Coordinate Bethe Ansatz for Spin s XXX Model[*]

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Abstract. We compute the eigenfunctions and eigenvalues of the periodic integrable spin s XXX model using the coordinate Bethe ansatz. To do so, we compute explicitly the Hamiltonian of the model. These results generalize what has been obtained for spin $\frac{1}{2}$ and spin 1 chains.

Key words: coordinate Bethe ansatz; spin chains

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1 Introduction

The resolution of Heisenberg spin chain [1] was initiated in H. Bethe's seminal paper [2] where he used a method called now coordinate Bethe ansatz. Since this work, several new methods appeared: algebraic Bethe ansatz [3, 4], functional Bethe ansatz (or separation of variables) [5] or analytical Bethe ansatz [6]. These more elaborated techniques allowed one to go further: new solvable models have been discovered and new results have been computed such as correlation functions. As a consequence, the coordinate Bethe ansatz was neglected. However, this method is the simplest one and gives a very efficient way to construct explicitly eigenfunctions, but it is believed that it works only for simple models. In this note, we show that actually it can be applied also to more complicated models as the spin s XXX model.

This paper is organized as follows. In Section 2, we compute the Hamiltonian of the spin s chain we want to solve. To our knowledge, the explicit form of the entries of the Hamiltonian are written for the first time. We also compte the $su(2)$ symmetry algebra and the pseudo-vacuum, a particular (reference) eigenstate. In Section 3, we present the coordinate Bethe ansatz and get the Bethe equations obtained previously by the algebraic or analytical Bethe ansatz. We conclude, in Section 4, on the advantages of this method and on open problems.

2 Integrable periodic spin s chain

2.1 Hamiltonian of the spin s chain

The Hamiltonian of the periodic integrable spin s chain has been computed in [7] thanks to a fusion procedure. This Hamiltonian has been expressed as a polynomial of the invariant

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of $su(2)$ (see for example also [8] for a review). For our purpose, we need to give an explicit expression of the Hamiltonian entries. Namely, the Hamiltonian is the following matrix acting on $(\mathbb{C}^{2s+1})^{\otimes L}$

$$\mathcal{H} = \sum_{j=1}^{L} h_{j,j+1} \tag{2.1}$$

with the periodic condition $L + 1 = 1$ and the subscript $(j, j + 1)$ stands for the two spaces where the $(2s + 1)^2 \times (2s + 1)^2$-matrix h acts non-trivially. We choose to enumerate the basis of $\mathbb{C}^{2s+1}$ as follows: $|s\rangle, |s - 1\rangle, \ldots, |-s\rangle$, where $|m\rangle \equiv |s, m\rangle$ denotes[1] the spin s state with $\mathfrak{s}^z$-component equals to m.

The non-vanishing entries of the matrix h may be parameterized by three integer parameters m_1, m_2, n

$$h = \sum_{m_1,m_2=0}^{2s} \sum_{n=-\min(m_1,2s-m_2)}^{\min(m_2,2s-m_1)} \beta_{m_1,m_2}^n \, |s - m_1 - n\rangle\langle s - m_1| \otimes |s - m_2 + n\rangle\langle s - m_2|.$$

From the results found in [7], one may prove that β_{m_1,m_2}^n can be factorised as:

$$\beta_{m_1,m_2}^n = \frac{(-1)^{n-1}}{n} \sqrt{\frac{\dbinom{M_1 + n}{M_1} \dbinom{M_2}{n}}{\dbinom{2s - M_1}{n} \dbinom{2s - M_2 + n}{n}}} \qquad \text{for } n > 0, \tag{2.2}$$

where we have introduced the notation

$$M_1 = \min(m_1, 2s - m_2) \qquad \text{and} \qquad M_2 = \min(m_2, 2s - m_1). \tag{2.3}$$

The remaining β's are given by the relations:

$$\beta_{m_1,m_2}^0 = -\sum_{\ell=0}^{M_1-1} \frac{1}{2s - \ell} - \sum_{\ell=0}^{M_2-1} \frac{1}{2s - \ell},$$

$$\beta_{m_2,m_1}^{-n} = \beta_{m1,m_2}^n.$$

2.2 $su(2)$ symmetry

At each site, we have a spin s representation, and the expression of the $su(2)$ generators in this representation reads:

$$\mathfrak{s}^- = \sum_{n=-s+1}^{s} \sqrt{(s + n)(s - n + 1)} |n - 1\rangle\langle n|,$$

$$\mathfrak{s}^+ = \sum_{n=-s}^{s-1} \sqrt{(s - n)(s + n + 1)} |n + 1\rangle\langle n|,$$

$$\mathfrak{s}^z = \sum_{n=-s}^{s} n |n\rangle\langle n|.$$

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[1] Let us stress that since all the sites have spin s, we do not mention it, and write only the $\mathfrak{s}^z$ value of the states.

They obey

$$[\mathfrak{s}^+, \mathfrak{s}^-] = 2\mathfrak{s}^z, \qquad [\mathfrak{s}^z, \mathfrak{s}^\pm] = \pm\mathfrak{s}^\pm, \qquad c_2 = (\mathfrak{s}^z)^2 + \frac{1}{2}(\mathfrak{s}^+\mathfrak{s}^- + \mathfrak{s}^-\mathfrak{s}^+) = s(s+1).$$

We will note $\mathfrak{s}_j^\alpha$, $\alpha = z, \pm$ and $j = 1, \ldots, L$, the generators acting on site j. Let us stress that these local operators do not commute with the Hamiltonian $\mathcal{H}$ given in (2.1). However, there is a global $su(2)$ symmetry. The generators of this $su(2)$ symmetry take the form

$$\mathcal{S}^z = \sum_{j=1}^L \mathfrak{s}_j^z \qquad \text{and} \qquad \mathcal{S}^\pm = \sum_{j=1}^L \mathfrak{s}_j^\pm.$$

They obey the $su(2)$ commutation relations $[\mathcal{S}^z, \mathcal{S}^\pm] = \pm\mathcal{S}^\pm$, $[\mathcal{S}^+, \mathcal{S}^-] = 2\mathcal{S}^z$. Remark that the Casimir operator $C_2 = (\mathcal{S}^z)^2 + \frac{1}{2}(\mathcal{S}^+\mathcal{S}^- + \mathcal{S}^-\mathcal{S}^+)$, although central, is not proportional to the identity, since we are considering the tensor product of L spin s representations, a reducible representation.

It is a simple calculation to show that $[\mathfrak{s}_j^\alpha + \mathfrak{s}_{j+1}^\alpha, h_{j,j+1}] = 0$, $\alpha = z, \pm$. It amounts to check the following recursion relations on the coefficients β_{m_1,m_2}^n:

$$\sqrt{(m_1+1)(2s-m_1)}\beta_{m_1+1,m_2}^n = \sqrt{(2s+n-m_2+1)(m_2-n)}\beta_{m_1,m_2}^{n+1}$$
$$- \sqrt{(m_2+1)(2s-m_2)}\beta_{m_1,m_2+1}^{n+1} + \sqrt{(2s-n-m_1)(n+m_1+1)}\beta_{m_1,m_2}^n,$$
$$\sqrt{m_1(2s-m_1+1)}\beta_{m_1-1,m_2}^n = \sqrt{(2s+n-m_2)(m_2-n+1)}\beta_{m_1,m_2}^{n-1}$$
$$- \sqrt{m_2(2s-m_2+1)}\beta_{m_1,m_2-1}^{n-1} + \sqrt{(2s-n-m_1+1)(n+m_1)}\beta_{m_1,m_2}^n.$$

Hence, $\mathcal{S}^\alpha$ commutes with $\mathcal{H}$.

Due to this $su(2)$ symmetry, the wave functions can be characterized by their energy, their spin and their $\mathcal{S}^z$ component. In other words, one can diagonalize the Hamiltonian in a sector where the operators $\mathcal{S}^z$ has a fixed value $S^z = Ls - m$. This is done in the next section.

2.3   Pseudo-vacuum and pseudo-excitations

We wish to present the construction of the Hamiltonian eigenfunction in the framework of coordinate Bethe ansatz for spin s. The spin $s = \frac{1}{2}$ case is the Heisenberg chain, solved in [2]. It gave the name to the method. For the case $s = 1$, the method has been generalized in [9].

The first step of the coordinate Bethe ansatz [2] consists in finding a particular eigenvector, called the pseudo-vacuum, for the Hamiltonian. It is usually chosen as the vector with the highest spin. In the present case, it is the unique vector in the sector $S^z = Ls$:

$$|\varnothing\rangle = |s\rangle \otimes |s\rangle \cdots \otimes |s\rangle.$$

Using the explicit forms for the β's, we get $h_{12}|s\rangle \otimes |s\rangle = 0$. Thus, $|\varnothing\rangle$ is a $\mathcal{H}$-eigenvector with vanishing eigenvalue.

The second step consists in adding pseudo-excitations. These pseudo-excitations are not physical excitations (hence the name pseudo-excitation). They are obtained by acting with a creation operator e_j^-, conjugated to $\mathfrak{s}_j^-$, on the pseudo-vacuum $|\varnothing\rangle$. Let us remark that this operator in this finite representation does not satisfy $(e^-)^2 = 0$ but rather $(e^-)^{2s+1} = 0$. This explains the supplementary difficulties to deal with $s > \frac{1}{2}$. Indeed, in the case $s = \frac{1}{2}$, no more than one pseudo-excitation can be at the same site: we have strict exclusion. In the general case of spin s, we have a weaker exclusion. More precisely, we can have up to $2s$ pseudo-excitations at the same site. This behavior appears already for $s = 1$.

3 Coordinate Bethe ansatz for general spin s

We define a state in the sector $S^z = Ls - m$, for $1 \le x_1 \le x_2 \le \cdots \le x_m \le L$

$$|x_1, x_2, \ldots, x_m\rangle = e^-_{x_1} e^-_{x_2} \cdots e^-_{x_m}|\varnothing\rangle, \qquad (3.1)$$

where e^- is conjugated to $\mathfrak{s}^-$:

$$e^- = \sum_{n=-s+1}^{s} \sqrt{\frac{s+n}{s-n+1}}\, |n-1\rangle\langle n| = g\mathfrak{s}^- g^{-1} \qquad \text{with} \quad g = \sum_{n=-s}^{s} \frac{(2s)!}{(s-n)!}|n\rangle\langle n|.$$

This choice for e^- is for later convenience. The set of such non-vanishing vectors (i.e. $x_{j+2s} > x_j$ for $1 \le j \le L - 2s$) provides a basis for this sector. As already noticed, contrarily to the usual case $(s = \frac{1}{2})$, several (up to $2s$) pseudo-excitations at the same site are allowed, that is to say, some x_j's can be equal. The restriction that no more than $2s$ particles are on the same site is implemented by the fact that $(e^-)^{2s} = 0$. Using the explicit form of e^-, we can rewrite the excited states as follows:

$$|x_1, x_2, \ldots, x_m\rangle = \alpha_{m_1} \cdots \alpha_{m_k} \underbrace{|s\rangle \otimes \cdots \otimes |s\rangle}_{x_1 - 1} \otimes |s - m_1\rangle \otimes \underbrace{|s\rangle \otimes \cdots \otimes |s\rangle}_{x_{m_1+1} - x_{m_1} - 1} \otimes |s - m_2\rangle \otimes \cdots,$$

where m_j is the number of times x_j appears and

$$\alpha_m = \sqrt{\binom{2s}{m}},$$

where $\binom{z}{k} = \frac{z(z-1)\cdots(z-k+1)}{k!}$ is the binomial coefficient. Let us remark that, if $m_j > 2s$ the vector $|s - m_j\rangle$ has no meaning but the normalization α_{m_j} vanishes.

Any eigenvector in the sector $S^z = Ls - m$ is a linear combination of the vectors (3.1). Then, let us introduce the vector

$$\Psi_m = \sum_{x_1 \le x_2 \le \cdots \le x_m} a(x_1, x_2, \ldots, x_m)|x_1, x_2, \ldots, x_m\rangle,$$

where $a(x_1, x_2, \ldots, x_m)$ are complex-valued functions to be determined. As in the case of $s = \frac{1}{2}$, we assume a plane wave decomposition for these functions (Bethe ansatz)

$$a(x_1, \ldots, x_m) = \sum_{P \in \mathfrak{S}_m} A_P(\mathbf{k}) \exp\left\{i(k_{P1}x_1 + \cdots + k_{Pm}x_m)\right\},$$

where $\mathfrak{S}_m$ is the permutation group of m elements and $A_P(\mathbf{k})$ are functions on the symmetric group algebra depending on some parameters k which will be specified below[2]. Using the fact that the states $|x_1, x_2, \ldots, x_m\rangle$ form a basis, we can project the eigenvalue equation

$$\mathcal{H}\Psi_m = E\Psi_m \qquad (3.2)$$

on these different basis vectors to determine the $A_P(\mathbf{k})$ parameters.

Since $\mathcal{H}$ is a sum of operators acting on two neighbouring sites only, one has to single out the cases where the x's obey the following constraints:

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[2]In the following, to lighten the presentation, the $\mathbf{k}$-dependence will not be written explicitly.

- all the x_j's are far away one from each other and are not on the boundary sites 1 and L (this case will be called generic),

- $x_j + 1 = x_{j+1}$ for some j,

- $x_j = x_{j+1}$ for some j,

- $x_j = x_{j+1} = \cdots = x_{j+m_1}$ and $x_j + 1 = x_{j+m_1+1} = \cdots = x_{j+m_1+1+m_2}$ for some positive integers m_1 and m_2,

- $x_1 = 1$, or $x_m = L$.

As the eigenvalue problem is a linear problem, it is enough to treat the cases where at most one of the particular cases appears: more complicated cases just appear as superposition of these 'simple' cases.

Projection on $|x_1, x_2, \ldots, x_m\rangle$ with $x_j + 1 < x_{j+1}$, $\forall j$, $x_1 > 1$ and $x_m < L$. As usual, we start by projecting (3.2) on a generic vector $|x_1, x_2, \ldots, x_m\rangle$. This leads to

$$\sum_{P \in \mathfrak{S}_m} A_P \left(\sum_{j=1}^m (\beta_{1,0}^0 + \beta_{0,1}^0 + \beta_{1,0}^{-1} e^{ik_{Pj}} + \beta_{0,1}^1 e^{-ik_{Pj}}) - E \right) \exp(i(k_{P1}x_1 + \cdots + k_{Pm}x_m)) = 0$$

which must be true for any choice of generic x's. Therefore, we get for the energy (using the explicit forms of the β's given in Section 2)

$$E = -\frac{1}{2s} \sum_{j=1}^m \left(2 - e^{ik_j} - e^{-ik_j} \right).$$

After the change of variable

$$e^{ik_j} = \frac{\lambda_j + is}{\lambda_j - is}, \tag{3.3}$$

the energy becomes

$$E = -\sum_{j=1}^m \frac{2s}{\lambda_j + s^2}.$$

This form for the energy is the one obtained by algebraic Bethe ansatz [10].

Projection on $|x_1, x_2, \ldots, x_m\rangle$ with $x_j + 1 = x_{j+1}$ for some j. Let us consider now the projection of (3.2) when two pseudo-excitations are nearest neighbours. Using the form of the energy previously found, we get

$$\sum_{P \in \mathfrak{S}_m} A_P \left(\beta_{1,1}^0 - \beta_{1,0}^0 - \beta_{0,1}^0 + (\beta_{1,1}^{-1}\alpha_2 - \beta_{1,0}^{-1})e^{ik_{Pj}} + (\beta_{1,1}^1\alpha_2 - \beta_{0,1}^1)e^{-ik_{P(j+1)}} \right)$$
$$\times e^{i(\ldots k_{Pj}x_j + k_{P(j+1)}(1+x_j)\ldots)} = 0.$$

This equation is trivially satisfied for $s > \frac{1}{2}$ since using explicit values we find $\beta_{1,1}^0 - \beta_{1,0}^0 - \beta_{0,1}^0 = 0$, $\beta_{1,1}^{-1}\alpha_2 - \beta_{1,0}^{-1} = 0$ and $\beta_{1,1}^1\alpha_2 - \beta_{0,1}^1 = 0$. For the case $s = \frac{1}{2}$, we find a constraint between A_P and A_{PT_j} (where T_j is the transposition of j and $j+1$). Explicitly, it is given by (3.5) with $s = \frac{1}{2}$.

Projection on $|x_1, x_2, \ldots, x_m\rangle$ with $x_j = x_{j+1}$ for some j. For $s > \frac{1}{2}$, we must also consider the case when several particles are on the same site. Defining S_j, the shift operator adding 1 to the jth variable, we get the following relation, when two particles are on the same site

$$\frac{1}{2s(1-2s)}(1 + S_i^{-1}S_{i+1}^{-1})\left(S_iS_{i+1} + (2s-1)S_i - (2s+1)S_{i+1} + 1\right)a(\ldots, x_i, x_i, \ldots) = 0$$

$$\Rightarrow \quad \big(S_i S_{i+1} + (2s-1)S_i - (2s+1)S_{i+1} + 1\big) a(\dots, x_i, x_i, \dots) = 0. \tag{3.4}$$

Using the plane waves decomposition, we get the following constraint

$$A_{PT_j} = \sigma\big(e^{ik_{Pj}}, e^{ik_{P(j+1)}}\big) A_P, \tag{3.5}$$

where T_j is the transposition of j and $j+1$, and we have introduced the scattering matrix

$$\sigma(u,v) = -\frac{uv + (2s-1)u - (2s+1)v + 1}{uv + (2s-1)v - (2s+1)u + 1}. \tag{3.6}$$

As in the case $s = \frac{1}{2}$, relation (3.5) allows us to express all the A_P's in terms of only one, for instance A_{Id} (where Id is the identity of $\mathfrak{S}_m$). More precisely, one expresses $P \in \mathfrak{S}_m$ as a product of T_i, and then uses (3.5) recursively to express A_P in terms of A_{Id}. At this point, one must take into account that the expression of P in terms of T_i is not unique, because of the relations

$$T_i^2 = \mathrm{Id}, \qquad [T_j, T_i] = 0 \quad (|j - i| > 1) \qquad \text{and} \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

Therefore, for the construction to be consistent, the function σ has to satisfy the relations

$$\sigma(u,v)\sigma(v,u) = 1, \qquad [\sigma(u,v), \sigma(w,z)] = 0,$$
$$\sigma(u_1, u_2)\sigma(u_1, u_3)\sigma(u_2, u_3) = \sigma(u_2, u_3)\sigma(u_1, u_3)\sigma(u_1, u_2).$$

By direct computation, we can show that (3.6) indeed satisfies these relations. We can solve the recursive defining relations for A_P and we find, with a particular choice of normalisation for A_{Id}, the following explicit form, for any $P \in \mathfrak{S}_m$

$$A_P = \prod_{j<k} \left(1 + \frac{1}{2s} \frac{(e^{ik_{Pj}} - 1)(e^{ik_{Pk}} - 1)}{e^{ik_{Pj}} - e^{ik_{Pk}}}\right).$$

This scattering matrix becomes after the change of variables $u = \frac{\lambda + is}{\lambda - is}$ and $v = \frac{\mu + is}{\mu - is}$

$$\sigma(\lambda, \mu) = \frac{\lambda - \mu - i}{\lambda - \mu + i}.$$

Let us remark that after this change of variables the scattering matrix $\sigma(\lambda, \mu)$ does not depend on the value of spin and is similar to the one obtained for $s = \frac{1}{2}$.

Projection on $|x_1, \dots, \underbrace{x_i, \dots, x_i}_{m_1}, \underbrace{x_i + 1, \dots, x_i + 1}_{m_2}, \dots, x_m\rangle$. One can compute

$$\Big[P_{m_1,m_2}(S_i, \dots, S_{i+m_1-1}; S_{i+m_1}, \dots, S_{i+m_1+m_2-1})$$
$$+ P_{m_2,m_1}\big(S_{i+m_1+m_2-1}^{-1}, \dots, S_{i+m_1}^{-1}; S_{i+m_1-1}^{-1}, \dots, S_i^{-1}\big)\Big]$$
$$\times a(\dots, x_i, \dots, x_i, x_i + 1, \dots, x_i + 1, \dots) = 0, \tag{3.7}$$

where

$$P_{m_1,m_2}(\mathbf{y}; \mathbf{z}) = \sum_{n=1}^{m_2} \alpha_n \alpha_{m_2-n} \alpha_{m_1} \beta_{0,m_2}^n z_{m_2-n+1} \cdots z_{m_2}$$
$$+ \sum_{n=1}^{M_1} \alpha_{m_1-n} \alpha_{m_2+n} \beta_{m_1,m_2}^{-n} y_{m_1-n+1} \cdots y_{m_1}$$

$$- \alpha_{m_1} \alpha_{m_2} \sum_{j=1}^{m_1} \left(\frac{\beta_{1,0}^0 + \beta_{0,1}^0}{2} + \beta_{1,0}^{-1} y_j \right) - \alpha_{m_1} \alpha_{m_2} \sum_{j=1}^{m_2} \left(\frac{\beta_{1,0}^0 + \beta_{0,1}^0}{2} + \beta_{1,0}^{-1} z_j \right)$$

$$+ \frac{1}{2} \alpha_{m_1} \alpha_{m_2} (\beta_{m_1,m_2}^0 + \beta_{0,m_1}^0 + \beta_{0,m_2}^0),$$

and we have used the notation (2.3) and

$$\mathbf{y} = (y_1, y_2, \ldots, y_{m_1}) \qquad \text{with} \qquad y_k = S_{i+k-1}, \qquad 1 \le k \le m_1,$$
$$\mathbf{z} = (z_1, z_2, \ldots, z_{m_2}) \qquad \text{with} \qquad z_k = S_{i+m_1+k-1}, \qquad 1 \le k \le m_2.$$

Relation (3.7) is implied by (3.4). The sketch of the proof goes as follows.

First, one can check directly that[3] $P_{0,2}(S_j, S_{j+1}) = 0$, $\forall j$, since $P_{0,2}(S_j, S_{j+1})$ corresponds to relation (3.4). The same is true for $P_{2,0}(S_{j+1}^{-1}, S_j^{-1})$ (after multiplication by $S_{j+1}^{-1} S_j^{-1}$).

Next, we rewrite (3.7) as

$$P_{m_1,m_2}(\mathbf{y}; \mathbf{z}) + P_{m_2,m_1}(\overline{\mathbf{z}}; \overline{\mathbf{y}}) = 0, \tag{3.8}$$

where we have defined

$$\overline{\mathbf{y}} = \left(y_{m_1}^{-1}, \ldots, y_2^{-1}, y_1^{-1} \right), \qquad \text{i.e.} \qquad \overline{y}_k = S_{i+m_1-k}^{-1}, \qquad 1 \le k \le m_1,$$
$$\overline{\mathbf{z}} = \left(z_{m_2}^{-1}, \ldots, z_2^{-1}, z_1^{-1} \right), \qquad \text{i.e.} \qquad \overline{z}_k = S_{i+m_1+m_2-k}^{-1}, \qquad 1 \le k \le m_2.$$

These variables are such that if $P_{0,2}(z_j, z_{j+1}) = 0 \; \forall j$, then we have also $P_{0,2}(\overline{z}_j, \overline{z}_{j+1}) = 0$. Hence, a property valid for $P_{m_1,m_2}(\mathbf{y}; \mathbf{z})$ will be also valid for $P_{m_2,m_1}(\overline{\mathbf{z}}; \overline{\mathbf{y}})$.

We first focus on

$$P_{0,m}(\mathbf{z}) = \sum_{n=1}^{m} \alpha_n \alpha_{m-n} \beta_{0,m}^n z_{m-n+1} \cdots z_m - \alpha_m \beta_{1,0}^{-1} \sum_{j=1}^{m} z_j + \alpha_m \left(\beta_{0,m}^0 - m \beta_{1,0}^0 \right).$$

If we suppose that we have variables z_j such that $P_{0,2}(z_j, z_{j+1}) = 0$, $\forall j$, then from expression (2.2), and after some calculation, one can show that

$$z_1 z_2 \cdots z_m = 1 - m + \sum_{j=1}^{m} \chi_j^{(m)} z_j,$$

$$\chi_j^{(m)} = (-1)^{m+j} \prod_{k=1}^{m-j} \left(\frac{2s}{k} - 1 \right) \prod_{\ell=1}^{j-1} \left(\frac{2s}{\ell} + 1 \right).$$

Thus, the polynomial $P_{0,m}(\mathbf{z})$ becomes a linear function of the z_j's. Looking at the coefficient of z_j and at the constant term, one checks that they identically vanish, so that $P_{0,m}(\mathbf{z}) = 0$.

Looking at the general polynomial $P_{m_1,m_2}(\mathbf{y}, \mathbf{z})$ and using the relation

$$\alpha_{m_1-n} \alpha_{m_2+n} \beta_{m_1,m_2}^{-n} = \alpha_{M_2} \alpha_{M_1-n} \alpha_n \beta_{0,M_1}^n,$$

one can rewrite it as

$$P_{m_1,m_2}(\mathbf{y}, \mathbf{z}) = \alpha_{m_1} P_{0,m_2}(\mathbf{z}) + \alpha_{M_2} P_{0,M_1}(y_{m_1-M_1+1}, \ldots, y_{m_1}) + \alpha_{M_1} \alpha_{M_2} R_{m_1,m_2}(\mathbf{y}),$$

$$R_{m_1,m_2}(\mathbf{y}) = - \sum_{j=1}^{m_1-M_1} \left(\frac{\beta_{1,0}^0 + \beta_{0,1}^0}{2} + \beta_{1,0}^{-1} y_j \right) + \frac{1}{2} \left(\beta_{0,M_2}^0 - \beta_{M_1,0}^0 + \beta_{0,m_1}^0 - \beta_{0,m_2}^0 \right).$$

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[3]Multiplication on the right by $a(\ldots, x_i, \ldots, x_i, x_i + 1, \ldots, x_i + 1, \ldots)$ will be understood during the proof.

Thus, to prove relation (3.8), it is enough to show that

$$R_{m_1,m_2}(\mathbf{y}) + R_{m_2,m_1}(\overline{\mathbf{z}}) = 0. \tag{3.9}$$

Using the expression of M_1 and M_2, see (2.3), it is easy to see that

$$m_1 - M_1 = m_2 - M_2 \equiv \mathfrak{m}_{12} \geq 0.$$

Two cases have to be distinguished: $\mathfrak{m}_{12} = 0$ or $\mathfrak{m}_{12} > 0$. In the first case, relation (3.9) is trivially satisfied. In the second case, equation (3.9) can be rewritten as

$$\sum_{j=1}^{\mathfrak{m}_{12}} \left(y_j + \overline{z}_j\right) = 2\mathfrak{m}_{12},$$

which is obeyed if

$$S_{i+j-1} + S^{-1}_{i+2s+j-1} = 2.$$

To prove this last relation, we use recursively (3.4) to show

$$S_k\, S_{k+\ell} + 1 + \left(\frac{2s}{\ell} - 1\right) S_k - \left(\frac{2s}{\ell} + 1\right) S_{k+\ell} = 0 \qquad \forall\, j.$$

Taking $k = i + j - 1$, $\ell = 2s$ and using $m_1 + m_2 = 2s + \mathfrak{m}_{12}$ gives the result.

Hence, relation (3.8) is satisfied if the variables are such that $P_{0,2}(z_j, z_{j+1}) = P_{0,2}(y_j, y_{j+1}) = 0$, $\forall\, j$. This ends the proof.

This step concludes the bulk part of the problem, the other possible equations being fulfilled by linearity. It remains to take into account the periodic boundary condition. It is done through the following projection.

Projection on $|1, x_2, \ldots, x_m\rangle$. As usual, this leads to a constraint on the parameters k_j. It is not surprising since these parameters can be interpreted as momenta: we are quantizing them since we are on a line with (periodic) boundary conditions. Namely, this leads to

$$\sum_{P \in \mathfrak{S}_m} A_P \big(\exp(i(k_{P2}x_2 + \cdots + k_{Pm}x_m)) $$
$$- \exp(i(k_{P1}x_2 + k_{P2}x_3 + \cdots + k_{P(m-1)}x_m + k_{Pm}L))\big) = 0.$$

Now, we first perform the change of variable in the summation $P \to PT_1 \cdots T_{m-1}$ in the second term of the previous relation. Then, using recursively relation (3.5) and projecting on independent exponential functions, we get the quantization of the momenta via the so-called Bethe equations

$$e^{iLk_j} = \prod_{\ell \neq j} \sigma\big(e^{ik_\ell}, e^{ik_j}\big) \qquad \text{for} \quad j = 1, 2, \ldots, m. \tag{3.10}$$

Since these equations express the periodicity of the chain, they are equivalent to the ones obtained through projection on $|x_1, \ldots, x_{m-1}, L\rangle$ (as it can be checked explicitly). Thus, we do not have any new independent equations through projections, and the eigenvalue problem has been solved (up to the resolution of the Bethe equations).

Note that using the change of variables (3.3) and the expression (3.6) for the scattering matrix, equations (3.10) can be rewritten as

$$\frac{\lambda_j + is}{\lambda_j - is} = -\prod_{\ell=1}^{m} \frac{\lambda_\ell - \lambda_j - i}{\lambda_\ell - \lambda_j + i}.$$

One recognizes the usual Bethe equations of the spin s chain [10, 11].

Action of $su(2)$ generators

Since the $su(2)$ generators commute with the Hamiltonian, from any eigenfunction Ψ_m, one can construct (possibly) new eigenfunctions by application of $\mathcal{S}^\alpha$, $\alpha = z, \pm$ on Ψ_m. As already mentioned, it is a straightforward calculation to check that

$$\mathcal{S}^z \Psi_m = (Ls - m)\Psi_m.$$

Moreover, it is part of the ansatz to suppose that the eigenvector Ψ_m is a highest weight vector of the $su(2)$ symmetry algebra,

$$\mathcal{S}^+ \Psi_m = 0.$$

Let us stress that for Ψ_m to be an eigenvector, one has to assume that the rapidities λ_j have to obey the Bethe equations. In the same way, Ψ_m is a highest weight vector only when the Bethe equations are fulfilled. In the context of coordinate Bethe ansatz, there exists no general proof (for generic spin s) of it (at least to our knowledge). Note however that for spin $\frac{1}{2}$, the proof was given in [12]. Nevertheless, one can check the highest weight property on different cases, and we illustrate it below by the calculation of $\mathcal{S}^+\Psi_1$, $\mathcal{S}^+\Psi_2$ and $\mathcal{S}^+\Psi_3$. We also show on the last example where the proof used by Gaudin does not work anymore for spin $s > \frac{1}{2}$.

The Ψ_m vectors should be also related to the ones obtained through algebraic Bethe ansatz (ABA). Such a correspondence, for the case of spin $\frac{1}{2}$, has been done in [13] using an iteration trick based on the comultiplication [14, 15]. Let us note that in [13] they used the relation $(T_{12})^2 = 0$ which is not true anymore for $s > \frac{1}{2}$. Their proof must be generalized to apply in our case. Let us also notice the other method using the Drinfel'd twist [16]. Moreover, since it is known that the ABA construction leads to $su(2)$ highest weight vectors, and assuming the same property for the coordinate Bethe approach, it is clear that the two methods should lead to the same vectors, up to a normalisation.

For instance, considering Ψ_1, its ABA "counterpart" takes the form

$$\Phi_1 = \sum_{x=1}^{L} T_{11}^{(1)}(\lambda_1) \cdots T_{11}^{(x-1)}(\lambda_1) T_{12}^{(x)}(\lambda_1) T_{22}^{(x+1)}(\lambda_1) \cdots T_{22}^{(L)}(\lambda_1)|\varnothing\rangle,$$

where $T^{(j)}(\lambda)$ is the representation of the monodromy matrix at site j:

$$T^{(j)}(\lambda) = \frac{1}{\lambda - is} \begin{pmatrix} \lambda + i\,\mathfrak{s}_j^z & i\mathfrak{s}_j^- \\ i\mathfrak{s}_j^+ & \lambda - i\,\mathfrak{s}_j^z \end{pmatrix}$$

and λ_1 is the Bethe parameter. This leads to

$$\Phi_1 = \frac{i}{\lambda_1 + is} \sum_{x=1}^{L} \left(\frac{\lambda_1 + is}{\lambda_1 - is}\right)^x \mathfrak{s}_x^- |\varnothing\rangle,$$

that has to be compared with

$$\Psi_1 = \sum_{x=1}^{L} e^{ik_1 x}|x\rangle = \sum_{x=1}^{L} e^{ik_1 x} e_x^- |\varnothing\rangle.$$

Using the change of variable (3.3), it is clear that, apart from a normalisation factor, the two vectors are equal.

Calculation of $\mathcal{S}^+\Psi_1$ and $\mathcal{S}^+\Psi_2$. A direct calculation leads to

$$\mathcal{S}^+\Psi_1 = 2s\frac{y}{1-y}\left(1 - y^L\right)|\varnothing\rangle \qquad \text{with} \qquad y = e^{ik_1},$$

which is identically zero using the Bethe equation $y^L = 1$. Hence, Ψ_1 is indeed a highest weight vector for the $su(2)$ symmetry.

In the same way, one can compute

$$\mathcal{S}^+\Psi_2 = \sum_{P\in\mathfrak{S}_2} A_P(k_1, k_2)$$

$$\times \left\{ \sum_{x=1}^{L}(2s-1)e^{i(k_1+k_2)x}|x\rangle + \sum_{1\le x_1 < x_2 \le L} 2s\, e^{i(k_{P1}x_1 + k_{P2}x_2)}\big(|x_1\rangle + |x_2\rangle\big) \right\}.$$

Using the relation

$$A_{T_1}(k_1, k_2) = \sigma(k_1, k_2)A_{\mathrm{Id}}(k_1, k_2)$$

and the normalisation $A_{\mathrm{Id}}(k_1, k_2) = 1$, one gets

$$\mathcal{S}^+\Psi_2 = \sum_{x=1}^{L}\left\{ (2s-1)(y_1y_2)^x\big(1+\sigma(y_1,y_2)\big) + 2s\left[\frac{y_2^{x+1} - y_2^{L+1}}{1-y_2}y_1^x \right.\right.$$

$$\left.\left. + \sigma(y_1,y_2)\frac{y_1^{x+1} - y_1^{L+1}}{1-y_1}y_2^x + \frac{y_1 - y_1^x}{1-y_1}y_2^x + \sigma(y_1,y_2)\frac{y_2 - y_2^x}{1-y_2}y_1^x\right]\right\}|x\rangle,$$

where $y_j = e^{ik_j}$, $j = 1, 2$.

Now, from the Bethe equations

$$y_1^L = \sigma(y_1, y_2) \qquad \text{and} \qquad y_2^L = \sigma(y_2, y_1),$$

one simplifies it as

$$\mathcal{S}^+\Psi_2 = \sum_{x=1}^{L}(y_1y_2)^x\left\{ 2s - 1 + 2s\left(\frac{y_2}{1-y_2} - \frac{1}{1-y_1}\right) \right.$$

$$\left. + \sigma(y_1,y_2)\left[2s - 1 + 2s\left(\frac{y_1}{1-y_1} - \frac{1}{1-y_2}\right)\right]\right\}|x\rangle.$$

Finally, the form of the scattering matrix σ ensures that the quantity within brackets $\{\cdots\}$ vanishes.

Calculation of $\mathcal{S}^+\Psi_3$. Performing the same kind of calculation on Ψ_3, we get

$$\mathcal{S}^+\Psi_3 = \sum_{P\in\mathfrak{S}_3} A_P(\mathbf{k})\left\{ \sum_{x=1}^{L}(y_1y_2)^x\left[(2s-2)y_3^x + 2s\frac{y_3^{x+1} - y_3^{L+1}}{1-y_3} \right.\right.$$

$$\left. + 2s\sigma_{23}\sigma_{13}\frac{(y_3 - y_3^x)}{1-y_3}\right]|x, x\rangle + \sum_{1\le x_1 < x_2 \le L} y_1^{x_1}y_2^{x_2}\left[(2s-1)\sigma_{23}y_3^{x_1} + (2s-1)y_3^{x_2}\right.$$

$$\left.\left. + 2s\frac{y_3^{x_2+1} - y_3^{L+1}}{1-y_3} + 2s\sigma_{23}\sigma_{13}\frac{y_3 - y_3^{x_1}}{1-y_3} + 2s\sigma_{23}\frac{y_3^{x_1+1} - y_3^{x_2}}{1-y_3}\right]|x_1, x_2\rangle\right\},$$

where[4]

$$y_j = e^{ik_{Pj}}, \quad j = 1, 2, 3 \qquad \text{and} \qquad \sigma_{j\ell} = \sigma(y_j, y_\ell), \quad 1 \le j \ne \ell \le 3.$$

---

[4]Let us stress the dependence on $P \in \mathfrak{S}_3$ in the definition of y_j: it is used below.

After use of the Bethe equation, $y_3^L = \sigma_{23}\sigma_{13}$, it can be recast as

$$
\mathcal{S}^+\Psi_3 = \sum_{P\in\mathfrak{S}_3} A_P(\mathbf{k})\Bigg\{ \sum_{x=1}^{L}(y_1y_2y_3)^x\left[2s-2+\frac{2s}{1-y_3}(y_3-\sigma_{23}\sigma_{13})\right]|x,x\rangle
$$

$$
+ \sum_{1\leq x_1<x_2\leq L} y_1^{x_1}y_2^{x_2}\left[\sigma_{23}y_3^{x_1}\left(2s-1+2s\frac{y_3}{1-y_3}-\frac{2s}{1-y_3}\sigma_{13}\right)\right.
$$

$$
\left. + y_3^{x_2}\left(2s-1+2s\frac{y_3}{1-y_3}-\frac{2s}{1-y_3}\sigma_{23}\right)\right]|x_1,x_2\rangle\Bigg\}.
$$

Using the sum on P to relabel the variables y_j, one can rewrite this equality as

$$
\mathcal{S}^+\Psi_3 = \sum_{P\in\mathfrak{S}_3} A_P(\mathbf{k})\Bigg\{ \sum_{x=1}^{L}\frac{1}{6}(y_1y_2y_3)^x
$$

$$
\times\Bigg[(2s-2)\big(1+\sigma_{12}+\sigma_{23}+\sigma_{12}\sigma_{13}+\sigma_{23}\sigma_{13}+\sigma_{12}\sigma_{13}\sigma_{23}\big)
$$

$$
+\frac{2s}{1-y_3}\big(y_3(1-\sigma_{12})-\sigma_{23}\sigma_{13}-\sigma_{12}\sigma_{13}\sigma_{23}\big)
$$

$$
+\frac{2s}{1-y_2}\big(y_2\sigma_{23}(1-\sigma_{13})-\sigma_{12}-\sigma_{12}\sigma_{13}\big)
$$

$$
+\frac{2s}{1-y_1}\big(y_1\sigma_{12}\sigma_{13}(1-\sigma_{23})-1-\sigma_{23}\big)\Bigg]|x,x\rangle
$$

$$
+\sum_{1\leq x_1<x_2\leq L}(y_1y_2)^{x_1}y_3^{x_2}\left[(1+\sigma_{12})(2s-1)+2s\left(\frac{y_2}{1-y_2}-\frac{1}{1-y_1}\right)\right.
$$

$$
\left. +2s\sigma_{12}\left(\frac{y_1}{1-y_1}-\frac{1}{1-y_2}\right)\right]|x_1,x_2\rangle
$$

$$
+\sum_{1\leq x_1<x_2\leq L}y_1^{x_1}(y_2y_3)^{x_2}\left[(1+\sigma_{23})(2s-1)+2s\left(\frac{y_3}{1-y_3}-\frac{1}{1-y_2}\right)\right.
$$

$$
\left. +2s\sigma_{23}\left(\frac{y_2}{1-y_2}-\frac{1}{1-y_3}\right)\right]|x_1,x_2\rangle\Bigg\}.
$$

The term inside the square bracket $[\cdots]$ in factor of $(y_1y_2)^{x_1}y_3^{x_2}$ on the one hand, and in factor of $y_1^{x_1}(y_2y_3)^{x_2}$ on the other hand, identically vanishes. It is in fact the same identity as the one used to show that $\mathcal{S}^+\Psi_2 = 0$. It is also the identity used by Gaudin [12] to prove, for spin $\frac{1}{2}$, that Ψ_m, $\forall m$, is a highest weight vector.

When the spin is higher than $\frac{1}{2}$, it remains the term in factor of $|x,x\rangle$, which is a state that does not exist when $s = \frac{1}{2}$. The square bracket in front of $|x,x\rangle$ also identically vanishes, another identity due to the form of the scattering matrix σ, and we get $\mathcal{S}^+\Psi_3 = 0$.

When $s = 1$ this new identity is sufficient (together with the one used for $\mathcal{S}^+\Psi_2$) to prove that Ψ_m, $\forall m$, is a highest weight vector. However, for $s > 1$, to prove that $\mathcal{S}^+\Psi_4 = 0$, one needs to consider the state $|x,x,x\rangle$, that we will lead to another identity of the scattering matrix, and so on: for spin s, one needs $2s$ identities to prove that Ψ_m, $\forall m$, is a highest weight vector. Hence the difficulty to get a generic proof of it.

4   Conclusions

In previous studies, the eigenfunctions of the spin s chain studied in this paper were known thanks to the algebraic Bethe ansatz. This later construction allows one to compute the correlation functions [17, 18, 19]. Prior to that computation, the coordinate Bethe ansatz allowed Gaudin [20] to show, for spin $\frac{1}{2}$ chains, orthogonality relations for the Bethe eigenfunctions, and prove a closure property for these functions. The explicit form of the eigenfunctions computed in this note is a first step toward a generalisation to spin s chains. The same method can also be applied to spin chains associated to higher rank algebras, for which less is known.

In the same way, the spin 1 XXX chain with open (diagonal) boundaries has been studied in [21]: there is no doubt that their results can be generalized to spin s, using the present approach. The advantage of this method lies in the fact that we do not need to solve the reflection equation before computing the spectrum. We may start with general boundary conditions and find the ones for which the method is still consistent. In this way, the boundaries which keep the model solvable are classified.

We also believe that the method presented here can be applied to solve the XXZ model with higher spin in the case of periodic boundary conditions. These cases have been treated through algebraic Bethe ansatz, see e.g. [22, 23]. More interestingly, general XXZ models with open boundary conditions can also be treated in this way, see e.g. [24] where a first account has been given.

To conclude, we hope that this paper convinced the reader that the coordinate Bethe ansatz is a very powerful method and can be applied to solve a rather large class of integrable models.

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