The Relation Between the Associate Almost Complex Structure to HM' and (HM', S, T)-Cartan Connections

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Abstract. In the present paper, the (HM', S, T)-Cartan connections on pseudo-Finsler manifolds, introduced by A. Bejancu and H.R. Farran, are obtained by the natural almost complex structure arising from the nonlinear connection HM'. We prove that the natural almost complex linear connection associated to a (HM', S, T)-Cartan connection is a metric linear connection with respect to the Sasaki metric G. Finally we give some conditions for (M', J, G) to be a Kähler manifold.

Key words: almost complex structure; Kähler and pseudo-Finsler manifolds; (HM', S, T)-Cartan connection

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1 Introduction

Almost complex structures are important structures in differential geometry [8, 9, 11]. These structures have found many applications in physics. H.E. Brandt has shown that the spacetime tangent bundle, in the case of Finsler spacetime manifold, is almost complex [4, 5, 6]. Also he demonstrated that in this case the spacetime tangent bundle is complex provided that the gauge curvature field vanishes [3]. In [1, 2], for a pseudo-Finsler manifold $F^m = (M, M', F^*)$ with a nonlinear connection HM' and any two skew-symmetric Finsler tensor fields of type $(1, 2)$ on F^m, A. Bejancu and H.R. Farran introduced a notion of Finsler connections which named "(HM', S, T)-Cartan connections". If, in particular, HM' is the canonical nonlinear connection GM' of F^m and $S = T = 0$, the Finsler connection is called the Cartan connection and it is denoted by $FC^* = (GM', \nabla^*)$ (see [1]). They showed that ∇^* is the projection of the Levi-Civita connection of the Sasaki metric G on the vertical vector bundle. Also they proved that the associate linear connection $\mathcal{D}^*$ to the Cartan connection FC^* is a metric linear connection with respect to G [1]. In this paper we obtain the (HM', S, T)-Cartan connections by using the natural almost complex structure arising from the nonlinear connection HM', then the natural almost complex linear connection associated to a (HM', S, T)-Cartan connection is defined. We prove that the natural almost complex linear connection associated to a (HM', S, T)-Cartan connection is a metric linear connection with respect to the Sasaki metric G. Kähler and para-Kähler structures associated with Finsler spaces and their relations with flag curvature were studied by M. Crampin and B.Y. Wu (see [7, 12]). They have found some interesting results on this matter. In [12], B.Y. Wu gives some equivalent statements to the Kählerity of (M', G, J). In the present paper we give other conditions for the Kählerity of (M', G, J), which extend the previous results.

2 The associate almost complex structure to HM'

Let M be a real m-dimensional smooth manifold and TM be the tangent bundle of M. Let M' be a nonempty open submanifold of TM such that $\pi(M') = M$ and $\theta(M) \cap M' = \varnothing$, where θ is the zero section of TM. Suppose that $F^m = (M, M', F^*)$ is a pseudo-Finsler manifold where $F^* : M' \longrightarrow \mathbb{R}$ is a smooth function which in any coordinate system $\{(\mathcal{U}', \Phi') : x^i, y^i\}$ in M', the following conditions are fulfilled:

- F^* is positively homogeneous of degree two with respect to $(y^1, \ldots, y^m)$, i.e., we have

$$F^*(x^1, \ldots, x^m, ky^1, \ldots, ky^m) = k^2 F^*(x^1, \ldots, x^m, y^1, \ldots, y^m)$$

  for any point $(x, y) \in (\Phi', \mathcal{U}')$ and $k > 0$.

- At any point $(x, y) \in (\Phi', \mathcal{U}')$, g_{ij} are the components of a quadratic form on $\mathbb{R}^m$ with q negative eigenvalues and $m - q$ positive eigenvalues, $0 < q < m$ (see [1]).

Consider the tangent mapping $\pi_* : TM' \longrightarrow TM$ of the submersion $\pi : M' \longrightarrow M$ and define the vector bundle $VM' = \ker \pi_*$. A complementary distribution HM' to VM' in TM' is called a nonlinear connection or a horizontal distribution on M'

$$TM' = HM' \oplus VM'.$$

A nonlinear connection HM' enables us to define an almost complex structure on M' as follows:

$$J : \Gamma(TM') \longrightarrow \Gamma(TM'),$$
$$J\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i},$$

where $\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y)\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\}$ is assumed as a local frame field of TM' and $\Gamma(TM')$ is the space of smooth sections of the vector bundle TM'. We call J the associate almost complex structure to HM'. Obviously we have $J^2 = -Id_{TM'}$, also we can assume the conjugate of J, $J' = -J$, as an almost complex structure. Now we give the following proposition which was proved by B.Y. Wu [12].

Proposition 1. *Let $F^m = (M, M', F)$ be a Finsler manifold. Then the following statements are mutually equivalent:*

1) *$F^m = (M, M', F)$ has zero flag curvature;*
2) *J is integrable;*
3) *$\nabla J = 0$, where ∇ is the Levi-Civita connection of the Sasaki metric G;*
4) *(M', J, G) is Kählerian.*

Corollary 1. *Let the associate almost complex structure to J (or J') be a complex structure; then we have*

$$\frac{\delta N_j^k}{\delta x^i} = \frac{\delta N_i^k}{\delta x^j}, \qquad \frac{\partial N_j^k}{\partial y^i} = \frac{\partial N_i^k}{\partial y^j}.$$

So in this case the horizontal distribution is integrable.

3 (HM', S, T)-Cartan connection by using the associate almost complex structure J

In this section we give another way to define (HM', S, T)-Cartan connection by using the associate almost complex structure J on M'. Then we study the Kählerity of (M', J, G), where G is the Sasaki metric and $F^m = (M, M', F)$ is a Finsler manifold.

Let $F^m = (M, M', F^*)$ be a pseudo-Finsler manifold. Then a Finsler connection on F^m is a pair $FC = (HM', \nabla)$ where HM' is a nonlinear connection on M' and ∇ is a linear connection on the vertical vector bundle VM' (see [1]).

Theorem 1. *Let ∇ be a FC on M'. The differential operator $\mathcal{D}$ defined by*

$$\mathcal{D}_X Y = \nabla_X vY - J\nabla_X JhY \qquad \forall\, X, Y \in \Gamma(TM')$$

is a linear connection on M'. Also J is parallel with respect to $\mathcal{D}$, that is

$$(\mathcal{D}_X J)Y = 0 \qquad \forall\, X, Y \in \Gamma(TM').$$

We call $\mathcal{D}$ the natural almost complex linear connection associated to FC ∇ on M'.

Proof. For any $X, Y, Z \in \Gamma(TM')$ and $f \in \mathcal{C}^\infty(M')$ we have

$$\begin{aligned}
\mathcal{D}_{fX+Y} Z &= f\nabla_X vZ + \nabla_Y vZ - J(f\nabla_X JhZ + \nabla_Y JhZ) \\
&= f(\nabla_X vZ - J\nabla_X JhZ) + \nabla_Y vZ - J\nabla_Y JhZ = f\mathcal{D}_X Z + \mathcal{D}_Y Z, \\
\mathcal{D}_X(fY + Z) &= Xf(vY + hY) + f(\nabla_X vY - J\nabla_X JhY) + \nabla_X vZ - J\nabla_X JhZ \\
&= (Xf)Y + f\mathcal{D}_X Y + \mathcal{D}_X Z.
\end{aligned}$$

Therefore $\mathcal{D}$ is a linear connection on M'.

Also we have

$$\begin{aligned}
(\mathcal{D}_X J)(Z) &= \mathcal{D}_X(J(Z)) - J(\mathcal{D}_X Z) \\
&= \nabla_X vJ(Z) - J\nabla_X J(h(J(Z))) - J\nabla_X vZ - \nabla_X JhZ \\
&= \nabla_X\left(-Z^i\frac{\partial}{\partial y^i}\right) - J\nabla_X\left(-\tilde{Z}^i\frac{\partial}{\partial y^i}\right) - J\nabla_X\left(\tilde{Z}^i\frac{\partial}{\partial y^i}\right) - \nabla_X\left(-Z^i\frac{\partial}{\partial y^i}\right) = 0,
\end{aligned}$$

where in local coordinates $Z = Z^i\frac{\delta}{\delta x^i} + \tilde{Z}^i\frac{\partial}{\partial y^i}$. ■

Note that the torsion of $\mathcal{D}$ is given by the following expression:

$$T^{\mathcal{D}}(X, Y) = (\nabla_X vY - \nabla_Y vX - v[X, Y]) - J(\nabla_X JhY - \nabla_Y JhX - Jh[X, Y]). \qquad (1)$$

Theorem 2. *Let HM' be a nonlinear connection on M' and S and T be any two skew-symmetric Finsler tensor fields of type $(1, 2)$ on F^m. Then there exists a unique linear connection ∇ on VM' satisfying the conditions:*

(i) ∇ *is a metric connection;*

(ii) $T^{\mathcal{D}}$, S *and* T *satisfy*

$$\text{(a)}\quad T^{\mathcal{D}}(vX, vY) = S(vX, vY), \qquad \text{(b)}\quad hT^{\mathcal{D}}(hX, hY) = JT(JhX, JhY)$$

for any $X, Y \in \Gamma(TM')$, where J is the associate almost complex structure to HM'.

Proof. This proof is similar to [1]. We define a linear connection ∇ on VM' by using g, h, v, J, S and T in the following way. For any $X, Y, Z \in \Gamma(TM')$ let

$$\begin{aligned} 2g(\nabla_{vX}vY, vZ) &= vX(g(vY, vZ)) + vY(g(vZ, vX)) - vZ(g(vX, vY)) \\ &+ g(vY, [vZ, vX]) + g(vZ, [vX, vY]) - g(vX, [vY, vZ]) + g(vY, S(vZ, vX)) \\ &+ g(vZ, S(vX, vY)) - g(vX, S(vY, vZ)) \end{aligned} \tag{2}$$

and

$$\begin{aligned} 2g(\nabla_{hX}JhY, JhZ) &= hX(g(JhY, JhZ)) + hY(g(JhZ, JhX)) \\ &- hZ(g(JhX, JhY)) + g(JhY, Jh[hZ, hX]) + g(JhZ, Jh[hX, hY]) \\ &- g(JhX, Jh[hY, hZ]) + g(JhY, T(JhZ, JhX)) \\ &+ g(JhZ, T(JhX, JhY)) - g(JhX, T(JhY, JhZ)). \end{aligned} \tag{3}$$

Then for any $X, Y, Z \in \Gamma(TM')$ we have

$$\begin{aligned} (\nabla_X g)(vY, vZ) &= (\nabla_{vX+hX} g)(vY, vZ) \\ &= vX(g(vY, vZ)) - g(\nabla_{vX}vY, vZ) - g(vY, \nabla_{vX}vZ) + hX(g(vY, vZ)) \\ &- g(\nabla_{hX}vY, vZ) - g(vY, \nabla_{hX}vZ) = 0. \end{aligned}$$

The above computation shows that the connection ∇ defined by (2) and (3) is a metric connection.

Locally we set $\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} = F^k_{ij}(x,y)\frac{\partial}{\partial y^k}$, $\nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = C^k_{ij}(x,y)\frac{\partial}{\partial y^k}$, $S(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}) = S^k_{ij}\frac{\partial}{\partial y^k}$ and $T(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}) = T^k_{ij}\frac{\partial}{\partial y^k}$.

Now in (2) let $X = \frac{\partial}{\partial y^j}$, $Y = \frac{\partial}{\partial y^i}$ and $Z = \frac{\partial}{\partial y^l}$. After performing some computations we obtain the following expression for the coefficients C^m_{ij}:

$$C^m_{ij} = \frac{1}{2}\left\{\frac{\partial g_{il}}{\partial y^j} + \frac{\partial g_{lj}}{\partial y^i} - \frac{\partial g_{ji}}{\partial y^l} + S^h_{jl}g_{ih} + S^h_{ij}g_{lh} - S^h_{li}g_{jh}\right\} g^{lm}.$$

Also in (3) let $X = \frac{\delta}{\delta x^j}$, $Y = \frac{\delta}{\delta x^i}$ and $Z = \frac{\delta}{\delta x^l}$. Then we can obtain the following expression for the coefficients F^m_{ij}:

$$F^m_{ij} = \frac{1}{2}\left\{\frac{\delta g_{il}}{\delta x^j} + \frac{\delta g_{lj}}{\delta x^i} - \frac{\delta g_{ji}}{\delta x^l} - T^h_{jl}g_{ih} - T^h_{ij}g_{lh} + T^h_{li}g_{jh}\right\} g^{lm}.$$

By using the relations $J \circ v = h \circ J$, $v \circ J = J \circ h$ and (1) we have

$$T^{\mathcal{D}}(vX, vY) = \nabla_{vX}vY - \nabla_{vY}vX - [vX, vY], \tag{4}$$
$$hT^{\mathcal{D}}(hX, hY) = J(\nabla_{hY}JhX - \nabla_{hX}JhY + Jh[hX, hY]). \tag{5}$$

Suppose that $X, Y \in \Gamma(TM')$ are two arbitrary vector fields on M'. In local coordinates, let $X = X^i\frac{\delta}{\delta x^i} + \tilde{X}^i\frac{\partial}{\partial y^i}$ and $Y = Y^i\frac{\delta}{\delta x^i} + \tilde{Y}^i\frac{\partial}{\partial y^i}$, after performing some computations we have:

$$T^{\mathcal{D}}\left(\tilde{X}^i\frac{\partial}{\partial y^i}, \tilde{Y}^i\frac{\partial}{\partial y^i}\right) = S\left(\tilde{X}^i\frac{\partial}{\partial y^i}, \tilde{Y}^i\frac{\partial}{\partial y^i}\right), \tag{6}$$
$$hT^{\mathcal{D}}\left(X^i\frac{\delta}{\delta x^i}, Y^i\frac{\delta}{\delta x^i}\right) = JT\left(J\left(X^i\frac{\delta}{\delta x^i}\right), J\left(Y^i\frac{\delta}{\delta x^i}\right)\right). \tag{7}$$

The relations (6) and (7) show that ∇ satisfies (ii) of Theorem 2.

Now let $\tilde{\nabla}$ be another linear connection on VM' which satisfies (i) and (ii). By using the relations (i), (ii), (4) and (5) for $\tilde{\nabla}$ we have the following expressions:

$$\begin{aligned} &vX(g(vY,vZ)) + vY(g(vZ,vX)) - vZ(g(vX,vY)) \\ &\quad = g(\tilde{\nabla}_{vX}vY + \tilde{\nabla}_{vX}vY - T^{\mathcal{D}}(vX,vY) - [vX,vY],vZ) \\ &\quad + g(T^{\mathcal{D}}(vX,vZ) + [vX,vZ],vY) + g(T^{\mathcal{D}}(vY,vZ) + [vY,vZ],vX), \end{aligned} \tag{8}$$

$$\begin{aligned} &hX(g(vJY,vJZ)) + hY(g(vJZ,vJX)) - hZ(g(vJX,vJY)) \\ &\quad = g(\tilde{\nabla}_{hX}JhY + \tilde{\nabla}_{hX}JhY - JT(JhX,JhY) - Jh[hX,hY],JhZ) \\ &\quad + g(JT(JhX,JhZ) + Jh[hX,hZ],JhY) + g(JT(JhY,JhZ) + Jh[hY,hZ],JhX). \end{aligned} \tag{9}$$

The relations (8) and (9) show that $\tilde{\nabla}$ satisfies (2) and (3), respectively. Therefore $\nabla = \tilde{\nabla}$. ■

The Finsler connection $FC = (HM',\nabla)$ where ∇ is given by Theorem 2 is called the (HM',S,T)-Cartan connection (see [1, 2]) which in this case is obtained by the associate almost complex structure to HM'. If, in particular, HM' is just the canonical nonlinear connection GM' of $\mathbb{F}^m$ (for more details about GM' see [1]) and $S = T = 0$, the FC is called the Cartan connection and it is denoted by $FC^* = (GM',\nabla^*)$.

By means of the pseudo-Riemannian metric g on VM' we consider a pseudo-Riemannian metric on the vector bundle TM' similar to the Sasaki one and denote it by G, that is

$$G = g_{ij}(x,y)dx^i dx^j + g_{ij}(x,y)\delta y^i \delta y^j,$$

where $\delta y^i = dy^i + N^i_j(x,y)dx^j$. Denote by ∇' the Levi-Civita connection on M' with respect to G. A. Bejancu and H.R. Farran showed ∇^* is the projection of the Levi-Civita connection ∇' on the vertical vector bundle also they proved the following theorem (see [1]).

Theorem 3. *The associate linear connection $\mathcal{D}^*$ to the Cartan connection $FC^* = (GM',\nabla^*)$ is a metric linear connection with respect to G.*

Now we give the following theorem which shows the natural almost complex linear connections associated to (HM',S,T)-Cartan connections are metric linear connections with respect to G.

Theorem 4. *The natural almost complex linear connection $\mathcal{D}$ associated to a (HM',S,T)-Cartan connection $FC = (HM',\nabla)$ is a metric linear connection with respect to G.*

Proof. For any $X, X_1, X_2 \in \Gamma(TM')$ we have

$$\begin{aligned} \mathcal{D}_X G(X_1,X_2) &= XG(X_1,X_2) - G(\mathcal{D}_X X_1, X_2) - G(X_1, \mathcal{D}_X X_2) \\ &= X(G(X_1,X_2)) - G(\nabla_X vX_1, X_2) + G(J\nabla_X JhX_1, X_2) \\ &\quad - G(X_1, \nabla_X vX_2) + G(X_1, J\nabla_X JhX_2). \end{aligned} \tag{10}$$

By (10) and this fact that S and T are skew-symmetric we have:

$$\begin{aligned} &\mathcal{D}_{\frac{\partial}{\partial y^i}} G\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\right) = \mathcal{D}_{\frac{\delta}{\delta x^i}} G\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\right) = 0, \\ &\mathcal{D}_{\frac{\partial}{\partial y^i}} G\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = \mathcal{D}_{\frac{\partial}{\partial y^i}} G\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = \frac{\partial g_{jk}}{\partial y^i} - C^h_{ji} g_{hk} - C^h_{ki} g_{jh} = 0, \\ &\mathcal{D}_{\frac{\delta}{\delta x^i}} G\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = \mathcal{D}_{\frac{\delta}{\delta x^i}} G\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = \frac{\delta g_{jk}}{\delta x^i} - F^h_{ji} g_{hk} - F^h_{ki} g_{jh} = 0. \end{aligned}$$

Therefore $\mathcal{D}_X G = 0$ for any $X \in \Gamma(TM')$. ■

Let $F^m = (M, M', F)$ be a Finsler manifold. We can easily check that the pair (J, G) defines an almost Hermitian metric on M'. In the following theorem we give a sufficient condition for Finsler tensor fields S and T such that $\mathcal{D}$ be the Levi-Civita connection arising from G.

Theorem 5. *The natural almost complex linear connection $\mathcal{D}$ associated to a (HM', S, T)-Cartan connection $FC = (HM', \nabla)$ is the Levi-Civita connection arising from G if $T^{\mathcal{D}}(X, Y) = 0$ for any $X, Y \in \Gamma(TM')$ or equivalently if*

$$S = T = 0, \qquad C^k_{ij} = R^k_{ij} = 0, \qquad F^k_{ij} = \frac{\partial N^k_j}{\partial y^i},$$

where $R^k_{ij} = \frac{\delta N^k_i}{\delta x^j} - \frac{\delta N^k_j}{\delta x^i}$.

Proof. By Theorem 4, $\mathcal{D}$ is a metric linear connection with respect to G. Therefore if $T^{\mathcal{D}} = 0$ then $\mathcal{D}$ is the Levi-Civita connection. In local coordinates we have

$$T^{\mathcal{D}}\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = S^k_{ij}\frac{\partial}{\partial y^k},$$

$$T^{\mathcal{D}}\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = C^k_{ji}\frac{\delta}{\delta x^k} + \left(\frac{\partial N^k_j}{\partial y^i} - F^k_{ij}\right)\frac{\partial}{\partial y^k},$$

$$T^{\mathcal{D}}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = T^k_{ij}\frac{\delta}{\delta x^k} + \left(\frac{\delta N^k_j}{\delta x^i} - \frac{\delta N^k_i}{\delta x^j}\right)\frac{\partial}{\partial y^k}.$$

Therefore the proof is completed. ■

Corollary 2. *If $T^{\mathcal{D}} = 0$ then (M', J, G) is a Kähler manifold.*

Proof. If $T^{\mathcal{D}} = 0$ then $\mathcal{D}$ is the Levi-Civita connection of G. Also J is parallel with respect to $\mathcal{D}$. Therefore $\mathcal{D}$ (the Levi-Civita connection of G) is almost complex. Consequently by using Theorem 4.3 of [10], (M', J, G) is a Kähler manifold. ■

We know that the almost Hermitian manifold (M', J, G) is an almost Kähler manifold if and only if the fundamental 2-form Φ is closed (Φ is defined by $\Phi(X, Y) = G(X, JY)$ for all $X, Y \in \Gamma(TM')$). Therefore we can give the following theorem.

Theorem 6. *The almost Hermitian manifold (M', J, G) is an almost Kähler manifold if and only if*

$$\frac{\delta g_{ik}}{\delta x^j} + \frac{\partial N^h_k}{\partial y^i}g_{hj} - \left(\frac{\delta g_{ij}}{\delta x^k} + \frac{\partial N^h_j}{\partial y^i}g_{hk}\right) = 0 \tag{11}$$

and

$$R^h_{ij}g_{hk} - R^h_{ik}g_{hj} + R^h_{jk}g_{hi} = 0. \tag{12}$$

Proof. Let $X_0, X_1, X_2 \in \Gamma(TM')$. Then we have

$$\begin{aligned} d\Phi(X_0, X_1, X_2) = {} & X_0G(X_1, JX_2) - X_1G(X_0, JX_2) + X_2G(X_0, JX_1) \\ & - G([X_0, X_1], JX_2) + G([X_0, X_2], JX_1) - G([X_1, X_2], JX_0). \end{aligned}$$

By using the above relation in local coordinates we have:

$$d\Phi\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = d\Phi\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\right) = 0,$$

$$d\Phi\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right)=\frac{\delta g_{ik}}{\delta x^j}+\frac{\partial N^h_k}{\partial y^i}g_{hj}-\left(\frac{\delta g_{ij}}{\delta x^k}+\frac{\partial N^h_j}{\partial y^i}g_{hk}\right),$$

$$d\Phi\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right)=\left(\frac{\delta N^h_i}{\delta x^j}-\frac{\delta N^h_j}{\delta x^i}\right)g_{hk}-\left(\frac{\delta N^h_i}{\delta x^k}-\frac{\delta N^h_k}{\delta x^i}\right)g_{hj}+\left(\frac{\delta N^h_j}{\delta x^k}-\frac{\delta N^h_k}{\delta x^j}\right)g_{hi}.$$

Therefore the fundamental 2-form Φ is closed if and only if the equations (11) and (12) are confirmed. ■

Now, by using Proposition 1 and Corollary 2, we have the following corollary.

Corollary 3. *Let $F^m = (M, M', F)$ be a Finsler manifold. If $T^{\mathcal{D}} = 0$ then,*

1) *$F^m = (M, M', F)$ has zero flag curvature;*
2) *J is integrable;*
3) *$\nabla J = 0$, where ∇ is the Levi-Civita connection of the Sasaki metric G;*
4) *(M', J, G) is Kählerian.*

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