Compact Riemannian Manifolds with Homogeneous Geodesics*

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Abstract. A homogeneous Riemannian space $(M = G/H, g)$ is called a geodesic orbit space (shortly, GO-space) if any geodesic is an orbit of one-parameter subgroup of the isometry group G. We study the structure of compact GO-spaces and give some sufficient conditions for existence and non-existence of an invariant metric g with homogeneous geodesics on a homogeneous space of a compact Lie group G. We give a classification of compact simply connected GO-spaces $(M = G/H, g)$ of positive Euler characteristic. If the group G is simple and the metric g does not come from a bi-invariant metric of G, then M is one of the flag manifolds $M_1 = SO(2n+1)/U(n)$ or $M_2 = Sp(n)/U(1) \cdot Sp(n-1)$ and g is any invariant metric on M which depends on two real parameters. In both cases, there exists unique (up to a scaling) symmetric metric g_0 such that (M, g_0) is the symmetric space $M = SO(2n+2)/U(n+1)$ or, respectively, $\mathbb{C}P^{2n-1}$. The manifolds M_1, M_2 are weakly symmetric spaces.

Key words: homogeneous spaces, weakly symmetric spaces, homogeneous spaces of positive Euler characteristic, geodesic orbit spaces, normal homogeneous Riemannian manifolds, geodesics

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1 Introduction

A Riemannian manifold (M, g) is called a manifold with homogeneous geodesics or geodesic orbit manifold (shortly, GO-manifold) if all its geodesic are orbits of one-parameter groups of isometries of (M, g). Such manifold is a homogeneous manifold and can be identified with a coset space $M = G/H$ of a transitive Lie group G of isometries. A Riemannian homogeneous space $(M = G/H, g^M)$ of a group G is called a space with homogeneous geodesics (or geodesic orbit space, shortly, GO-space) if any geodesic is an orbit of a one-parameter subgroup of the group G. This terminology was introduced by O. Kowalski and L. Vanhecke in [20], who initiated a systematic study of such spaces.

Recall that homogeneous geodesics correspond to "relative equilibria" of the geodesic flow, considered as a hamiltonian system on the cotangent bundle. Due to this, GO-manifolds can be characterized as Riemannian manifolds such that all integral curves of the geodesic flow are relative equilibria.

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GO-spaces may be considered as a natural generalization of symmetric spaces, classified by É. Cartan [10]. Indeed, a simply connected symmetric space can be defined as a Riemannian manifold (M, g) such that any geodesic $\gamma \subset M$ is an orbit of one-parameter group g_t of transvections, that is one-parameter group of isometries which preserves γ and induces the parallel transport along γ. If we remove the assumption that g_t induces the parallel transport, we get the notion of a GO-space.

The class of GO-spaces is much larger then the class of symmetric spaces. Any homogeneous space $M = G/H$ of a compact Lie group G admits a metric g^M such that (M, g^M) is a GO-space. It is sufficient to take the metric g^M which is induced with a bi-invariant Riemannian metric g on the Lie group G such that $(G, g) \to (M = G/H, g^M)$ is a Riemannian submersion with totally geodesic fibres. Such GO-space $(M = G/H, g^M)$ is called a **normal homogeneous space**.

More generally, any naturally reductive manifold is a geodesic orbit manifold. Recall that a Riemannian manifold (M, g^M) is called **naturally reductive** if it admits a transitive Lie group G of isometries with a bi-invariant pseudo-Riemannian metric g, which induces the metric g^M on $M = G/H$, see [18, 8]. The first example of non naturally reductive GO-manifold had been constructed by A. Kaplan [16]. An important class of GO-spaces consists of weakly symmetric spaces, introduced by A. Selberg [22]. A homogeneous Riemannian space $(M = G/H, g^M)$ is a **weakly symmetric space** if any two points $p, q \in M$ can be interchanged by an isometry $a \in G$. This property does not depend on the particular invariant metric g^M. Weakly symmetric spaces $M = G/H$ have many interesting properties (for example, the algebra of G-invariant differential operators on M is commutative, the representation of G in the space $L^2(M)$ of function is multiplicity free, the algebra of G-invariant Hamiltonians on T^*M with respect to Poisson bracket is commutative) and are closely related with spherical spaces, commutative spaces and Gelfand pairs etc., see the book by J.A. Wolf [26]. The classification of weakly symmetric reductive homogeneous spaces was given by O.S. Yakimova [28], see also [26].

In [20], O. Kowalski and L. Vanhecke classified all GO-spaces of dimension ≤ 6. C. Gordon [14] reduced the classification of GO-spaces to the classification of GO-metrics on nilmanifolds, compact GO-spaces and non-compact GO-spaces of non-compact semisimple Lie group. She described GO-metrics on nilmanifolds. They exist only on two-step nilponent nilmanifolds. She also presented some constructions of GO-metrics on homogeneous compact manifolds and non compact manifolds of a semisimple group.

Many interesting results about GO-spaces one can find in [7, 12, 27, 23, 24], where there are also extensive references.

Natural generalizations of normal homogeneous Riemannian manifolds are δ-homogeneous Riemannian manifolds, studied in [3, 4, 5]. Note that the class of δ-homogeneous Riemannian manifolds is a proper subclass of the class of geodesic orbit spaces with non-negative sectional curvature (see the quoted papers for further properties of δ-homogeneous Riemannian manifolds).

In [1], a classification of non-normal invariant GO-metrics on flag manifolds $M = G/H$ was given. The problem reduces to the case when the (compact) group G is simple. There exist only two series of flag manifolds of a simple group which admit such metric, namely weakly symmetric spaces $M_1 = SO(2n+1)/U(n)$ and $M_2 = Sp(n)/U(1) \cdot Sp(n-1)$, equipped with any (non-normal) invariant metric (which depends on two real parameters). Moreover, there exists unique (up to a scaling) invariant metric g_0, such that the Riemannian manifolds (M_i, g_0) are isometric to the symmetric spaces $SO(2n+2)/U(n+1)$ and $\mathbf{C}P^{2n-1} = SU(2n)/U(2n-1)$, respectively.

The main goal of this paper is a generalization of this result to the case of compact homogeneous manifolds of positive Euler characteristic. We prove that the weakly symmetric manifolds M_1, M_2 exhaust all simply connected compact irreducible Riemannian non-normal GO-manifolds of positive Euler characteristic.

We indicate now the idea of the proof. Let $(M = G/H, g^M)$ be a compact irreducible non-normal GO-space of positive Euler characteristic. Then the stability subgroup H has maximal rank, which implies that G is simple. We prove that there is rank 2 regular simple subgroup G' of G (associated with a rank 2 subsystem R' of the root system R of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$) such that the orbit $M' = G'o = G'/H'$ of the point $o = eH \in M$ (with the induced metric) is a non-normal GO-manifold. Using [1, 3], we prove that the only such manifold M' is $SU(5)/U(2)$. This implies that the root system R is not simply-laced and admits a "special" decomposition $R = R_0 \cup R_1 \cup R_2$ into a disjoint union of three subsets, which satisfies some properties. We determine all such special decompositions of irreducible root systems and show that only root systems of type B_n and C_n admit special decomposition and associated homogeneous manifolds are M_1 and M_2.

The structure of the paper is the following. We fix notations and recall basic definitions in Section 2. Some standard facts about totally geodesic submanifolds of a homogeneous Riemannian spaces are collected in Section 3. We discuss some properties of compact GO-spaces in Section 4. These results are used in Section 5 to derive sufficient conditions for existence and non-existence of a non-normal GO-metric on a homogeneous manifold of a compact group. Section 6 is devoted to classification of compact GO-spaces with positive Euler characteristic.

2 Preliminaries and notations

Let $M = G/H$ be a homogeneous space of a compact connected Lie group G. We will denote by $b = \langle\cdot,\cdot\rangle$ a fixed Ad_G-invariant Euclidean metric on the Lie algebra $\mathfrak{g}$ of G (for example, the minus Killing form if G is semisimple) and by

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \tag{1}$$

the associated b-orthogonal reductive decomposition, where $\mathfrak{h} = \mathrm{Lie}(H)$. An invariant Riemannian metric g^M on M is determined by an Ad_H-invariant Euclidean metric $g = (\cdot,\cdot)$ on the space $\mathfrak{m}$ which is identified with the tangent space T_oM at the initial point $o = eH$.

If $\mathfrak{p}$ is a subspace of $\mathfrak{m}$, we will denote by $X_{\mathfrak{p}}$ the b-orthogonal projection of a vector $X \in \mathfrak{g}$ onto $\mathfrak{p}$, by $b_{\mathfrak{p}}$ the restriction of the symmetric bilinear form to $\mathfrak{p}$ and by $A^{\mathfrak{p}} = \mathrm{pr}_{\mathfrak{p}} \circ A \circ \mathrm{pr}_{\mathfrak{p}}$ the projection of an endomorphism A to $\mathfrak{p}$. If g is a Ad_H-invariant metric, the quotient

$$A = b_{\mathfrak{m}}^{-1} \circ g$$

is an Ad_H-equivariant symmetric positively defined endomorphism on $\mathfrak{m}$, which we call the **metric endomorphism**. Conversely, any such equivariant positively defined endomorphism A of $\mathfrak{m}$ defines an invariant metric $g = b \circ A = b(A\cdot,\cdot)$ on $\mathfrak{m}$, hence an invariant Riemannian metric g^M on M.

Lemma 1. *Let $(M = G/H, g^M)$ be a compact homogeneous Riemannian space with metric endomorphism A and*

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_k, \tag{2}$$

the A-eigenspace decomposition such that $A|_{\mathfrak{m}_i} = \lambda_i \cdot \mathbf{1}_{\mathfrak{m}_i}$. Then

$$(\mathfrak{m}_i, \mathfrak{m}_j) = \langle \mathfrak{m}_i, \mathfrak{m}_j \rangle = 0 \tag{3}$$

and Ad_H-modules $\mathfrak{m}_i$ satisfy $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}$ for $i \neq j$.

Proof. Since A commute with Ad_H, eigenspaces $\mathfrak{m}_i$ are Ad_H-invariants and for $X \in \mathfrak{m}_i$, $Y \in \mathfrak{m}_j$, $i \neq j$, we get

$$\lambda_i \langle X, Y \rangle = \langle AX, Y \rangle = (X, Y) = \langle X, AY \rangle = \lambda_j \langle X, Y \rangle.$$

This implies (3). The inclusion $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}$ follows from the fact that $\mathfrak{m}_j$ is Ad_H-invariant and $\langle [\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{h} \rangle = \langle \mathfrak{m}_i, [\mathfrak{m}_j, \mathfrak{h}] \rangle = 0$. ■

For any subspace $\mathfrak{p} \subset \mathfrak{m}$ we will denote by $\mathfrak{p}^\perp$ its orthogonal complement with respect to the metric g and by $\mathbf{1}_\mathfrak{p}$ the identity operator on $\mathfrak{p}$.

Recall that Ad_H-submodules $\mathfrak{p}$, $\mathfrak{q}$ are called **disjoint** if they have no non-zero equivalent submodules. If Ad_H-module $\mathfrak{m}$ is decomposed into a direct sum

$$\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$$

of disjoint submodules, then any Ad_H-invariant metric g and associated metric endomorphism A have the form

$$g = g_{\mathfrak{m}_1} \oplus \cdots \oplus g_{\mathfrak{m}_k}, \qquad A = A^{\mathfrak{m}_1} \oplus \cdots \oplus A^{\mathfrak{m}_k}.$$

Let $(M = G/H, g^M)$ be a compact homogeneous Riemannian space with the reductive decomposition (1) and metric endomorphism $A \in \mathrm{End}(\mathfrak{m})$.

We identify elements $X, Y \in \mathfrak{g}$ with Killing vector fields on M. Then the covariant derivative $\nabla_X Y$ at the point $o = eH$ is given by

$$\nabla_X Y(o) = -\tfrac{1}{2}[X, Y]_\mathfrak{m} + U(X_\mathfrak{m}, Y_\mathfrak{m}), \tag{4}$$

where the bilinear symmetric map $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is given by

$$2(U(X, Y), Z) = (\mathrm{ad}_Z^\mathfrak{m} X, Y) + (X, \mathrm{ad}_Z^\mathfrak{m} Y) \tag{5}$$

for any $X, Y, Z \in \mathfrak{m}$ and $X_\mathfrak{m}$ is the $\mathfrak{m}$-part of a vector $X \in \mathfrak{g}$ [8].

Definition 1. A homogeneous Riemannian space $(M = G/H, g^M)$ is called a space with homogeneous geodesics shortly, **GO-space** if any geodesic γ of M is an orbit of 1-parameter subgroup of G. The invariant metric g^M is called **GO-metric**.

If G is the full isometry group, then GO-space is called a **manifold with homogeneous geodesics** or **GO-manifold**.

Definition 2. A GO-space $(M = G/H, g^M)$ of a simple compact Lie group G is called a **proper GO-space** if the metric g^M is not G-normal, i.e. the metric endomorphism A is not a scalar operator.

Lemma 2 ([1]). *A compact homogeneous Riemannian space $(M = G/H, g^M)$ with the reductive decomposition (1) and metric endomorphism A is GO-space if and only if for any $X \in \mathfrak{m}$ there is $H_X \in \mathfrak{h}$ such that one of the following equivalent conditions holds:*

i) $[H_X + X, A(X)] \in \mathfrak{h}$;

ii) $([H_X + X, Y]_\mathfrak{m}, X) = 0$ *for all* $Y \in \mathfrak{m}$.

This lemma shows that the property to be GO-space depends only on the reductive decomposition (1) and the Euclidean metric g on $\mathfrak{m}$. In other words, if $(M = G/H, g^M)$ is a GO-space, then any locally isomorphic homogeneous Riemannian space $(M' = G'/H', g^{M'})$ is a GO-space. Also a direct product of Riemannian manifolds is a manifold with homogeneous geodesics if and only if each factor is a manifold with homogeneous geodesics.

3 Totally geodesic orbits in a homogeneous Riemannian space

In this section we deal with totally geodesic submanifolds of compact homogeneous Riemannian spaces. This is a useful tool for study of GO-spaces due to the following

Proposition 1 ([3, Theorem 11]). *Every closed totally geodesic submanifold of a Riemannian manifold with homogeneous geodesics is a manifold with homogeneous geodesics.*

Let $(M = G/H, g^M)$ be a compact Riemannian homogeneous space with the reductive decomposition (1).

Definition 3. A subspace $\mathfrak{p} \subset \mathfrak{m}$ is called **totally geodesic** if it is the tangent space at o of a totally geodesic orbit $Ko \subset G/H = M$ of a subgroup $K \subset G$.

Proposition 2. *A subspace $\mathfrak{p} \subset \mathfrak{m}$ is totally geodesic if and only if the following two conditions hold:*

a) *$\mathfrak{p}$ generates a subalgebra of the form $\mathfrak{k} = \mathfrak{h}' + \mathfrak{p}$, where $\mathfrak{h}'$ is a subalgebra of $\mathfrak{h}$;*

b) *the endomorphism $\mathrm{ad}_Z^{\mathfrak{p}} \in \mathrm{End}(\mathfrak{p})$ for $Z \in \mathfrak{p}^{\perp}$ is g-skew-symmetric or, equivalently,*

$$U(\mathfrak{p}, \mathfrak{p}) \subset \mathfrak{p}.$$

Proof. If $\mathfrak{p}$ is the tangent space of the orbit $Ko = K/H'$, then $\mathrm{Lie}(K) = \mathfrak{k} = \mathfrak{h}' + \mathfrak{p}$, where $\mathfrak{h}' = \mathrm{Lie}(H')$ is a subalgebra of $\mathfrak{h}$. Moreover, the formulas (4) and (5) imply $U(\mathfrak{p}, \mathfrak{p}) \subset \mathfrak{p}$. Conversely, the conditions a) and b) imply that $\mathfrak{p}$ is the tangent space of the totally geodesic orbit Ko of the subgroup K generated by the subalgebra $\mathfrak{k}$. ■

Corollary 1.

i) *A subspace $\mathfrak{p} \subset \mathfrak{m}$ is totally geodesic if a) holds and $A\mathfrak{p} = \mathfrak{p}$.*

ii) *If a totally geodesic subspace $\mathfrak{p}$ is $\mathrm{ad}_{\mathfrak{h}}$-invariant and A-invariant, then*

$$[\mathfrak{h} + \mathfrak{p}, \mathfrak{p}^{\perp}] \subset \mathfrak{p}^{\perp}.$$

Proof. i) Assume that $A\mathfrak{p} = \mathfrak{p}$. Then $A\mathfrak{p}^{\perp} = \mathfrak{p}^{\perp}$ and $\langle \mathfrak{p}, \mathfrak{p}^{\perp} \rangle = 0$. From i) and $A\mathfrak{p} = \mathfrak{p}$ we get $\langle Z, [X, AX] \rangle = 0$ for any $X \in \mathfrak{p}$ and $Z \in \mathfrak{p}^{\perp}$. This implies

$$\begin{aligned} 0 = \langle [Z, X], AX \rangle &= \langle [Z, X]_{\mathfrak{m}}, AX \rangle = ([Z, X]_{\mathfrak{m}}, X) \\ &= ([Z, X]_{\mathfrak{p}}, X) = (U(X, X), Z). \end{aligned}$$

ii) follows from the fact that the endomorphisms $\mathrm{ad}_{\mathfrak{h}+\mathfrak{p}}$ are b-skew-symmetric and preserves the subspace $\mathfrak{p}$. Hence, they preserve its b-orthogonal complement $\mathfrak{p}^{\perp}$. ■

Corollary 2. *Let $(M = G/H, g)$ be a compact Riemannian homogeneous space and K a connected subgroup of G. The orbit $P = Ko = K/H'$ is a totally geodesic submanifold if and only if the Lie algebra $\mathfrak{k}$ is consistent with the reductive decomposition (1) (that is $\mathfrak{k} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{k} \cap \mathfrak{m} = \mathfrak{h}' + \mathfrak{p}$) and*

$$U(\mathfrak{p}, \mathfrak{p}) \subset \mathfrak{p}$$

or, equivalently, the endomorphisms $\mathrm{ad}_Z^{\mathfrak{p}} \in \mathrm{End}(\mathfrak{p})$, $Z \in \mathfrak{p}^{\perp}$ are g-skew-symmetric.

4 Properties of GO-spaces

Lemma 3. *Let* $(M = G/H, g^M)$ *be a GO-space with the reductive decomposition* (1) *and* $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$ *a* g*-orthogonal* Ad_H*-invariant decomposition. Then*

$$U(\mathfrak{p},\mathfrak{p}) \subset \mathfrak{p}, \qquad U(\mathfrak{q},\mathfrak{q}) \subset \mathfrak{q}$$

and the endomorphisms $\mathrm{ad}^{\mathfrak{q}}_{\mathfrak{p}}$, $\mathrm{ad}^{\mathfrak{p}}_{\mathfrak{q}}$ *are skew-symmetric.*

Proof. For $X \in \mathfrak{p}$, $Y \in \mathfrak{q}$ we have

$$0 = ([Y + H_Y, X]_{\mathfrak{m}}, Y) = -\,(\mathrm{ad}_X Y, Y) = -\,(U(Y,Y), X)\,,$$

where H_Y is as in Lemma 2. This shows that $\mathrm{ad}^{\mathfrak{q}}_X$ is skew-symmetric and $U(\mathfrak{q},\mathfrak{q}) \subset \mathfrak{q}$. ■

Lemma 3 together with Proposition 2 implies

Proposition 3. *Let* $(M = G/H, g^M)$ *be a GO-space with the reductive decomposition* (1)*. Then any connected subgroup* $K \subset G$ *which contains* H *has the totally geodesic orbit* $P = Ko = K/H$ *which is GO-space* (*with respect to the induced metric*)*. Moreover, if the space* $\mathfrak{p} := \mathfrak{k} \cap \mathfrak{m}$ *is* A*-invariant, then*

$$[\mathfrak{k}, \mathfrak{m}^{\perp}] \subset \mathfrak{m}^{\perp}$$

and the metric $\bar{g} := g|_{\mathfrak{p}^{\perp}}$ *is* Ad_K*-invariant and defines an invariant GO-metric* g^N *on the homogeneous manifolds* $N = G/K$*. The projection* $\pi : G/H \to G/K$ *is a Riemannian submersion with totally geodesic fibers such that the fibers and the base are GO-spaces.*

Proof. The first claim follows from Lemma 3, Lemma 2 and Proposition 2. If $A\mathfrak{p} = \mathfrak{p}$, then $\mathfrak{m} = \mathfrak{p} + \mathfrak{p}^{\perp}$ is a b-orthogonal decomposition and since the metric b is Ad_G-invariant, $\mathrm{Ad}_K \mathfrak{p}^{\perp} = \mathfrak{p}^{\perp}$. Then Lemma 3 shows that the metric $g|_{\mathfrak{p}^{\perp}}$ is Ad_K-invariant and defines an invariant metric g^N on $N = G/K$ such that N becomes GO-space. ■

Note that a subgroup $K \supset H$ is compatible with any invariant metric on G/H if Ad_H-modules $\mathfrak{p}$ and $\mathfrak{m}/\mathfrak{p}$ are strictly disjoint. This remark implies

Proposition 4. *Let* $(M = G/H, g)$ *be a compact homogeneous Riemannian space. Then the connected normalizer* $N_0(Z)$ *of a central subgroup* Z *of* H *and the connected normalizer* $N_0(H)$ *are subgroups consistent with any invariant metric on* M*.*

Proposition 5. *Let* $(M = G/H, g^M)$ *be a compact GO-space with metric endomorphism* A*.*

i) *Let* $X, Y \in \mathfrak{m}$ *be eigenvectors of the metric endomorphism* A *with different eigenvalues* λ, μ*. Then*

$$[X, Y] = \frac{\lambda}{\lambda - \mu}[H, X] + \frac{\mu}{\lambda - \mu}[H, Y]$$

for some $H \in \mathfrak{h}$*.*

ii) *Assume that the vectors* X, Y *belong to the* λ*-eigenspace* $\mathfrak{m}_\lambda$ *of* A *and* X *is* g*-orthogonal to the subspace* $[\mathfrak{h}, Y]$*. Then*

$$[X, Y] \in \mathfrak{h} + \mathfrak{m}_\lambda.$$

Proof. *i*) Let $X, Y \in \mathfrak{m}$ be eigenvectors of A with different eigenvalues λ, μ and $H = H_{X+Y} \in \mathfrak{h}$ the element defined in Lemma 2. Then

$$\begin{aligned}[H + X + Y, A(X + Y)] &= [H + X + Y, \lambda X + \mu Y] \\ &= \lambda[H, X] + \mu[H, Y] + (\mu - \lambda)[X, Y] \in \mathfrak{h}.\end{aligned}$$

By Lemma 1, $[H, X], [H, Y], [X, Y] \in \mathfrak{m}$ and the right hand side is zero.

ii) Assume now that $X, Y \in \mathfrak{m}_\lambda$ satisfy conditions *ii*) and Z is an eigenvector of A with an eigenvalue $\mu \neq \lambda$. Then we have

$$\begin{aligned}([X, Y]_{\mathfrak{m}}, Z) &= \mu\langle [X, Y], Z\rangle = \mu\langle X, [Y, Z]\rangle = \frac{\mu}{\lambda}(X, [Y, Z]_{\mathfrak{m}}) \\ &= \frac{\mu}{\lambda}\left(X, \frac{\lambda}{\lambda - \mu}[H, Y] + \frac{\mu}{\lambda - \mu}[H, Z]\right) = 0.\end{aligned}$$

This shows that $[X, Y] \in \mathfrak{h} + \mathfrak{m}_\lambda$. ■

Corollary 3. *Let* $(M = G/H, g^M)$ *be a compact GO-space with the reductive decomposition* (1) *and metric endomorphism* A *and*

$$\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k \tag{6}$$

the A*-eigenspace decomposition such that* $A|_{\mathfrak{m}_i} = \lambda_i 1_{\mathfrak{m}_i}$. *Then for any* Ad_H*-submodules* $\mathfrak{p}_i \subset \mathfrak{m}_i$, $\mathfrak{p}_j \subset \mathfrak{m}_j$, $i \neq j$, *we have*

$$[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_i + \mathfrak{p}_j.$$

Moreover, if $\mathfrak{p}$, $\mathfrak{p}'$ *are* g*-orthogonal* Ad_H*-submodules of* $\mathfrak{m}_i$ *then*

$$[\mathfrak{p}, \mathfrak{p}'] \subset \mathfrak{h} + \mathfrak{m}_i.$$

5 Some applications

5.1 A sufficient condition for non-existence of GO-metric

Here we consider some applications of results of the previous section.

Definition 4. Let $(M = G/H, g^M)$ be a compact homogeneous Riemannian space. A connected closed Lie subgroup $K \subset G$ which contains H is called **compatible with the metric** g^M if the subspace $\mathfrak{p} = \mathfrak{k} \cap \mathfrak{m}$ of $\mathfrak{m}$ is invariant under the metric endomorphism A.

Let K, K' be two subgroups of G which are compatible with the metric of a homogeneous Riemannian space $(M = G/H, g^M)$. Then we can decompose the space $\mathfrak{m}$ into a g-orthogonal sum of A-invariant Ad_H-modules

$$\mathfrak{m} = \mathfrak{q} + \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{n} \tag{7}$$

where

$$\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}', \qquad \mathfrak{p} = \mathfrak{k} \cap \mathfrak{m} = \mathfrak{q} + \mathfrak{p}_1, \qquad \mathfrak{p}' = \mathfrak{k}' \cap \mathfrak{m} = \mathfrak{q} + \mathfrak{p}_2$$

and $\mathfrak{n}$ is the orthogonal complement to

$$\mathfrak{p} + \mathfrak{p}' = \mathfrak{q} + \mathfrak{p}_1 + \mathfrak{p}_2$$

in $\mathfrak{m}$.

Proposition 6. *Let $(M = G/H, g^M)$ be a homogeneous Riemannian space, K, K' two subgroups of G which are compatible with g^M and (7) the associated decomposition as above. Then $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{n}$ are $\mathrm{Ad}_{\widetilde{H}}$-modules, where $\widetilde{H} = K \cap K'$ is the Lie group with the Lie algebra $\widetilde{\mathfrak{h}} = \mathfrak{h} + \mathfrak{q}$, and*

$$[\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{n}.$$

Moreover, if $(M = G/H, g^M)$ is a GO-space, then the restriction $A^{\widetilde{\mathfrak{m}}}$ of the metric endomorphism to $\widetilde{\mathfrak{m}} = \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{n}$ commutes with $\mathrm{Ad}_{\widetilde{H}}|_{\widetilde{\mathfrak{m}}}$ and for any $\widetilde{H}$-irreducible submodules $\mathfrak{p}_1' \subset \mathfrak{p}_1$, and $\mathfrak{p}_2' \subset \mathfrak{p}_2$ such that

$$[\mathfrak{p}_1', \mathfrak{p}_2'] \neq 0,$$

the metric endomorphism A is a scalar on the space

$$\mathfrak{p}_1' + \mathfrak{p}_2' + [\mathfrak{p}_1', \mathfrak{p}_2'].$$

Proof. Since the decomposition (7) is b-orthogonal, we conclude that it is $\mathrm{Ad}_{\widetilde{H}}$-invariant and

$$[\mathfrak{p}, \mathfrak{p}^{\perp}] = [\mathfrak{p}, \mathfrak{p}_2 + \mathfrak{n}] \subset \mathfrak{p}_2 + \mathfrak{n},$$
$$[\mathfrak{p}', (\mathfrak{p}')^{\perp}] = [\mathfrak{p}', \mathfrak{p}_1 + \mathfrak{n}] \subset \mathfrak{p}_1 + \mathfrak{n},$$

by Proposition 2. This implies

$$[\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{n}.$$

If (M, g^M) is a GO-space, then by Proposition 3 the metric endomorphism $A^{\widetilde{\mathfrak{m}}}$ is $\widetilde{H}$-invariant. If modules $\mathfrak{p}_1'$, $\mathfrak{p}_2'$ belong to A-eigenspaces with different eigenvalues, then by Corollary 3,

$$[\mathfrak{p}_1', \mathfrak{p}_2'] \subset \mathfrak{p}_1 + \mathfrak{p}_2.$$

Together with the previous inclusion, it implies $[\mathfrak{p}_1', \mathfrak{p}_2'] = 0$. If these modules belong to the same eigenspace $\mathfrak{m}_\lambda$, then by Corollary 3,

$$[\mathfrak{p}_1', \mathfrak{p}_2'] \subset \mathfrak{m}_\lambda. \qquad \blacksquare$$

As a corollary, we get the following sufficient condition that a homogeneous manifold $M = G/H$ does not admit a proper GO-metric.

Proposition 7. *Let $M = G/H$ be a homogeneous space of a compact group G with the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Assume that the Lie algebra $\mathfrak{g}$ has two subalgebras $\mathfrak{k} = \mathfrak{h} + \mathfrak{p}$, $\mathfrak{k}' = \mathfrak{h} + \mathfrak{p}'$ which contain $\mathfrak{h}$ and generate $\mathfrak{g}$. Let*

$$\mathfrak{m} = \mathfrak{q} + \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{n}, \qquad \mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}' \tag{8}$$

be the associated b-orthogonal decomposition. Assume that there is no commuting $\mathrm{ad}_{\mathfrak{h}+\mathfrak{q}}$ submodules of $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Then for any GO-metric, defined by an operator A which preserves this decomposition, A is a scalar operator on $\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{n}$. In particular, if $\mathfrak{q}$ is trivial and Ad_H-modules $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{n}$ are strictly non-equivalent, then the only GO-metric on M is the normal metric.

Proof. Let A be an operator on $\mathfrak{m}$ which preserves the decomposition (8) and defines a GO-metric. Then by Proposition 6,

$$A|_{\mathfrak{p}_1+\mathfrak{p}_2+[\mathfrak{p}_1,\mathfrak{p}_2]} = \lambda \cdot \mathbf{1}$$

for some λ. Now $\mathfrak{p}_1$ and $[\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{n}$ are two g-orthogonal submodules of the A-eigenspace $\mathfrak{m}_\lambda$. Applying Corollary 3, we conclude that

$$[\mathfrak{p}_1, [\mathfrak{p}_1, \mathfrak{p}_2]] \subset \mathfrak{m}_\lambda.$$

Iterating this process, we prove that

$$\mathfrak{n} = [\mathfrak{p}_1, \mathfrak{p}_2] + [\mathfrak{p}_1, [\mathfrak{p}_1, \mathfrak{p}_2]]_{\mathfrak{n}} + [\mathfrak{p}_2, [\mathfrak{p}_1, \mathfrak{p}_2]]_{\mathfrak{n}} + \cdots \subset \mathfrak{m}_\lambda$$

and $A = \lambda \cdot \mathbf{1}$ on $\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{n}$. $\blacksquare$

5.2 A sufficient condition for existence of GO-metric

Lemma 4. *Let $M = G/H$ be a homogeneous space of a compact Lie group with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Assume that Ad_H-module $\mathfrak{m}$ has a decomposition*

$$\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$$

into invariant submodules, such that for any $i < j$

$$[\mathfrak{m}_i, \mathfrak{m}_j] = 0$$

or this condition valid with one exception $(i, j) = (1, 2)$ and in this case

$$[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2$$

and for any $X \in \mathfrak{m}_1$, $Y \in \mathfrak{m}_2$ there is $H \in \mathfrak{h}$ such that $\mathrm{ad}_H Y = \mathrm{ad}_X Y$ and

$$\mathrm{ad}_H(\mathfrak{m}_1 + \mathfrak{m}_3 + \cdots + \mathfrak{m}_k) = 0.$$

Then any metric endomorphism of the form $A = \sum x_i \cdot \mathbf{1}_{\mathfrak{m}_i}$ defines a GO-metric on M.

Proof. Under the assumptions of lemma, for $H \in \mathfrak{h}$ and $X_i \in \mathfrak{m}_i$ we have

$$\begin{aligned}\left[H + \sum X_i, \sum x_i X_i\right] &= \sum_{i<j}(x_j - x_i)[X_i, X_j] + \sum_i x_i \,\mathrm{ad}_H X_i \\ &= (x_2 - x_1)\,\mathrm{ad}_{X_1} X_2 + x_2 \,\mathrm{ad}_H X_2 + \mathrm{ad}_H\left(x_1 X_1 + \sum_{k\geq 3} x_k X_k\right).\end{aligned}$$

The right-hand side is zero if H is chosen as in the lemma (where $Y = x_2 X_2$ and $X = (x_1 - x_2)X_1$). Now, it suffices to apply Lemma 2. ■

Example 1. The homogeneous space $M = SU_{p+q}/SU_p \times SU_q$ is a GO-space with respect to any invariant metric.

We have the reductive decomposition

$$\mathfrak{su}_{p+q} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{su}_p + \mathfrak{su}_q) + (\mathbb{R}a + \mathfrak{p}),$$

where $\mathfrak{p} \simeq \mathbb{C}^p \otimes \mathbb{C}^q$ and $\mathrm{ad}_a|_{\mathfrak{p}} = i \cdot \mathbf{1}_{\mathfrak{p}}$. Any metric endomorphism $A = \lambda \cdot \mathbf{1}_{\mathbb{R}a} + \mu \cdot \mathbf{1}_{\mathfrak{p}}$ defines a GO-metric by above lemma since for any $X \in \mathfrak{p}$ there is $H \in \mathfrak{h}$ such that $\mathrm{ad}_H X = iX$. Note that for $p \neq q$ these manifolds are weakly symmetric spaces [26].

5.3 GO-metrics on a compact group G

Proposition 8. *A compact Lie group G with a left-invariant metric g is a GO-space if and only if the corresponding Euclidean metric $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$ is bi-invariant.*

Proof. The condition that (G, g) is a GO-space can be written as

$$0 = (X, [X, Y]) = -(\mathrm{ad}_Y X, X) = 0.$$

This shows that the metric $(\cdot, \cdot)$ is bi-invariant. ■

Note that a compact Lie group G can admit a non-bi-invariant left-invariant metrics g with homogeneous geodesics. But the corresponding GO-space will have the form L/H where the group L will contain G as a proper subgroup. See [11] for details.

6 Homogeneous GO-spaces with positive Euler characteristic

6.1 Basic facts about homogeneous manifolds of positive Euler characteristic

Here we recall some properties of homogeneous spaces with positive Euler characteristic (see, for example, [21] or [4]). A homogeneous space $M = G/H$ of a compact connected Lie group G has positive Euler characteristic $\chi(M) > 0$ if and only if the stabilizer H has maximal rank ($\mathrm{rk}(H) = \mathrm{rk}(G)$).

If the group G acts on M almost effectively, then it is semisimple and the universal covering $\widetilde{M} = \widetilde{G}/\widetilde{H}$ is a direct product

$$\widetilde{M} = G_1/H_1 \times \cdots \times G_k/H_k,$$

where $\widetilde{G} = G_1 \times G_2 \times \cdots \times G_k$ is the decomposition of the group $\widetilde{G}$ (which is a covering of G) into a direct product of simple factors and $H_i = \widetilde{H} \cap G_i$.

Any invariant metric g^M on M defines an invariant metric $g^{\widetilde{M}}$ on $\widetilde{M}$ and the homogeneous Riemannian space $(\widetilde{M} = \widetilde{G}/\widetilde{H}, g^{\widetilde{M}})$ is a direct product of homogeneous Riemannian spaces $(M_i = G_i/H_i, g^{M_i})$, $i = 1, \dots, k$, of simple compact Lie groups G_i, see [19]. We have

Proposition 9 ([19]). *A compact almost effective homogeneous Riemannian space $(M = G/H, g^M)$ of positive Euler characteristic is irreducible if and only if the group G is simple. If the group G acts effectively on M, it has trivial center.*

This proposition shows that a simply connected compact GO-space $(M = G/H, g^M)$ of positive Euler characteristic is a direct product of simply connected GO-spaces $(M_i = G_i/H_i, g^{M_i})$ of simple Lie groups with positive Euler characteristic. So it is sufficient to classify simply connected GO-spaces of a simple compact Lie group with positive Euler characteristic.

A description of homogeneous spaces G/H of positive Euler characteristic reduces to description of connected subgroups H of maximal rank of G or equivalently, subalgebras of maximal rank of a simple compact Lie algebra $\mathfrak{g}$, see [9] and also Section 8.10 in [25]. An important subclass of compact homogeneous spaces of positive Euler characteristic consists of **flag manifolds**. They are described as adjoint orbits $M = \mathrm{Ad}_G\, x$ of a compact connected semisimple Lie group G or, in other terms as quotients $M = G/H$ of G by the centrelizer $H = Z_G(T)$ of a non-trivial torus $T \subset G$.

Note that every compact naturally reductive homogeneous Riemannian space of positive Euler characteristic is necessarily normal homogeneous with respect to some transitive semisimple isometry group [4].

6.2 The main theorem

Let G be a simple compact connected Lie group, $H \subset K \subset G$ its closed connected subgroups. We denote by $b = \langle \cdot, \cdot \rangle$ the minus Killing form on the Lie algebra $\mathfrak{g}$ and consider the following b-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_2$$

is the Lie algebra of the group K. Obviously, $[\mathfrak{m}_2, \mathfrak{m}_1] \subset \mathfrak{m}_1$. Let $g^M = g_{x_1,x_2}$ be a G-invariant Riemannian metric on $M = G/H$, generated by the Euclidean metric $g = (\cdot, \cdot)$ on $\mathfrak{m}$ of the form

$$g = x_1 \cdot b_{\mathfrak{m}_1} + x_2 \cdot b_{\mathfrak{m}_2}, \tag{9}$$

where x_1 and x_2 are positive numbers, or, equivalently, by the metric endomorphism

$$A = x_1 \cdot \mathbf{1}_{\mathfrak{m}_1} + x_2 \cdot \mathbf{1}_{\mathfrak{m}_2}. \tag{10}$$

We consider two examples of such homogeneous Riemannian spaces $(M = G/H, g_{x_1,x_2})$:

a) $(G, K, H) = (SO(2n + 1), U(n), SO(2n))$, $n \geq 2$. The group $G = SO(2n + 1)$ acts transitively on the symmetric space $Com(\mathbb{R}^{2n+2}) = SO(2n + 2)/U(n)$ of complex structures in $\mathbb{R}^{2n+2}$ with stabilizer $H = U(n)$, see [15]. So we can identify $M = G/H$ with this symmetric space, but the metric g_{x_1,x_2} is not $SO(2n + 2)$-invariant if $x_2 \neq 2x_1$ [17].

b) $(G, K, H) = (Sp(n), Sp(1) \cdot Sp(n - 1), U(1) \cdot Sp(n - 1))$, $n \geq 2$. The group $G = Sp(n)$ acts transitively on the projective space $\mathbb{C}P^{2n-1} = SU(2n + 2)/U(2n + 1)$ with stabilizer $H = U(1) \cdot Sp(n - 1)$. So we can identify $M = G/H$ with $\mathbb{C}P^{2n-1}$, but the metric g_{x_1,x_2} is not $SU(2n + 2)$-invariant if $x_2 \neq 2x_1$, see [15, 17].

Now we can state the main theorem about compact GO-spaces of positive Euler characteristic.

Theorem 1. *Let $(M = G/H, g^M)$ is a simply connected proper GO-space with positive Euler characteristic and simple compact Lie group G. Then $M = G/H = SO(2n + 1)/U(n)$, $n \geq 2$, or $G/H = Sp(n)/U(1) \times Sp(n - 1)$, $n \geq 2$, and $g^M = g_{x_1,x_2}$ is any G-invariant metric which is not G-normal homogeneous. The metric g^M is G-normal homogeneous (respectively, symmetric) when $x_2 = x_1$ (respectively, $x_2 = 2x_1$). Moreover, these homogeneous spaces are weakly symmetric flag manifolds.*

The non-symmetric metrics g_{x_1,x_2} have G as the full connected isometry group of the considered GO-spaces $(M = G/H, g_{x_1,x_2})$, see discussion in [17, 21, 1]. The claim that all these homogeneous Riemannian spaces are weakly symmetric spaces was proved in [27]. Note also that Theorem 1 allows to simplify some arguments in the paper [3].

6.3 Proof of the main theorem

Using results from [1] and [3], we reduce the proof to a description of some special decompositions of the root system of the Lie algebra $\mathfrak{g}$ of the isometry group G.

Let $M = G/H$ be a homogeneous space of a compact simple Lie group of positive characteristic and

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

associated reductive decomposition. The subgroup H contains a maximal torus T of G. We consider the root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

of the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$, where $\mathfrak{t}^{\mathbb{C}}$ is the Cartan subalgebra associated with T and R is the root system.

For any subset $P \subset R$ we denote by

$$\mathfrak{g}(P) = \sum_{\alpha \in P} \mathfrak{g}_\alpha$$

the subspace spanned by corresponding root space $\mathfrak{g}_\alpha$. Then H-module $\mathfrak{m}^{\mathbb{C}}$ is decomposed into a direct sum

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}(R_1) + \cdots + \mathfrak{g}(R_k)$$

of disjoint submodules, where $R = R_1 \cup \cdots \cup R_k$ is a disjoint decomposition of R and subsets R_i are symmetric, i.e. $-R_i = R_i$. Moreover, real H-modules $\mathfrak{g} \cap \mathfrak{g}(R_i)$ are irreducible. Any invariant metric on M is defined by the metric endomorphism A on $\mathfrak{m}$ whose extension to $\mathfrak{m}^{\mathbb{C}}$ has the form

$$A = \operatorname{diag}(x_1 \cdot \mathbf{1}_{\mathfrak{p}_1}, \dots, x_\ell \cdot \mathbf{1}_{\mathfrak{p}_\ell}),$$

where x_i are arbitrary positive numbers, $x_i \neq x_j$ and $\mathfrak{p}_i$ is a direct sum of modules $\mathfrak{g}(R_m)$.

We will assume that A is not a scalar operator (i.e. $\ell > 1$) and it defines an invariant metric with homogeneous geodesics. We say that a root α corresponds to eigenvalue x_i of A if $\mathfrak{g}_\alpha \subset \mathfrak{p}_i$.

Lemma 5. *There are two roots α, β which correspond to different eigenvalues of A such that $\alpha + \beta$ is a root.*

Proof. If it is not the case, $[\mathfrak{p}_1, \mathfrak{p}_i] = 0$ for $i \neq 1$ and $\mathfrak{g}_1 = \mathfrak{p}_1 + [\mathfrak{p}_1, \mathfrak{p}_1]$ would be a proper ideal of a simple Lie algebra $\mathfrak{g}$. ■

Now, consider the roots α and β as in the previous lemma. Since $R(\alpha, \beta) := R \cap \operatorname{span}\{\alpha, \beta\}$ is a rank 2 root system, we can always choose roots $\alpha, \beta \in R$ which form a basis of the root system $R(\alpha, \beta)$. Then the subalgebra

$$\mathfrak{g}_{\alpha,\beta} := \mathfrak{t}^{\mathbb{C}} + \sum_{\gamma \in R(\alpha,\beta)} \mathfrak{g}_\gamma$$

of $\mathfrak{g}^{\mathbb{C}}$ is the centralizer of the subalgebra $\mathfrak{t}' = \ker \alpha \cap \ker \beta \subset \mathfrak{t}^{\mathbb{C}}$.

Then the orbit $G_{\alpha,\beta} o \subset M$ of the corresponding subgroup $G_{\alpha,\beta} = T' \cdot G'_{\alpha,\beta} \subset G$ is a totally geodesic submanifold (see Corollary 2), hence a proper GO-space with the effective action of the rank two simple group $G'_{\alpha,\beta}$ associated with the root system $R(\alpha, \beta)$ (see Proposition 1). Note that it has positive Euler characteristic since the stabilizer of the point o contains the two-dimensional torus generated by vectors $H_\alpha, H_\beta \in \mathfrak{t}^{\mathbb{C}}$ associated with roots α, β. Recall that $H_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} b^{-1} \cdot \alpha$.

Proposition 10. *Every proper GO-space $(M = G/H, g^M)$ with positive Euler characteristic of a simple group G of rank 2 is locally isometric to the manifold $M = SO(5)/U(2)$ with the metric defined by the metric endomorphism*

$$A = x_1 \cdot \mathbf{1}_{\mathfrak{g}(R^s)} + x_2 \cdot \mathbf{1}_{\mathfrak{g}(R^\ell)}, \qquad x_1 \neq x_2 > 0$$

where

$$R^s = \{\pm\epsilon_1, \pm\epsilon_2\}, \qquad R^\ell = \{\pm\epsilon_1 \pm \epsilon_2\},$$

are the sets of short and, respectively, long roots of the Lie algebra $\mathfrak{so}(5)$. We may assume also that

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}(R^s \cup \{\epsilon_1 + \epsilon_2\}) \qquad \textit{and} \qquad \mathfrak{h}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \mathfrak{g}_{\epsilon_1 - \epsilon_2}.$$

Proof. Proof of this proposition follows from results of the papers [1] and [3]. Indeed, the group G has the Lie algebra $\mathfrak{g}$ isomorphic to $su(3) = A_2$, $so(5) = sp(2) = B_2 = C_2$ or g_2. Since the universal Riemannian covering of a GO-space is a GO-space (Lemma 2), we may assume without loss of generality that G/H is simply connected.

If $\mathfrak{g} = su(3)$, then $G/H = SU(3)/S(U(2) \times U(1))$ (a symmetric space) or $G/H = SU(3)/T^2$, where T^2 is a maximal torus in $SU(3)$. Both these spaces are flag manifolds, and results of [1] show that any GO-metric on these spaces is $SU(3)$-normal homogeneous.

If $\mathfrak{g} = so(5) = sp(2)$, then $(\mathfrak{g}, \mathfrak{h}) = (so(5), \mathbb{R}^2)$, $(\mathfrak{g}, \mathfrak{h}) = (so(5), \mathbb{R} \oplus su(2)_l)$, $(\mathfrak{g}, \mathfrak{h}) = (so(5), \mathbb{R} \oplus su(2)_s)$, or $(\mathfrak{g}, \mathfrak{h}) = (so(5), su(2)_l \oplus su(2)_l)$, where $su(2)_l$, (respectively, $su(2)_s$) stands for a three-dimensional subalgebras generated by all long (respectively, short) roots of $\mathfrak{g}$. The last pair corresponds to the irreducible symmetric space $SO(5)/SO(4)$, which admits no non-normal invariant metric. All other spaces are flag manifolds. Results of [1] implies that the only possible pair is $(\mathfrak{g}, \mathfrak{h}) = (so(5), \mathbb{R} \oplus su(2)_l)$, which corresponds to the space $SO(5)/U(2) = Sp(2)/U(1) \cdot Sp(1)$.

For $\mathfrak{g} = g_2$ the statement of proposition is proved in [3, Proposition 23]. ■

Corollary 4. *Let G be a simple compact Lie group and $M = G/H$ a proper GO-space with positive Euler characteristic. Then the root system R of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ admits a disjoint decomposition*

$$R = R_0 \cup R_1 \cup R_2,$$

where R_0 is the root system of the complexified stability subalgebra $\mathfrak{h}^{\mathbb{C}}$,with the following properties:

- *i) If $\alpha \in R_1$, $\beta \in R_2$ and $\alpha + \beta \in R$ then $\alpha - \beta \in R$ and the rank 2 root system $R(\alpha, \beta)$ has type $B_2 = C_2$.*
- *ii) Moreover, if α, β is a basis of $R(\alpha, \beta)$ (that is $\langle \alpha, \beta \rangle < 0$), then one of the roots α, β is short and the other is long and one of the long roots $\alpha \pm \beta$ belongs to R_0 and second one belongs to $R_1 \cup R_2$.*
- *iii) If both roots α, β are short, then one of the long roots $\alpha \pm \beta$ belongs to R_0 and the other belongs to $R_1 \cup R_2$.*
- *iv) If $\alpha \in R_1$ and $\beta \in R_2$ are long roots, then $\alpha \pm \beta \notin R$.*

We will call a decomposition with the above properties a **special decomposition**. Corollary 4 implies

Corollary 5. *There is no proper GO-spaces of positive Euler characteristic with simple isometry group $G = SU(n)$, $SO(2n)$, E_6, E_7, E_8 (these are all simple Lie algebras with all roots of the same length (simply-laced root system)).*

Corollary 6 ([3, Proposition 23]). *Any GO-space $(G/H, \mu)$ of positive Euler characteristic with $G = G_2$ is normal homogeneous.*

Now, we describe all **special decompositions** of the root systems of types B_n, C_n, F_4. We will use notation from [13] for root systems and simple roots.

Lemma 6. *The root system*

$$R(F_4) = \{\pm\epsilon_i,\ 1/2(\pm\epsilon_1 + \mp\epsilon_2 \pm \epsilon_3 + \pm\epsilon_4, \pm\epsilon_i \pm \epsilon_j),\ i, j = 1, 2, 3, 4,\ i \neq j\}$$

does not admit a special decomposition.

Proof. Assume that such a decomposition exists. Then we can choose roots $\alpha \in R_1$, $\beta \in R_2$ such that $\alpha \pm \beta$ is a root. Then α, β has different length and we may assume that $|\alpha| < |\beta|$ and $\langle \alpha, \beta \rangle < 0$. Then we can include α, β into a system of simple roots δ, α, β, γ, see [13]. Since all such systems are conjugated, we may assume that $\alpha = \epsilon_4$, $\beta = -\epsilon_4 + \epsilon_3$, see [13]. Then we get contradiction, since $\alpha - \beta$ is not a root. ■

Now we describe two special decompositions for the root systems

$$R(B_n) = \{\pm\epsilon_i,\ \pm\epsilon_i \pm \epsilon_j,\ i,j = 1,\dots,n\}$$

and

$$R(C_n) = \{\pm 2\epsilon_i,\ \pm\epsilon_i \pm \epsilon_j,\ i,j = 1,\dots,n\}$$

of types B_n and C_n. Note that in both cases $R_A = \{\pm(\epsilon_i - \epsilon_j)\}$ is a closed subsystem. We set $R_A^+ = \{\pm(\epsilon_i + \epsilon_j)\}$.

We denote by R^+ the standard subsystem of positive roots of a root system R and by R^s and R^ℓ the subset of short and, respectively, long roots of R. Then there is a special decomposition $R = R_0 \cup R_1 \cup R_2$ of the systems $R(B_n)$, $R(C_n)$ which we call the standard decomposition:

$$R(B_n) = R_A \cup R^s \cup R_A^+,$$
$$R(C_n) = R_A \cup R^\ell \cup R_A^+.$$

These decompositions define the following reductive decompositions of the homogeneous spaces $SO(2n+1)/U(n)$ and $Sp(n)/U(n)$:

$$\mathfrak{so}(2n+1) = \mathfrak{h} + (\mathfrak{m}_1 + \mathfrak{m}_2) = \mathfrak{g}(R_A) + (\mathfrak{g}(R^s) + \mathfrak{g}(R_A^+)),$$
$$\mathfrak{sp}(n) = \mathfrak{h} + (\mathfrak{m}_1 + \mathfrak{m}_2) = \mathfrak{g}(R_A) + (\mathfrak{g}(R^\ell) + \mathfrak{g}(R_A^+)),$$

where $\mathfrak{m}_1$, $\mathfrak{m}_2$ are irreducible submodules of $\mathfrak{m}$. It is known [1] that any metric endomorphism $A = \operatorname{diag}(x_1 \cdot \mathbf{1}_{\mathfrak{m}_1}, x_2 \cdot \mathbf{1}_{\mathfrak{m}_2})$ defines a metric with homogeneous geodesics on the corresponding manifold $M = G/H$ (see a discussion before the statement of Theorem 1). Now, the proof of Theorem 1 follows from the following proposition.

Proposition 11. *Any special decomposition of the root systems R_B, R_C is conjugated to the standard one.*

Proof. We give a proof of this proposition for $R(B_n)$. The proof for $R(C_n)$ is similar.

Let

$$R(B_n) = R_0 \cup R_1 \cup R_2$$

be a special decomposition of $R(B_n)$. We may assume that there are roots $\alpha \in R_1$ and $\beta \in R_2$ with $\langle\alpha,\beta\rangle < 0$ and $|\alpha| < |\beta|$. Then we can include α,β into a system of simple roots, which, without loss of generality, can be written as

$$\alpha_1 = \epsilon_1 - \epsilon_2,\ \dots,\ \epsilon_{n-2} - \epsilon_{n-1},\ \epsilon_{n-1} + \epsilon_n = \beta,\ -\epsilon_n = \alpha.$$

Then $(\epsilon_{n-1} - \epsilon_n) \in R_0$. We need the following lemma.

Lemma 7. *Let $R(B_n) = R_0 \cup R_1 \cup R_2$ be a special decomposition as above, $V' = \epsilon_n^\perp$ the orthogonal complement of the vector ϵ_n and $R(B_{n-1}) = R' := R \cap V'$ the root system induced in the hyperspace V'. Then the induced decomposition $R' = R'_0 \cup R'_1 \cup R'_2$, where $R'_i := R_i \cap V'$, is a special decomposition.*

Proof. It is sufficient to check that subsets R'_1, R'_2 are not empty.

We say that two roots γ, δ are R_0-**equivalent** ($\gamma \sim \delta$) if their difference belongs to R_0. The equivalent roots belong to the same component R_i. The root $\epsilon_{n-1} = \epsilon_n - (\epsilon_{n-1} - \epsilon_n)$ is R_0-equivalent to $\alpha = \epsilon_n$. Hence it belongs to R_1.

We say that a pair of roots γ, δ with $\langle\gamma,\delta\rangle < 0$ is **special** if one of the roots belongs to R_1 and another to R_2. Then they have different length (say, $|\gamma| < |\delta|$). Moreover, the root $\gamma+\delta$ is short and it belongs to the same part R_i, $i = 1, 2$ as the short root δ and the root $2\gamma+\delta$ is long and it belongs to R_0.

Consider the roots $\sigma_\pm = \pm\epsilon_{n-2} + \epsilon_{n-1}$. They have negative scalar product with $\epsilon_{n-1} \in R_1$ and $\beta = \epsilon_{n-1} + \epsilon_n \in R_2$. They can not belong to R_1 since then we get a special pair $\delta_\pm, \beta$ which consists of long roots. They both can not belong to R_0 since otherwise the root $\epsilon_{n-2} \sim \epsilon_{n-1} \in R_1$ and $\pm\epsilon_{n-2} + \epsilon_n \sim \epsilon_{n-1} + \epsilon_n \in R_2$ and we get a special pair

$$\gamma = \epsilon_{n-2} \in R_1, \qquad \delta = -\epsilon_{n-2} + \epsilon_n \in R_2,$$

such that $2\gamma + \delta \in R_0$, which is impossible. We conclude that one of the roots $\sigma_\pm = \pm\epsilon_{n-2} + \epsilon_{n-1} \in R'$ must belongs to R_2. Since the root $\epsilon_{n-1} \in R'$ belongs to R_1, the lemma is proved. ■

Now we prove the proposition by induction on n. The claim is true for $n = 2$ by Proposition 10. Assume that it is true for $R(B(n-1))$ and let $R(B_n) = R_0 \cup R_1 \cup R_2$ be a special decomposition as above. By lemma, the decomposition $R' = R'_0 \cup R'_1 \cup R'_2$, indiced in the hyperplane $V' = e_n \perp$, is a special decomposition. By inductive hypothesis we may assume that it has the standard form:

$$R_0 = \{\pm(\epsilon_i - \epsilon_j)\}, \qquad R_1 = \{\pm\epsilon_i\}, \qquad R_2 = \{\pm(\epsilon_i + \epsilon_j),\ i, j = 1, \dots, n-1\}.$$

This implies that the initial decomposition is also standard. ■

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