A Dual Mesh Method for a Non-Local Thermistor Problem

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Abstract. We use a dual mesh numerical method to study a non-local parabolic problem arising from the well-known thermistor problem.

Key words: non-local thermistor problem; joule heating; box scheme method

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1 Introduction

In this work we propose a dual mesh numerical scheme for analysis of the following non-local parabolic problem coming from conservation law of electric charges:

$$\frac{\partial u}{\partial t} - \nabla\cdot(k(u)\nabla u) = \lambda \frac{f(u)}{\left(\int_\Omega f(u)\,dx\right)^2} \quad \text{in} \quad \Omega\times]0;T[,$$
$$u = 0 \quad \text{on} \quad \partial\Omega\times]0;T[, \qquad u/_{t=0} = u_0 \quad \text{in} \quad \Omega, \tag{1}$$

where ∇ denotes the gradient with respect to the x-variables. The nonlinear problem (1) is obtained, under some simplificative conditions, by reducing the well-known thermistor problem (cf., e.g., [13, 14, 15]), which consists of the heat equation, with joule heating as a source, and subject to current conservation:

$$u_t = \nabla\cdot(k(u)\nabla u) + \sigma(u)\,|\nabla\varphi|^2, \qquad \nabla\cdot(\sigma(u)\nabla\varphi) = 0, \tag{2}$$

where the domain $\Omega \subset \mathbb{R}^2$ occupied by the thermistor is a bounded convex polygonal; $\varphi = \varphi(x,t)$ and $u = u(x,t)$ are, respectively, the distributions of the electric potential and the temperature in Ω; $\sigma(u)$ and $k(u)$ are, respectively, the temperature-dependant electrical and thermal conductivities; $\sigma(u)\,|\nabla\varphi|^2$ is the joule heating. The literature on problem (2) is vast (see e.g. [2, 6, 7, 8, 9, 10, 11, 16, 17]). With respect to numerical approximation results to problem (2) we are aware of [1, 11, 12, 18]: in [18] a numerical analysis of the non-steady thermistor problem by a finite element method is discussed; in [12] the authors study a spatially and completely discrete finite element model; in [11] a semi-discretization by the backward Euler scheme is given for the special case $k = Id$; in [1] a box approximation scheme is presented and analyzed. A completely discrete scheme based on the backward Euler method with semi-implicit linearization to (2) is presented in [12] for the special case $k(u) = 1$. Existence and uniqueness of solutions to the problem (1) were proved in [10].

Finite volume methods emerged recently and seem to have a significant role on concrete applications, because they have very interesting properties in view of the subjacent physical problems: in particular in conservation of flows. An equation coming from a conservation law has a good chance to be correctly discretized by the finite volume method. We also recall that these schemes have been widely used to approximate solutions of the heat linear equation, semi-linear or parabolic equations. Since we consider data f with lack of regularity when compared to previous work, we need a new way to discretize (1). We present a dual mesh method capable of handling the non local term $\frac{\lambda f(u)}{\left(\int_\Omega f(u)\,dx\right)^2}$ which is a noticeable feature of (1), by generalizing the results of [1]. A box approximation scheme for discretizing (1) with the case k being different from the identity is obtained. Speed of convergence is directly related with regularity of the continuous problem. When one increases regularity of the second term and data, the solution see its regularity increasing in parallel, and precise speed of convergence can be established. In the existing literature (see e.g. [5, 12]) the error estimates for both the finite element or volume element method are usually derived for solutions that are sufficiently smooth. Because the domain is polygonal, special attention has to be paid to regularity of the exact solution. We give sufficient conditions in terms of data and the solution u that yield error estimates (see hypothesis (H1) below).

The text is organized as follows. In Section 2 we set up the notation and the functional spaces used throughout the paper. Section 3 introduces a box scheme model for problem (1), and existence and uniqueness of the solution of the approximating problem (12) is obtained from the fixed point theorem and equivalence of norms in the finite dimensional space S_h^0. Finally, in Section 4, under some regularity assumptions, we prove error estimates.

2 Notation and functional spaces

Let $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm in $L^2(\Omega)$; $H_0^1(\Omega) = \left\{u \in H^1(\Omega), u/\partial\Omega = 0\right\}$; $\|\cdot\|_s$, $\|\cdot\|_{s,p}$ denote the $H^s(\Omega)$ and the $W^{s,p}(\Omega)$ norm respectively; T_h denote a triangulation of Ω; T_v^h be the set of vertices of a quasi-uniform triangulation T_h; and $\left\{S_h^0\right\}_{h>0}$ be the family of approximating subspaces of $H_0^1(\Omega)$ defined by

$$S_h^0 = \left\{v \in H_0^1(\Omega) : v/e \text{ is a linear function for all } e \in T_h\right\}.$$

In the remainder of this paper we denote by c various constants that may depend on the data of the problem, and that are not necessarily the same at each occurrence. We assume that the family of triangulations is such that the following estimates [4] hold for all $v \in S_h^0$:

$$\begin{aligned} &\|v\|_{\beta,q} \le ch^{r-\beta-2\max\{0,1/p-1/q\}}\|v\|_{r,p}, \qquad 0 \le r \le \beta \le 1, \quad 1 \le p,q \le \infty,\\ &\|v\|_{0,\infty} \le c|\ln h|^{\frac{1}{2}}\|v\|_1. \end{aligned} \tag{3}$$

Let $P_h : L^2(\Omega) \to S_h^0$ be the standard L^2-projection. One has [4]:

$$\begin{aligned} &\|v - P_hv\| + h\|v - P_hv\|_1 \le ch^2\|v\|_2,\\ &\|v - P_hv\|_{0,\infty} \le ch\|v\|_2, \qquad \|P_hv\|_{1,\infty} \le c\|v\|_{1,\infty}. \end{aligned} \tag{4}$$

We construct the box scheme B_h (dual mesh) employed in the discretization as follows. From a given triangle $e \in T_h$, we choose a point $q \in \overline{e}$ as the intersection of the perpendicular bisectors of the three edges of e. Then, we connect q by straight-line segments to the edge midpoints of e. To each vertex $p \in T_v^h$, we associate the box $b_p \in B_h$, consisting of the union of subregions which have p as a corner (see Fig. 1). For the piecewise constant interpolation operator I_h, defined by

$$I_h : \mathcal{C}(\Omega) \to L^2(\Omega), \qquad I_hv = v(p), \quad \text{on} \;\; b_p \in B_h, \quad \forall\, p \in T_v^h,$$

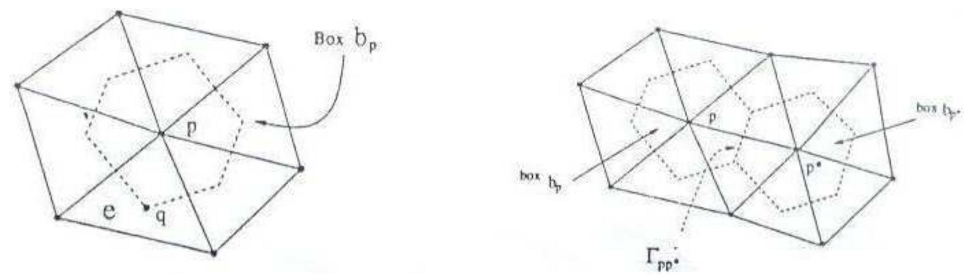

Figure 1. Construction of the dual mesh.

we have the following standard error estimates [1, 3]:

$$\begin{aligned} &c^{-1}\|v\| \le \|I_h v\| \le c\|v\|, \quad \forall\, v \in S_0^h,\\ &\|v - I_h v\| \le ch\|v\|_1, \quad \forall\, v \in S_0^h. \end{aligned} \tag{5}$$

We denote by $N_h(p)$ the set of the neighboring vertices of $p \in T_v^h$, $\partial b = \bigsqcup_{p \in T_v^h} \partial b_p$, $\partial b_p = \bigsqcup_{p* \in N_h(p)} \{\Gamma_{pp*}\}$, where $\Gamma_{pp*} = \partial b_p \bigcap \partial b_{p*}$ (see Fig. 1). Let $l_{\partial b} : \partial b \to \mathbb{R}^+$ be defined as follows: for $p \in T_v^h$ and $b_p \in B_h$,

$$l_{\partial b}/\Gamma_{pp*} = |p - p*| \quad \text{for} \quad p* \in N_h(p).$$

For $b \in B_h$, we denote the jump in w across ∂b at x by $[w]_{\partial b}(x) = w(x+0) - w(x-0)$, where $w(x \pm 0)$ are the outside and inside limit values of $w(x)$ along the normal directions for ∂b.

We now collect from the literature [1, 3] some important lemmas and trace results, that are needed in the sequel.

Lemma 1. *Assume that B_h is a dual mesh. If v is a piecewise linear function, and x is not a vertex, then*

$$[I_h v]/_{\partial b_p}(x) = \frac{\partial v}{\partial n} l_{\partial b}/\Gamma_{pp*}, \quad x \in \Gamma_{pp*}, \quad \forall\, b \in B_h,$$

where n is the unit outward normal vector on ∂b.

The h-dependent norms are defined as follows:

$$\|v\|_{1,h} = \left(\sum_{l \in \partial b} |[I_h v]_l|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|v\|_{0,h} = \|I_h v\|\,.$$

Lemma 2. *There exists a constant $c > 0$ such that*

$$\begin{aligned} &c^{-1}\,\|\nabla v\| \le \|v\|_{1,h} \le c\|\nabla v\|, \quad \forall\, v \in S_h^0,\\ &c^{-1}\|v\| \le \|v\|_{0,h} \le c\|v\|, \quad \forall\, v \in S_h^0. \end{aligned}$$

Lemma 3. *For any $a \in \mathcal{C}(\overline{\Omega})$ there exists a positive constant c such that*

$$\left| -\sum_{b \in B_h} \int_{\partial b} a \frac{\partial u}{\partial n} I_h v \right| \le c\|u\|_1\|v\|_1, \quad \forall\, u, v \in S_h^0. \tag{6}$$

Moreover, if there exists a constant $a_0 > 0$ such that $a \ge a_0$ in Ω, then

$$c^{-1}\|v\|_1^2 \le -\sum_{b \in B_h} \int_{\partial b} a \frac{\partial v}{\partial n} I_h v, \quad \forall\, v \in S_h^0. \tag{7}$$

Let $Q_h : H^2(\Omega) \to S_h^0$ be defined by $Q_h u - i_h u \in S_h^0$, and

$$-\sum_{b\in B_h} \int_{\partial b} a \frac{\partial(u - Q_h u)}{\partial n} I_h v = 0, \quad \forall\, v \in S_h^0, \tag{8}$$

where $i_h : \mathcal{C}(\Omega) \to S_h^0$ is the Lagrangian interpolation operator and $u \in H^2(\Omega)$.

Lemma 4. *Assume that $a \in L^\infty(\Omega)$, with $a \geq a_0$ for some constant $a_0 > 0$. Then, there exists $c > 0$ such that for $u \in H^2(\Omega)$*

$$\|u - Q_h u\|_1 \leq ch\|u\|_2. \tag{9}$$

Moreover, if $u \in H^2(\Omega) \bigcap W^{1,\infty}(\Omega)$, then

$$\|Q_h u\|_{1,\infty} \leq c\left(\|u\|_{1,\infty} + \|u\|_2\right). \tag{10}$$

Lemma 5. *For each $b \in B_h$ one has*

$$h^{\frac{1}{2}} \|v\|_{L^2(\partial b)} \leq c\left(\|v\|_{L^2(b)} + h\|v\|_{H^1(b)}\right), \quad \forall\, v \in H^1(b).$$

Throughout this work, we assume that the following hypotheses on the solution and data of problem (1) are satisfied:

(H1) $u \in L^\infty(H_0^1(\Omega) \bigcap H^2(\Omega))$, $u_t \in L^2(H^1(\Omega))$;

(H2) $c^{-1} \leq k(s) \leq c$;

(H3) there exist positive constants c_1, c_2 and ν, such that $\nu \leq f(\xi) \leq c_1|\xi| + c_2$ for all $\xi \in \mathbb{R}$;

(H4) $|f(\xi) - f(\xi')| + |k(\xi) - k(\xi')| \leq c\,|\xi - \xi'|$.

3 Existence and uniqueness result for the box scheme method

Let u be the solution of (1). Integrating over an element b in B_h we obtain:

$$\int_b u_t - \int_{\partial b} k(u)\frac{\partial u}{\partial n} = \frac{\lambda}{\left(\int f(u)\,dx\right)^2} \int_b f(u), \quad \forall\, b \in B_h. \tag{11}$$

We consider a box scheme defined as follows: find $u_h \in S_h^0$ such that

$$\left(I_h u_t^h, I_h v\right) - \sum_{b\in B_h} \int_{\partial b} k\left(u^h\right) \frac{\partial u^h}{\partial n} I_h v = \frac{\lambda}{\left(\int_\Omega f(u^h)\,dx\right)^2}\left(f\left(u^h\right), I_h v\right), \quad \forall\, v \in S_h^0, \tag{12}$$

where $u^h(0) = P_h u_0$ and I_h is the interpolation operator.

Theorem 1. *Let* (H1)–(H4) *be satisfied. Then, for each $h > 0$, there exists $t_0(h)$ such that* (12) *possesses a unique solution u^h for $0 \leq t \leq t_0(h)$.*

Proof. We begin by proving existence of solution. We define a nonlinear operator G from S_h^0 to S_h^0 as follows. For each $u^h \in S_h^0$, $w^h = G(u^h)$ is obtained as the unique solution of the following problem:

$$\left(I_h w_t^h, I_h v\right) - \sum_{b\in B_h} \int_{\partial b} k\left(u^h\right) \frac{\partial w^h}{\partial n} I_h v = \frac{\lambda}{\left(\int_\Omega f(u^h)\,dx\right)^2}\left(f\left(u^h\right), I_h v\right), \quad \forall\, v \in S_h^0. \tag{13}$$

We remark that G is well defined. Using $v = w^h$ as a test function in (13), hypotheses (H2) and (H3), and Holder's inequality, we can write:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|I_h w^h\|^2 + c\|w^h\|_1^2 &\le c\big(f(u^h), I_h w^h\big) \le c\int \big(|u^h|+1\big)|I_h w^h| \\ &\le c\|u^h\|_{L^2}\|I_h w^h\|_{L^2} + c\|I_h w^h\| \le c\|u^h\|_1\|I_h w^h\|_1 + c\|I_h w^h\| \\ &\le \frac{c}{2}\|I_h w^h\|_1^2 + c\|u^h\|_1^2 + c.\end{aligned}$$

Thus, we have

$$\frac{d}{dt}\|I_h w^h\|^2 + c\|w^h\|_1^2 \le c\|u^h\|_1^2 + c\,. \tag{14}$$

Integrating (14) with respect to t and using the equivalency of $\|I_h \cdot\|$ and $\|\cdot\|$ in S_h^0 (see (5)) yields

$$\begin{aligned}\|w^h\|^2 + c\int_0^t \|w^h\|_1^2 &\le c\|I_h P_h u_0\|^2 + c\int_0^t \|u^h\|_1^2\,dx + ct \\ &\le c\|u_0\|^2 + c\int_0^t \|u^h\|_1^2\,dx + ct.\end{aligned}$$

Define now the following set

$$D = \left\{ u^h \in S_h^0, \|u^h\|^2 + c\int_0^t \|u^h\|_1^2 \le c\left(\|u_0\|^2 + 1\right)\right\}.$$

We can easily see that D is closed subset of $L^\infty(0,t,L^2(\Omega))$ with its natural norm. We conclude that there exists $t > 0$ such that $G(D) \subset D$. To obtain that G has a fixed point $w^h = G(w^h)$, we prove that G is a contraction. Conclusion follows from Banach's fixed point theorem. For this purpose, let u_1^h and $u_2^h \in S_h^0 \times S_h^0$ such that $Gu_1^h = w_1^h$ and $Gu_2^h = w_2^h$. We have, from the equation (13) verified by w_1^h and w_2^h, that

$$\begin{aligned}&\big(I_h(w_{1t}^h - w_{2t}^h), I_h v\big) - \sum_{b\in B_h}\int_{\partial b} k\big(u_1^h\big)\frac{\partial w_1^h}{\partial n} I_h v + \sum_{b\in B_h}\int_{\partial b} k\big(u_2^h\big)\frac{\partial w_2^h}{\partial n} I_h v \\ &\quad = \frac{\lambda}{\left(\int_\Omega f(u_1^h)\,dx\right)^2}\big(f\big(u_1^h\big), I_h v\big) - \frac{\lambda}{\left(\int_\Omega f(u_2^h)\,dx\right)^2}\big(f\big(u_2^h\big), I_h v\big).\end{aligned}$$

On the other hand, one has

$$\begin{aligned}&-\sum_{b\in B_h}\int_{\partial b} k\big(u_1^h\big)\frac{\partial w_1^h}{\partial n} I_h v + \sum_{b\in B_h}\int_{\partial b} k\big(u_2^h\big)\frac{\partial w_2^h}{\partial n} I_h v \\ &\quad = -\sum_{b\in B_h}\int_{\partial b} k(u_1^h)\frac{\partial\big(w_1^h - w_2^h\big)}{\partial n} I_h v + \sum_{b\in B_h}\int_{\partial b}\big(k\big(u_2^h\big) - k\big(u_1^h\big)\big)\frac{\partial w_2^h}{\partial n} I_h v.\end{aligned}$$

Schwartz inequality implies that

$$\sum_{b\in B_h}\int_{\partial b}\big(k\big(u_1^h\big) - k\big(u_2^h\big)\big)\frac{\partial w_2^h}{\partial n} I_h v \ge -c\|w_2^h\|_{1,\infty}\left(\sum_{b\in B_h}\|u_1^h - u_2^h\|_{0,\partial b}\right)\|v\|.$$

By Lemma 5, we have

$$h^{\frac{1}{2}}\sum_{b\in B_h}\|u_1^h - u_2^h\|_{0,\partial b} \le c\sum_{b\in B_h}\left(\|u_1^h - u_2^h\|_{L^2(b)} + h\|u_1^h - u_2^h\|_{H^1(b)}\right) \le c\|u_1^h - u_2^h\|_1.$$

Thus, from the inverse estimate (3),

$$\sum_{b\in B_h}\int_{\partial b}\left(k\left(u_1^h\right)-k\left(u_2^h\right)\right)\frac{\partial w_2^h}{\partial n}I_h v \geq -c(h)\|w_2^h\|_1\|u_1^h-u_2^h\|_1\|v\|_1. \tag{15}$$

On the basis of hypotheses (H1)–(H4), we have:

$$\begin{aligned}
&\frac{\lambda}{\left(\int_\Omega f(u_1^h)\,dx\right)^2}\left(f\left(u_1^h\right),I_h v\right)-\frac{\lambda}{\left(\int_\Omega f\left(u_2^h\right)\,dx\right)^2}\left(f\left(u_2^h\right),I_h v\right)\\
&\quad=\frac{\lambda}{\left(\int_\Omega f\left(u_1^h\right)\,dx\right)^2}\left(f\left(u_1^h\right)-f\left(u_2^h\right),I_h v\right)\\
&\quad+\lambda\left(\frac{1}{\left(\int_\Omega f\left(u_1^h\right)\,dx\right)^2}-\frac{1}{\left(\int_\Omega f\left(u_2^h\right)\,dx\right)^2}\right)\left(f\left(u_2^h\right),I_h v\right)\\
&\quad\leq c\|u_1^h-u_2^h\|\,\|v\|+\lambda\frac{\left(\int_\Omega f\left(u_2^h\right)-f\left(u_1^h\right)\right)\left(\int_\Omega f\left(u_2^h\right)+f\left(u_1^h\right)\right)}{\left(\int_\Omega f\left(u_2^h\right)\,dx\right)^2\left(\int_\Omega f\left(u_1^h\right)\,dx\right)^2}\left(f\left(u_2^h\right),I_h v\right)\\
&\quad\leq c\|u_1^h-u_2^h\|\,\|v\|+c\|I_h v\|_{L^2(\Omega)}\|u_1^h-u_2^h\|_{L^1(\Omega)}\\
&\quad\leq c\|u_1^h-u_2^h\|\,\|v\|\leq c\|u_1^h-u_2^h\|_1\,\|v\|_1.
\end{aligned}\tag{16}$$

It follows from (15) and (16) that

$$\begin{aligned}
&\left(I_h\left(w_{1t}^h-w_{2t}^h\right),I_h v\right)-\sum_{b\in B_h}\int_{\partial b}k(u_1^h)\frac{\partial\left(w_1^h-w_2^h\right)}{\partial n}I_h v\\
&\leq c(h)\|u_1^h-u_2^h\|_1\|v\|_1-\sum_{b\in B_h}\int_{\partial b}\left(k\left(u_1^h\right)-k\left(u_2^h\right)\right)\frac{\partial w_2^h}{\partial n}I_h v\leq c(h)\|u_1^h-u_2^h\|_1\|v\|_1.
\end{aligned}\tag{17}$$

Now, using $v=w_1^h-w_2^h$ as a test function in (17), we obtain from (7):

$$\frac{1}{2}\frac{d}{dt}\left\|I_h\left(w_1^h-w_2^h\right)\right\|^2+c\|w_1^h-w_2^h\|_1^2\leq c(h)\|u_1^h-u_2^h\|_1\|w_1^h-w_2^h\|_1. \tag{18}$$

With use of the Holder's inequality and equivalency of $\|I_h\cdot\|$ and $\|\cdot\|$, integration of (18) with respect to time gives:

$$\left\|\left(w_1^h-w_2^h\right)\right\|^2\leq c\left\|I_h\left(w_1^h-w_2^h\right)\right\|^2\leq c(h)\int_0^t\left\|\left(u_1^h-u_2^h\right)\right\|_1^2 ds.$$

Thus G is a contraction. We prove now uniqueness. Following the same arguments as before, we have

$$\left(I_h\left(u_{1t}^h-u_{2t}^h\right),I_h v\right)-\sum_{b\in B_h}\int_{\partial b}k(u_1^h)\frac{\partial(u_1^h-u_2^h)}{\partial n}I_h v\leq c(h)\|u_1^h-u_2^h\|_1\|v\|. \tag{19}$$

Choosing $v=u_1^h-u_2^h$ as test function in (19), using again (7) and integrating, we obtain

$$\left\|\left(u_1^h-u_2^h\right)\right\|^2\leq c(h)\int_0^t\left\|\left(u_1^h-u_2^h\right)\right\|^2 ds,$$

which gives, by Gronwall's Lemma, uniqueness of solution. ■

4 Error analysis

In this section we prove error estimates under certain assumptions on regularity of the exact solution u.

Theorem 2. *Under assumptions* (H1)–(H4), *if* (u, u^h) *are solutions of* (11)–(12) *for* $0 \le t \le t_0(h)$, *then*

$$\|u^h - u\|_{L^\infty(L^2)} + \|u^h - u\|_{L^2(H^1)} \le ch.$$

Proof. From (11) and (12) we obtain

$$\begin{aligned}
&\big(I_h(u^h - P_h u)_t, I_h v\big) - \sum_{b\in B_h}\int_{\partial b} k\big(u^h\big)\frac{\partial\left(u_1^h - P_h u\right)}{\partial n} I_h v \\
&= \frac{\lambda}{\left(\int_\Omega f\left(u^h\right)\,dx\right)^2}\big(f\big(u^h\big), I_h v\big) - \frac{\lambda}{\left(\int_\Omega f(u)\,dx\right)^2}\big(f(u), I_h v\big) + \sum_{b\in B_h}\int_{\partial b} k\big(u^h\big)\frac{\partial\left(P_h u - u\right)}{\partial n} I_h v \\
&- \sum_{b\in B_h}\int_{\partial b}\big(k(u) - k\big(u^h\big)\big)\frac{\partial u}{\partial n} I_h v + \big((I - P_h)u_t, I_h v\big) + \big((I - I_h)P_h u_t, I_h v\big).
\end{aligned} \tag{20}$$

We now estimate, separately, the terms on the right-hand side of (20). We have from (6) and (4) that

$$\left|\sum_{b\in B_h}\int_{\partial b} k\big(u^h\big)\frac{\partial(P_h u - u)}{\partial n} I_h v\right| \le c\|P_h u - u\|_1\|v\|_1 \le ch\|u\|_2\|v\|_1 \le ch\|v\|_1, \tag{21}$$

$$\begin{aligned}
&\left|\sum_{b\in B_h}\int_{\partial b}\big(k(u) - k\big(u^h\big)\big)\frac{\partial u}{\partial n} I_h v\right| \\
&\qquad \le c\|v\|_1\left(\sum_{b\in B_h}\left(\int_{\partial b}|u - u^h|\left|\frac{\partial u}{\partial n}\right|\right)^2\right)^{\frac12} \le ch^{\frac12}\|v\|_1\left(\sum_{b\in B_h}\|u - u^h\|_{0,\partial b}^2\right)^{\frac12}.
\end{aligned} \tag{22}$$

By Lemma 5, inverse inequality (3) and (4), we have:

$$\begin{aligned}
\sum_{b\in B_h}\|u^h - u\|_{0,\partial b}^2 &\le 2\sum_{b\in B_h}\|u^h - P_h u\|_{0,\partial b}^2 + 2\sum_{b\in B_h}\|P_h u - u\|_{0,\partial b}^2 \\
&\le c\big(h^2 + \|u^h - P_h u\|^2\big).
\end{aligned} \tag{23}$$

Consequently, we obtain from (23) that

$$\left|\sum_{b\in B_h}\int_{\partial b}\big(k(u) - k\big(u^h\big)\big)\frac{\partial u}{\partial n} I_h v\right| \le c\big(h + \|u^h - P_h u\|\big)\|v\|_1. \tag{24}$$

Based on our earlier development in (16), we also know:

$$\frac{\lambda}{\left(\int_\Omega f(u^h)\,dx\right)^2}\big(f\big(u^h\big), I_h v\big) - \frac{\lambda}{\left(\int_\Omega f(u)\,dx\right)^2}\big(f(u), I_h v\big) \le c\|u^h - u\|\|v\|_1. \tag{25}$$

Let $v = u^h - P_h u$ be a test function in (20). Using Lemma 3, it follows from (21)–(25) that

$$\frac12\frac{d}{dt}\|I_h(u^h - P_h u)\|^2 + c\|u^h - P_h u\|_1^2$$

$$\leq c\|u^h-u\|\|u^h-P_hu\|_1+ch\|u^h-P_hu\|_1+c\big(h+\|u^h-P_hu\|\big)\|u^h-P_hu\|_1$$
$$+c(\|(I-P_h)u_t\|+\|(I-I_h)P_hu_t\|)\|u^h-P_hu\|$$
$$\leq c(\|(I-P_h)u_t\|+\|(I-I_h)P_hu_t\|)\|u^h-P_hu\|$$
$$+c\big(h+\|u^h-P_hu\|\big)\|u^h-P_hu\|_1+\|P_hu-u\|\|u^h-P_hu\|_1.$$

By properties (4) and Cauchy's inequality, it follows:

$$\frac{d}{dt}\big\|I_h\big(u^h-P_hu\big)\big\|^2+c\|u^h-P_hu\|_1^2\leq c\|P_hu-u\|_1\|u^h-P_hu\|_1$$
$$+c\big\{h+\|(I-P_h)u_t\|+\|(I-I_h)P_hu_t\|+\|u^h-P_hu\|\big\}\|u^h-P_hu\|_1$$
$$\leq c\big\{h^2+\|(I-P_h)u_t\|_1^2+\|(I-I_h)P_hu_t\|^2+\|u^h-P_hu\|^2\big\}+\frac{c}{2}\|u^h-P_hu\|_1^2$$
$$\leq c\big\{h^2+h^2\|u_t\|_2^2+ch^2\|P_hu_t\|_1^2\big\}+c\|u^h-P_hu\|^2+\frac{c}{2}\|u^h-P_hu\|_1^2.$$

Hence,

$$\frac{d}{dt}\big\|I_h\big(u^h-P_hu\big)\big\|^2+c\big\|u^h-P_hu\big\|_1^2\leq ch^2+c\|u^h-P_hu\|^2. \tag{26}$$

Integrating (26) and applying Gronwall Lemma and using again the equivalency of $\|\cdot\|$ and $\|I_h\cdot\|$, we get that

$$\|u^h-P_hu\|^2+c\int_0^t\|u^h-P_hu\|_1^2\leq ch^2.$$

Then, by the triangular inequality, we conclude with the intended result. ■

Under more restrictive hypotheses on the data, it is possible to derive the following error estimate.

Theorem 3. *Assume* (H1)–(H4). *If* $k(s)=1$ *and* $u_0\in H_0^1(\Omega)\bigcap H^2(\Omega)$, *then*

$$\|u^h-u\|_{L^\infty(H^1)}\leq ch.$$

Proof. From equations (11) and (12), we have:

$$\big(I_hu_t^h-u_t,I_hv\big)-\sum_{b\in B_h}\int_{\partial b}\frac{\partial u^h}{\partial n}I_hv+\sum_{b\in B_h}\int_{\partial b}\frac{\partial u}{\partial n}I_hv$$
$$=\frac{\lambda}{\big(\int_\Omega f(u^h)\,dx\big)^2}\big(f\big(u^h\big),I_hv\big)-\frac{\lambda}{\big(\int_\Omega f(u)\,dx\big)^2}(f(u),I_hv).$$

Using the definition (8) of Q_h, we get

$$\big(I_hu_t^h-u_t,I_hv\big)-\sum_{b\in B_h}\int_{\partial b}\frac{\partial\left(u^h-Q_hu\right)}{\partial n}I_hv$$
$$=\frac{\lambda}{\big(\int_\Omega f(u^h)\,dx\big)^2}\big(f\big(u^h\big),I_hv\big)-\frac{\lambda}{\big(\int_\Omega f(u)\,dx\big)^2}(f(u),I_hv),$$

and it follows that

$$\big(I_h\big(u_t^h-Q_hu\big)_t,I_hv\big)-\sum_{b\in B_h}\int_{\partial b}\frac{\partial(u^h-Q_hu)}{\partial n}I_hv$$

$$= \frac{\lambda}{\left(\int_\Omega f(u^h)\,dx\right)^2}\big(f(u^h), I_h v\big) - \frac{\lambda}{\left(\int_\Omega f(u)\,dx\right)^2}(f(u), I_h v) + ((I - Q_h)u_t, I_h v) + ((I - I_h)Q_h u_t, I_h v). \tag{27}$$

In order to estimate the right hand side of the last inequality, we treat both terms separately. By similar arguments to those used in (16),

$$\left| \frac{\lambda}{\left(\int_\Omega f(u^h)\,dx\right)^2}\big(f(u^h), I_h v\big) - \frac{\lambda}{\left(\int_\Omega f(u)\,dx\right)^2}(f(u), I_h v) \right| \leq c\|u^h - u\|\|v\|.$$

Taking a function test $v = (u^h - Q_h u)_t$ in (27), by (9) and (10) we have

$$\begin{aligned}
&\|I_h(u^h - Q_h u)_t\|^2 - \sum_{b \in B_h} \int_{\partial b} \frac{\partial (u^h - Q_h u)}{\partial n} I_h\big(u^h - Q_h u\big)_t \\
&\quad \leq c\big\{h + \|(I - I_h)Q_h u_t\| + \|(I - Q_h)u_t\| + \|Q_h u - u^h\|\big\}\|I_h(u^h - Q_h u)_t\| \\
&\quad \leq c\big\{h + ch\|u_t\|_2 + ch\|Q_h u_t\|_1 + \|Q_h u - u^h\|\big\}\|I_h(u^h - Q_h u)_t\| \\
&\quad \leq c\big\{h + \|Q_h u - u^h\|\big\}\|I_h(u^h - Q_h u)_t\| \\
&\quad \leq ch^2 + c\|Q_h u - u^h\|^2 + \frac{1}{2}\|I_h(u^h - Q_h u)_t\|^2.
\end{aligned} \tag{28}$$

Integrating (28), we arrive to

$$\begin{aligned}
\|u^h - Q_h u\|_1^2 &\leq c\left(h^2 + \int_0^t \|u^h - Q_h u\|^2\right) \\
&\leq c\left(h^2 + \int_0^t \|u^h - Q_h u\|_1^2\right) = ch^2 + c\|u^h - Q_h u\|^2_{L^2(H^1(\Omega))},
\end{aligned}$$

and Theorem 2 gives

$$\|u^h - Q_h u\|_1^2 \leq ch^2.$$

On the other hand, by triangular inequality, (9) and the regularity of the exact solution u, we have

$$\|u^h - u\|_1^2 \leq 2\|u^h - Q_h u\|_1^2 + 2\|Q_h u - u\|_1^2 \leq ch^2\|u\|_2^2 + ch^2 \leq ch^2.$$

We conclude then with the desired error estimate. ■

5 Conclusion

In this paper a dual mesh numerical scheme was proposed for a nonlocal thermistor problem. We have showed the existence and uniqueness of the approximate solution via Banach's fixed point theorem. We have also proved H^1-error bounds under minimal regularity assumptions. We only obtain first-order estimates: higher order estimates are difficult to obtain due to the nonstandard nonlocal term. Optimal error analysis to the present context, under appropriate smoothness assumptions on data, can be derived by application of the techniques of [5], but this needs further developments.

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