\big)z\big(C(z) - (z-\beta_0)D(z)\big)\big\}H_{q^{-1}}(S(u^{(1)}))(z)\\ &= \gamma_1^2 D(z)S(u^{(1)})(z)h_{q^{-1}}(S(u^{(1)}))(z) + \gamma_1\big\{((q^{-1}+1)z - 2\beta_0)D(z) - C(z)\big\}S(u^{(1)})(z)\\ &\quad + B(z) + \big(q^{-1}z - \beta_0\big)(z-\beta_0)D(z) - \big(q^{-1}z-\beta_0\big)C(z) - (h_{q^{-1}}\Phi)(z).\end{aligned}$$

Therefore the $q$-Riccati equation satisfied by $S(u^{(1)})$

$$\big(h_{q^{-1}}\Phi^{(1)}\big)H_{q^{-1}}S(u^{(1)}) = B^{(1)}S\big(u^{(1)}\big)h_{q^{-1}}S\big(u^{(1)}\big) + C^{(1)}S\big(u^{(1)}\big) + D^{(1)}, \tag{4.8}$$

where

$$\begin{aligned}&K\Phi^{(1)}(x) = \Phi(x) + (q-1)x\{(qx-\beta_0)(h_qD)(x) - (h_qC)(x)\},\\ &KB^{(1)}(x) = \gamma_1 D(x), \qquad KC^{(1)}(x) = \gamma_1\big\{\big(\big(q^{-1}+1\big)x - 2\beta_0\big)D(x) - C(x)\big\},\\ &KD^{(1)}(x) = B(x) + (q^{-1}x-\beta_0)(x-\beta_0)D(x) - (q^{-1}x-\beta_0)C(x) - (h_{q^{-1}}\Phi)(x).\end{aligned} \tag{4.9}$$

$u^{(1)}$ is then a $q$-Laguerre–Hahn form.

Moreover, the regular form $u^{(1)}$ fulfils the $q$-difference equation

$$H_q\big(\Phi^{(1)}u^{(1)}\big) + \Psi^{(1)}u^{(1)} + B^{(1)}\big(x^{-1}u^{(1)}h_qu^{(1)}\big) = 0, \tag{4.10}$$

with

$$\Psi^{(1)} = -q^{-1}\big(C^{(1)} + H_{q^{-1}}\Phi^{(1)}\big). \tag{4.11}$$

Likewise, it is straightforward to prove that the class of $u^{(1)}$ is also $s$. ■

**Example 2.** If $u$ is a $q$-classical form satisfying the $q$-analog of the distributional equation of Pearson type (4.6) then the associated $u^{(1)}$ of $u$ is a $q$-Laguerre–Hahn form of class zero. $u^{(1)}$ and the formal Stieltjes function $S(u^{(1)})$ satisfy, respectively, the $q$-difference equation (4.10) and the $q$-Riccati equation (4.8) where on account of (4.9) and (4.11)

$$
\begin{aligned}
K\Phi^{(1)}(x) &= q\frac{\phi''(0)}{2}x^2 + \left\{q\phi'(0) + (q-1)\left(q\psi(0) + \beta_0\left(\frac{\phi''(0)}{2} + q\psi'(0)\right)\right)\right\}x + \phi(0),\\
K\Psi^{(1)}(x) &= -q^{-1}\left\{(q+1)\frac{\phi''(0)}{2} - \psi'(0))x + (q+1)\phi'(0)\right.\\
&\qquad\qquad \left. + q^2\psi(0) + \left(q^2 - q + 2\right)\left(\frac{\phi''(0)}{2} + q\psi'(0)\right)\beta_0\right\},\\
KB^{(1)}(x) &= -\gamma_1\left(\frac{\phi''(0)}{2} + q\psi'(0)\right),\\
KC^{(1)}(x) &= \gamma_1\left\{-\psi'(0)x + \beta_0(\phi''(0) + 2q\psi'(0)) + q\psi(0) + \phi'(0)\right\},\\
KD^{(1)}(x) &= \psi(\beta_0)x - \phi(\beta_0) - q\beta_0\psi(\beta_0).
\end{aligned}
$$

### 4.3 The inverse of a $q$-Laguerre–Hahn form

Let $u$ be a regular form and $\{P_n\}_{n\geq 0}$ its corresponding MOPS satisfying (1.1). Let $\{P_n^{(1)}\}_{n\geq 0}$ be its associated sequence of the first kind fulfilling (4.6) and orthogonal with respect to the regular form $u^{(1)}$. The inverse form of $u$ satisfies [20, equation (5.27)]

$$x^2u^{-1} = -\gamma_1 u^{(1)}. \tag{4.12}$$

The following results can be found in [2]

$$u^{-1} = \delta - \left(u^{-1}\right)_1\delta' - \gamma_1 x^{-2}u^{(1)}. \tag{4.13}$$

In general, the form $u^{-1}$ given by (4.13) is regular if and only if $\Delta_n \neq 0$, $n \geq 0$, with

$$\Delta_n = \langle u^{(1)}, \left(P_n^{(1)}\right)^2\rangle\left\{\gamma_1 + \sum_{\nu=0}^{n}\frac{\left(\gamma_1 P_{\nu-1}^{(2)}(0) - (u^{-1})_1 P_\nu^{(1)}(0)\right)^2}{\langle u^{(1)}, (P_\nu^{(1)})^2\rangle}\right\}, \qquad n \geq 0,$$

where $\{P_n^{(2)}\}_{n\geq 0}$ is the associated sequence of $\{P_n^{(1)}\}_{n\geq 0}$. In this case, the orthogonal sequence $\{P_n^{(-)}\}_{n\geq 0}$ relative to $u^{-1}$ is given by

$$
\begin{aligned}
&P_0^{(-)}(x) = 1, \qquad P_1^{(-)}(x) = P_1^{(1)}(x) + b_0,\\
&P_{n+2}^{(-)}(x) = P_{n+2}^{(1)}(x) + b_{n+1}P_{n+1}^{(1)}(x) + a_n P_n^{(1)}(x), \qquad n \geq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
&b_0 = \beta_1 - \left(u^{-1}\right)_1,\\
&b_{n+1} = \beta_{n+2} - \frac{\left((u^{-1})_1 P_n^{(1)}(0) - \gamma_1 P_{n-1}^{(2)}(0)\right)\left((u^{-1})_1 P_{n+1}^{(1)}(0) - \gamma_1 P_n^{(2)}(0)\right)}{\Delta_n}, \qquad n \geq 0,\\
&a_n = \frac{\Delta_{n+1}}{\Delta_n}, \qquad n \geq 0.
\end{aligned}
$$

Also, the sequence $\{P_n^{(-)}\}_{n\geq 0}$ satisfies the three-term recurrence relation

$$P_0^{(-)}(x) = 1, \qquad P_1^{(-)}(x) = x - \beta_0^{(-)},$$

$$P^{(-)}_{n+2}(x) = \big(x-\beta^{(-)}_{n+1}\big)P^{(-)}_{n+1}(x) - \gamma^{(-)}_{n+1}P^{(-)}_n(x), \qquad n\ge 0,$$

with

$$\beta^{(-)}_0 = \big(u^{-1}\big)_1, \qquad \beta^{(-)}_{n+1} = \beta_{n+2} + b_n - b_{n+1}, \qquad n\ge 0,$$
$$\gamma^{(-)}_1 = -\Delta_0, \qquad \gamma^{(-)}_2 = \gamma_1\frac{\Delta_1}{\Delta_0^2}, \qquad \gamma^{(-)}_{n+3} = \frac{\Delta_{n+2}\Delta_n}{\Delta^2_{n+1}}\gamma_{n+2}, \qquad n\ge 0.$$

In particular, when $\gamma_1>0$ and $u^{(1)}$ is positive definite, then $u^{-1}$ is regular. When $u^{(1)}$ is symmetrical, then $u^{-1}$ is a symmetrical regular form and we have

$$a_{2n} = \frac{\gamma_1\Lambda_n+1}{\gamma_1\Lambda_{n-1}+1}\gamma_{2n+2}, \qquad a_{2n+1} = \gamma_{2n+3}, \qquad n\ge 0, \tag{4.14}$$
$$\gamma^{(-)}_1 = -\gamma_1, \qquad \gamma^{(-)}_{2n+2} = a_{2n}, \qquad \gamma^{(-)}_{2n+3} = \frac{\gamma_{2n+2}\gamma_{2n+3}}{a_{2n}}, \qquad n\ge 0, \tag{4.15}$$

with

$$\Lambda_{-1} = 0, \qquad \Lambda_n = \sum_{\nu=0}^{n}\left(\prod_{k=0}^{\nu}\frac{\gamma_{2k+1}}{\gamma_{2k+2}}\right), \quad n\ge 0, \qquad \gamma_0 = 1. \tag{4.16}$$

**Proposition 11.** *If $u$ is a $q$-Laguerre–Hahn form of class $s$, then, when $u^{-1}$ is regular, $u^{-1}$ is a $q$-Laguerre–Hahn form of class at most $s+2$.*

**Proof.** Let $u$ be a $q$-Laguerre–Hahn form of class $s$ satisfying (1.17). It is seen in Proposition 10 that $u^{(1)}$ is also a $q$-Laguerre–Hahn form of class $s$ satisfying the $q$-difference equation (4.10) with polynomials $\Phi^{(1)}$, $\Psi^{(1)}$, $B^{(1)}$ respecting (4.9) and (4.11).

Let us suppose $u^{-1}$ is regular that is to say $\Delta_n\neq 0$, $n\ge 0$. Multiplying (4.10) by $(-\gamma_1)$ and on account of (4.12) and (1.7), the $q$-difference equation (4.10) becomes

$$H_q\big(x^2\Phi^{(1)}(x)u^{-1}\big) + x^2\Psi^{(1)}(x)u^{-1} - q^{-2}\gamma_1^{-1}B^{(1)}\big(x^{-1}\big(x^2u^{-1}\big)\big(x^2h_qu^{-1}\big)\big) = 0.$$

Consequently, the form $u^{-1}$ satisfies the following $q$-difference equation

$$H_q\big(\Phi^{(-)}u^{-1}\big) + \Psi^{(-)}u^{-1} + B^{(-)}\big(x^{-1}u^{-1}h_qu^{-1}\big) = 0, \tag{4.17}$$

with

$$\begin{aligned} K\Phi^{(-)}(x) &= x^2\big\{\Phi^{(1)}(x) + (1-q)\gamma_1^{-1}x(qx-\beta_0)(h_qB^{(1)})(x)\big\},\\ K\Psi^{(-)}(x) &= x\Big\{\big(q^{-1}+1\big)\big(\big(h_{q^{-1}}\Phi^{(1)}\big)(x) - q^{-1}\Phi^{(1)}(x)\big) - q^{-3}x(H_{q^{-1}}\Phi^{(1)})(x)\\ &\quad + \gamma_1^{-1}x\big(\big(2q^{-1}+q^{-2}-q^{-3}\big)x - \big(1+2q^{-2}-q^{-3}\big)\beta_0\big)B^{(1)}(x)\\ &\quad - \big(q^{-2}-1\big)\gamma_1^{-1}x(qx-\beta_0)\big(h_qB^{(1)}\big)(x)\\ &\quad - q^{-4}x^2(1-q)\gamma_1^{-1}(qx-\beta_0)\big(H_qB^{(1)}\big)(x) - xC^{(1)}(x)\Big\}, \qquad (4.18)\\ KB^{(-)}(x) &= -\gamma_1^{-1}q^{-2}x^4B^{(1)}(x). \end{aligned}$$ ■

**Example 3.** Let $\mathcal{Y}(b,q^2)$ be the form of Brenke type which is symmetrical $q$-semiclassical of class one such that [14, equation (3.22), $q\leftarrow q^2$]

$$H_q\big(x\mathcal{Y}\big(b,q^2\big)\big) - (b(q-1))^{-1}\big(q^{-2}x^2+b-1\big)\mathcal{Y}\big(b,q^2\big) = 0 \tag{4.19}$$

for $q\in\widetilde{\mathbb{C}}$, $b\neq 0$, $b\neq q$, $b\neq q^{-2n}$, $n\ge 0$ and its MOPS $\{P_n\}_{n\ge 0}$ satisfying (1.1) with [7]

$$\beta_n = 0,$$

$$\gamma_{2n+1} = q^{2n+2}\big(1 - bq^{2n}\big), \qquad \gamma_{2n+2} = bq^{2n+2}\big(1 - q^{2n+2}\big), \qquad n \geq 0. \tag{4.20}$$

Denoting $\mathcal{Y}^{(1)}(b, q^2)$ its associated form and $\mathcal{Y}^{-1}(b, q^2)$ its inverse one. Taking into account (4.19) we have

$$\Phi(x) = x, \qquad \Psi(x) = -(b(q-1))^{-1}\big(q^{-2}x^2 + b - 1\big), \qquad B(x) = 0. \tag{4.21}$$

Also, by virtue of (3.2) and (4.21) we get

$$C(x) = (b(q-1))^{-1}q^{-1}x^2 + q(q-1)^{-1}\big(1 - b^{-1}\big) - 1, \qquad D(x) = (bq(q-1))^{-1}x. \tag{4.22}$$

According to Proposition 10 the form $\mathcal{Y}^{(1)}(b, q^2)$ is $q$-Laguerre–Hahn of class one satisfying the $q$-difference equation (4.10) and its formal Stieltjes function satisfies the $q$-Riccati equation (4.8) where on account of (4.20)–(4.22) we obtain for (4.9), (4.11)

$$\begin{aligned}
&K\Phi^{(1)}(x) = b^{-1}x,\\
&K\Psi^{(1)}(x) = -q^{-2}(b(q-1))^{-1}x^2 + q(q-1)^{-1}\big(1 - b^{-1}\big) - (qb)^{-1} - 1,\\
&KB^{(1)}(x) = \big(b^{-1} - 1\big)q(q-1)^{-1}x,\\
&KC^{(1)}(x) = q^{-2}(b(q-1))^{-1}x^2 + 1 - q(q-1)^{-1}\big(1 - b^{-1}\big),\\
&KD^{(1)}(x) = q^{-2}(b(q-1))^{-1}x.
\end{aligned} \tag{4.23}$$

On the one hand, $\mathcal{Y}^{(1)}(b, q^2)$ is a symmetrical regular form, then $\mathcal{Y}^{-1}(b, q^2)$ is also a symmetrical regular form and we have for (4.14)–(4.16) according to (4.20)

$$\begin{aligned}
&\Lambda_{-1} = 0, \qquad \Lambda_0 = \frac{b^{-1} - 1}{1 - q^2}, \qquad \Lambda_n = \sum_{\nu=1}^{n+1} b^{-\nu}\frac{(b; q^2)_\nu}{(q^2; q^2)_\nu}, \qquad n \geq 1,\\
&\gamma_1^{(-)} = q^2(b-1), \qquad \gamma_{2n+2}^{(-)} = bq^{2n+2}\big(1 - q^{2n+2}\big)\frac{1 + q^2(1-b)\Lambda_n}{1 + q^2(1-b)\Lambda_{n-1}}, \qquad n \geq 0,\\
&\gamma_{2n+3}^{(-)} = q^{2n+4}(1 - bq^{2n+2})\frac{1 + q^2(1-b)\Lambda_{n-1}}{1 + q^2(1-b)\Lambda_n}, \qquad n \geq 0,
\end{aligned}$$

with [7]

$$(a; q)_0 = 1, \qquad (a; q)_n = \prod_{k=1}^{n}\big(1 - aq^{k-1}\big), \qquad n \geq 1.$$

On the other hand, according to Proposition 11, (4.18) and (4.23), the inverse form $\mathcal{Y}^{-1}(b, q^2)$ is symmetrical $q$-Laguerre–Hahn satisfying the $q$-difference equation (4.17) where

$$\begin{aligned}
&K\Phi^{(-)}(x) = b^{-1}x^3\big(1 - qx^2\big),\\
&K\Psi^{(-)}(x) = b^{-1}(q-1)^{-1}x^2\big(b - q - q^{-3}(q-1) + \big(-2q^{-4} + 2q^{-3} + q^{-2} - q^{-1} + q\big)x^2\big),\\
&KB^{(-)}(x) = -b^{-1}q^{-3}(q-1)^{-1}x^5.
\end{aligned}$$

Thus, according to (2.17) it is possible to simplify by $x$ one time uniquely. Consequently, by virtue of (2.16) the inverse form $\mathcal{Y}^{-1}(b, q^2)$ is $q$-Laguerre–Hahn of class two fulfilling the $q$-difference equation

$$\begin{aligned}
&H_q\big(x^2\big(x^2 - q^{-1}\big)\mathcal{Y}^{-1}\big(b, q^2\big)\big) - q^{-1}x\big\{1 + q(q-1)^{-1}\big(b - q - q^{-3}(q-1)\big)\\
&\quad + \big(q(q-1)^{-1}\big(-2q^{-4} + 2q^{-3} + q^{-2} - q^{-1} + q\big) - q\big)x^2\big\}\mathcal{Y}^{-1}\big(b, q^2\big)\\
&\quad + q^{-3}(q-1)^{-1}x^4\big(x^{-1}\mathcal{Y}^{-1}\big(b, q^2\big)h_q\mathcal{Y}^{-1}\big(b, q^2\big)\big) = 0.
\end{aligned}$$

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